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SOME NEW RESULTS ON RÉNYI ENTROPY OF RESIDUAL LIFE AND INACTIVITY TIME

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This article deals with Rényi entropies for the residual life and the inactivity time. Monotonic properties of the entropy in order statistics, record values, and weighted distributions are investigated, and the comparison on weighted random variables is studied in terms of residual Rényi entropy as well.

1. INTRODUCTION AND PRELIMINARIES

Let X be a nonnegative random variable with density function f, distribution function F, hazard rate λ_F, and reversed hazard rate μ_F. The Rényi entropy of order α for X is

$$H_\alpha(f) = \frac{1}{1-\alpha} \log \int_0^\infty f^\alpha(x)\,dx, \qquad \alpha > 0, \alpha \neq 1.$$

As a measure of complexity and uncertainty of chaotic systems, the Rényi entropy has been playing a key role in various areas such as physics, electronics, engineering, and so forth (Kurths et al. [16]) ever since it was introduced. For more on Rényi entropy, one can refer to Rényi [22,23] and Song [25], among others. It is well known

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that the Rényi entropy reduces to Shannon's entropy as the order tends to 1; that is,

$$\lim_{\alpha\to 1} H_\alpha(f) = -\int_0^\infty f(x)\log f(x)\,dx = H(f).$$

Ebrahimi [12] studied Shannon's entropy of the residual life $X_t^+ = [X - t | X \geq t]$ (called the residual entropy):

$$H^+(f;t) = -\int_0^\infty f_t^+(x)\log f_t^+(x)\,dx,$$

where $f_t^+(x) = f(t+x)/\bar{F}(t)$ is the density function of X_t^+. Later, Abraham and Sankaran [1] further considered Rényi's entropy of order α for X_t^+:

$$\begin{aligned} H_\alpha^+(f;t) &= \frac{1}{1-\alpha}\log\int_0^\infty \left[f_t^+(x)\right]^\alpha\,dx \\ &= \log\bar{F}(t) + \frac{1}{1-\alpha}\log \mathsf{E}[f^{\alpha-1}(X)|X\geq t] \\ &= \frac{\log\alpha}{\alpha-1} - \frac{1}{1-\alpha}\log \mathsf{E}[\lambda_F^{\alpha-1}(X_\alpha)|X_\alpha \geq t], \end{aligned} \tag{1.1}$$

where $\lambda_F(x) = f(x)/\bar{F}(x)$ is the hazard rate function of X and X_α has the survival function $\bar{F}^\alpha(x)$.

Due to the duality, it is also natural to consider the entropy of the inactivity time $X_t^- = [t - X | X \leq t]$. (Block, Savits, and Singh [9]; Chandra and Roy [10]). In fact, Di Crescenzo and Longobardi [11] studied the inactivity entropy

$$H^-(f;t) = -\int_0^\infty f_t^-(x)\log f_t^-(x)\,dx,$$

where $f_t^-(x) = f(x-t)/F(t)$ is the density function of X_t^-. Afterward, Gupta and Nanda [13] considered the Rényi entropy for inactivity time:

$$\begin{aligned} H_\alpha^+(f;t) &= \frac{1}{1-\alpha}\log\int_0^t \left(f_t^-(x)\right)^\alpha\,dx \\ &= \log F(t) + \frac{1}{1-\alpha}\log \mathsf{E}[f^{\alpha-1}(X)|X\leq t] \\ &= \frac{\log\alpha}{\alpha-1} - \frac{1}{1-\alpha}\log \mathsf{E}[\mu_F^{\alpha-1}(X_\alpha^*)|X_\alpha^* \leq t], \end{aligned} \tag{1.2}$$

where $\mu_F(x) = f(x)/F(x)$ is the reversed hazard rate function of X and X_α^* has the distribution function $F^\alpha(x)$. The following two notions are closely related to Rényi entropies for residual life and inactivity time, respectively.

DEFINITION 1.1 (Abraham and Sankaran [1]): *A random variable X has decreasing α order Rényi entropy for residual life* (DRERL(α)) *if $H_\alpha^+(f;t)$ is decreasing in $t > 0$.*

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Definition 1.2 (Nanda and Paul [18]): *A nonnegative random variable X is said to have increasing uncertainty of inactivity time of order* α (IUIT (α)) *if* $H_\alpha^+(f;t)$ *is increasing in* $t \geq 0$.

For independent random variables $X_1, X_2, \ldots, X_n$ with common distribution function F, let $X_{1:n} \leq \cdots \leq X_{n:n}$ be their order statistics. It is well known that $X_{k:n}$ represents the lifetime of a $(n-k+1)$-out-of-n system. Particularly, $X_{1:n}$ and $X_{n:n}$ give the lifetimes of the series system and the parallel system, respectively. $X_{k:n}$ has its density function, distribution function, and hazard rate function as follows:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} \bar{F}(x)^{n-k} f(x),$$

$$F_{k:n}(x) = \sum_{i=k}^{n} \binom{n}{i} F(x)^i \bar{F}(x)^{n-i},$$

and

$$\lambda_{F_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!} \frac{[F(x)/\bar{F}(x)]^{k-1}}{\sum_{i=0}^{k-1} \binom{n}{i} [F(x)/\bar{F}(x)]^i} \lambda_F(x).$$

For more detailed discussions on order statistics, one can refer to Arnold, Balakrishnan, and Nagaraja [2].

For a sequence of independent random variables $X_1, X_2, \ldots$ with a common distribution function F, X_n is called an upper record value if $X_n > X_i$ for all $i = 1, 2, \ldots, n-1$. By convention, X_1 is a record value. The upper record times at which record values occur may be defined recursively is $L_1 = 1, L_n = \min\{k : k > L_{n-1}, X_k > X_{L_{n-1}}\}$, $n \geq 2$.$\{X_{L_n}; n \geq 1\}$ is the corresponding sequence of record values. The nth upper record value X_{L_n} has its probability density, survival function, and hazard rate function as follows:

$$f_{L_n}(t) = \frac{\Lambda_F^{n-1}(t)}{(n-1)!} f(t), \qquad \bar{F}_{L_n}(t) = \bar{F}(t) \sum_{k=0}^{n-1} \frac{\Lambda_F^k(t)}{k!},$$

$$\lambda_{F_{L_n}}(t) = \frac{\Lambda_F^{n-1}(t)/(n-1)!}{\sum_{k=0}^{n-1} \Lambda_F^k(t)/k!} \lambda_F(t),$$

where $\Lambda_F(t) = -\log \bar{F}(t)$ is the cumulative hazard. For more details on record values, one can refer to Arnold, Balakrishnan, and Nagaraja [3].

In the past decade, many authors devoted themselves to investigating entropies of order statistics and record values. For example, Asadi and Ebrahimi [4] studied order statistics and record values related DURL (decreasing uncertainty of residual life) distributions, Kundu, Nanda, and Hu [15] made a parallel study on IUIT distributions, and Baratpour, Ahmadi, and Arghami [6] established several characterizations of distributions based on the Rényi entropy of order statistics and record values.

Let w be a nonnegative real function such that $0 < \mathsf{E}(w(X)) < \infty$. Then the weighted version X_w associated to X and w has its density function, survival function, and hazard rate function as follows:

$$f_w(t) = \frac{w(t)}{\mathsf{E}(w(X))} f(t), \qquad \bar{F}_w(t) = \frac{\mathsf{E}(w(X)|X \geq t)}{\mathsf{E}(w(X))} \bar{F}(t),$$
$$\lambda_{F_w}(t) = \frac{w(t)}{\mathsf{E}(w(X)|X \geq t)} \lambda_F(t).$$

For more on the weighted distribution, we refer readers to Rao [21] and Patil and Rao [20], among others. In the literature, there are also some research works on the entropy of weighted distribution. See, for example, Belzunce, Navarro, Ruiz, and Aguila [8] and Navarro, Aguila, and Asadi [19].

The article is a further study on Rényi entropy for both the residual life and the inactivity time. Section 2 proves that the DRERL property of a stochastically smaller (in the sense of likelihood ratio order) random variable is preserved by a larger one, and based on this result, it is shown that the DRERL property is preserved by both the formation of parallel systems and the record value. In Section 3 we prove that the IUIT property of a stochastically larger (in the sense of likelihood ratio order) random variable is preserved by a smaller one and, hence, it is also preserved by the formation of series systems. As an application, Section 4 addresses some comparisons between a random variable and its weighted version.

2. ON RESIDUAL LIFE

In reliability theory, some aging properties can be characterized through a stochastic comparison between the total life and its residual life. For example, X is NBU (new better than used) if and only if X is larger than X_t^+ in the usual stochastic order for all $t > 0$, and it is DMRL (decreasing mean residual life) if and only if X is larger than X_t^+ in terms of the mean residual life order for all $t \geq 0$. Readers can refer to Shaked and Shanthikumar [24] for more on stochastic orders. The first theorem presents a similar characterization for DRERL.

THEOREM 2.1: *A random variable X with density function f is* DRERL(α) *if and only if*

$$H_\alpha^+(f_s^+; t) \leq H_\alpha^+(f; t) \quad \textit{for all } s, t \geq 0. \tag{2.1}$$

PROOF: By definition, X is DRERL(α) if and only if

$$H_\alpha^+(f; s + t) \leq H_\alpha^+(f; t) \quad \text{for all } s, t \geq 0. \tag{2.2}$$

Note that

$$\bar{F}_s(x) = P(X - s > x | X \geq s) = \frac{\bar{F}(s + x)}{\bar{F}(s)}, \qquad f_s^+(x) = \frac{f(s + x)}{\bar{F}(s)},$$

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and we have, for all $s, t \geq 0$,

$$\begin{aligned} H_\alpha^+(f_s^+; t) &= \frac{1}{1-\alpha} \log \int_t^\infty \frac{(f_s^+(x))^\alpha}{\bar{F}_s^\alpha(t)}\, dx \\ &= \frac{1}{1-\alpha} \log \int_t^\infty \frac{f^\alpha(s+x)}{\bar{F}^\alpha(s+t)}\, dx \\ &= \frac{1}{1-\alpha} \log \int_{s+t}^\infty \frac{f^\alpha(y)}{\bar{F}^\alpha(s+t)}\, dy \\ &= H_\alpha^+(f; s+t) \end{aligned}$$

It is obvious that (2.2) is equivalent to (2.1). This completes the proof. ■

Let Y be another continuous random variable with probability density function $g(x)$, distribution function $G(x)$, survival function $\bar{G}(x)$, hazard rate $\lambda_G(x)$ and reversed hazard rate $\mu_G(x)$. Recall that X is said to be smaller than Y in the likelihood ratio order, denoted as $X \leq_{\rm lr} Y$, if $g(x)/f(x)$ is increasing in x over the union of their supports. Readers can refer Shaked and Shanthikumar [24] for more stochastic orders.

THEOREM 2.2: *Let $X \leq_{\rm lr} Y$ and $\lambda_G(t)/\lambda_F(t)$ be increasing in $t \geq 0$. Then Y is also* DRERL(α) *if X is.*

PROOF: Case for $\alpha > 1$. Denote $\theta(x) = \lambda_G(t)/\lambda_F(t)$. Due to (1.1), it holds that

$$\begin{aligned} H_\alpha^+(g; t) - \frac{\log \alpha}{\alpha - 1} &= -\frac{\log \mathsf{E}[\lambda_G^{\alpha-1}(Y_\alpha) | Y_\alpha \geq t]}{\alpha - 1} \\ &= -\frac{\log \mathsf{E}[\lambda_F^{\alpha-1}(Y_\alpha)\theta^{\alpha-1}(Y_\alpha) | Y_\alpha \geq t]}{\alpha - 1}. \end{aligned}$$

Thus, we need to prove that $\Delta(t) = \mathsf{E}[\lambda_F^{\alpha-1}(Y_\alpha)\theta^{\alpha-1}(Y_\alpha) | Y_\alpha \geq t]$ is increasing in $t \geq 0$. In view of

$$\begin{aligned} \Delta'(t) &= \left(\int_t^\infty \frac{\alpha g(x) \bar{G}^{\alpha-1}(x)}{\bar{G}^\alpha(t)} \lambda_F^{\alpha-1}(x) \theta^{\alpha-1}(x)\, dx \right)' \\ &= \frac{\alpha g(t) \bar{G}^{\alpha-1}(t) \left[\begin{array}{c} -\lambda_F^{\alpha-1}(t)\theta^{\alpha-1}(t)\bar{G}^\alpha(t) \\ + \int_t^\infty \alpha g(x)\bar{G}^{\alpha-1}(x)\lambda_F^{\alpha-1}(x)\theta^{\alpha-1}(x)\, dx \end{array} \right]}{\bar{G}^{2\alpha}(t)} \\ &= -\alpha\lambda_G(t)\lambda_F^{\alpha-1}\theta^{\alpha-1}(t) + \alpha\lambda_G(t)\Delta(t) \\ &= \alpha\lambda_G(t) \left[\Delta(t) - \lambda_F^{\alpha-1}(t)\theta^{\alpha-1}(t) \right], \end{aligned}$$

it suffices for us to show $\Delta(t) \geq \lambda_F^{\alpha-1}(t)\theta^{\alpha-1}(t)$ for all $t \geq 0$.

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For all $t \geq 0$, it holds that

$$\begin{aligned}
&\Delta(t) - \lambda_F^{\alpha-1}(t)\theta^{\alpha-1}(t) \\
&\quad = \int_t^\infty \frac{\alpha g(x)\bar{G}^{\alpha-1}(x)}{\bar{G}^\alpha(t)} \left[\lambda_F^{\alpha-1}(x)\theta^{\alpha-1}(x) - \lambda_F^{\alpha-1}(t)\theta^{\alpha-1}(t)\right] dx \\
&\quad = \int_t^\infty \frac{\alpha g(x)\bar{G}^{\alpha-1}(x)}{\bar{G}^\alpha(t)}\theta^{\alpha-1}(x) \left[\lambda_F^{\alpha-1}(x) - \lambda_F^{\alpha-1}(t)\right] dx \\
&\qquad + \int_t^\infty \frac{\alpha g(x)\bar{G}^{\alpha-1}(x)}{\bar{G}^\alpha(t)}\lambda_F^{\alpha-1}(t) \left[\theta^{\alpha-1}(x) - \theta^{\alpha-1}(t)\right] dx \\
&\quad \triangleq I_1(t) + I_2(t).
\end{aligned}$$

Denote $m(t) = \mathsf{E}[\lambda_F^{\alpha-1}(X_\alpha)|X_\alpha \geq t]$. By assumption, $H_\alpha^+(f, t)$ is decreasing in $t \geq 0$. Equivalently, $m(t)$ is increasing in $t \geq 0$. Note that $m'(t) = -\alpha\lambda_F(t)(\lambda_F^{\alpha-1}(t) - m(t))$ and we have $m(t) = \lambda_F^{\alpha-1}(t)$ for all $t \geq 0$; that is,

$$\int_t^\infty \alpha f(x)\bar{F}^{\alpha-1}(x)\left[\lambda_F^{\alpha-1}(x) - \lambda_F^{\alpha-1}(t)\right] dx \geq 0, \qquad t \geq 0. \tag{2.3}$$

$X \leq_{\mathrm{lr}} Y$ implies that $g(x)/f(x)$ is increasing in $x \geq 0$. Applying Lemma 7.1(i) of Barlow and Proschan [7] to (2.3), we immediately have, for all $t \geq 0$,

$$\begin{aligned}
I_1(t) &= \frac{1}{\bar{G}^\alpha(t)} \int_t^\infty \frac{\alpha g(x)\bar{G}^{\alpha-1}(x)\theta^{\alpha-1}(x)}{\alpha f(x)\bar{F}^{\alpha-1}(x)}\alpha f(x)\bar{F}^{\alpha-1}(x)\left[\lambda_F^{\alpha-1}(x) - \lambda_F^{\alpha-1}(t)\right] dx \\
&= \frac{1}{\bar{G}^\alpha(t)} \int_t^\infty \frac{g^\alpha(x)}{f^\alpha(x)}\alpha f(x)\bar{F}^{\alpha-1}(x)\left[\lambda_F^{\alpha-1}(x) - \lambda_F^{\alpha-1}(t)\right] dx \\
&\geq 0.
\end{aligned}$$

On the other hand, since $\theta(x)$ is nonnegative and increasing, it holds that, for all $t \geq 0$,

$$I_2(t) = \int_t^\infty \frac{\alpha g(x)\bar{G}^{\alpha-1}(x)}{\bar{G}^\alpha(t)}\lambda_F^{\alpha-1}(t)\left[\theta^{\alpha-1}(x) - \theta^{\alpha-1}(t)\right] dx \geq 0.$$

Now, we can conclude that $\Delta(t) \geq \lambda_F^{\alpha-1}(t)\theta^{\alpha-1}(t)$ for all $t \geq 0$.

For the case with $0 < \alpha < 1$, note that $H_\alpha^+(g; t)$ is decreasing $t \geq 0$ in is equivalent to $\Delta(t)$ is decreasing in $t \geq 0$; to get the desired result, we only need to reverse all of the above inequalities. ■

Asadi and Ebrahimi [4] proved that DURL is preserved under the formation of parallel system and record values. As a direct consequence of Theorem 2.2, DRERL can be proved to be preserved under the formation of parallel systems.

Corollary 2.3: *If X is* DRERL(α), *then $X_{n:n}$ is also* DRERL(α).

PROOF: Note that $F_{n:n}(x) = 1 - F^n(x)$ and $f_{n:n}(x) = nf(x)F^{n-1}(x)$ and we have $\lambda_{F_{n:n}(x)} = \theta(x)\lambda_F(x)$ with

$$\theta(x) = \frac{n}{\sum_{i=0}^{n-1} F^{i-(n-1)}(x)}.$$

It is not difficult to verify that $\theta(x)$ is increasing in $x \geq 0$ and $X \leq_{\rm lr} X_{n:n}$. Therefore, the desired result stems immediately from Theorem 2.2. ■

One might wonder whether DRERL is also preserved under the formation of series systems. The next example serves as a negative answer.

Example 2.4: Suppose X is the random variable with survival function

$$\bar{F}(x) = \begin{cases} 1 - \dfrac{x^2}{2} & \text{if } 0 \leq x < 1 \\ \dfrac{2}{3} - \dfrac{x^2}{6} & \text{if } 1 \leq x < 2 \\ 0 & \text{if } x \geq 2. \end{cases}$$

Then its residual Rényi entropy of order α is

$$H_\alpha^+(f;t) = \begin{cases} \dfrac{1}{1-\alpha} \log \dfrac{6^\alpha(1-t^{\alpha+1}) + 2^\alpha(2^{\alpha+1}-1)}{3^\alpha(\alpha+1)(2-t^2)^\alpha} & \text{if } 0 \leq t < 1 \\ \dfrac{1}{1-\alpha} \log \dfrac{2^\alpha(2^{\alpha+1}-t^{\alpha+1})}{(\alpha+1)(4-t^2)^\alpha} & \text{if } 1 \leq t < 2. \end{cases}$$

For $\alpha = 0.5, H_\alpha^+(f;t)$ is decreasing in t (see Fig. 1a); that is, X is DRERL(0.5). However, for $1 \leq t < 2$, the residual Rényi entropy of order α corresponding to a series

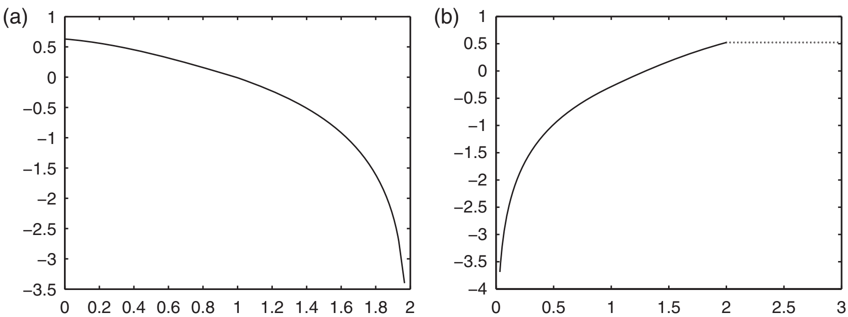

FIGURE 1. Curves of Renyi entropies: (a) curve $H_{0.5}^+(f;t)$ in Example 2.4; (b) curve $H_2^-(f;t)$ in Example 3.3.

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system with n independent and identical components is

$$H_\alpha^+(f_{1:n};t) = \frac{1}{1-\alpha}\log\frac{(2n)^\alpha\int_t^2 x^\alpha(4-x^2)^{\alpha(n-1)}\,dx}{(4-t^2)^{\alpha n}}.$$

It is easy to check that for $n = 25$ and $\alpha = 0.5$,

$$\begin{aligned}H_\alpha^+(f_{1:n};1.86) = -3.9201 > H_\alpha^+(f_{1:n};1.88) = -4.2699\\ < H_\alpha^+(f_{1:n};1.89) = -4.1088;\end{aligned}$$

that is, $H_\alpha^+(f_{1:n};t)$ is not monotone in t. Therefore, $X_{1:n}$ is not DREFL(0.5).

Remark 2.5: In Example 2.4, $\lambda_{F_{1:n}}(X)/\lambda_F(x) = n$ is both increasing and decreasing, whereas $f_{1:n}(x)/f(x) = n\bar{F}^{n-1}(x)$ is decreasing in x; that is, $X \geq_{\rm lr} X_{1:n}$. This actually tells us that the condition $X \leq_{\rm lr} Y$ can not be dropped in Theorem 2.2. However, as can be seen in the proof of Lemma 2.1 in Asadi and Ebrahimi [4], the condition $X \leq_{\rm lr} Y$ in Theorem 2.2 can be relaxed to $X \leq_{\rm hr} Y$(the hazard rate order) at the cost of adding the other condition $\lim_{t\to\infty}\bar{G}(t)/\bar{F}(t) < \infty$, which is not always the case. For example,

$$\lim_{t\to\infty}\frac{\bar{F}_{L_n}(t)}{\bar{F}(t)} = \lim_{t\to\infty}\sum_{k=0}^{n-1}\frac{[\Lambda_F(t)]^k}{k!} = \infty.$$

In Asadi and Ebrahimi [4], it was proved that $X_{k+1:n}, X_{k:n-1}$ and $X_{k+1:n+1}$ are also DURL if $X_{k:n}$ is and that X_{L_n} is also DURL if X is. The following two corollaries deal with the corresponding Rényi entropy.

COROLLARY 2.6: *If* $X_{k:n}$ *is* DRERL(α), *then* $X_{k+1:n}, X_{k:n-1}$ *and* $X_{k+1:n+1}$ *are also* DRERL(α).

PROOF: Let $X_{k_1:n_1}$ and $X_{k_2:n_2}$ be order statistics of two sets of independent random variables with common distribution function F; also let $\lambda_{F_{k_1:n_1}}(x)$ and $\lambda_{F_{k_2:n_2}}(x)$ be their respective hazard rate functions. Then

$$\frac{\lambda_{F_{k_2:n_2}}(x)}{\lambda_{F_{k_1:n_1}}(x)}\alpha\left(\frac{F(x)}{\bar{F}(x)}\right)^{k_2-k_1}\frac{\sum_{i=0}^{k_1-1}\binom{n_1}{i}(F(x)/\bar{F}(x))^i}{\sum_{j=0}^{k_2-1}\binom{n_2}{j}(F(x)/\bar{F}(x))^j}.$$

Note that

$$\frac{f_{k+1:n}(x)}{f_{k:n}(x)} = \frac{(n-k)F(x)}{k\bar{F}(x)}, \qquad \frac{f_{k:n-1}(x)}{f_{k:n}(x)} = \frac{(n-k)}{n\bar{F}(x)}, \qquad \frac{f_{k+1:n+1}(x)}{f_{k:n}(x)} = \frac{nF(x)}{k}$$

are all increasing in x; it holds that

$$X_{k:n} \leq_{\rm lr} X_{k+1:n}, \qquad X_{k:n} \leq_{\rm lr} X_{k:n-1}, \qquad X_{k:n} \leq_{\rm lr} X_{k+1:n+1}.$$

On the other hand, according to Nagaraja [17], $\lambda_{F_{k_2:n_2}}(x)/\lambda_{F_{k_1:n_1}}(x)$ is increasing in x in the following three cases: (i) $n_1 = n_2 = n, k_1 = k$, and $k_2 = k+1$; (ii) $n_1 = n$,

$n_2 = n - 1$, and $k_1 = k_2 = k$; (iii) $n_1 = n, n_2 = n + 1, k_1 = k$, and $k_2 = k + 1$. Thus, from Theorem 2.2, our claim follows immediately. ∎

COROLLARY 2.7: *If X is* DRERL(α), *then* X_{L_n} *is also* DRERL(α).

PROOF: Denote $\Lambda_F(t) = -\log \bar{F}(t)$. Note that

$$\frac{\lambda_{F_{L_n}}(t)}{\lambda_F(t)} = \frac{[\Lambda_F(t)]^{n-1}/(n-1)!}{\sum_{k=0}^{n-1}(1/k!)[\Lambda_F(t)]^k}$$

is increasing in t and

$$\frac{f_{L_n}(t)}{f(t)} = \frac{[\Lambda_F(t)]^{n-1}}{(n-1)!}$$

is also increasing in t; the conclusion follows from Theorem 2.2 directly. ∎

In fact, DRERL property extends from X_{L_n} to $X_{L_{n+1}}$.

COROLLARY 2.8: *If* X_{L_n} *is* DRERL(α), *then* $X_{L_{n+1}}$ *is also* DRERL(α).

PROOF: By Kochar [14]

$$\frac{\lambda_{F_{L_{n+1}}}(t)}{\lambda_{F_{L_n}}(t)} = \frac{\sum_{k=0}^{n-1}(1/k!)[\Lambda_F(t)]^{k+1}}{n\sum_{k=0}^{n}(1/k!)[\Lambda_F(t)]^k}$$

is increasing in $t \geq 0$.Since $X_{L_n} \leq_{\rm lr} X_{L_{n+1}}$, we reach the conclusion by Theorem 2.2 again. ∎

3. ON INACTIVITY TIME

The first theorem of this section, which helps to deduce those upcoming conclusions, asserts that IUIT property of a stochastically larger random variable can be preserved by the smaller one.

THEOREM 3.1: *Let* $X \geq_{\rm lr} Y$ *and* $\mu_G(t)/\mu_F(t)$ *be decreasing in* $t \geq 0$. *Then* Y *is also* IUIT(α) *if* X *is.*

PROOF: Case for $\alpha > 1$. Denote $\eta(x) = \mu_G(x)/\mu_F(x)$. By (1.2),

$$H_\alpha^-(g;t) - \frac{\log\alpha}{\alpha-1} = \frac{\log \mathsf{E}[\mu_G^{\alpha-1}(Y_\alpha^*)|Y_\alpha^* \leq t]}{\alpha-1} = \frac{\log \mathsf{E}[\mu_F^{\alpha-1}(Y_\alpha^*)\eta^{\alpha-1}(Y_\alpha^*)|Y_\alpha^* \leq t]}{\alpha-1},$$

it suffices to prove $\Delta_*(t) = \mathsf{E}[\mu_F^{\alpha-1}(Y_\alpha^*)\eta^{\alpha-1}(Y_\alpha^*)|Y_\alpha^* \leq t]$ is decreasing in t. Due to

$$\Delta_*'(t) = \alpha\mu_G(t)[\mu_F^{\alpha-1}(t)\eta^{\alpha-1}(t) - \Delta_*(t)],$$

this is equivalent to showing $\Delta_*(t) \geq \mu_F^{\alpha-1}(t)\eta^{\alpha-1}(t)$ for all $t \geq 0$.

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For any $t \geq 0$, we have

$$\begin{aligned}
&\Delta_*(t) - \mu_F^{\alpha-1}(t)\eta^{\alpha-1}(t) \\
&\quad = \int_0^t \frac{\alpha g(x) G^{\alpha-1}(x)}{G^\alpha(t)} [\mu_F^{\alpha-1}(x)\eta^{\alpha-1}(x) \\
&\qquad - \mu_F^{\alpha-1}(t)\eta^{\alpha-1}(t)]\, dx \\
&\quad = \int_0^t \frac{\alpha g(x) G^{\alpha-1}(x)}{G^\alpha(t)} \eta^{\alpha-1}(x)[\mu_F^{\alpha-1}(x) - \mu_F^{\alpha-1}(t)]\, dx \\
&\qquad + \int_0^t \frac{\alpha g(x) G^{\alpha-1}(x)}{G^\alpha(t)} \mu_F^{\alpha-1}(t)[\eta^{\alpha-1}(x) - \eta^{\alpha-1}(t)]\, dx \\
&\quad \triangleq I_1^*(t) + I_2^*(t).
\end{aligned}$$

Let $m_*(t) = \mathsf{E}[\mu_F^{\alpha-1}(X_\alpha^*)|X_\alpha^* \leq t]$. Since $H_\alpha^-(f,t)$ is increasing in $t \geq 0$, $m_*(t)$ is decreasing in $t \geq 0$. Note that

$$m_*'(t) = \alpha \mu_F(t)[\mu_F^{\alpha-1}(t) - m_*(t)] \text{ and}$$

and we have $m_*(t) \geq \mu_F^{\alpha-1}(t)$ for all $t \geq 0$; that is,

$$\int_0^t \alpha f(x) F^{\alpha-1}(x)[\mu_F^{\alpha-1}(x) - \mu_F^{\alpha-1}(t)]\, dx \geq 0. \tag{3.1}$$

Since $X \geq_{\mathrm{lr}} Y$, $g(x)/f(x)$ is decreasing in $x \geq 0$. Applying Lemma 7.1(ii) of Barlow and Proschan [7] to (3.1), we have

$$\begin{aligned}
I_1^*(t) &= \frac{1}{G^\alpha(t)} \int_0^t \frac{\alpha g(x) G^{\alpha-1}(x)}{\alpha f(x) F^{\alpha-1}(x)} \eta^{\alpha-1}(x) \alpha f(x) F^{\alpha-1}(x)(\mu_F^{\alpha-1}(x) - \mu_F^{\alpha-1}(t))\, dx \\
&= \frac{1}{G^\alpha(t)} \int_0^t \frac{g^\alpha(x)}{f^\alpha(x)} \alpha f(x) F^{\alpha-1}(x)(\mu_F^{\alpha-1}(x) - \mu_F^{\alpha-1}(t))\, dx \\
&\geq 0.
\end{aligned}$$

On the other hand, since $\eta(x)$ is nonnegative decreasing, it holds that

$$I_2^* = \int_0^t \frac{\alpha g(x) G^{\alpha-1}(x)}{G^\alpha(t)} \mu_F^{\alpha-1}(t)(\eta^{\alpha-1}(x) - \eta^{\alpha-1}(t))\, dx \geq 0.$$

Therefore, we have $\Delta_*(t) \geq \mu_F^{\alpha-1}(t)\eta^{\alpha-1}(t)$ for all $t \geq 0$.

For the case with $0 < \alpha < 1$, note that $H_\alpha^-(g;t)$ is increasing in $t \geq 0$ is equivalent to $\Delta_*(t)$ is increasing in $t \geq 0$. By reversing all of the above inequalities, we reach the desired result. ■

Kundu et al. [15] proved that the IUIT property is preserved under the formation of a series system. As a direct consequence of Theorem 3.1, we get the corresponding version on IUIT(α).

COROLLARY 3.2: *If X is* IUIT(α), *then* $X_{1:n}$ *is also* IUIT(α).

PROOF: Since $F_{1:n}(x) = 1 - \bar{F}^n(x)$ and $f_{1:n}(x) = nf(x)\bar{F}^{n-1}(x)$, we have $\mu_{F_{1:n}}(x) = \eta(x)\mu_F(x)$ with

$$\eta(x) = \frac{n}{\sum_{i=0}^{n-1} \bar{F}^{i-(n-1)}(x)},$$

which can be easily proved to be decreasing. Note that $f_{1:n}(x)/f(x) = n\bar{F}^{n-1}(x)$ is decreasing; that is, $X \geq_{\mathrm{lr}} X_{1:n}$ and the conclusion follows directly from Theorem 3.1. ■

The following example shows that IUIT(α) is not preserved under the formation of parallel systems.

Example 3.3: Consider again the distribution in Example 2.4. The inactive Rényi entropy of order α for X is

$$H_\alpha^-(f;t) = \begin{cases} \dfrac{1}{1-\alpha}\log\dfrac{2^\alpha t^{-\alpha+1}}{\alpha+1} & \text{if } 0 \leq t < 1 \\ \dfrac{1}{1-\alpha}\log\dfrac{2^\alpha(3^\alpha + t^{\alpha+1} - 1)}{(\alpha+1)(t^2+2)^\alpha} & \text{if } 1 \leq t < 2 \\ \dfrac{1}{1-\alpha}\log\dfrac{3^\alpha + 2^{\alpha+1} - 1}{3^\alpha(\alpha+1)} & \text{if } t \geq 2. \end{cases}$$

For $\alpha = 2, H_\alpha^-(f;t)$ is increasing in t (see Fig. 1b). Therefore, X is IUIT(2). However, the inactive Rényi entropy of order α for the corresponding parallel system is

$$H_\alpha^-(f_{n:n};t) = \frac{1}{1-\alpha}\log\frac{3^{\alpha n}(2n)^\alpha/(2n\alpha - \alpha + 1) + (2n)^\alpha\int_1^t x^\alpha(x^2+2)^{\alpha(n-1)}\,dx}{(t^2+2)^{\alpha n}},$$

$$\times\ 1 \leq t < 2.$$

It is easy to check that for $n = 30$ and $\alpha = 2$,

$$H_\alpha^-(f_{n:n};1.1) = -2.3579 > H_\alpha^-(f_{n:n};1.5) = -2.3604 < H_\alpha^-(f_{n:n};1.9) = -2.3221;$$

that is, $H_\alpha^-(f_{n:n};t)$ is not monotone in t. Therefore, $X_{n:n}$ is not IUIT(2).

Remark 3.4: Note that $\mu F_{n:m}(x)/\mu_F(x) = n$ in Example 3.3 is both decreasing and increasing and $f_{n:n}(x)/f(x) = nF^{n-1}(x)$ is increasing; that is, $X \leq_{\mathrm{lr}} X_{n:n}$ and $X \geq_{\mathrm{lr}} Y$ in Theorem 3.1 cannot be dropped in general. However, as can be seen in the proof of

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Lemma 2.3 in Kundu et al. [15], $X \geq_{\rm lr} Y$ in Theorem 3.1 can be relaxed to $X \geq_{\rm rh} Y$ (reversed hazard rate order) at the cost of further requiring $\lim_{x\to 0} G(x)/F(x) < \infty$.

Kundu et al. [15]. proved that $X_{k-1:n}, X_{k:n+1}$, and $X_{k-1:n-1}$ are also IUIT if $X_{k:n}$ is. Corollary 3.5 extends it to the case for the Rényi entropy.

COROLLARY 3.5: *If $X_{k:n}$ is* IUIT(α) *then $X_{k-1:n}, X_{k:n+1}$, and $X_{k-1:n-1}$ are also* IUIT(α).

PROOF: Let $\mu_{F_{k_1:n_1}}(x)$ and $\mu_{F_{k_2:n_2}}(x)$ be the respective reversed hazard rate functions of $X_{k1:n1}$ and $X_{k2:n2}$, order statistics from a population F. Then

$$\frac{\mu_{F_{k_2:n_2}}(x)}{\mu_{F_{k_1:n_1}}(x)} \alpha \left(\frac{F(x)}{\bar F(x)}\right)^{k_2-k_1} \frac{\sum_{i=k_1}^{n_1}\binom{n_1}{i}(F(x)/\bar F(x))^i}{\sum_{j=k_2}^{n_2}\binom{n_2}{j}(F(x)/\bar F(x))^j}.$$

According to Kundu et al. [15], $\mu_{F_{k_2:n_2}}(x)/\mu_{F_{k_1:n_1}}(x)$ is decreasing in x in the following cases: (i) $n_1 = n_2 = n, k_1 = k$, and $k_2 = k-1$; (ii) $n_1 = n, n_2 = n+1$, and $k_1 = k_2 = k$; (iii) $n_1 = n, n_2 = n-1, k_1 = k$, and $k_2 = k-1$. Note that $X_{k:n} \geq_{\rm lr} X_{k-1:n}, X_{k:n} \geq_{\rm lr} X_{k:n+1}$, and $X_{k:n} \geq_{\rm lr} X_{k-1:n-1}$; the conclusion follows from Theorem 3.1 immediately. ■

4. ON WEIGHTED DISTRIBUTION

In the literature, there are some studies on the comparison between a random variable and its weighted version in terms of the residual Rényi entropy—for example, Belzunce et al. [8] and Navarro et al. [19]. Here, we present some further comparison results in this line of research; some of them are applications of the main results in previous sections.

THEOREM 4.1: *Suppose $\mathsf{E}(w(X)|X \geq t)$ or $w(t)$ is decreasing (increasing). If X or X_w is DFR (decreasing failure rate), then $H_\alpha^+(f;t) \geq (\leq) H_\alpha^+(f_w;t)$ for all $t \geq 0$.*

PROOF: If $w(t)$ is decreasing (increasing) in $t \geq 0$, then $\mathsf{E}(w(X)|X \geq t) \leq (\geq) w(t)$ and hence $\lambda_F(t) \leq (\geq) \lambda_{F_w}(t)$ for all $t \geq 0$. Let $m_w(t) = \mathsf{E}(w(X)|X \geq t)$. Since, $m_w'(t) = \lambda_F(t)(m_w(t) - w(t))$, if $\mathsf{E}(w(X)|X \geq t)$ is decreasing (increasing) in $t \geq 0$, we also have $\mathsf{E}(w(X)|X \geq t) \leq (\geq) w(t)$ and thus $\lambda_F(t) \leq (\geq) \lambda_{F_w}(t)$ for all $t \geq 0$. Now, by Theorem 4 of Asadi, Ebrahimi, and Soofi [5] we complete the proof. ■

For a random variable X, its equilibrium version X_e has the probabilistic density function $f_e(t) = \bar F(t)/\mathsf{E}X$ and its length-biased version X_ℓ has the probabilistic density function $f_\ell(t) = tf(t)/\mathsf{E}X$. It is well known that X_e and X_ℓ are weighted versions of X with weight function $w_e(t) = 1/\lambda_F(t)$ and $w_\ell(t) = t$, respectively. As a direct consequence of Theorem 4.1, we have Corollary 4.2.

COROLLARY 4.2: *If X is* DFR, *then $H_\alpha^+(f;t) \leq H_\alpha^+(f_e;t)$ and $H_\alpha^+(f;t) \leq H_\alpha^+(f_\ell;t)$ for all $t \geq 0$.*

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Belzunce et al. [8] investigated the preservation property of DURL under the weighted transforms. In view of

$$\frac{f_w(t)}{f(t)} = \frac{w(t)}{\mathsf{E}(w(X))}, \qquad \frac{\lambda_{F_w}(t)}{\lambda_F(t)} = \frac{w(t)}{\mathsf{E}[w(X)|X \geq t]}, \qquad \frac{\mu_{F_w}(t)}{\mu_F(t)} = \frac{w(t)}{\mathsf{E}[w(X)|X \leq t]},$$

the following two corollaries parallel to DRERL(α) and IUIT(α) follow directly from Theorem 2.2 and Theorem 3.1, respectively.

COROLLARY 4.3: *Suppose $w(t)$ is increasing (decreasing) and $\mathsf{E}(w(X)|X \geq t)/w(t)$ is decreasing (increasing). Then $X_w(X)$ is also* DRERL(α) *if $X(X_w)$ is.*

COROLLARY 4.4: *Suppose $w(t)$ is decreasing (increasing) and $\mathsf{E}(w(X)|X \leq t)/w(t)$ is increasing (decreasing). Then $X_w(X)$ is also* IUIT(α) *if $X(X_w)$ is.*

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