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LOCALLY GRADED GROUPS WITH ALL SUBGROUPS NORMAL-BY-FINITE

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Abstract

In a paper published in this journal [1], J. T. Buckley, J. C. Lennox, B. H. Neumann and the authors considered the class of CF-groups, that is, groups G such that $|H : \mathrm{Core}_G(H)|$ is finite for all subgroups H. It is shown that locally finite CF-groups are abelian-by-finite and BCF, that is, there is an integer n such that $|H : \mathrm{Core}_G(H)| \leq n$ for all subgroups H. The present paper studies these properties in the class of locally graded groups, the main result being that locally graded BCF-groups are abelian-by-finite. Whether locally graded CF-groups are BCF remains an open question. In this direction, the following problem is posed. Does there exist a finitely generated infinite periodic residually finite group in which all subgroups are finite or of finite index? Such groups are locally graded and CF but not BCF.

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1. Introduction

A group G is said to be a CF-group if every subgroup of G has finite index over its core, that is, H/H_G is finite for all subgroups H. If there is an integer k such that $|H/H_G| \leq k$ for all H, then G is said to be a BCF-group. It was proved in [1] that every locally finite CF-group is abelian-by-finite and BCF; this is a partial dual to a result of B. H. Neumann [3], which states that a group in which every subgroup has finite index in its normal closure is finite-by-abelian. However, the duality is imperfect. It was pointed out in [1] that the Tarski monsters of prime exponent p are BCF and a long way from being abelian-by-finite. The aim of this article is to obtain a positive result along these lines, by imposing a relatively weak further hypothesis. A group is *locally graded* if every finitely generated nontrivial subgroup has a nontrivial finite image. We prove the following result.

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THEOREM 1. *Every locally graded* BCF-*group is abelian-by-finite.*

We have been unable to decide whether every locally graded CF-group is abelian-by-finite. Further discussion of this problem, which leads to some questions of independent interest, is postponed until Section 3. Our second result is easy to prove, and provides a hint as to what might go wrong in the general case.

THEOREM 2. *Let G be a* CF-*group such that every periodic image of G is locally finite. Then G is abelian-by-finite.*

The class of groups satisfying the extra requirement of Theorem 2 is, of course, quite large. For instance, we see that every locally radical CF-group is abelian-by-finite. A result from [1] tells us that every finitely generated soluble CF-group is abelian-by-finite and BCF. Finite generation is important here. Let A be the direct product of an infinite cyclic group and a p-quasicyclic group, and G the splitting extension of A by the automorphism of order 2 centralizing the cyclic group and inverting all elements of the quasicycle. It is easy to check the G is CF but not BCF. Now G is metabelian and of rank 3, and is even hypercentral when $p = 2$, and so some obvious conjectures arising from [1] are disposed of. However, for nilpotent groups the situation is quite different.

THEOREM 3. *Every nilpotent* CF-*group is* BCF *and abelian-by-finite.*

2. Proofs

We begin with a couple of lemmas that reduce the proofs of Theorems 1 and 2 to a few lines.

LEMMA 1. *Suppose that G is a locally nilpotent* CF-*group. Then G is abelian-by-finite.*

PROOF. Let N be the subgroup generated by all normal infinite cyclic subgroups of G. By Lemma 4.3 of [1], N is abelian and its centralizer C has index at most 2 in G. Since G is locally nilpotent, this means that $C = G$: no element of infinite order can be conjugate to its inverse. Let A be a torsionfree subgroup of N such that N/A is periodic. Then G/A is locally finite and hence abelian-by-finite, by [1]. Because of this, we may assume that G/A is abelian and thus that G' is torsionfree. Let T be the torsion subgroup of G. Then G/T is locally nilpotent, torsionfree and abelian-by-periodic, and hence abelian. Thus G' is periodic and therefore trivial, so G is abelian, and the proof is complete.

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LEMMA 2. *Let G be a locally graded periodic* BCF-*group. Then G is locally finite.*

PROOF. If the lemma is false, we may assume that G is finitely generated and infinite. Let H be the locally finite radical of G. By [1] and the CF-property, there exists a G-invariant abelian subgroup K of finite index in H. By Lemmas 3 and 7 of [4], G/H is then locally graded, and so we may assume that $H = 1$. Since G is BCF, there is a positive integer k such that every subgroup of G has index at most k over its core. Thus, for every g in G, $\langle g^{k!} \rangle \lhd G$. But g has a finite order so $g^{k!} = 1$, since the locally finite radical has been assumed trivial. Thus G has exponent at most $k!$.

At this point we depart from simplicity and use the fact that the restricted Burnside problem has a positive solution for all exponents; this is a consequence of the Classification Theorem and Zel'manov's celebrated solutions for prime-power exponents [5, 6]. (Added in proof: We have now found a way of obviating reference to these deep results, so that there is a self-contained 'simple' proof.) What it means here is that there is a positive integer n such that every finite image of G has order at most n. But then the finite residual R of G has index at most n in G, and is therefore finitely generated. Since G is locally graded, it follows that $R = 1$ and G is finite. This contradiction completes the proof.

We turn now to the proof of Theorem 1. Let N be as defined in the proof of Lemma 1, and let C be the centralizer of N in G. By Lemma 1 of [4], C/N is locally graded and hence, by Lemma 2 and [1], it is locally finite and abelian-by-finite. Thus G is nilpotent-by-finite, and Lemma 1 applies to give the result.

The proof of Theorem 2 is very similar, and we omit it.

For Theorem 3, let G be a nilpotent CF-group and Z the centre of G. By Lemma 1, there is a normal abelian subgroup A of finite index n, say, in G. Since $[A, {}_cG] = 1$ for some positive integer c, induction on c gives that $[A^{n'}, G] = 1$, where $n' = n^{c-1}$; it follows that G/Z has exponent dividing n^c. We need to show that G is BCF, and we begin by reducing to the case where n is a prime-power. Suppose that $G = G_1G_2$, where G_1 and G_2 are normal subgroups of G such that $A \leq G_1 \cap G_2$, and that there exist integers k_1 and k_2 such that $|H/\operatorname{Core}_{G_i} H| \leq k_i$, $i = 1, 2$, for every subgroup H of A. Let L be an arbitrary subgroup of A, and write $K = \operatorname{Core}_{G_1} L$, so that $|L/K| \leq k_1$. Also, $\operatorname{Core}_G K = \operatorname{Core}_{G_2} K$, so that this core has index at most k_2 in K. It follows that L has index at most k_1k_2 over its G-core. This (together with an obvious induction) allows us to assume that $n = |G/A| = p^k$, for some prime-power p^k.

Let B denote the p'-component of A, suppose that there exists an integer m such that every subgroup of G/B is of index at most m over its core, and let H be any subgroup of A. Then there is a normal subgroup K of G such that $B \leq K \leq HB$ and $|HB/K| \leq m$. Now $K = B(K \cap H)$ and $K \cap H$ has index at most m in H, so

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in order to show that $|H/H_G|$ is bounded, we may assume that HB is normal in G. Take x in $H^G \cap B$, so that $x = h\sigma$ for some $h \in H$, $\sigma \in [H, G] \leq [A, G]$. Then $\sigma^{n^c} = 1$ and $x^{n^c} = h^{n^c} \in H \cap B$; since B is a p'-group we see that $x \in H \cap B$ and hence that $H^G \cap B \leq H$. Thus $H^G = H^G \cap HB = H(H^G \cap B) = H$, so that $H \lhd G$. Factoring by B, we may thus assume that the torsion subgroup T or A is a p-group. The argument splits into two cases depending on the finiteness or otherwise of the exponent of T.

If T has finite exponent, then it is a direct factor of A (see [2]), and we may write $A = T \times U$, where U is torsionfree. Using the CF-property and arguing as in the proof of Lemma 2.1 of [1], we may assume that every subgroup of T is normal in G. Let H be any subgroup of A; then $H = (T \cap H) \times V$ for some torsionfree subgroup V of H. If U has finite rank, then V also has finite rank s, say. By the CF-property, V is then finite over a normal subgroup W, which is central since $[A, G]$ is periodic. Thus $V = \langle W, a_1, \dots, a_s\rangle$ for suitable $a_1, \dots, a_s$; since each $\langle a_i\rangle$ has index at most n^c over its core and $T \cap H \lhd G$, this means that H has bounded index over its core. Next, suppose that T is finite. Then G is centre-by-finite, since U is finite over a central subgroup, and altogether we may assume that T and U are both of infinite rank. We shall prove once more that G is centre-by-finite. If not, it is clear that there exists a countably infinite direct product $\langle a_1\rangle \times \langle a_2\rangle \times \cdots$ of cyclic subgroups of T such that no $\langle a_i\rangle$ is central in G. Let $u_1, u_2, \dots$ be free generators of a free abelian subgroup of $U \cap Z(G)$ of countably infinite rank, and set $h_i = u_i a_i$ for each i. Then $\langle h_i\rangle$ is not normal in G, since $h_i^g = u_i a_i^g \notin \langle u_i a_i\rangle$ since $[a_i, g] \neq 1$. However, $\langle h_i\rangle^G \leq \langle u_i, a_i\rangle$. Writing $S = \langle h_1, h_2, \dots\rangle$, we see that S does not have finite index over its core, a contradiction. Thus G is centre-by-finite and therefore BCF. We may now assume that T is of infinite exponent.

Let B be a basic subgroup of T. If B has finite exponent, then A splits over T, so that $A = T \times U$ for some U, and $T = B \times D$, where D is divisible [2]. Clearly, D is central in G and, as above, B is a finite extension of a central subgroup and G is yet again centre-by-finite. Thus we may assume that B is of infinite exponent. As in the proof of Lemma 2.1 of [1], B contains a subgroup B_1 of finite index all of whose subgroups are normal; arguing as in the proof of Lemma 3.8 of [1], we deduce that B_1 is central in G and hence that $[B, G]$ is finite. For $a \in T$, $y \in G$ we have $a = a_0^{n^c} b$ for some $a_0 \in T$, $b \in B$. Thus $[a, y] = [b, y]$ and $[T, G] = [B, G]$, so $[T, G]$ is finite. Our aim is to prove that $[A, G]$ is finite, but let us assume that this is so at this point. Then G is finite-by-(centre-by-finite), and hence G' is finite, in particular G is FC. But G is in any case abelian-by-finite, so it is centre-by-finite and hence BCF.

Assume for a contradiction that $[A, G]$ is infinite. Since $[T, G]$ is finite, we may assume that $[T, G] = 1$. Note that $[a, G]$ is finite for every a in A, since $C_G(a) \geq A$ and a has only finitely many conjugates and $[A, G]$ is periodic. There exists a_1 in A such that $|[a_1, G]| = n_1 > 1$; setting $I_1 = I_A(\langle a_1\rangle)$ for the isolator

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of $\langle a_1 \rangle$ in A, we have that A/I_1 is torsionfree. Furthermore, I_1T/T is torsionfree of rank 1, so that, since $Z \geq T$ and G/Z is of finite exponent, I_1Z/Z is finite. Thus $[I_1, G]$ is finite since I_1 is finite mod $I_1 \cap Z$ and $[a, G]$ is finite for every a in A. The next step is to choose a_2 in A/I_1 such that $[a_2, G] \not\leq [I_1, G]$, and set $I_2 = I_A(\langle a_1, a_2 \rangle)$. Then $|[\langle a_1, a_2 \rangle, G]| = n_2 > n_1$, A/I_2 is torsionfree and $[I_2, G]$ is finite by arguments like those just used for $[I_1, G]$. Continuing in the obvious way, we find a subgroup $A_0 = \langle a_1, a_2, \dots \rangle$ such that $[A_0, G]$ is infinite and A_0T/T is free. But then $A_0T = T \times V$, for some free abelian subgroup V, which, as before, contains a G-invariant and hence central subgroup of finite index. This gives the contradiction that $[A_0T, G]$ is finite, and the proof of Theorem 3 is complete.

3. Concluding remarks

As we observed in the introduction, we do not know whether every locally graded CF-group is abelian-by-finite. If G is such a group and N is defined as in the proof of Lemma 1, then once again G/N is locally graded. Let H/N be the locally finite radical of G/N; then, by Theorem 2, H is abelian-by-finite. If $H \neq G$, then G/H has no nontrivial normal locally finite subgroups, and hence the CF-property gives that all locally finite subgroups of G/H are finite. As in the proof of Lemma 2, G/H is locally graded.

These considerations result in the following observation. If there exists a locally graded CF-group which is not abelian-by-finite, then there exists a group G of this sort that is finitely generated and periodic, in which every locally finite subgroup is finite. Clearly, the finite residual of G must have infinite index; since factoring by a maximal normal abelian subgroup or a finite normal subgroup preserves residual finiteness, we may suppose that G is residually finite and hence that every locally finite subgroup is finite. It is not clear whether we my factor by a normal subgroup M which is maximal with respect to G/M being infinite, since local gradedness could be lost. However, all of this indicates a special case that needs consideration, and we ask:

QUESTION 1. Let G be a finitely generated, periodic, residually finite group in which every subgroup is either finite or of finite index in G. Is G finite?

As a special case of this, we have:

QUESTION 2. Suppose that G is a p-group satisfying the hypotheses of Question 1. Is G finite?

Note that no infinite group G satisfying the hypotheses of Question 1 can have an infinite abelian subgroup. It is not known, see [7, Problem 11.56], whether every

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finitely generated infinite residually finite group has an infinite abelian subgroup. Finally, one might hope to reduce the general question on locally graded CF-groups to one where the groups involve only finitely many primes, perhaps only one. We offer one last problem here.

QUESTION 3. Let G be a finitely generated, residually finite, periodic CF-group in which every p-subgroup is finite, for all primes p. Is G finite?

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