

## A COUNTEREXAMPLE TO A RESULT OF JABERI AND MAHMOODI

A. SAHAMI  and S. F. SHARIATI 

(Received 25 June 2023; accepted 9 July 2023; first published online 10 August 2023)

### Abstract

We show that  $\ell^1(\mathbb{N}_\wedge)$  is  $\varphi$ -amenable for each multiplicative linear functional  $\varphi : \ell^1(\mathbb{N}_\wedge) \rightarrow \mathbb{C}$ . This is a counterexample to the final corollary of Jaberi and Mahmoodi [‘On  $\varphi$ -amenability of dual Banach algebras’, *Bull. Aust. Math. Soc.* **105** (2022), 303–313] and shows that the final theorem in that paper is not valid.

2020 *Mathematics subject classification*: primary 46H25; secondary 43A07.

*Keywords and phrases*:  $\varphi$ -amenability, Banach algebra.

### 1. Introduction and preliminaries

The cohomological notion of amenability was introduced and studied in the pioneering work of Johnson [5]. A Banach algebra  $\mathcal{A}$  is amenable if every continuous derivation from  $\mathcal{A}$  into a dual Banach  $\mathcal{A}$ -bimodule  $E^*$  is inner. A modification of amenability depending on multiplicative linear functionals was introduced and studied by Kaniuth *et al.* [6]. A Banach algebra  $\mathcal{A}$  is called  $\varphi$ -amenable if there exists an element  $m$  in  $\mathcal{A}^{**}$  such that  $m(\varphi) = 1$  and  $m(f \cdot a) = \varphi(a)m(f)$  for every  $a \in \mathcal{A}$  and  $f \in \mathcal{A}^*$ , where  $\varphi$  is a multiplicative linear functional on  $\mathcal{A}$ . For a locally compact group  $G$ , the Fourier algebra  $\mathcal{A}(G)$  is always  $\varphi$ -amenable. Moreover, the Segal algebra  $S^1(G)$  is  $\varphi$ -amenable if and only if  $G$  is amenable (see [1, 6]).

Jaberi and Mahmoodi [4] introduced the new concept of  $\varphi$ -injectivity for the category of dual Banach algebras, where  $\varphi$  is a  $wk^*$ -continuous multiplicative linear functional on  $\mathcal{A}$ . A dual Banach algebra  $\mathcal{A}$  is  $\varphi$ -injective if whenever  $\pi : \mathcal{A} \rightarrow \mathcal{L}(E)$  is a  $wk^*$ -continuous unital representation on a reflexive Banach space  $E$ , then there is a projection  $Q : \mathcal{L}(E) \rightarrow \pi(\mathcal{A})^\varphi$  such that  $Q(STU) = SQ(T)U$  for  $S, U \in \pi(\mathcal{A})^c$  and  $T \in \mathcal{L}(E)$ , where  $\pi(\mathcal{A})^\varphi = \{T \in \mathcal{L}(E) : \pi(a)T = \varphi(a)T \quad (a \in \mathcal{A})\}$ . They proved that  $\varphi$ -injectivity is equivalent to  $\varphi$ -amenability [4, Theorem 3.6].

There is an important category of dual Banach algebras, called enveloping dual Banach algebras. Let  $\mathcal{A}$  be a Banach algebra and let  $E$  be a Banach  $\mathcal{A}$ -bimodule.



An element  $x \in E$  is called weakly almost periodic if the module maps  $\mathcal{A} \rightarrow E$  given by  $a \mapsto a \cdot x$  and  $a \mapsto x \cdot a$  are weakly compact. The set of all weakly almost periodic elements of  $E$  is denoted by  $WAP(E)$  [7, Definition 4.1]. Runde observed that  $WAP(\mathcal{A}^*)^*$  is a canonical dual Banach algebra associated to an arbitrary Banach algebra  $\mathcal{A}$  [7, Theorem 4.10]. By means of the new notion of  $\varphi$ -injectivity, Jaberi and Mahmoodi investigated  $\varphi$ -amenability of the enveloping dual Banach algebra  $WAP(\mathcal{A}^*)^*$  [4, Theorem 4.8]. In a short final section of the paper, they claimed that  $WAP(\ell^1(\mathbb{N}_\wedge))^*$  is not  $\tilde{\varphi}$ -amenable, where  $\tilde{\varphi}$  is the unique extension of the augmentation character  $\varphi$  on the semigroup algebra  $\ell^1(\mathbb{N}_\wedge)$  [4, Theorem 5.4]. From this result, they concluded that  $\ell^1(\mathbb{N}_\wedge)$  is not  $\varphi$ -amenable, where  $\varphi$  is the augmentation character [4, Corollary 5.5].

On the contrary, we show that  $\ell^1(\mathbb{N}_\wedge)$  is  $\varphi$ -amenable for each multiplicative linear functional  $\varphi : \ell^1(\mathbb{N}_\wedge) \rightarrow \mathbb{C}$  and comment on the reason for this counterexample to the result stated in [4].

### 2. $\varphi$ -amenability of $\ell^1(\mathbb{N}_\wedge)$

Let  $S = \mathbb{N}$ . With the semigroup product  $m \wedge n = \min\{m, n\}$ , for  $m, n \in S$ , the set  $S$  becomes a semigroup. It is known that  $\Delta(\ell^1(S))$  consists of all the functions  $\varphi_n : \ell^1(S) \rightarrow \mathbb{C}$  given by  $\varphi_n(\sum_{i=1}^\infty \alpha_i \delta_i) = \sum_{i=n}^\infty \alpha_i$ , for  $n \in S$  (see [2, page 32]). Suppose that  $m = \delta_1$ . Then  $\varphi_1(m) = \varphi_1(\delta_1) = 1$  and

$$am = a\delta_1 = \left( \sum_{i=1}^\infty a_i \right) \delta_1 = \varphi_1(a)\delta_1 = \varphi_1(a)m, \quad \text{where } a = \sum_{i=1}^\infty a_i \delta_i \in \ell^1(S).$$

It follows that  $\ell^1(S)$  is  $\varphi_1$ -amenable. For  $n > 1$ , define  $m_n = \delta_n - \delta_{n-1}$ . Then,

$$\varphi_n(m_n) = \varphi_n(\delta_n - \delta_{n-1}) = 1 - 0 = 1$$

and

$$am_n = a(\delta_n - \delta_{n-1}) = \sum_{i=n}^\infty a_i(\delta_n - \delta_{n-1}) = \varphi_n(a)(\delta_n - \delta_{n-1}) = \varphi_n(a)m_n,$$

where  $a = \sum_{i=1}^\infty a_i \delta_i \in \ell^1(S)$ . It follows that  $\ell^1(S)$  is  $\varphi$ -amenable with respect to each multiplicative linear functional  $\varphi : \ell^1(S) \rightarrow \mathbb{C}$ . Thus, [4, Corollary 5.5] is not true.

This counterexample to [4, Corollary 5.5] shows that [4, Theorem 5.4] is also not true. The mistake is the assertion in the second sentence of the proof of Theorem 5.4 that ‘there is an isometric isomorphism  $\Theta$  from  $\rho(\ell^1(\mathbb{N}_\wedge))^c$  onto  $\rho(\ell^1(\mathbb{N}_\wedge))^\varphi$ ’. An example showing that  $\Theta$  cannot be isometric can be constructed using [3, Theorem 7.6]. Take  $\|\sum_{n=1}^\infty a_n \delta_n\| = \sup_F \|\sum_{n \in F} a_n \delta_n\|$ , where  $F$  is a finite subset of  $\mathbb{N}$ . Take indices 1 and  $2n + 1$  so that the corresponding basis elements belong to distinct summands. Set  $A$  to be the diagonal matrix having ones at indices 1 and  $2n + 1$  and zero otherwise. Set  $B$  to have ones at indices 1 and  $2n + 1$  in the first row and zeros otherwise. Then  $B = \Theta(A)$  and

$$\begin{aligned} \|A\| &= \sup \left\{ \|a_1\delta_n + a_{2n+1}\delta_{2n+1}\| : \left\| \sum a_n\delta_n \right\| = 1 \right\} \\ &= \sum \{(a_1^2 + a_{2n+1}^2)^{1/2} : (a_1^2 + a_{2n+1}^2)^{1/2} = 1\} = 1, \end{aligned}$$

while

$$\begin{aligned} \|B\| &= \sup \left\{ |a_1 + a_{2n+1}| : \left\| \sum a_n\delta_n \right\| = 1 \right\} \\ &= \sum \{|a_1 + a_{2n+1}| : (a_1^2 + a_{2n+1}^2)^{1/2} = 1\} = \sqrt{2}. \end{aligned}$$

Consequently,  $\Theta$  is not isometric. By taking  $k$  summands in a similar way, it can be shown that  $\Theta$  is unbounded on diagonal elements of finite support.

### Acknowledgements

The authors would like to thank the anonymous referee for very careful reading and valuable comments that improved the presentation of the manuscript and pinpointed the error in [4]. Also the first author thanks Ilam university for its support.

### References

- [1] M. Alaghmandan, R. Nasr Isfahani and M. Nemati, 'Character amenability and contractibility of abstract Segal algebras', *Bull. Aust. Math. Soc.* **82** (2010), 274–281.
- [2] H. G. Dales and R. J. Loy, *Approximate Amenability of Semigroup Algebras and Segal Algebras*, *Dissertationes Mathematicae (Rozprawy Mat.)*, 474 (Instytut Matematyczny Polskiej Akademii Nauk, Warszawa, 2010).
- [3] M. Daws, 'Dual Banach algebras: representations and injectivity', *Stud. Math.* **178** (2007), 231–275.
- [4] A. Jaber and A. Mahmoodi, 'On  $\varphi$ -amenability of dual Banach algebras', *Bull. Aust. Math. Soc.* **105** (2022), 303–313.
- [5] B. E. Johnson, *Cohomology in Banach Algebras*, *Memoirs of the American Mathematical Society*, 127 (American Mathematical Society, Providence, RI, 1972).
- [6] E. Kaniuth, A. T. M. Lau and J. Pym, 'On  $\varphi$ -amenability of Banach algebras', *Math. Proc. Cambridge Philos. Soc.* **144** (2008), 85–96.
- [7] V. Runde, 'Dual Banach algebras: Connes-amenability, normal, virtual diagonals and injectivity of the predual bimodule', *Math. Scand.* **95** (2004), 124–144.

A. SAHAMI, Department of Mathematics,  
Faculty of Basic Sciences, Ilam University, P.O. Box 69315-516 Ilam, Iran  
e-mail: [a.sahami@ilam.ac.ir](mailto:a.sahami@ilam.ac.ir)

S. F. SHARIATI, Department of Mathematics and Computer Sciences,  
Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran  
e-mail: [f.shariati@aut.ac.ir](mailto:f.shariati@aut.ac.ir)