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A NOTE ON MONOTONE COMPARATIVE STATICS FOR MONETARY DIRECTED SEARCH MODELS

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This note uses monotone methods to derive two sets of comparative statics results for monetary directed search models. First, it characterizes the impact of a higher inflation rate or a higher cost of using credit on market outcomes, regardless of the choice of matching function. Second, the seller-to-buyer ratio, output level, and money demand increase as the matching function becomes more efficient in a log-supermodular sense. I also consider an extension with endogenous search intensity and show that search intensity and trade volume always decrease in the nominal interest rate.

Keywords: Directed Search, Money, Log-Supermodular, Monotone Comparative Statics

1. INTRODUCTION

This note can be considered an extension of the monetary model Lagos and Rocheteau (2005) and Rocheteau and Wright (2005) that is finding many applications these days. It does this by applying monotone methods to a generalization of the monetary model to allow the use of credit at a cost and endogenous search intensity. Directed search theory, as exemplified by Montgomery (1991), Moen (1997), and Burdett et al. (2001), analyzes how buyers search for sellers under perfect information about prices. As discussed in recent surveys by Lagos et al. (2017) and Wright et al. (2017), directed search models are well suited for studying monetary exchange.[1]

Directed search models are easy to apply because they can often be formulated as unconstrained maximization problems—and thereby one can do comparative statics by studying how the set of maximizers varies with parameters. However, the maximization problems induced by directed search theory are often not concave and thus traditional methods are not useful.[2] Monotone methods were first introduced into economics by Vives (1990) and Milgrom and Shannon (1994). It is well known that the use of monotone methods shows how the set of maximizers varies with parameters even if the objective function is not concave.[3] Recently

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Lagos and Rocheteau (2005), Gonzalez and Shi (2010), and Menzio et al. (2013) use advanced monotone methods to analyze dynamic directed search models.

This note has two goals. First, it generalizes a well-known result—previous research shows the amount of monetary exchange falls in the nominal interest rate by making assumptions about the elasticity of the matching function. I show these assumptions are not needed. I also strengthen the standard results by deriving the impact of an increase in the cost of using credit on market outcomes. The second goal is to clarify the role of search frictions in monetary exchanges. When will a more efficient matching function increase market tightness and trade volume? This is a rarely explored question in the directed search literature. Intuitively, when matching opportunities become more abundant, the volume of trade should increase. But buyers also become choosier and this effect might discourage sellers from entering the market and hence reduce market tightness. I show that more sellers enter the market and more trades take place when the matching function increases in a log-supermodular sense. Finally, I study a version of the model where buyers' search intensity is endogenous. As the nominal interest rate rises, buyers search less and the volume of trade drops. This result is in contrast with the findings in Lagos and Rocheteau (2005).

1.1. Monotone Comparative Statics

Since directed search models usually assume differentiability, I use a version of monotone methods in which all functions are differentiable. Generally, neither differentiability nor continuity is necessary to apply monotone comparative statics. Let $X = X_1 \times \cdots \times X_m$ be a set of m choice variables, where $X_i \subset \mathcal{R}$ for $i = 1, \ldots, m$. Let $Y \subset \mathcal{R}$ be a parameter. Let $F(X \times Y) \to \mathcal{R}$ be an objective function. Let $\phi : Y \to X$ be the solution correspondence for the maximization problem, that is, $\phi(y) = \arg\max_{x \in X} F(x, y)$. Theorem 1 shows that when all cross partial derivatives of F are weakly positive and the cross partial derivatives of F with respect to x_i and y are strictly positive for all i, then every selection in $\phi(y)$ is nondecreasing in y. If some of the cross partial derivatives between x_i and y are weakly positive, then there exists a selection in $\phi(y)$ that is nondecreasing in y. The following is proved in Theorem 2.3 in Vives (2001).

THEOREM 1. *Suppose* $\partial^2 F(x, y)/\partial x_i \partial y \geq 0$ *for* $i \in \{1, \ldots, m\}$; *and* $\partial^2 F(x, y)/\partial x_i \partial x_k \geq 0$ *for* $i, k \in \{1, \ldots, m\}$ *such that* $i \neq k$. *Then, for* $y^L, y^H \in Y$ *such that* $y^H > y^L$ *and for* $x^L \in \phi(y^L)$ *and* $x^H \in \phi(y^H)$, *either (i)* $x_i^H \geq x_i^L$ *for all i, or (ii)* x^H *and* x^L *are solutions for both parameters. If* $\partial^2 F(x, y)/\partial x_i \partial y > 0$ *for a given i, then* $x_i^H \geq x_i^L$.

2. MONEY AND COSTLY CREDIT

2.1. Nominal Interest Rate and Entry Cost

Consider the monetary directed search model Rocheteau and Wright (2005). There is a continuum of buyers and sellers and they meet randomly to trade a

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divisible good. The sellers post a pair of (z, q) where z is a real payment in money and q is the quantity of goods. Buyers see (z, q) in each submarket and then decide where to search. Let $u(q)$ be a buyer's utility from consuming q units of goods and $c(q)$ be a seller's cost of production. Assume $u', c', c'' > 0$ and $u'' < 0$. Let i be the nominal interest rate. A buyer must give up a forgone interest iz to carry z units of money. In equilibrium, sellers maximize buyers' expected payoff subject to the participation condition, that is

$$\max_{z,q,n} \alpha(n)[u(q) - z] - iz \quad s.t. \quad \frac{\alpha(n)}{n}[z - c(q)] = k, \tag{1}$$

where n is the seller-to-buyer ratio, α is the matching function, and k is the sellers' expected payoff. The matching probability for a buyer is $\alpha(n)$ and that for a seller is $\alpha(n)/n$ due to a constant return to scale matching technology. Assume $\alpha(n)$ and $\alpha(n)/n$ are, respectively, strictly increasing and strictly decreasing in n. Also $\alpha''(n) \leq 0$. If there is free entry of sellers, then k equals their outside option payoff. If the number of sellers is fixed, then k is endogenous. I assume there is a fixed measure of buyers and free entry of sellers so that k is a seller's entry cost. Define q^* as the efficient level of output so that $u'(q^*) = c'(q^*)$. It is without loss of generality to restrict the choice set of q to $[0, q^*]$, as a seller will never post a (z, q) with $q > q^*$. Assume $k < u(q^*) - c(q^*)$ so that some submarkets are open.

A standard question is the comparative statics of (z, q, n) with respect to i or k. Define

$$F(q, n, i, k) \equiv \alpha(n)[u(q) - c(q)] - nk - i\left[\frac{kn}{\alpha(n)} + c(q)\right]. \tag{2}$$

If z in (1) is eliminated by the constraint, then (1) becomes $\max_{q,n} F(q, n, i, k)$. Let $\phi(i, k) = \arg\max_{q,n} F(q, n, i, k)$.

Intuitively, as the nominal interest rate increases, buyers carry less money and purchase fewer goods per trade. Because buyers carry less money, fewer sellers enter the market. Therefore, z, q, and n fall in i. As the entry cost increases, there are fewer sellers in the market and they produce fewer goods per trade. Therefore, n and q fall in k. The effect of an increase in k on the payment z is ambiguous.[4] To derive these results, previous studies have assumed that the elasticity of the matching function weakly decreases in n in order to apply the implicit function theorem. Proposition 1 shows that this assumption is unnecessary when using monotone methods.

PROPOSITION 1. *For any $i_2 > i_1$ and k, if $(n_1, q_1, z_1) \in \phi(i_1, k)$ and $(n_2, q_2, z_2) \in \phi(i_2, k)$, then $n_2 \leq n_1$, $q_2 \leq q_1$ and $z_2 \leq z_1$. For any $k_2 > k_1$ and i, if $(n_1, q_1, z_1) \in \phi(i, k_1)$ and $(n_2, q_2, z_2) \in \phi(i, k_2)$, then $n_2 \leq n_1$ and $q_2 \leq q_1$.*

Proof. Compute $\partial^2 F/\partial n \partial i = -k\left[1 - n\alpha'(n)/\alpha(n)\right]/\alpha(n) < 0$ and $\partial^2 F/\partial q \partial i = -c'(q) < 0$. Moreover, $\partial^2 F/\partial q \partial n = \alpha'(n)[u'(q) - c'(q)] \geq 0$, as $q \in [0, q^*]$. By Theorem 1, any selection of n and q decreases in i. Since $z = kn/\alpha(n) + c(q)$, z falls in i as well.

Because $\partial^2 F/\partial q\partial n \geq 0$, $\partial^2 F/\partial n\partial k < 0$, and $\partial^2 F/\partial q\partial k = 0$, by Theorem 1, as k increases either (i) $n_2 \leq n_1$ and $q_2 \leq q_1$ or (ii) both (n_1, q_1) and (n_2, q_2) are in $\phi(i, k_1)$ and $\phi(i, k_2)$. If (ii) is true, then $F(n_1, q_1, i, k_1) - F(n_2, q_2, i, k_1) = F(n_1, q_1, i, k_2) - F(n_2, q_2, i, k_2) = 0$; however by (2), this is true only if $n_1 = n_2$. Since $\alpha(n)[u'(q) - c'(q)] = ic'(q)$, $q_1 = q_2$ if $n_1 = n_2$. Therefore, $n_2 \leq n_1$ and $q_2 \leq q_1$ under both (i) and (ii). □

Proposition 1 provides comparative statics that are missing in Rocheteau and Wright (2005) and strengthens the predictions of other monetary directed search models. For example, Rocheteau et al. (2018) use (1) to study open market operations. They derive the effects of monetary policies by restricting the elasticity of the matching function to be constant. Proposition 1 suggests that their conclusion is in fact robust to the choice of matching function. See Dong (2010) for another example.

2.2. Costly Credit

Now let us introduce costly credit into (1). This extension is useful because nowadays most retailers allow buyers to use money or credit for transactions, and hence a comprehensive analysis of monetary policy should consider inflation as well as credit conditions. Again sellers post a pair of (p, q) where now p is a real payment in money or credit.[5] In equilibrium, sellers solve

$$\max_{p,n,q,z} \alpha(n)[u(q) - p - g(p - z)] - iz \quad s.t. \quad \frac{\alpha(n)}{n}[p - c(q)] = k \tag{3}$$

where $g(p - z)$ is the cost of using $p - z$ units of credit. One can interpret $g(p - z)$ as a monitoring cost, a record-keeping cost or a tax on transactions. Assume $g(x) = 0$ for $x \leq 0$, $g'(0) = 0$ and $g''(x) > 0$ for $x > 0$. If the cost of using credit explodes, then (3) reduces to (1). Next, eliminate p from the objective function in (3) by the constraint. Define

$$F(q, n, z, i, k) \equiv \alpha(n)\left\{u(q) - c(q) - g\left[\frac{nk}{\alpha(n)} + c(q) - z\right]\right\} - kn - iz \tag{4}$$

so that (3) is the same as $\max_{n,q,z} F(q, n, z, i, k)$. Let $\phi(i, k) = \arg\max_{n,q,z} F(q, n, z, i, k)$.

As the nominal interest rate i increases, buyers carry less money and use more credit. As the entry cost k increases, there are fewer sellers in the market and buyers use more credit because it is more difficult to meet with a seller and thus relatively more costly to carry money.

PROPOSITION 2. *For any $i_2 > i_1$ and k, if $(n_1, q_1, z_1) \in \phi(i_1, k)$ and $(n_2, q_2, z_2) \in \phi(i_2, k)$, then $n_2 \leq n_1$, $q_2 \leq q_1$, $z_2 \leq z_1$, $p_2 \leq p_1$ and $p_2 - z_2 \geq p_1 - z_1$. For any $k_2 > k_1$ and i, if $(n_1, q_1, z_1) \in \phi(i, k_1)$ and $(n_2, q_2, z_2) \in \phi(i, k_2)$, then $n_2 \leq n_1$, $q_2 \leq q_1$ and $p_2 - z_2 \geq p_1 - z_1$.*

Since the cross partial derivative of F in (4) with respect to q and n is not always positive, one cannot directly apply Theorem 1. Instead, the proof of Proposition 2 denotes (3) as $\max_{n,z}\{\max_q F(q,n,z,i,k)\}$ and applies Theorem 1 only to the outer problem.

Proof of Proposition 2. Changing variable $y \equiv z - kn/\alpha(n)$ in (4), then the problem $\max_{n,q,z} F(q,n,z,i,k)$ becomes

$$\max_{n,q,y} \left\{\alpha(n)\left\{u(q)-c(q)-g[c(q)-y]\right\} - kn - i\left[\frac{kn}{\alpha(n)}+y\right]\right\}.$$

Next, let $J(y) \equiv \max_q\{u(q)-c(q)-g[c(q)-y]\}$ and

$$\tilde{F}(n,y,i,k) \equiv \alpha(n)J(y) - kn - i\left[\frac{kn}{\alpha(n)}+y\right].$$

Then (4) becomes $\max_{n,y}\tilde{F}(n,y,i,k)$. I will use standard methods to characterize $J(y)$ and then use Theorem 1 to solve $\max_{n,y}\tilde{F}(n,y,i,k)$. Since $u''<0$, $c', g'', c''>0$ and $g'\geq 0$, the function $u(q)-c(q)-g[c(q)-y]$ is strictly concave in q. Thus, q solves the first-order condition $u'(q)-c'(q)-g'[c(q)-y]c'(q)=0$. By the implicit function theorem, q rises and $c(q)-y$ falls in y. Since $p-z=c(q)-y$ by the definition of y and the participation condition (3), $p-z$ falls in y. By the envelope theorem, $J'(y)=g'(c(q)-y)\geq 0$ and $J''(y)=-g''[c(q)-y]\left[c'(q)\partial q/\partial y-1\right]>0$. Therefore, the derivatives of $\tilde{F}$ are

$$\frac{\partial\tilde{F}}{\partial n} = \alpha'(n)J(y) - k - \frac{ik}{\alpha(n)}\left[1-\frac{n\alpha'(n)}{\alpha(n)}\right] \quad\text{and}\quad \frac{\partial\tilde{F}}{\partial y} = \alpha(n)J'(y) - i. \tag{5}$$

It follows that the cross partials are

$$\frac{\partial^2\tilde{F}}{\partial n\partial y} = \alpha'(n)J'(y)\geq 0, \quad \frac{\partial^2\tilde{F}}{\partial i\partial n} = -\frac{k}{\alpha(n)}\left[1-\frac{n\alpha'(n)}{\alpha(n)}\right]<0, \text{ and } \frac{\partial^2\tilde{F}}{\partial i\partial y} = -1.$$

The second inequality is true as $\alpha(n)/n$ strictly falls in n. Given the sign of these cross partials, every selection of n and y falls in i by Theorem 1. Since q rises and $p-z$ falls in y, q falls and $p-z$ rises in i. Moreover, $z=kn/\alpha(n)+y$ falls as n and y fall in i. Finally p falls in i as n and q fall in i and $p=c(q)+nk/\alpha(n)$ by the participation condition.

Next, apply Theorem 1 to derive how n and y vary with k. Since

$$\frac{\partial^2\tilde{F}}{\partial k\partial n} = -1-\frac{i}{\alpha(n)}\left[1-\frac{n\alpha'(n)}{\alpha(n)}\right]<0 \quad\text{and}\quad \frac{\partial^2\tilde{F}}{\partial k\partial y}=0,$$

either (i) $n_2\leq n_1$ and $y_2\leq y_1$ or (ii) $\tilde{F}(n_1,y_1,i,k_1)=\tilde{F}(n_2,y_2,i,k_1)$ and $\tilde{F}(n_1,y_1,i,k_2)=\tilde{F}(n_2,y_2,i,k_2)$ by Theorem 1. If (ii) is true, then

$$\tilde{F}(n_1,y_1,i,k)-\tilde{F}(n_2,y_2,i,k) = -k\left[n_1-n_2+i\left(\frac{n_1}{\alpha(n_1)}-\frac{n_2}{\alpha(n_2)}\right)\right]=0 \text{ for } k=k_1,k_2.$$

Since this holds for $k=k_2, k_1$ and $k_2 \neq k_1$, it must be the case that $n_1 = n_2$. Since $\partial \tilde{F}/\partial y = \alpha(n)J'(y) - i = 0$ by the envelope theorem and $J''(y) > 0$, y is unique given n. Thus $n_1 = n_2$ implies $y_1 = y_2$. Altogether, $n_2 \leq n_1$ and $y_2 \leq y_1$ under both (i) and (ii). Since q rises and $p-z$ falls in y, $q_2 \leq q_1$ and $p_2 - z_2 \geq p_1 - z_1$. □

Next, consider the impact of a change in credit conditions. Define a ranking by saying $g_2(\cdot)$ is more costly than $g_1(\cdot)$ when $g_2^{-1}(\xi)$ is weakly flatter than $g_1^{-1}(\xi)$ $\forall \xi > 0$. Proposition 3 shows that buyers naturally use less credit and carry more money as g becomes more costly. But perhaps surprisingly, a higher money holding attracts more sellers to enter and raises the amount of trade q and payment p. See Bethune et al. (2019) for more discussions on this issue and for a quantitative analysis on the impact of a change in credit conditions. Let $\phi(g)$ be the set of maximizers of (4).

LEMMA 1. *If g_2 is more costly than g_1, then $g_2(g_2'^{-1}(a)) < g_1(g_1'^{-1}(a))$ $\forall a \geq 0$.*

PROPOSITION 3. *For any g_2 that is more costly than g_1, if $(n_1, q_1, z_1) \in \phi(g_1)$ and $(n_2, q_2, z_2) \in \phi(g_2)$, then $n_2 \geq n_1$, $q_2 \geq q_1$, $z_2 \geq z_1$, $p_2 \geq p_1$ and $p_2 - z_2 \leq p_1 - z_1$.*

Proof. Let $a \equiv g'(p-z)$ so $p = g'^{-1}(a) + z$. I will exploit the fact that the choice of q, n, and z only depend on a. Since the optimizers satisfy the first-order necessary condition with respect to q, namely $u'(q)/c'(q) = 1 + g'(p-z)$, q can be represented as an implicit function of a solving $u'[q(a)]/c'[q(q)] = 1 + a$. Next, from the first-order necessary condition with respect to z, namely $i = \alpha(n)g'(p-z)$, one can express n as an implicit function $n(a)$ where $i = \alpha[n(a)]a$. Next, by free entry, $z = p - g'^{-1}(a) = c[q(a)] - g'^{-1}(a) + kn(a)/\alpha[n(a)]$. Substitute $a, q(a), n(a)$ and $z = c[q(a)] - g'^{-1}(a) + kn/\alpha(n)$ into (4) to get

$$\max_a \left\{ \frac{i}{a} \left\{ u[q(a)] - c[q(a)] - g[g'^{-1}(a)] \right\} - kn(a) - i\left(c[q(a)] - g'^{-1}(a) + \frac{kn(a)}{\alpha[n(a)]} \right) \right\}.$$

Let $F_j(a)$ be the function in braces when the cost function is g_j for $j = 1, 2$. Differentiate F_j and use the envelope conditions to get

$$\frac{\partial F_2(a)}{\partial a} - \frac{\partial F_1(a)}{\partial a} = \frac{i}{a^2} \left[g_2(g_2'^{-1}(a)) - g_1(g_1'^{-1}(a)) \right] < 0.$$

The last inequality uses Lemma 1. By Theorem 1, a falls strictly in j. It follows that n rises as g becomes more costly by $i = \alpha(n)a$. Also, q rises as a falls because $u'(q)/c'(q) = 1 + a$ where u is concave and c is convex. Moreover, the surplus for sellers $p - c(q)$ rises by free entry, and thus p rises. As the cost of using credit rises, the total expenditure on credit $g(p-z) = g(g'^{-1}(a))$ falls because a falls and $g(g'^{-1}(a))$ falls $\forall a$ by Lemma 1. Then debt $p-z$ falls because $g(\cdot)$ becomes more costly but $g(p-z)$ falls in equilibrium. Since p rises and $p-z$ falls, z rises. □

3. SEARCH FRICTIONS

3.1. Advances in Matching Technology

In the last few decades, the number of matching opportunities increased dramatically due to advances in information technology. For example, it is now much easier for buyers to search for sellers in online platforms or search engines. But since matching opportunities become more abundant, buyers become choosier and this effect could discourage sellers from entering the market. Will a more efficient matching technology lead to more sellers, trades, and output?[6] To answer this, index the matching function by $v \in \mathbb{R}$ so that $\alpha(n, v)$ rises strictly in v for all $n > 0$. Although matching is easier as v rises, it may not be efficient to add more sellers to the market. This is because the contribution of surplus by the marginal seller can rise or fall in v. The next result states that if $\alpha(n, v)$ is log-supermodular in (n, v), then there are more sellers in the market as v rises. Let $\phi(v) = \arg\max_{n,q,z} F(q, n, z, i, k, v)$.

PROPOSITION 4. *Assume $\alpha(n, v)$ strictly rises in v for $n > 0$ and is log-supermodular in (n, v). For any $v_2 > v_1$, if $(n_1, q_1, z_1) \in \phi(v_1)$ and $(n_2, q_2, z_2) \in \phi(v_2)$, then $n_2 \geq n_1$, $q_2 \geq q_1$, $z_2 \geq z_1$ and $p_2 \geq p_1$.*

Proof. By the proof logic in Proposition 1, the function F in (2) satisfies $\partial^2 F/\partial n \partial q \geq 0$. Also by (2), $\partial F/\partial q = \alpha(n, v)[u'(q) - c'(q)] - ic'(q)$ rises strictly in v because $\alpha(n, v)$ rises strictly in v. Moreover,

$$\frac{\partial F}{\partial n} = \alpha_n(n, v)[u(q) - c(q)] - k - \frac{ik}{\alpha(n, v)}\left[1 - \frac{n\alpha_n(n, v)}{\alpha(n, v)}\right].$$

Since $\alpha(n, v)$ is log-supermodular, $\alpha_n(n, v)/\alpha(n, v)$ rises in v and hence the last bracketed term falls in v. Since the elasticity $n\alpha_n(n, v)/\alpha(n, v) \in (0, 1)$, the last bracketed term is positive and hence the last term falls strictly in v. Since $\alpha(n, v)$ rises in v and is log-supermodular, it is supermodular in (n, v) or equivalently $\alpha_n(n, v)$ rises strictly in v. Hence, the first term rises strictly in v. Altogether the right side rises strictly in v. By Theorem 1, n and q rise in v. By the free entry condition, z also rises in v. □

Intuitively, the log-supermodularity of $\alpha(n, v)$ ensures the elasticity $n\alpha'(n, v)/\alpha(n, v)$ rises in v for any given n. Since the matching function becomes more elastic, sellers can create more surplus by entering the market and thus n rises. Since it is easier to match with a seller, the buyers carry more money z and thus the output q rises.[7] It is easy to satisfy the log-supermodularity condition. For example, if $\alpha(n, v) = v\alpha(n)$, then $\alpha(n, v)$ satisfies the premise in Proposition 4. Since the classic directed search model Moen (1997) is a special case of Rocheteau and Wright (2005) (by setting $i = 0$), Proposition 4 extends to Moen's environment.

3.2. Endogenous Search Intensity

Consider the directed search model Lagos and Rocheteau (2005) where buyers can choose their search intensity. By Lemma 3(b) in Lagos and Rocheteau (2005), the market outcome solves

$$\max_{n,e,q,z}\{e\alpha(n)[u(q)-z]-\psi(e)-iz\} \quad s.t. \quad \frac{\alpha(n)}{n}[z-c(q)]=k, \tag{6}$$

where $e>0$ is buyers' search intensity and $\psi(e)$ is the utility cost of choosing e. Assume $\psi', \psi''>0$ for $e>0$ and $\psi(0)=\psi'(0)=0$. Let $\phi(i)$ be the set of maximizers of (6).

Lagos and Rocheteau (2005) assume a fixed measure of agents and show the search intensity e and the total trade volume $e\alpha(n)$ rise in i around $i=0$. They apply this result to explain the *hot potato effect*.[8] The next result shows that with free entry of sellers e and n fall in i provided that α has decreasing elasticity. Hence, the total trade volume falls in i.

PROPOSITION 5. *For any* $i_2>i_1$ *and* k, *if* $(n_1,e_1,q_1,z_1)\in\phi(i_1)$ *and* $(n_2,e_2,q_2,z_2)\in\phi(i_2)$, *then* $q_2\le q_1$, $z_2\le z_1$ *and* $e_2\le e_1$. *If* α *has decreasing elasticity, then* $n_2\le n_1$.

Proof. By the free entry condition, $q=c^{-1}[z-nk/\alpha(n)]$. Define $v(x)\equiv u[c^{-1}(x)]$. Then we can rewrite (6) as $\max_{e,z}\{\max_n F(e,n,z,i,k)\}$ where

$$F(e,n,z,i,k)\equiv e\alpha(n)\left[v\left(z-\frac{nk}{\alpha(n)}\right)-z\right]-\psi(e)-iz.$$

Consider $\max_n F(e,n,z,i,k)$. Note that the choice of n is independent of e and only depends on z. The solution of $n(z)$ solves the first-order necessary condition:

$$\frac{v[z-nk/\alpha(n)]-z}{v'[z-nk/\alpha(n)]}=\frac{k}{\alpha'(n)}\left[1-\frac{n\alpha'(n)}{\alpha(n)}\right]. \tag{7}$$

Now consider the choice of e and z. Using the envelope condition above

$$\frac{\partial F}{\partial z}=e\,\alpha[n(z)]\left[v'\left(z-\frac{n(z)k}{\alpha[n(z)]}\right)-1\right]-i,\quad \frac{\partial^2 F}{\partial e\partial z}=\alpha[n(z)]\left[v'\left(z-\frac{n(z)k}{\alpha[n(z)]}\right)-1\right]\ge 0.$$

It is easy to check that $\partial^2F/\partial i\partial z=-1<0$ and $\partial^2F/\partial i\partial e=0$. Hence by Theorem 1, z falls in i. As z falls, e falls by the first-order condition $\partial F/\partial e=0$. If n falls in i, then q falls in i by the first-order condition $i=e\alpha(n)[v'(q)-1]$. If n rises in i, then q also falls in i by the free entry condition $q=c^{-1}[z-nk/\alpha(n)]$. Altogether q falls in i. Finally, the left side of (7) rises strictly in z and falls strictly in n. The right side rises in n provided that α has decreasing elasticity. Hence $n(z)$ rises in z, and thus n falls in i provided that α has decreasing elasticity. □

4. CONCLUSION

As shown above, one can often convert directed search models into unconstrained maximization problems and then apply monotone methods to study comparative statics. Even when the unconstrained problem violates some necessary conditions for Theorem 1, one can still make progress using a combination of monotone methods and traditional techniques, as demonstrated by the proof of Propositions 2, 3, and 5. If the model has nonlinear constraints, such as those implied by asymmetric information (see Guerrieri et al. (2010) and Li and Rocheteau (2010)), then usually it cannot be represented as an unconstrained problem. In this case one might need more advanced techniques such as those in Quah (2007).

NOTES

1. Some recent developments include Rocheteau and Wright (2005), Faig and Jerez (2006), Huangfu (2007), Dong (2010, 2011), Dutu et al. (2011), Menzio et al. (2013), Bethune et al. (2019), and Rocheteau et al. (2018).

2. As discussed in Rocheteau and Wright (2005), the equilibrium of monetary directed search models is potentially discontinuous in parameter values and sometimes there are multiple equilibria.

3. See the survey by Amir (2005) for details.

4. The payment z can go up or down as k increases. For example, if $i = 0$, it is easy to show that z increases (decreases) in k if the elasticity of the matching function decreases (increases) in n.

5. Since the use of money is subject to the inflation tax, while credit involves transaction costs, in principle both can be used in equilibrium. This approach follows a tradition in reduced-form monetary economics of assuming costly credit. See Bethune et al. (2019) and Wang et al. (2016) for recent developments of this approach.

6. Martellini and Menzio (2018) study a related problem in the labor market and show that the unemployment rate can remain unchanged while matching technology improves, provided that the productivities are Pareto distributed.

7. Despite huge improvements in information technology in the last few decades, there seems to be no obvious increase in money demand according to Lucas and Nicolini (2015). One explanation is that alternative payment technologies such as credit cards and e-money are getting more popular, and hence money demand falls. Another explanation is that the Internet enables buyers to direct their search to low-price sellers and thus buyers carry less money (see Bethune et al. (2019)).

8. Liu et al. (2011) and Nosal (2011) also study the hot potato effect in a related search-theoretic framework.

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APPENDIX A: OMITTED PROOF

Proof of Lemma 1: Since $g_2^{-1}(b)$ is flatter than $g_1^{-1}(b)$ $\forall b > 0$,

$$\frac{\partial g_2^{-1}(b)}{\partial b} < \frac{\partial g_1^{-1}(b)}{\partial b} \Leftrightarrow \frac{1}{g_2'(g_2^{-1}(b))} < \frac{1}{g_1'(g_1^{-1}(b))} \Leftrightarrow g_1'(g_1^{-1}(b)) < g_2'(g_2^{-1}(b)).$$

Since $g'(d)$ and $g^{-1}(b)$ are increasing functions, $g'(g^{-1}(b))$ rises with b. Thus, the last inequality implies $\forall b_1, b_2,\ g_2'(g_2^{-1}(b_2)) = g_1'(g_1^{-1}(b_1)) \Rightarrow b_1 > b_2$. Equivalently $a = g_2'(g_2^{-1}(b_2)) = g_1'(g_1^{-1}(b_1)) \Rightarrow g_2(g_2'^{-1}(a)) = b_2 < b_1 = g_1(g_1'^{-1}(a))$. Therefore, given $a \geq 0$, $g(g'^{-1}(a))$ falls as g^{-1} becomes flatter. □

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