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Existence results of solitons in discrete non-linear Schrödinger equations†

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The discrete non-linear Schrödinger equation is one of the most important inherently discrete models, having a crucial role in the modelling of a great variety of phenomena, ranging from solid-state and condensed-matter physics to biology. In this paper, a class of discrete non-linear Schrödinger equations are considered. Using critical point theory in combination with periodic approximations, we establish some new sufficient conditions on the existence results for solitons of the equation. The classical Ambrosetti–Rabinowitz superlinear condition is improved.

Key words: existence, solitons, discrete non-linear Schrödinger equations, critical point theory

1 Introduction

Below $\mathbf{N}$, $\mathbf{Z}$ and $\mathbf{R}$ denote the sets of all natural numbers, integers and real numbers respectively. l^2 denotes the space of all real functions whose second powers are summable on $\mathbf{Z}$. Also, $*$ denotes the transpose of a vector.

In this paper, we consider the following discrete non-linear Schrödinger (DNLS) equation:

$$i\dot{\psi}_n = -\Delta\psi_n + \varepsilon_n\psi_n - f_n(\psi_n),\ n \in \mathbf{Z}, \tag{1.1}$$

where $\Delta\psi_n = \psi_{n+1} + \psi_{n-1} - 2\psi_n$ is discrete Laplacian operator, ε_n is real valued for each $n \in \mathbf{Z}$, $\varepsilon_{n+T} = \varepsilon_n$, $f_n \in C(\mathbf{R}, \mathbf{R})$, $f_{n+T}(\cdot) = f_n(\cdot)$. Here, T is a positive integer. We assume that $f_n(0) = 0$ and the non-linearity $f_n(u)$ is gauge invariant, that is,

$$f_n\left(e^{i\theta}u\right) = e^{i\theta}f_n(u),\ \theta \in \mathbf{R}. \tag{1.2}$$

Since solitons are spatially localized time-periodic solutions and decay to zero at infinity. Thus, ψ_n has the form

$$\psi_n = u_n e^{-i\omega t},$$

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and

$$\lim_{|n|\to\infty} \psi_n = 0,$$

where ψ_n is real valued for each $n \in \mathbf{Z}$ and $\omega \in \mathbf{R}$ is the temporal frequency. Then, (1.1) becomes

$$-\Delta u_n + \varepsilon_n u_n - \omega u_n = f_n(u_n), \ n \in \mathbf{Z}, \tag{1.3}$$

and

$$\lim_{|n|\to\infty} u_n = 0 \tag{1.4}$$

holds.

The DNLS equation is a non-linear lattice system that appears in many areas of physics such as non-linear optics [7], biomolecular chains [17] and Bose–Einstein condensates [18]. Fundamental states supported by the DNLS equations are discrete solitons. For example, experimental observations of two-dimensional discrete solitons have been reported in [8].

In the past decade, the existence of solitons of the DNLS equations has drawn a great deal of interest [15, 16, 21–24, 31–35]. The existence for the periodic DNLS equations with superlinear non-linearity [21–24] and with saturable non-linearity [34, 35] has been studied. And the existence results of solitons of the DNLS equations without periodicity assumptions were established in [15, 16, 31, 32]. As for the existence of the homoclinic orbits of non-linear Schrödinger equations, we refer to [5, 26–29].

Actually, we consider a more general equation:

$$Lu_n - \omega u_n = f_n(u_n), \ n \in \mathbf{Z}, \tag{1.5}$$

with the same boundary condition (1.4). Here, L is the Jacobi operator (see [30]) given by

$$Lu_n = a_n u_{n+1} + a_{n-1}u_{n-1} + b_n u_n,$$

where a_n and b_n are real valued for each $n \in \mathbf{Z}$, $a_{n+T} = a_n$, $b_{n+T} = b_n$. When $a_n \equiv -1$ and $b_n \equiv 2 + \varepsilon_n$, we obtain (1.3). As usual, we say that a solution $u = \{u_n\}$ of (1.5) is homoclinic (to 0) if (1.4) holds. Naturally, if we look for solitons of (1.1), we just need to get the homoclinic solutions of (1.5).

Let $F_n(u) = \int_0^u f_n(t)dt, t \in \mathbf{R}$ and

$$\underline{\lambda} = \min_{n\in\mathbf{Z}} \left(b_n - |a_{n-1}| - |a_n|\right) > \omega, \ \bar{\lambda} = \max_{n\in\mathbf{Z}} \left(b_n + |a_{n-1}| + |a_n|\right).$$

Our main results are the following theorems.

Theorem 1.1 *Suppose that the following hypotheses are satisfied:*
(L) $b_n - |a_{n-1}| - |a_n| > 0$, *for all* $n \in \boldsymbol{Z}$;
(F_1) *there exist positive constants* ϱ *and* $a < \frac{\underline{\lambda}-\omega}{2}$ *such that*

$$|F_n(u)| \leqslant au^2 \text{ for all } n \in \boldsymbol{Z} \text{ and } |u| \leqslant \varrho;$$

(F_2) *there exist constants* $\rho, c > \frac{\bar{\lambda}-\omega}{2}$ *and* b *such that*

$$F_n(u) \geqslant cu^2 + b \text{ for all } n \in \boldsymbol{Z} \text{ and } |u| \geqslant \rho;$$

(F_3) $f_n(u)u - 2F_n(u) > 0$, *for all* $n \in \boldsymbol{Z}$ *and* $u \in \boldsymbol{R} \setminus \{0\}$;

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(F_4) *$f_n(u)u - 2F_n(u) \to +\infty$ as $|u| \to +\infty$.*
Then, (1.5) has a non-trivial homoclinic solution.

Remark 1.1 *Since $f_{n+T}(u) = f_n(u)$, from (F_1), it is easy to see that*
(f) *$|f_n(u)| \leqslant 2a|u|$, for all $n \in \boldsymbol{Z}$ and $|u| \leqslant \varrho$.*

Remark 1.2 *By (F_2), it is easy to see that there exists a constant $\zeta > 0$ such that*
(F_2') *$F_n(u) \geqslant cu^2 + b - \zeta,\ \forall (n,u) \in \boldsymbol{Z} \times \boldsymbol{R}$.*

As a matter of fact, let $\zeta = \max\left\{\left|F_n(u) - cu^2 - b\right| : n \in \boldsymbol{Z},\ |u| \leqslant \rho\right\}$, we can easily get the desired result.

Remark 1.3 *A crucial role that the classical Ambrosetti–Rabinowitz condition plays is to ensure the boundedness of Palais–Smale sequences. This is very crucial in applying the critical-point theory. In many studies (see e.g. [23,24]), the following classical Ambrosetti–Rabinowitz condition is assumed.*
($\boldsymbol{AR}$) there exists a constant $\beta > 2$ such that

$$0 < \beta F_n(u) \leqslant u f_n(u) \text{ for all } n \in \boldsymbol{Z} \text{ and } u \in \boldsymbol{R} \setminus \{0\}.$$

It is easily checked that $(\boldsymbol{AR})$ satisfies $(F_2) - (F_4)$. Thus, $(F_2) - (F_4)$ improve $(\boldsymbol{AR})$.

Example 1.1 Let

$$f_n(u) = \frac{\gamma u^3}{1 + u^2},$$

and

$$F_n(u) = \frac{1}{2}\gamma \left[u^2 - \ln\left(1 + u^2\right)\right],$$

where $\gamma > \bar{\lambda}$. If (L) is satisfied, then it is easy to verify all the assumptions of Theorem 1.1 are satisfied. Consequently, a non-trivial homoclinic solution is obtained.

Theorem 1.2 *Suppose that (L), $(F_1) - (F_4)$ and the following hypothesis are satisfied:*
(F_5) *$a_{-n} = a_n,\ b_{-n} = b_n,\ f_{-n}(\cdot) = f_n(\cdot)$.*
Then, (1.5) has a non-trivial even homoclinic solution.

The main idea in this paper is an application of Mountain Pass Lemma combined with an approximation technique. This idea has been employed in [20]. We mention that critical-point theory is a powerful tool to deal with the homoclinic solutions of differential equations [9–14] and is used to study homoclinic solutions of discrete systems in recent years [1–4, 6, 19, 20, 33, 34]. We should emphasize that the results are obtained without the classical Ambrosetti–Rabinowitz condition [23, 24].

2 Preliminaries

In this section, we shall present some definitions and lemmas that will be used in the proof of our results.

Let S be the set of sequences $u = (\ldots, u_{-n}, \ldots, u_{-1}, u_0, u_1, \ldots, u_n, \ldots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{\{u_n\} | u_n \in \mathbf{R},\ n \in \mathbf{Z}\}.$$

For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then, S is a vector space.

For any fixed positive integers m and T, E_m is defined as a subspace of S by

$$E_m = \{u \in S | u_{n+2mT} = u_n,\ n \in \mathbf{Z}\}.$$

Clearly, E_m is isomorphic to $\mathbf{R}^{2mT}$. E_m can be equipped with the inner product

$$\langle u, v \rangle = \sum_{j=-mT}^{mT-1} u_j v_j,\ u, v \in E_m, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=-mT}^{mT-1} u_j^2 \right)^{\frac{1}{2}}, u \in E_m, \tag{2.2}$$

respectively. We also define a norm $\|\cdot\|_\infty$ in E_m by

$$\|u\|_\infty = \max_{j \in \mathbf{Z}} |u_j|,\ u \in E_m.$$

Consider the functional J on E_m defined by

$$J(u) = \frac{1}{2} \sum_{n=-mT}^{mT-1} Lu_n \cdot u_n - \frac{\omega}{2} \sum_{n=-mT}^{mT-1} u_n^2 - \sum_{n=-mT}^{mT-1} F_n(u_n). \tag{2.3}$$

Then,

$$\langle J'(u), v \rangle = \sum_{n=-mT}^{mT-1} [Lu_n \cdot v_n - \omega u_n v_n - f_n(u_n) v_n],\ u, v \in E_m. \tag{2.4}$$

Since $\{a_n\}$ and $\{b_n\}$ are T-periodic, it is easy to see that the critical points of J in E_m are exactly $2mT$-periodic solutions of equation (1.5).

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais–Smale condition (P.S. condition for short) if any sequence $\{u_n\} \subset E$ for which $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ $(n \to \infty)$ possesses a convergent subsequence in E.

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Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Mountain Pass Lemma [25]). *Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*
(J_1) there exist constants ρ, $\alpha > 0$ such that $J|_{\partial B_\rho} \geqslant \alpha$, and
(J_2) there exists $e \in E \setminus B_\rho$ such that $J(e) \leqslant 0$.
Then, J possesses a critical value $c \geqslant \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} J(g(s)), \tag{2.5}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0, \ g(1) = e\}. \tag{2.6}$$

Lemma 2.2 *The following inequality is true:*

$$\underline{\lambda}\|u\|^2 \leqslant \sum_{n=-mT}^{mT-1} Lu_n \cdot u_n \leqslant \bar{\lambda}\|u\|^2. \tag{2.7}$$

Proof Let

$$\sum_{n=-mT}^{mT-1} Lu_n \cdot u_n = \langle P_m u, u \rangle, \tag{2.8}$$

where $u = (u_{-mT}, \ldots, u_{-1}, u_0, u_1, \ldots, u_{mT-1})^*$,

$$P_m = \begin{pmatrix} b_{-mT} & a_{-mT} & 0 & \cdots & 0 & a_{-mT-1} \\ a_{-mT} & b_{-mT+1} & a_{-mT+1} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & b_{mT-2} & a_{mT-2} \\ a_{mT-1} & 0 & 0 & \cdots & a_{mT-2} & b_{mT-1} \end{pmatrix}_{2mT \times 2mT}.$$

By (L), P_m is positive definite. Suppose that the eigenvalues of P_m are $\lambda_{-mT}, \lambda_{-mT+1}, \ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_{mT-2}, \lambda_{mT-1}$, then they are all greater than zero. So, by (2.2) and (2.8), we get

$$\underline{\lambda}\|u\|^2 \leqslant \sum_{n=-mT}^{mT-1} Lu_n \cdot u_n \leqslant \bar{\lambda}\|u\|^2.$$

The proof of Lemma 2.2 is complete. □

3 Proofs of theorems

In this section, we shall prove our main results by using the critical-point method.

Lemma 3.1 *Suppose that (L) and $(F_1) - (F_4)$ are satisfied. Then, J satisfies the P.S. condition.*

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Proof Assume that $\{u^{(i)}\}_{i\in\mathbf{N}}$ in E_m is a sequence such that $\{J(u^{(i)})\}_{i\in\mathbf{N}}$ is bounded. Then, there is a positive constant K such that $-K \leqslant J(u^{(i)})$. By (2.7) and (F_2'), we have

$$-K \leqslant J\left(u^{(i)}\right) \leqslant \frac{\bar{\lambda}-\omega}{2}\left\|u^{(i)}\right\|^2 - \sum_{n=-mT}^{mT-1}\left[c\left(u_n^{(i)}\right)^2 + b - \zeta\right]$$

$$\leqslant \left(\frac{\bar{\lambda}-\omega}{2} - c\right)\left\|u^{(i)}\right\|^2 + 2mT(\zeta - b).$$

Therefore,

$$\left(c - \frac{\bar{\lambda}-\omega}{2}\right)\left\|u^{(i)}\right\|^2 \leqslant 2mT(\zeta - b) + K. \tag{3.1}$$

Since $c > \frac{\bar{\lambda}-\omega}{2}$, (3.1) implies that $\{u^{(i)}\}_{i\in\mathbf{N}}$ is bounded in E_m. Thus, $\{u^{(i)}\}_{i\in\mathbf{N}}$ possesses a convergence subsequence in E_m. The desired result follows. □

Lemma 3.2 *Suppose that* (L) *and* (F_1)–(F_4) *are satisfied. Then, for which u, (1.5) possesses a* $2mT$*-periodic solution* $u^{(m)} \in E_m$.

Proof In our case, it is clear that $J(0) = 0$. By Lemma 3.1, J satisfies the P.S. condition. By (F_1), we have

$$J(u) \geqslant \frac{\lambda}{2}\sum_{n=-mT}^{mT-1} u_n^2 - \frac{\omega}{2}\sum_{n=-mT}^{mT-1} u_n^2 - a\sum_{n=-mT}^{mT-1} u_n^2$$

$$= \frac{\lambda - \omega - 2a}{2}\|u\|^2.$$

Taking $\alpha = \frac{\lambda-\omega-2a}{2}\varrho^2 > 0$, we obtain

$$J(u)|_{\partial B_\varrho} \geqslant \alpha > 0,$$

which implies that J satisfies the condition (J_1) of the Mountain Pass Lemma.

Next, we shall verify the condition (J_2).

There exists a sufficiently large number $\varepsilon > \max\{\varrho, \rho\}$ such that

$$\left(c - \frac{\bar{\lambda}-\omega}{2}\right)\varepsilon^2 \geqslant |b|. \tag{3.2}$$

Let $e \in E_m$ and

$$e_n = \begin{cases} \varepsilon, \text{ if } n = 0, \\ 0, \text{ if } n \in \{j \in \mathbf{Z} : -mT \leqslant j \leqslant mT - 1 \text{ and } j \neq 0\}. \end{cases}$$

Then,

$$F_n(e_n) = \begin{cases} F_n(\varepsilon), & \text{if } n = 0, \\ 0, & \text{if } n \in \{j \in \mathbf{Z} : -mT \leqslant j \leqslant mT - 1 \text{ and } j \neq 0\}. \end{cases}$$

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With (3.2), we have

$$J(e) = \frac{1}{2}\sum_{n=-mT}^{mT-1} Le_n \cdot e_n - \frac{\omega}{2}\sum_{n=-mT}^{mT-1} e_n^2 - \sum_{n=-mT}^{mT-1} F_n(e_n)$$

$$\leqslant \frac{\bar{\lambda}}{2}\|e\|^2 - \frac{\omega}{2}\|e\|^2 - c\|e\|^2 - b$$

$$= -\left(c - \frac{\bar{\lambda}-\omega}{2}\right)\varepsilon^2 - b \leqslant 0.$$

All the assumptions of the Mountain Pass Lemma have been verified. Consequently by Lemmas 2.1 and 3.1, J possesses a critical value c_m given by (2.5) and (2.6) with $E = E_m$ and $\Gamma = \Gamma_m$, where $\Gamma_m = \{g_m \in C([0,1], E_m) | g_m(0) = 0,\ g_m(1) = e,\ e \in E_m \backslash B_\varepsilon\}$. Let $u^{(m)}$ denote the corresponding critical point of J in E_m. Note that $\left\|u^{(m)}\right\| \neq 0$ since $c_m > 0$. □

Lemma 3.3 *Suppose that* (L) *and* $(F_1) - (F_4)$ *are satisfied. Then, there exist positive constants* ϱ *and* η *independent of* m *such that*

$$\varrho \leqslant \left\|u^{(m)}\right\|_\infty \leqslant \eta. \tag{3.3}$$

Proof The continuity of $F_n(u)$ with respect to the variable u implies that there exists a constant $\tau > 0$ such that $|F_n(u)| \leqslant \tau$ for $|u| \leqslant \varrho$. It is clear that

$$J\left(u^{(m)}\right) \leqslant \max_{0\leqslant s\leqslant 1}\left\{\frac{1}{2}\sum_{n=-mT}^{mT-1}\left|L(se)_n \cdot (se)_n - \omega(se)_n^2\right| - \sum_{n=-mT}^{mT-1} F_n((se)_n)\right\}$$

$$\leqslant \frac{\bar{\lambda}+\omega}{2}\|e\|^2 + \tau$$

$$= \frac{(\bar{\lambda}+\omega)\varepsilon^2}{2} + \tau.$$

Let $\xi = \frac{(\bar{\lambda}+\omega)\varepsilon^2}{2} + \tau$, we have that $J\left(u^{(m)}\right) \leqslant \xi$, which is independent of m. From (2.5) and (2.6), we have

$$J\left(u^{(m)}\right) = \frac{1}{2}\sum_{n=-mT}^{mT-1} f_n\left(u_n^{(m)}\right)u_n^{(m)} - \sum_{n=-mT}^{mT-1} F_n\left(u_n^{(m)}\right) \leqslant \xi.$$

By (F_3) and (F_4), there exists a constant $\eta > 0$ such that

$$\tfrac{1}{2} f_n(v)v - F_n(v) > \xi, \text{ for all } |v| \geqslant \eta,$$

which implies that $\left|u_n^{(m)}\right| \leqslant \eta$ for all $n \in \mathbf{Z}$, that is, $\left\|u^{(m)}\right\|_\infty \leqslant \eta$.

From the definition of J, we have

$$0 = \left\langle J'\left(u^{(m)}\right), u^{(m)}\right\rangle \geqslant (\underline{\lambda} - \omega)\sum_{n=-mT}^{mT-1}\left|u_n^{(m)}\right|^2 - \sum_{n=-mT}^{mT-1} f_n\left(u_n^{(m)}\right)u_n^{(m)}.$$

Therefore, combined with (F_2), we get

$$(\underline{\lambda}-\omega)\left\|u^{(m)}\right\|^2 \leqslant \sum_{n=-mT}^{mT-1} f_n\left(u_n^{(m)}\right)u_n^{(m)} \leqslant \left\{\sum_{n=-mT}^{mT-1}\left[f_n\left(u_n^{(m)}\right)\right]^2\right\}^{\frac{1}{2}}\left\|u^{(m)}\right\|.$$

That is,

$$(\underline{\lambda}-\omega)\left\|u^{(m)}\right\| \leqslant \left\{\sum_{n=-mT}^{mT-1}\left[f_n\left(u_n^{(m)}\right)\right]^2\right\}^{\frac{1}{2}}.$$

Thus,

$$(\underline{\lambda}-\omega)^2\left\|u^{(m)}\right\|^2 \leqslant \sum_{n=-mT}^{mT-1}\left[f_n\left(u_n^{(m)}\right)\right]^2. \tag{3.4}$$

Combined with (F_1), we get

$$(\underline{\lambda}-\omega)^2\left\|u^{(m)}\right\|^2 \leqslant \sum_{n=-mT}^{mT-1}\left[2a\left|u_n^{(m)}\right|\right]^2 = 4a^2\left\|u^{(m)}\right\|^2.$$

Thus, we have $u^{(m)}=0$. But this contradicts $\left\|u^{(m)}\right\| \neq 0$, which shows that

$$\left\|u^{(m)}\right\|_\infty \geqslant \varrho,$$

and the proof of Lemma 3.3 is finished. □

Proof of Theorem 1.1. Consider the sequence $\left\{u_n^{(m)}\right\}_{n\in\mathbf{Z}}$ of $2mT$-periodic solutions found in Lemma 3.2. First, by (3.3), for any $m \in \mathbf{N}$, there exists a constant $n_m \in \mathbf{Z}$ independent of m such that

$$\left|u_{n_m}^{(m)}\right| \geqslant \varrho. \tag{3.5}$$

Since a_n, b_n are T-periodic in n, $\left\{u_{n+jT}^{(m)}\right\}$ $(\forall j \in \mathbf{N})$ is also $2mT$-periodic solution of (1.3). Hence, making such shifts, we can assume that $0 \leqslant n_m \leqslant T-1$ in (3.5). Moreover, passing to a subsequence of ms, we can even assume that $n_m = n_0$ is independent of m.

Next, we extract a subsequence, still denote by $u^{(m)}$, such that

$$u_n^{(m)} \to u_n,\ m\to\infty,\ \forall n \in \mathbf{Z}.$$

Inequality (3.5) implies that $|u_{n_0}| \geqslant \varrho$ and, hence, $u=\{u_n\}$ is a non-zero sequence. Moreover,

$$\begin{aligned}&Lu_n - \omega u_n - f_n(u_n)\\ &= \lim_{m\to\infty}\left[Lu_n^{(m)} - \omega u_n^{(m)} - f_n\left(u_n^{(m)}\right)\right] = 0.\end{aligned}$$

So $u=\{u_n\}$ is a solution of (1.5).

Finally, we show that $u \in l^2$. For $u_m \in E_m$, let

$$P_m = \left\{n \in \mathbf{Z} : \left|u_n^{(m)}\right| < \varrho, -mT \leqslant n \leqslant mT-1\right\},$$

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$$Q_m = \left\{ n \in \mathbf{Z} : \left| u_n^{(m)} \right| \geqslant \varrho, -mT \leqslant n \leqslant mT - 1 \right\}.$$

Since $f_n(u) \in C(\mathbf{R}, \mathbf{R})$, there exist constants $\bar{\xi} > 0$, $\underline{\xi} > 0$ such that

$$\max \{ |f_n(u)| : \varrho \leqslant |u| \leqslant \eta, \ n \in \mathbf{Z} \} \leqslant \bar{\xi},$$

$$\min \left\{ \frac{1}{2} f_n(u)u - F_n(u) : \varrho \leqslant |u| \leqslant \eta, \ n \in \mathbf{Z} \right\} \geqslant \underline{\xi}.$$

For $n \in Q_m$,

$$\left| f_n \left(u_n^{(m)} \right) \right| \leqslant \frac{\bar{\xi}}{\underline{\xi}} \left[\frac{1}{2} f_n \left(u_n^{(m)} \right) u_n^{(m)} - F_n \left(u_n^{(m)} \right) \right]. \tag{3.6}$$

By (F_1), (3.4) and (3.6), we have

$$\begin{aligned} (\underline{\lambda} - \omega)^2 \left\| u^{(m)} \right\|^2 &\leqslant \sum_{n \in P_m} \left[f_n \left(u_n^{(m)} \right) \right]^2 + \sum_{n \in Q_m} \left[f_n \left(u_n^{(m)} \right) \right]^2 \\ &\leqslant \sum_{n \in P_m} \left[2a \left| u_{n+1}^{(m)} \right| \right]^2 + \sum_{n \in Q_m} \left[\frac{1}{2} f_n \left(u_n^{(m)} \right) u_n^{(m)} - F_n \left(u_n^{(m)} \right) \right] \\ &\leqslant 4a^2 \left\| u^{(m)} \right\|^2 + \frac{\bar{\xi}\xi}{\underline{\xi}}. \end{aligned}$$

Thus,

$$\left\| u^{(m)} \right\|^2 \leqslant \frac{\bar{\xi}\xi}{\underline{\xi} \left[(\underline{\lambda} - \omega)^2 - 4a^2 \right]}.$$

For any fixed $D \in \mathbf{Z}$ and m large enough, we have that

$$\sum_{n=-D}^{D} \left| u_n^{(m)} \right|^2 \leqslant \left\| u^{(m)} \right\|^2 \leqslant \frac{\bar{\xi}\xi}{\underline{\xi} \left[(\underline{\lambda} - \omega)^2 - 4a^2 \right]}.$$

Since $\bar{\xi}$, $\underline{\xi}$, ξ, $\underline{\lambda}$, a and ω are constants independent of m, passing to the limit, we have that

$$\sum_{n=-D}^{D} |u_n|^2 \leqslant \frac{\bar{\xi}\xi}{\underline{\xi} \left[(\underline{\lambda} - \omega)^2 - 4a^2 \right]}.$$

Due to the arbitrariness of D, $u \in l^2$. Therefore, u satisfies $u_n \to 0$ as $|n| \to \infty$. □

Proof of Theorem 1.2. Consider the following boundary problem:

$$\begin{cases} Lu_n - \omega u_n - f_n(u_n) = 0, & -mT \leqslant \mathbf{Z} \leqslant mT, \\ a_{-mT} = a_{mT} = 0, \ b_{-mT} = b_{mT} = 0, & \\ a_{-n} = a_n, \ b_{-n} = b_n, & -mT \leqslant \mathbf{Z} \leqslant mT. \end{cases}$$

Let S be the set of sequences $u = (\ldots, u_{-n}, \ldots, u_{-1}, u_0, u_1, \ldots, u_n, \ldots) = \{u_n\}_{n=-\infty}^{+\infty}$, that is

$$S = \{ \{u_n\} | u_n \in \mathbf{R}, \ n \in \mathbf{Z} \}.$$

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For any $u, v \in S$, $a, b \in \mathbf{R}$, $au + bv$ is defined by

$$au + bv = \{au_n + bv_n\}_{n=-\infty}^{+\infty}.$$

Then, S is a vector space.

For any given positive integers m and T, $\tilde{E}_m$ is defined as a subspace of S by

$$\tilde{E}_m = \{u \in S | u_{-n} = u_n, \ \forall n \in \mathbf{Z}\}.$$

Clearly, $\tilde{E}_m$ is isomorphic to $\mathbf{R}^{2mT+1}$. $\tilde{E}_m$ can be equipped with the inner product

$$\langle u, v \rangle = \sum_{j=-mT}^{mT} u_j v_j, \ \forall u, v \in \tilde{E}_m$$

by which the norm $\| \cdot \|$ can be induced by

$$\|u\| = \left(\sum_{j=-mT}^{mT} u_j^2 \right)^{\frac{1}{2}}, \ \forall u \in \tilde{E}_m.$$

It is obvious that $\tilde{E}_m$ is Hilbert space with $2mT+1$-periodicity and linearly homeomorphic to $\mathbf{R}^{2mT+1}$.

Similarly to the proof of Theorem 1.1, we can also prove Theorem 1.2. For simplicity, we omit its proof. □

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