

UNIVERSAL MIXED ELLIPTIC MOTIVES

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(Received 2 March 2017; revised 5 March 2018; accepted 9 March 2018;
first published online 30 April 2018)

Abstract In this paper we construct a \mathbb{Q} -linear tannakian category \mathbf{MEM}_1 of universal mixed elliptic motives over the moduli space $\mathcal{M}_{1,1}$ of elliptic curves. It contains \mathbf{MTM} , the category of mixed Tate motives unramified over the integers. Each object of \mathbf{MEM}_1 is an object of \mathbf{MTM} endowed with an action of $\mathrm{SL}_2(\mathbb{Z})$ that is compatible with its structure. Universal mixed elliptic motives can be thought of as motivic local systems over $\mathcal{M}_{1,1}$ whose fiber over the tangential base point $\partial/\partial q$ at the cusp is a mixed Tate motive. The basic structure of the tannakian fundamental group of \mathbf{MEM} is determined and the lowest order terms of a set (conjecturally, a minimal generating set) of relations are deduced from computations of Brown. This set of relations includes the arithmetic relations, which describe the ‘infinitesimal Galois action’. We use the presentation to give a new and more conceptual proof of the Ihara–Takao congruences.

Keywords: mixed elliptic motive; mixed Tate motive; modular curve; modular form; elliptic polylogarithm; variation of mixed Hodge structure

2010 *Mathematics subject classification:* Primary 14F42; 14F35; 14H52
Secondary 11G55; 14C30; 19E20

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The research was supported in part by the Scientific Grants-in-Aid 19204002, 23244002, 15K13460, JST-CREST 151001, and by the Core-to-Core grant 18005 from the Japan Society for the Promotion of Science; and grants DMS-0706955, DMS-1005675 and DMS-1406420 from the National Science Foundation.

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1. Introduction

Among the principal goals of the theory of motives is to construct a \mathbb{Q} -linear tannakian category $\text{MM}(T)$ of mixed motivic sheaves over a smooth base T whose ext groups are the motivic cohomology groups of T :

$$\text{Ext}_{\text{MM}(T)}^\bullet(\mathbb{Q}, M) \cong H_{\text{mot}}^\bullet(T, M),$$

where M is a motive over T , such as $\mathbb{Q}(n)$. This goal was partially achieved by Levine [42], Hanamura [35] and Voevodsky [60], each of whom constructed a triangulated tensor category of mixed motives with the correct ext groups. Many obstructions remain to constructing tannakian categories of mixed motives including, most notably, the problem of establishing Beilinson–Soulé vanishing. One case where this goal has been achieved is that of mixed Tate motives over a ring of S integers in a number field. This was established by Levine [41] and Deligne–Goncharov [17] using the work of Borel [8], Beilinson [4] and the existence of a triangulated category of mixed motives. The theory of mixed Tate motives can be regarded as the story of motives associated to genus 0 curves and their moduli spaces, [9].

In this paper we take a step toward extending this story to genus 1 curves and their moduli spaces. We construct a \mathbb{Q} -linear tannakian category MEM_1 of *universal mixed elliptic motives* over $\mathcal{M}_{1,1/\mathbb{Z}}$, the moduli stack of elliptic curves, which contains MTM , the category of mixed Tate motives unramified over \mathbb{Z} . The ring of functions on its tannakian fundamental group is a Hopf algebra in the category of ind-objects of MTM which encodes relations between the periods of iterated integrals of Eisenstein series and periods classical modular forms of level 1.

Each object \mathbb{V} of MEM_1 consists of an object V of MTM whose Betti realization is endowed with an action of $\text{SL}_2(\mathbb{Z})$. This action is required to determine a compatible set of local systems (Betti, \mathbb{Q} -de Rham, Hodge, ℓ -adic) over $\mathcal{M}_{1,1}$ whose fibers over the canonical tangent vector $\mathfrak{t} := \partial/\partial q$ of $\overline{\mathcal{M}}_{1,1}$ at the cusp are the various realizations of V . An object of MEM_1 can be specialized to an elliptic curve E/T by pulling back its associated local systems along the structure map $T \rightarrow \mathcal{M}_{1,1}$ to obtain a family of local systems over T .

This hybrid approach to constructing a tannakian category of mixed motives over $\mathcal{M}_{1,1}$ is necessitated by the lack of understanding of motives associated to modular forms. In particular, it circumvents the problem of proving Beilinson–Soulé vanishing for the category of mixed motives generated by motives of classical modular forms. We expect

the local systems (Betti, de Rham, Hodge, ℓ -adic) associated to an object of \mathbf{MEM}_1 to be the set of compatible realizations of a mixed motivic sheaf over $\mathcal{M}_{1,1/\mathbb{Q}}$ in the sense of Ayoub [2] or Arapura [1]. If this is the case, then their pullbacks to T should be the set of realizations of a mixed elliptic motive over T in the sense of Goncharov [21]. We have not pursued this as our interests lie in the tannakian aspects of the theory, especially in the determination of the fundamental group of \mathbf{MEM}_1 and its relation to classical modular forms and mixed Tate motives.

The most basic object of \mathbf{MEM}_1 is the local system $\mathbb{H} := R^1\pi_*\mathbb{Q}$ associated to the universal elliptic curve $\pi : \mathcal{E}/\mathbb{Z} \rightarrow \mathcal{M}_{1,1/\mathbb{Z}}$. The simple objects of \mathbf{MEM}_1 are the Tate twists $S^m\mathbb{H}(r)$ of the symmetric powers of \mathbb{H} . The elliptic polylogarithms of Beilinson and Levin [7] produce basic objects of \mathbf{MEM}_1 . These provide, for each $n > 1$, a non-trivial extension

$$0 \rightarrow S^{2n-2}\mathbb{H}(2n-1) \rightarrow \mathbb{E} \rightarrow \mathbb{Q}(0) \rightarrow 0, \tag{1.1}$$

which corresponds to the Eisenstein series of weight $2n$.

The tannakian fundamental group of \mathbf{MEM}_1 is an extension of \mathbf{GL}_2 by a prounipotent group $\mathcal{U}_1^{\mathbf{MEM}}$. The main result of this paper (Theorem 25.1) is a partial presentation of the Lie algebra $\mathfrak{u}_1^{\mathbf{MEM}}$ of $\mathcal{U}_1^{\mathbf{MEM}}$. General results imply that this has a ‘minimal presentation’ whose generators project to a basis of $H_1(\mathfrak{u}_1^{\mathbf{MEM}})$ and where a minimal set of relations projects to a basis of $H_2(\mathfrak{u}_1^{\mathbf{MEM}})$. Existing results on modular forms and mixed Tate motives are used to compute $H^1(\mathfrak{u}_1^{\mathbf{MEM}})$ and to give a lower bound on the size of $H^2(\mathfrak{u}_1^{\mathbf{MEM}})$. Specifically, the extensions (1.1) comprise a basis of the ‘geometric part’ of $H^1(\mathfrak{u}_1^{\mathbf{MEM}})$. A complete basis is obtained by adding the extensions that correspond to the (motivic) odd zeta values $\zeta^m(2m+1)$, ($m > 0$). Using work of Brown [11] and Pollack [49], we construct a (conjecturally complete) set of minimal relations between generators dual to these extensions and determine their leading quadratic terms. These relations are dual to a linearly independent set of elements of $H^2(\mathfrak{u}_1^{\mathbf{MEM}})$. Each minimal relation corresponds to a Hecke eigenform, and each Hecke eigenform determines a countable set of minimal relations. If one assumes that the ‘regulator mapping’

$$\mathrm{Ext}_{\mathbf{MEM}_1}^2(\mathbb{Q}, S^{2n}\mathbb{H}(r)) \otimes \mathbb{R} \rightarrow H_{\mathcal{D}}^2(\mathcal{M}_{1,1/\mathbb{R}}, S^{2n}\mathbb{H}_{\mathbb{R}}(r))$$

is injective,¹ an analogue of Beilinson’s conjecture [6, Conjecture 8.4.1], then these relations generate all relations in $\mathfrak{u}_1^{\mathbf{MEM}}$. This partial presentation is discussed in more detail later in the introduction.

One goal of this work is to illuminate the relationship between cusp forms of $\mathbf{SL}_2(\mathbb{Z})$ and the depth filtration of the fundamental group of \mathbf{MTM} . In this vein, in §29, we show that the relations in $\mathfrak{u}_1^{\mathbf{MEM}}$ coming from cusp forms imply the congruences between the generators of the infinitesimal Galois action on the unipotent fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ in depth 2 that were discovered by Ihara and Takao [36] and made explicit by Goncharov [22] and Schneps [51]. Our proof gives a conceptual explanation of how and why cusp forms impose relations on the depth graded quotients of mixed Tate motives.

We now give a more detailed, but still informal, discussion of universal mixed elliptic motives. The full definition is given in §6. Suppose that r and n are non-negative integers

¹See Conjecture 17.1(i) for a more precise statement.

with $r + n > 0$. Denote the moduli stack over $\text{Spec } \mathbb{Z}$ of smooth projective curves of genus 1 with n marked points and r non-zero tangent vectors by $\mathcal{M}_{1,n+\vec{r}/\mathbb{Z}}$. So $\mathcal{M}_{1,1}$ is the moduli stack of elliptic curves, and $\mathcal{M}_{1,2}$ is the moduli stack of elliptic curves $(E, 0)$ together with an additional point $x \neq 0$. For $* \in \{1, \vec{1}, 2\}$ we construct a category MEM_* of universal mixed elliptic motives over $\mathcal{M}_{1,*/\mathbb{Z}}$. Objects of MEM_* will be called *universal mixed elliptic motives of type **.

The Tate curve $\mathcal{E}_{\text{Tate}} \rightarrow \text{Spec } \mathbb{Z}[[q]]$ defines a tangential base point

$$\vec{t} : \text{Spec } \mathbb{Z}((q)) \rightarrow \mathcal{M}_{1,1/\mathbb{Z}}.$$

The normalization of the fiber \overline{E}_0 over $q = 0$ is isomorphic to \mathbb{P}^1 and has a natural coordinate w , unique up to the involution $w \leftrightarrow w^{-1}$, that takes the value 1 at the identity and defines an isomorphism of the smooth points E_0 of \overline{E}_0 with \mathbb{G}_m/\mathbb{Z} . There is therefore a map

$$\text{Spec } \mathbb{Z}((t)) \rightarrow \text{Spec } \mathbb{Z}((q, v)) \rightarrow \mathcal{M}_{1,2/\mathbb{Z}}, \quad q \mapsto t, v \mapsto t,$$

where $v = w - 1$, which determines a tangential base point of $\mathcal{M}_{1,2/\mathbb{Z}}$ and the tangential base point

$$\text{Spec } \mathbb{Z}((t)) \rightarrow \text{Spec } \mathbb{Z}((q, v)) \rightarrow \mathcal{M}_{1,\vec{1}/\mathbb{Z}}$$

of $\mathcal{M}_{1,\vec{1}/\mathbb{Z}}$. We shall denote any of these distinguished base points, including \vec{t} , by \vec{v}_o .

Informally, a *mixed elliptic motive of type $* \in \{1, \vec{1}, 2\}$ over \mathbb{Z}* is a ‘motivic local system’ \mathbb{V} of \mathbb{Q} -vector spaces over $\mathcal{M}_{1,*/\mathbb{Z}}$ with a weight filtration W_\bullet by sublocal systems that satisfies:

- (i) each weight graded quotient of \mathbb{V} is a sum of the simple local systems $S^n \mathbb{H}(m)$, where $\mathbb{H} = R^1 \pi_* \mathbb{Q}(0)$ and $\pi : \mathcal{E} \rightarrow \mathcal{M}_{1,*}$ is the universal elliptic curve;
- (ii) the fiber V_o of \mathbb{V} over \vec{v}_o is an object of **MTM**; its weight filtration M_\bullet is the relative weight filtration associated to the weight filtration W_\bullet and the monodromy operator about the nodal cubic.

The fiber of \mathbb{H} over \vec{v}_o will be denoted by H . It is isomorphic to $\mathbb{Q}(0) \oplus \mathbb{Q}(-1)$ as an object of **MTM**.

Objects of MEM_* have compatible Betti, Hodge, ℓ -adic and \mathbb{Q} -de Rham realizations. The Hodge realization of a mixed elliptic motive \mathbb{V} is an admissible variation of mixed Hodge structure over $\mathcal{M}_{1,*}^{\text{an}}$ whose fiber over \vec{v}_o is the associated limit mixed Hodge structure. The ℓ -adic realization of \mathbb{V} is a lisse sheaf \mathbb{V}_ℓ over $\mathcal{M}_{1,*/\mathbb{Z}[1/\ell]}$ whose fiber over \vec{v}_o is the ℓ -adic realization of V_o . The \mathbb{Q} -de Rham realization of \mathbb{V} is a bifiltered bundle $(\mathcal{V}, F^\bullet, W_\bullet)$ over $\overline{\mathcal{M}}_{1,*}/\mathbb{Q}$ with a regular connection that underlies Deligne’s canonical extension of \mathbb{V} to $\overline{\mathcal{M}}_{1,1}^{\text{an}}$ and its Hodge and weight filtrations. There are natural functors

$$\text{MTM} \rightarrow \text{MEM}_1 \rightarrow \text{MEM}_2 \rightarrow \text{MEM}_{\vec{1}} \rightarrow \text{MTM}$$

whose composite is the identity.

Along with the pure motives $S^n \mathbb{H}(m)$, the simplest objects of MEM_* include the ‘geometrically constant’ motives. These are the objects of **MTM** that are pulled back along the structure map $\mathcal{M}_{1,*/\mathbb{Z}} \rightarrow \text{Spec } \mathbb{Z}$. Their Hodge realizations are constant variations of

mixed Hodge structure (MHS) over $\mathcal{M}_{1,*}^{\text{an}}$. More interesting objects of MEM_* can be constructed from the unipotent fundamental group of punctured elliptic curves. Denote by \mathfrak{p} the local system over $\mathcal{M}_{1,2}$ whose fiber over $(E; 0, x)$ is the Lie algebra $\mathfrak{p}(E', x)$ of the unipotent completion of $\pi_1(E', x)$, where E' denotes $E - \{0\}$. The local system \mathfrak{p} is a pro-object of MEM_2 . This is an elliptic analogue of the result of Deligne and Goncharov [17] that the Lie algebra of the unipotent fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ is a pro-object of MTM .

The category MEM_* , being a \mathbb{Q} -linear neutral tannakian category, is the category of representations of an affine group scheme, unique up to inner automorphism, that we shall denote by $\pi_1(\text{MEM}_*)$. One goal of this paper is to determine the structure of $\pi_1(\text{MEM}_*)$. While we do not find an explicit presentation of its Lie algebra, we are able to give a good idea of its ‘shape’. The first step in this direction is taken in § 16 where we show that $\pi_1(\text{MEM}_*)$ is an extension of $\text{GL}(H)$ by a prounipotent group and prove that there are natural isomorphisms

$$\pi_1(\text{MEM}_{\bar{t}}) \cong \pi_1(\text{MEM}_1) \times \mathbb{Q}(1) \quad \text{and} \quad \pi_1(\text{MEM}_2) \cong \pi_1(\text{MEM}_1) \times \pi_1^{\text{un}}(E'_{\bar{t}}, \vec{v}_o),$$

where $E_{\bar{t}}$ is the fiber of the universal elliptic curve over \bar{t} . This reduces the problem of finding presentations of $\pi_1(\text{MEM}_*)$ to the case $* = 1$ and to the problem of determining how it acts on the Lie algebra of $\pi_1^{\text{un}}(E'_{\bar{t}}, \vec{v}_o)$.

When trying to understand $\pi_1(\text{MEM}_*)$, it is convenient to define the geometric fundamental group $\pi_1^{\text{geom}}(\text{MEM}_*)$ of MEM_* to be the kernel of the natural surjection $\pi_1(\text{MEM}_*) \rightarrow \pi_1(\text{MTM})$. Taking the fiber over the tangent vector \vec{v}_o corresponding to the Tate curve gives a splitting $s_{\bar{t}} : \pi_1(\text{MTM}) \rightarrow \pi_1(\text{MEM}_*)$ of this homomorphism, so that one has an isomorphism

$$\pi_1(\text{MEM}_*) \cong \pi_1(\text{MTM}) \times \pi_1^{\text{geom}}(\text{MEM}_*).$$

A key technical point (established in § 23), which simplifies the problem of finding presentations of the Lie algebras of the $\pi_1(\text{MEM}_*)$, is that the de Rham realization of every object of MEM_* has a canonical (i.e., unique natural) bigrading that splits the Hodge filtration and both weight filtrations. This generalizes the canonical grading of the de Rham realization of a mixed Tate motive that splits its weight and Hodge filtrations. This bigrading allows us to canonically identify the de Rham realization V^{DR} of an object V of MEM_* with its associated bigraded module

$$\text{Gr } V^{\text{DR}} = \bigoplus_{m,n} \text{Gr}_{2m}^M \text{Gr}_n^W V^{\text{DR}}$$

and reduces the problem of finding a presentation of the Lie algebra of $\pi_1(\text{MEM}_*)$ to the problem of finding a presentation of its associated bigraded Lie algebra.

For this reason, we now specify the fiber functor to be the de Rham fiber functor. This allows us to identify the Lie algebra $\mathfrak{g}_*^{\text{MEM}}$ of $\pi_1(\text{MEM}_*)$ with the completion of its associated bigraded Lie algebra. Denote its pronilpotent radical by $\mathfrak{u}_*^{\text{MEM}}$. The canonical splitting of the weight filtration W_{\bullet} gives a canonical splitting of the extension

$$0 \rightarrow \mathfrak{u}_*^{\text{MEM}} \rightarrow \mathfrak{g}_*^{\text{MEM}} \rightarrow \mathfrak{gl}(H) \rightarrow 0.$$

So we can regard $\mathfrak{u}_*^{\text{MEM}}$ as a Lie algebra in the category of $\mathfrak{gl}(H)$ -modules and canonically identify $\mathfrak{g}_*^{\text{MEM}}$ with $\mathfrak{gl}(H) \times \mathfrak{u}_*^{\text{MEM}}$.

The computation of the simple extensions in Part 2 is used in §20 to find a set of generators \mathbf{e}_{2n} of the Lie algebra $\mathfrak{u}_1^{\text{geom}}$ of the prounipotent radical of $\pi_1^{\text{geom}}(\text{MEM}_*)$. The generator \mathbf{e}_{2n} is dual to the extension (1.1) associated to the Eisenstein series of weight $2n$. It is a highest weight vector in a copy of $S^{2n-2}H(2n-1)$ in $\mathfrak{u}_1^{\text{geom}}$.

A fundamental result [17] of Deligne and Goncharov asserts that $\pi_1(\text{MTM})$ is an extension of \mathbb{G}_m by a prounipotent group whose Lie algebra is freely generated (non-canonically) by a set $\{\sigma_{2m+1} : m \geq 1\}$. In Part 3, we show that $\mathfrak{u}_1^{\text{MEM}}$ has a bigraded presentation with generators

$$\mathbf{z}_{2m+1} \in \text{Gr}_{-4m-2}^W \text{Gr}_{-4m-2}^M \mathfrak{u}_1^{\text{MEM}} \quad \text{and} \quad \mathbf{e}_{2n} \in \text{Gr}_{-2n}^W \text{Gr}_{-2}^M \mathfrak{u}_1^{\text{MEM}}, \quad m > 0, n > 1,$$

as a Lie algebra in the category of $\mathfrak{sl}(H)$ -modules, where each \mathbf{z}_{2m+1} spans a copy of the trivial representation of $\mathfrak{sl}(H)$ and projects to the corresponding generator σ_{2m+1} of the Lie algebra of the prounipotent radical of $\pi_1(\text{MTM})$. The Lie algebra $\mathfrak{u}_1^{\text{geom}}$ is generated (topologically) by the set

$$\{\mathbf{e}_0^j \cdot \mathbf{e}_{2n} : n > 1, 0 \leq j \leq 2n - 2\},$$

where $\mathbf{e}_0 \in \mathfrak{sl}(H)$ lowers $\mathfrak{sl}(H)$ -weights by 2 and where $u \cdot v$ denotes the adjoint action of u on v .

There are two kinds of relations in $\mathfrak{u}_1^{\text{MEM}}$: namely,

geometric relations: the relations between the geometric generators $\{\mathbf{e}_{2n} : n \geq 0\}$,

arithmetic relations: which describe how the arithmetic generators \mathbf{z}_{2m+1} act on the geometric generators \mathbf{e}_{2n} .

In §25, we compute the quadratic leading terms of a minimal generating set of relations of $\mathfrak{u}_1^{\text{MEM}}$. There is one such geometric relation for each normalized Hecke eigen cusp form and each integer $d \geq 2$, the degree of the relation as an expression in the \mathbf{e}_{2n} , $n \geq 0$. Each Eisenstein series G_{2m} determines a countable set of arithmetic relations which express $[\mathbf{z}_{2m-1}, \mathbf{e}_{2n}]$ in terms of the geometric generators $\{\mathbf{e}_k : k \geq 0\}$. We conjecture that the regulator

$$\text{Ext}_{\text{MM}}^1(\mathbb{Q}, H_{\text{cusp}}^1(\mathcal{M}_{1,1}, S^{2n}\mathbb{H}(2n+d))) \otimes_{\mathbb{Q}} \mathbb{R} \rightarrow H_{\mathcal{D}}^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}}(2n+d))^{\overline{\mathcal{F}}_{\infty}}$$

should be an isomorphism. (Cf. Conjecture 17.1(i).) This is an analogue of Beilinson’s conjecture [6]. If true, it would imply that the relations we have found generate all relations in $\mathfrak{u}_*^{\text{MEM}}$.

In Part 3, we approach the problem of finding the geometric relations by studying the monodromy representation $\mathfrak{u}_1^{\text{geom}} \rightarrow \text{Der } \mathfrak{p}$, where \mathfrak{p} denotes the Lie algebra of $\pi_1^{\text{un}}(E'_t, \vec{v}_o)$. Relations between the images of the \mathbf{e}_{2n} in $\text{Der } \mathfrak{p}$ give an upper bound on the relations that hold between the \mathbf{e}_{2n} in $\mathfrak{u}_1^{\text{geom}}$. The problem of determining these relations can be studied by passing to the associated bigraded Lie algebras. Since \mathfrak{p} is free, it is isomorphic to the completion of its associated bigraded, which is the free Lie algebra $\mathbb{L}(H) \cong \mathbb{L}(A, T)$. The \mathbb{Q} -de Rham story gives a formula for the images of the \mathbf{e}_{2n} in $\text{Der } \mathbb{L}(A, T)$ via the elliptic

KZB equation [13, 27, 43, 44]. The image of the generator e_{2n} of u_1^{geom} is $2\epsilon_{2n}/(2n - 2)!$, where ϵ_{2n} is the unique derivation² of $\mathbb{L}(H)$ satisfying

$$\epsilon_{2n}(T) = -\text{ad}_T^{2n}(A) \quad \text{and} \quad \epsilon_{2n}([T, A]) = 0, \quad n > 1.$$

Pollack, in his undergraduate thesis [49], found all quadratic relations that hold between the $\{\epsilon_{2n} : n \geq 0\}$ in $\text{Der } \mathbb{L}(A, T)$ and showed that they correspond to cuspidal cocycles of $\text{SL}_2(\mathbb{Z})$ invariant under the ‘real Frobenius’ operator. He also found relations of each degree $d \geq 3$ that hold between the ϵ_{2n} modulo ‘depth 3’, one for each invariant cuspidal cocycle of $\text{SL}_2(\mathbb{Z})$. His results are summarized in § 24.

A lower bound on the relations in u_1^{MEM} is obtained from the computation of the cup products $(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}(r))$ in the real Deligne cohomology groups $H_D^\bullet(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}(r))$. These are deduced from the fundamental work of Brown [11] who computed periods of twice iterated integrals of Eisenstein series. His work is closely related to unpublished computations of Terasoma [58] who computed cup products in Deligne cohomology of the classes in higher Chow groups of Kuga–Sato varieties that correspond to Eisenstein series, generalizing to higher weight Beilinson’s computations [5] in weight 2.³ It turns out that the upper and lower bounds on the quadratic heads of the relations in u_1^{MEM} coincide, as we show in § 25, which allows us to determine the quadratic leading terms of all relations in u_1^{MEM} .

In related work, Enriquez [19] has defined an elliptic generalization of Drinfeld’s braided monoidal categories [18] and defined an affine \mathbb{Q} -group GRT_{ell} for which the scheme of elliptic associators is a torsor.⁴ It is a split extension

$$1 \rightarrow R_{\text{ell}} \rightarrow \text{GRT}_{\text{ell}} \rightarrow \text{GRT} \rightarrow 1,$$

where GRT is the affine \mathbb{Q} -group, defined by Drinfeld [18], and where the projection to GRT is induced by the map (27.3). One expects there to be a homomorphism $\pi_1(\text{MEM}_{\bar{1}}) \rightarrow \text{GRT}_{\text{ell}}$ with an appropriate choice of fiber functors that induces a map to the extension above from the extension

$$1 \rightarrow \pi_1^{\text{geom}}(\text{MEM}_{\bar{1}}) \rightarrow \pi_1(\text{MEM}_{\bar{1}}) \rightarrow \pi_1(\text{MTM}) \rightarrow 1.$$

Assuming that it exists, this homomorphism will be surjective if and only if Enriquez’s conjecture [19, Conjecture 9.1] holds and the standard homomorphism $\pi_1(\text{MTM}) \rightarrow \text{GRT}$ is surjective. It will be injective if and only if $u^{\text{MEM}} \rightarrow \text{Der } \mathfrak{p}$ is injective.⁵ The existence of this homomorphism will imply that the relations in u_1^{MEM} will also hold in the Lie algebra grt_{ell} of GRT_{ell} .

²These derivations were first studied by Tsunogai [59] and later used in this context by Calaque *et al.* [13].

³There is an inconsistency between the results of Brown and Terasoma. Our computations are consistent with Brown’s. Nonetheless, we believe that Terasoma’s results are basically correct.

⁴Implicit in [19] is that this group depends on the choice of a free Lie algebra of rank 2 over \mathbb{Q} together with an ordered pair of generators. Two natural choices are the Betti and de Rham realizations of the Lie algebra of $\pi_1^{\text{un}}(E_{\bar{1}}^{\vee}, \bar{v}_o)$ with a pair of generators which descends to a symplectic basis of its abelianization. Such a choice will correspond to the choice of a fiber functor.

⁵Brown’s result [10] implies that $\pi_1(\text{MTM}) \rightarrow \text{GRT}$ is injective.

This paper is in four parts. In the first part we define universal mixed elliptic motives. Before doing this, we give a detailed description of the local system \mathbb{H} over $\mathcal{M}_{1,1}$ and its fiber H over \vec{t} in all of its manifestations. We also consider the basic structure of $\pi_1(\text{MEM}_*)$ that one gets from formal considerations.

The second part is devoted to proving that the elliptic polylogarithmic extensions (1.1) lie in MEM_* and that they exhaust all extensions of \mathbb{Q} by a simple object $S^m\mathbb{H}(r)$ of MEM_1 when $m > 0$. For this we use the Hodge theory of the relative completion of $\text{SL}_2(\mathbb{Z})$, which is described in detail in [28]. Results from [29] allow us to relate extensions of admissible variations of mixed Hodge structure over $\mathcal{M}_{1,*}^{\text{an}}$ to the Deligne–Beilinson cohomology groups $H_{\mathcal{D}}^{\bullet}(\mathcal{M}_{1,*}^{\text{an}}, S^m\mathbb{H}(r))$. We also use weighted and crystalline completion to study the ℓ -adic aspects. This part concludes with a discussion of the relationship between the groups $\text{Ext}_{\text{MEM}_*}^2(\mathbb{Q}, S^m\mathbb{H}(r))$ and certain of the conjectures of Beilinson and Bloch–Kato on regulators associated to modular curves.

In Part 3 we consider the problem of finding a presentation of the Lie algebra $\mathfrak{u}_*^{\text{MEM}}$. We first show that the \mathbf{e}_{2n} and the \mathbf{z}_{2m+1} generate $\mathfrak{u}_1^{\text{MEM}}$. We then recall Pollack’s relations and use Brown’s period computations to show that they lift to relations in $\mathfrak{u}_*^{\text{MEM}}$.

In Part 4 we use the results of Part 3 to begin the study of the relationship between universal mixed elliptic motives and mixed Tate motives. This is achieved by specialization to the nodal cubic. As an application, we deduce the Ihara–Takao congruences from Pollack’s relations. We also work out the precise relationship between the depth filtration on $\pi_1(\text{MTM})$ and the pullback of the ‘elliptic weight filtration’ W_{\bullet} of $\text{Der } \mathfrak{p}$.

1.1. History

This paper evolved slowly over the past 8 years and owes much to its antecedents. These include the paper [7] of Beilinson and Levin on the elliptic polylogarithm, the work of Deligne [15] on the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$, and the paper of Deligne and Goncharov [17] on mixed Tate motives. The works of Calaque *et al.* [13] and Levin–Racinet [43] on the elliptic KZB connection proved to be useful, both in understanding the \mathbb{Q} -de Rham picture and in establishing formulas that play a key role in Part 4. Finally, Nakamura’s computation [45] of the action of the absolute Galois group $G_{\mathbb{Q}}$ on the fundamental group of $E'_{\vec{t}}$ was also a useful reference point and helped us understand the ℓ -adic picture.

This work originated in informal computations done in the Spring of 2007 in which we found (under optimistic assumptions) the basic form (described in §21) of a presentation of the ℓ -adic crystalline completion of $\pi_1(\mathcal{M}_{1,1/\mathbb{Z}[\ell^{-1}]}, \vec{t})$. Standard conjectures on the ranks of Selmer groups and an optimistic assumption on the size of the image of cup products in Galois cohomology – which now appears likely to be true – led us to predict that each cusp form of $\text{SL}_2(\mathbb{Z})$ should determine a sequence of relations in the unipotent radical of the crystalline completion, and thus between the derivations ϵ_{2n} discussed above. The first author gave the problem of finding some of these relations to Aaron Pollack, who was then an undergraduate. He found all the quadratic relations between the ϵ_{2n} and also relations of each degree ≥ 3 that held modulo depth ≥ 3 . This suggested that our optimistic assumptions were indeed true. However, the proof that Pollack’s

relations lifted to u_1^{MEM} had to wait until the period computations of Brown [11]. There is no ℓ -adic proof of this. It would be very interesting to have one.

2. Notation and conventions

All moduli spaces in this paper will be regarded as stacks and their underlying analytic spaces will be regarded as orbifolds (i.e., stacks in the category of complex analytic varieties). The dual $\text{Hom}_F(V, F)$ of a vector space V over a field F will be denoted by V^\vee .

2.1. Path multiplication

We use the topologists’ convention. The product $\alpha\beta$ of two paths $[0, 1] \rightarrow X$ in a topological space is defined when $\alpha(1) = \beta(0)$. The product $\alpha\beta$ traverses α first, then β . This is the opposite of the algebraists’ convention, where paths are multiplied in the ‘functional order’.

2.2. The topological fundamental group

Suppose that X is an algebraic variety over a subring R of \mathbb{C} . Denote the corresponding analytic variety by X^{an} . For $x \in X(\mathbb{C})$, we denote the topological fundamental group of X^{an} by $\pi_1^{\text{top}}(X, x)$.

If X is a Deligne–Mumford stack (hereafter, DM stack) over R , the associated analytic space X^{an} is an orbifold (i.e., a stack in the category of topological spaces). For each suitable base point x of X^{an} , we denote the orbifold fundamental group of X^{an} by $\pi_1^{\text{top}}(X, x)$. Fundamental groups of stacks are defined in [47], for example. Typically in this paper, X will be the stack/orbifold quotient of a smooth variety by a finite group. In this case the orbifold fundamental group is easy to describe directly. See, [26, § 3], for example.

Denote the orbifold quotient of a topological space \mathcal{X} by a discrete group Γ by $\Gamma \backslash \mathcal{X}$. If \mathcal{X} is simply connected, then for each choice of a base point $x_o \in \mathcal{X}$, there is a natural isomorphism

$$\pi_1^{\text{top}}(\Gamma \backslash \mathcal{X}, x_o) \cong \Gamma.$$

In addition, we can regard the projection $p : \mathcal{X} \rightarrow \Gamma \backslash \mathcal{X}$ as a base point. In this case Γ is naturally isomorphic to $\pi_1^{\text{top}}(\Gamma \backslash \mathcal{X}, p)$. For example

$$\pi_1^{\text{top}}(\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}, p) \cong \text{SL}_2(\mathbb{Z}).$$

2.3. The étale fundamental group

Denote the étale fundamental group of a scheme X over a ring R by $\pi_1(X, \bar{x})$, where \bar{x} is a geometric point of X . The étale fundamental group of a DM stack X can also be defined – see [47]. We will also denote it by $\pi_1(X, \bar{x})$. Suppose that R is a field k with algebraic closure \bar{k} . The fundamental group of $\text{Spec } k$ with respect to the geometric point $\text{Spec } \bar{k} \rightarrow \text{Spec } k$ is simply $\text{Gal}(\bar{k}/k)$. We denote it by G_k . The structure morphism $X \rightarrow \text{Spec } k$ induces a homomorphism $\pi_1(X, \bar{x}) \rightarrow G_k$. One has the canonical short exact

sequence

$$1 \rightarrow \pi_1(X \times_k \bar{k}, \bar{x}) \rightarrow \pi_1(X, \bar{x}) \rightarrow G_k \rightarrow 1. \tag{2.1}$$

2.4. Topological versus étale fundamental groups

When X is a DM stack over \mathbb{C} and $x \in X(\mathbb{C})$, there is a natural isomorphism⁶

$$\pi_1(X, x) \cong \pi_1^{\text{top}}(X, x)^\wedge$$

between the étale fundamental group of X and the profinite completion of the topological fundamental group of the associated analytic variety. When k is a subfield of \mathbb{C} , the exact sequence (2.1) becomes

$$1 \rightarrow \pi_1^{\text{top}}(X, x)^\wedge \rightarrow \pi_1(X, x) \rightarrow G_k \rightarrow 1.$$

2.5. Tannakian fundamental groups

Suppose that F is a field of characteristic zero. We will denote the tannakian fundamental group of an F -linear neutral tannakian category \mathcal{C} with respect to a fiber functor $\omega : \mathcal{C} \rightarrow \mathbf{Vec}_F$ by $\pi_1(\mathcal{C}, \omega)$. It is an affine group scheme that is an inverse limit of affine algebraic groups. When the context is clear, we will omit ω from the notation.

We will denote the category of linear representations of a group G over a field F by $\mathbf{Rep}_F(G)$ and the subcategory of finite dimensional representations by $\mathbf{Rep}_F^{\text{fte}}(G)$. If G is an affine F -group (scheme), we will write $\mathbf{Rep}(G)$ instead of $\mathbf{Rep}_F(G)$. If F has characteristic zero, then $\mathbf{Rep}(G)$ is equivalent to the category of ind-objects of $\mathbf{Rep}^{\text{fte}}(G)$.

Subsequently, the term ‘algebraic group’ will mean ‘affine algebraic group’. In particular, we will not refer to elliptic curves as algebraic groups.

2.6. Cohomology of affine group schemes

The cohomology of an affine group scheme G over a field F of characteristic zero with coefficients in a G -module V is defined by

$$H^\bullet(G, V) := \text{Ext}_{\mathbf{Rep}(G)}^\bullet(F, V),$$

where $\mathbf{Rep}(G)$ denotes the category of G -modules. The fact that every G -module is a direct limit of its finite dimensional submodules implies that if V is finite dimensional over F , then

$$H^\bullet(G, V) = \text{Ext}_{\mathbf{Rep}^{\text{fte}}(G)}^\bullet(F, V),$$

where $\mathbf{Rep}^{\text{fte}}(G)$ denotes the category of finite dimensional G -modules. If G is an extension of a reductive group R by a prounipotent group U , and if U has Lie algebra \mathfrak{u} , then there are natural isomorphisms

$$H^\bullet(G, V) \cong H^\bullet(U, V)^R \cong H^\bullet(\mathfrak{u}, V)^R.$$

For background and more details, see [37, Chapters 3, 4] and [29].

⁶Strictly speaking, our path multiplication convention implies that this is an anti-isomorphism. If one prefers, one can replace the étale fundamental group by its opposite group.

2.7. Free Lie algebras

The free Lie algebra generated by a vector space V will be denoted by $\mathbb{L}(V)$. The free Lie algebra generated by a set \mathcal{X} will be denoted by $\mathbb{L}(\mathcal{X})$. It is isomorphic to the free Lie algebra generated by the vector space with basis \mathcal{X} .

Part 1. Introduction to universal mixed elliptic motives

3. Mixed Tate motives

Deligne and Goncharov [17], using work of Voevodsky [60] and Levine [41, 42], constructed a \mathbb{Q} -tannakian category of *mixed Tate motives* $\text{MTM}(\mathcal{O}_{K,S})$ over a number field K , unramified outside a subset S of primes of \mathcal{O}_K . This category has simple objects $\mathbb{Q}(n)$ and has the property that

$$\text{Ext}_{\text{MTM}(\mathcal{O}_{K,S})}^j(\mathbb{Q}, \mathbb{Q}(n)) \cong \begin{cases} \mathbb{Q} & j = n = 0, \\ K_{2n-1}(\mathcal{O}_{K,S}) \otimes \mathbb{Q} & j = 1, n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

In this paper, we need only the case where $\mathcal{O}_{K,S} = \mathbb{Z}$, so for simplicity we restrict discussion to that case. Denote $\text{MTM}(\mathbb{Z})$ by MTM . Each object V of MTM has a weight filtration, which we denote by M_\bullet . There are realization functors on MTM that take the object V to its Betti realization (V^B, M_\bullet) , its de Rham realization $(V^{\text{DR}}, M_\bullet, F^\bullet)$, and for each prime number ℓ , its ℓ -adic realization (V_ℓ, M_\bullet) , which is a filtered $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ module, unramified outside ℓ and crystalline at ℓ . There are comparison isomorphisms

$$((V^{\text{DR}}, M_\bullet) \otimes_{\mathbb{Q}} \mathbb{C} \cong V^B, M_\bullet) \otimes_{\mathbb{Q}} \mathbb{C} \quad \text{and} \quad (V^B, M_\bullet) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong (V_\ell, M_\bullet).$$

The Betti and de Rham realizations combine to give the Hodge realization of V , which is a mixed Hodge structure whose weight graded quotients are of Tate type. One important property of MTM is that the functor that takes MTM to the category MHS of \mathbb{Q} - MHS is fully faithful.

Every simple object of MTM is isomorphic to $\mathbb{Q}(n)$ for some $n \in \mathbb{Z}$. For later use, we recall some basic facts. The weight filtration of $V = \mathbb{Q}(n)$ is

$$0 = M_{-2n-1}V \subset M_{-2n}V = V$$

and that the realizations of $\mathbb{Q}(n)$ are

$$V^B = \mathbb{Q}, \quad V^{\text{DR}} = \mathbb{Q}, \quad V_\ell = \mathbb{Q}_\ell.$$

The Galois action on V_ℓ is given by the n th power of the ℓ -adic cyclotomic character $\chi_\ell : G_K \rightarrow \mathbb{Z}_\ell^\times$. The Hodge filtration of V^{DR} is

$$V^{\text{DR}} = F^{-n}V^{\text{DR}} \supset F^{-n+1}V^{\text{DR}} = 0.$$

The comparison isomorphism $V^{\text{DR}} \otimes \mathbb{C} \rightarrow V^B \otimes \mathbb{C}$ is multiplication by $(2\pi i)^{-n}$. The Hodge realization is the one-dimensional Hodge structure of type $(-n, -n)$.

The category MTM^{ss} of semi-simple mixed Tate motives (i.e., those that are direct sum of copies of $\mathbb{Q}(m)$'s) is neutral tannakian and has fundamental group isomorphic to \mathbb{G}_m ; the mixed Tate motive $\mathbb{Q}(m)$ corresponds to the m th power of the standard character of \mathbb{G}_m . The functor $\text{Gr}_{\bullet}^M : \text{MTM} \rightarrow \text{MTM}^{\text{ss}}$ is a retraction of MTM onto MTM^{ss} .

Let

$$\omega^B : \text{MTM} \rightarrow \text{Vec}_{\mathbb{Q}}, \quad \omega^{\text{DR}} : \text{MTM} \rightarrow \text{Vec}_{\mathbb{Q}}, \quad \omega_{\ell} : \text{MTM} \rightarrow \text{Vec}_{\mathbb{Q}_{\ell}}$$

be the fiber functors that take V to V^B , V^{DR} and V_{ℓ} , respectively. For each of these, the inclusion $\text{MTM}^{\text{ss}} \rightarrow \text{MTM}$ induces a homomorphism $\pi_1(\text{MTM}, \omega) \rightarrow \mathbb{G}_m$. Denote its kernel by \mathcal{K} (or \mathcal{K}_{ω} when we want to identify the fiber functor). It is pronipotent. Since for every mixed Tate motive V , there is a natural isomorphism

$$V^{\text{DR}} \cong \bigoplus_n F^n V^{\text{DR}} \cap M_{2n} V^{\text{DR}}$$

of \mathbb{Q} -vector spaces, the functor Gr_{\bullet}^M induces a splitting of the homomorphism $\pi_1(\text{MTM}, \omega^{\text{DR}}) \rightarrow \pi_1(\text{MTM}^{\text{ss}}, \omega^{\text{DR}}) = \mathbb{G}_m$. The Lie algebra \mathfrak{k} of \mathcal{K} is thus a \mathbb{G}_m -module. It is (non-canonically) isomorphic to the completion of the free Lie algebra

$$\mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \dots),$$

where \mathbb{G}_m acts on σ_{2n+1} via the $(2n + 1)$ st power of the standard character.

4. Moduli spaces of elliptic curves

Suppose that r and n are non-negative integers satisfying $r + n > 0$. Denote the moduli stack over $\text{Spec } \mathbb{Z}$ of smooth projective curves of genus 1 with n marked points and r non-zero tangent vectors by $\mathcal{M}_{1,n+\bar{r}}$. Here the n marked points and the anchor points of the r tangent vectors are distinct. Taking each tangent vector to its anchor point defines a morphism $\mathcal{M}_{1,n+\bar{r}} \rightarrow \mathcal{M}_{1,n+r}$ that is a principal \mathbb{G}_m^r -bundle. The Deligne–Mumford compactification [39] of $\mathcal{M}_{1,n}$ will be denoted by $\overline{\mathcal{M}}_{1,n}$. It is also defined over $\text{Spec } \mathbb{Z}$. In this paper, we are primarily concerned with the cases $(n, r) = (1, 0), (0, 1)$ and $(2, 0)$. Then $\mathcal{M}_{1,1}$ is the moduli stack of elliptic curves $(E, 0)$, and $\mathcal{M}_{1,2}$ is the moduli stack of elliptic curves $(E, 0)$ with an additional point $x \neq 0$.

4.1. The moduli stacks $\mathcal{M}_{1,1}$, $\mathcal{M}_{1,\bar{1}}$ and $\mathcal{M}_{1,2}$

For a \mathbb{Z} -algebra A , we denote by $\mathcal{M}_{1,n+\bar{r}/A}$ the stack $\mathcal{M}_{1,n+\bar{r}} \times_{\text{Spec } \mathbb{Z}} \text{Spec } A$. Similarly, the pullback of $\overline{\mathcal{M}}_{1,n}$ to $\text{Spec } A$ will be denoted by $\overline{\mathcal{M}}_{1,n/A}$. The universal elliptic curve over $\mathcal{M}_{1,1}$ will be denoted by \mathcal{E} . Note that $\mathcal{M}_{1,2}$ is \mathcal{E} with its identity section removed. The extension $\overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$ of the universal elliptic curve to $\overline{\mathcal{M}}_{1,1}$ is obtained by glueing in the Tate curve $\mathcal{E}_{\text{Tate}} \rightarrow \text{Spec } \mathbb{Z}[[q]]$ (cf. [54, Chapter V]). It is simply $\overline{\mathcal{M}}_{1,2}$.

The unique cusp of $\overline{\mathcal{M}}_{1,1}$ is the moduli point of the nodal cubic. Denote it by e_o . The standard line bundle \mathcal{L} over $\overline{\mathcal{M}}_{1,1}$ is the conormal bundle of the zero section of $\overline{\mathcal{E}} \rightarrow \overline{\mathcal{M}}_{1,1}$. Sections of $\mathcal{L}^{\otimes n}$ over $\overline{\mathcal{M}}_{1,1}$ are modular forms of weight n , and those that vanish at the cusp e_o are cusp forms of weight n .

The restriction of the relative tangent bundle of $\bar{\mathcal{E}}$ to its identity section is the dual $\check{\mathcal{L}}$ of \mathcal{L} . The moduli space $\mathcal{M}_{1,\bar{1}}$ is $\check{\mathcal{L}}'$, the restriction of $\check{\mathcal{L}}$ to $\mathcal{M}_{1,1}$ with its zero section removed. This is isomorphic to the complement \mathcal{L}' of the zero section of \mathcal{L} .

We will also identify $\overline{\mathcal{M}}_{1,1}$ with the identity section of $\bar{\mathcal{E}}$. With this convention, e_o also denotes the identity of the nodal cubic in $\bar{\mathcal{E}}$ and also the corresponding point of the partial compactification $\check{\mathcal{L}}$ of $\mathcal{M}_{1,\bar{1}}$.

An explicit description of \mathcal{L}' over $\overline{\mathcal{M}}_{1,1}$ can be deduced from the discussion [38, Chapter 2] and the formulas in [53, Appendix A]; namely $\overline{\mathcal{M}}_{1,1}$ is the quotient stack $\mathbb{G}_m \backslash \mathcal{L}'$. When 2 and 3 are invertible in A , \mathcal{L}'_A is the scheme

$$\mathcal{L}'_A = \mathbb{A}_A^2 - \{0\} = \text{Spec } A[u, v] - \{0\}.$$

The point (u, v) corresponds to the plane cubic $y^2 = 4x^3 - ux - v$ and the abelian differential dx/y . The \mathbb{G}_m -action is $\lambda : (u, v) \mapsto (\lambda^{-4}u, \lambda^{-6}v)$. In this case, $\overline{\mathcal{M}}_{1,1}$ and $\mathcal{M}_{1,1}$ are the quotient stacks

$$\overline{\mathcal{M}}_{1,1/A} = \mathbb{G}_m \backslash \mathcal{L}'_A \quad \text{and} \quad \mathcal{M}_{1,1/A} = \mathbb{G}_m \backslash (\mathbb{A}_A^2 - D^{-1}(0)),$$

where $D = u^3 - 27v^2$ is (up to a factor of 4) the discriminant of the cubic.

Since some 2-pointed genus 1 curves have a non-trivial automorphism, $\overline{\mathcal{M}}_{1,2}$ is a stack, but not a scheme. When 2 and 3 are invertible in A , the stack $\mathcal{M}_{1,2/A}$ is the quotient of the scheme

$$\{(u, v, x, y) \in \mathbb{A}_A^2 \times \mathbb{A}_A^2 : y^2 = 4x^3 - ux - v, (u, v) \neq 0\}$$

by the \mathbb{G}_m -action

$$\lambda : (u, v, x, y) \mapsto (\lambda^{-4}u, \lambda^{-6}v, \lambda^{-2}x, \lambda^{-3}y).$$

The point (u, v, x, y) corresponds to the point (x, y) on the cubic $y^2 = 4x^3 - ux - v$ and the abelian differential dx/y . The action of λ multiplies dx/y by λ .

4.2. Tangent vectors

We use Deligne’s tangential base points [15, §15]. The Tate curve $E_{\text{Tate}} \rightarrow \text{Spec } \mathbb{Z}[[q]]$ corresponds to a morphism $\text{Spec } \mathbb{Z}[[q]] \rightarrow \overline{\mathcal{M}}_{1,1}$. The parameter q is a formal parameter of $\overline{\mathcal{M}}_{1,1}$ at the cusp e_o . The tangent vector

$$\partial/\partial q = \text{Spec } \overline{\mathbb{Q}}((q^{1/n} : n \geq 1))$$

of $\overline{\mathcal{M}}_{1,1}$ at the moduli point of the nodal cubic is integral and its reduction mod p is non-zero for all prime numbers p . Denote it by $\bar{\mathfrak{t}}$. Identify $\overline{\mathcal{M}}_{1,1}$ with the identity section of the completion $\overline{\mathcal{M}}_{1,2}$ of the universal elliptic curve over $\overline{\mathcal{M}}_{1,1}$. With this identification, e_o is the identity of the nodal cubic.

The fiber \bar{E}_0 of the Tate curve over $q = 0$ is an integral model of the nodal cubic. Its normalization is $\mathbb{P}_{\mathbb{Z}}^1$. Let E_0 be \bar{E}_0 with its double point removed. It is isomorphic to \mathbb{G}_m/\mathbb{Z} . Let w be a parameter on \bar{E}_0 whose pullback to the normalization of \bar{E}_0 takes the values 0 and ∞ on the inverse image of the double point, and the value 1 at the identity. It is unique up to $w \mapsto w^{-1}$. It determines the tangent vector $\bar{\mathfrak{w}}_o := \partial/\partial w$ of \bar{E}_0 at the identity e_o of E_0 . It is integrally defined and non-zero at all primes.

We thus have the tangent vector $\bar{\mathfrak{v}}_o := \bar{\mathfrak{t}} + \bar{\mathfrak{w}}_o$ of $\mathcal{M}_{1,\bar{1}}$ (and thus of $\mathcal{M}_{1,2}$ and \mathcal{E} as well) at the identity e_o of E_0 , which is non-zero at all primes.

4.3. Moduli spaces as complex orbifolds

To fix notation and conventions, we give a quick review of the construction of $\mathcal{M}_{1,1}^{\text{an}}$, \mathcal{E}^{an} and their Deligne–Mumford compactifications. All will be regarded as complex analytic orbifolds. This material is classical and very well known. A detailed discussion of these constructions and an explanation of the notation can be found in [26].

4.3.1. The orbifolds $\mathcal{M}_{1,1}^{\text{an}}$ and $\overline{\mathcal{M}}_{1,1}^{\text{an}}$. The moduli space $\mathcal{M}_{1,1}^{\text{an}}$ is the orbifold quotient $\mathcal{M}_{1,1}^{\text{an}} = \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$ of the upper half plane \mathfrak{h} by the standard $\text{SL}_2(\mathbb{Z})$ action. For $\tau \in \mathfrak{h}$, set $\Lambda_\tau := \mathbb{Z} \oplus \mathbb{Z}\tau$. The point $\tau \in \mathfrak{h}$ corresponds to the elliptic curve

$$(E_\tau, 0) := (\mathbb{C}/\Lambda_\tau, 0),$$

together with the symplectic basis \mathbf{a}, \mathbf{b} of $H_1(E_\tau, \mathbb{Z})$ that corresponds to the generators $1, \tau$ of Λ_τ via the canonical isomorphism $\Lambda_\tau \cong H_1(E_\tau, \mathbb{Z})$. The element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of $\text{SL}_2(\mathbb{Z})$ takes the basis \mathbf{a}, \mathbf{b} of $H_1(E_\tau, \mathbb{Z})$ to the basis

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \tag{4.1}$$

of $H_1(E_{\gamma\tau}, \mathbb{Z})$.

The orbifold $\overline{\mathcal{M}}_{1,1}^{\text{an}}$ underlying the Deligne–Mumford compactification $\overline{\mathcal{M}}_{1,1}$ of $\mathcal{M}_{1,1}$ is obtained by glueing in the quotient $C_2 \backslash \mathbb{D}$ of a disk \mathbb{D} of radius $e^{-2\pi}$ by a trivial action of the cyclic group $C_2 := \{\pm 1\}$. These are glued together by the diagram

$$C_2 \backslash \mathbb{D} \xleftarrow{\approx} \begin{pmatrix} \pm 1 & \mathbb{Z} \\ 0 & \pm 1 \end{pmatrix} \backslash \{\tau \in \mathfrak{h} : \text{Im}(\tau) > 1\} \longrightarrow \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h},$$

where the left-hand map takes τ to $q := \exp(2\pi i\tau)$ and where C_2 is included in $\begin{pmatrix} \pm 1 & \mathbb{Z} \\ 0 & \pm 1 \end{pmatrix}$ as the scalar matrices.

4.3.2. The line bundle \mathcal{L}^{an} . The restriction of \mathcal{L}^{an} to $\mathcal{M}_{1,1}^{\text{an}}$ is the orbifold quotient of the trivial line bundle $\mathbb{C} \times \mathfrak{h} \rightarrow \mathfrak{h}$ by the $\text{SL}_2(\mathbb{Z})$ action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (z, \tau) \mapsto ((c\tau + d)z, (a\tau + b)/(c\tau + d)).$$

It is an orbifold line bundle over $\mathcal{M}_{1,1}^{\text{an}}$. Its restriction (i.e., pullback) to the punctured q -disk

$$\mathbb{D}^* := \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \backslash \mathfrak{h}$$

is naturally isomorphic to the trivial bundle $\mathbb{C} \times \mathbb{D}^* \rightarrow \mathbb{D}^*$, and therefore extends naturally to a (trivial) line bundle over \mathbb{D} . The line bundle \mathcal{L}^{an} over $\overline{\mathcal{M}}_{1,1}$ is the unique extension of the line bundle above to $\overline{\mathcal{M}}_{1,1}^{\text{an}}$ that restricts to this trivial bundle over the q -disk.

4.3.3. Eisenstein series. To fix notation and normalizations we recall some basic facts from the analytic theory of elliptic curves.

Suppose that $k \geq 1$. The (normalized) Eisenstein series of weight $2k$ is defined by the series

$$G_{2k}(\tau) = \frac{1}{2} \frac{(2k-1)!}{(2\pi i)^{2k}} \sum_{\substack{\lambda \in \mathbb{Z} \oplus \mathbb{Z}\tau \\ \lambda \neq 0}} \frac{1}{\lambda^{2k}} = -\frac{B_{2k}}{4k} + \sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n,$$

where $\sigma_k(n) = \sum_{d|n} d^k$. When $k > 1$ it converges absolutely to a modular form of weight $2k$. When properly summed, it also converges when $k = 1$. In this case the logarithmic 1-form

$$\frac{d\xi}{\xi} - 2 \cdot 2\pi i G_2(\tau) d\tau \tag{4.2}$$

is $SL_2(\mathbb{Z})$ -invariant, where $SL_2(\mathbb{Z})$ acts on $\mathbb{C}^* \times \mathfrak{h}$ by $\gamma : (\xi, \tau) \mapsto (\xi/(c\tau + d), \gamma\tau)$.

The ring of modular forms of $SL_2(\mathbb{Z})$ is the polynomial ring $\mathbb{C}[G_4, G_6]$.

4.4. The Weierstrass \wp -function

For a lattice $\Lambda \subset \mathbb{C}$, the Weierstrass \wp -function is defined by

$$\wp_{\Lambda}(z) := \frac{1}{z^2} + \sum_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} \left[\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right].$$

For $\tau \in \mathfrak{h}$, set $\wp_{\tau}(z) = \wp_{\Lambda_{\tau}}(z)$. One has the expansion

$$\wp_{\tau}(z) = (2\pi i)^2 \left(\frac{1}{(2\pi i z)^2} + \sum_{m=1}^{\infty} \frac{2}{(2m)!} G_{2m+2}(\tau) (2\pi i z)^{2m} \right).$$

The function $\mathbb{C} \rightarrow \mathbb{P}^2(\mathbb{C})$ defined by $z \mapsto [(2\pi i)^{-2}\wp_{\tau}(z), (2\pi i)^{-3}\wp'_{\tau}(z), 1]$ induces an embedding of $\mathbb{C}/\Lambda_{\tau}$ into $\mathbb{P}^2(\mathbb{C})$. The image has affine equation

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau),$$

where

$$g_2(\tau) = 20G_4(\tau) \quad \text{and} \quad g_3(\tau) = \frac{7}{3}G_6(\tau).$$

This has discriminant the normalized cusp form of weight 12:

$$\Delta(\tau) := g_2(\tau)^3 - 27g_3(\tau)^2 = q \prod_{n \geq 1} (1 - q^n)^{24}.$$

The following statement is easily verified. For a proof of the last statement, see [27, Proposition 19.1].

Proposition 4.1. *The abelian differential dx/y corresponds to $d(2\pi i dz)$ and the differential $x dx/y$ of the second kind corresponds to $(2\pi i)^{-2} \wp_{\tau}(z) d(2\pi i z)$. These differentials form a symplectic basis of $H_{DR}^1(E_{\tau})$ as*

$$\int_{E_{\tau}} \frac{dx}{y} \smile \frac{x dx}{y} = 2\pi i.$$

4.5. The analytic space $\mathcal{M}_{1,1}^{\text{an}}$

Recall that $D(u, v) = u^3 - 27v^2$. The map

$$\mathbb{C}^* \times \mathfrak{h} \rightarrow \mathbb{C}^2 - D^{-1}(0), \quad (\xi, \tau) \mapsto (\xi^{-4}g_2(\tau), \xi^{-6}g_3(\tau))$$

induces a biholomorphism $\mathcal{L}^{\text{an}'} \rightarrow \mathcal{M}_{1,1}^{\text{an}} = \mathbb{C}^2 - D^{-1}(0)$. The point (ξ, τ) of $\mathbb{C}^* \times \mathfrak{h}$ corresponds to the point

$$(\mathbb{C}/\xi\Lambda_\tau, 2\pi i dz) \cong (\mathbb{C}/\Lambda_\tau, 2\pi i\xi dz)$$

of $\mathcal{M}_{1,1}^{\text{an}}$, which also corresponds to the curve $y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$ with the abelian differential $\xi dx/y$.

4.5.1. The universal elliptic curve \mathcal{E}^{an} . Define Γ to be the subgroup of $\text{GL}_3(\mathbb{Z})$ that consists of the matrices

$$\gamma = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix}$$

where $a, b, c, d, m, n \in \mathbb{Z}$ and $ad - bc = 1$. It is isomorphic to the semi-direct product of $\text{SL}_2(\mathbb{Z})$ and \mathbb{Z}^2 and acts on $X := \mathbb{C} \times \mathfrak{h}$ on the left via the formula $\gamma : (z, \tau) \mapsto (z', \tau')$, where

$$\begin{pmatrix} \tau' \\ 1 \\ z' \end{pmatrix} = (c\tau + d)^{-1} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix} \begin{pmatrix} \tau \\ 1 \\ z \end{pmatrix}.$$

The universal elliptic curve $\pi^{\text{an}} : \mathcal{E}^{\text{an}} \rightarrow \mathcal{M}_{1,1}^{\text{an}}$ is the orbifold quotient of the projection $\mathbb{C} \times \mathfrak{h} \rightarrow \mathfrak{h}$ by Γ , which acts on \mathfrak{h} via the quotient map $\Gamma \rightarrow \text{SL}_2(\mathbb{Z})$. The fiber of π^{an} over the orbit of τ is E_τ .

It can also be regarded as the orbifold quotient $\text{SL}_2(\mathbb{Z}) \backslash \mathcal{E}_{\mathfrak{h}}$ of the universal framed elliptic curve

$$\mathcal{E}_{\mathfrak{h}} := \mathbb{Z}^2 \backslash (\mathbb{C} \times \mathfrak{h}),$$

where $(m, n) : (z, \tau) \mapsto (z + m\tau + n, \tau)$, by the natural action

$$\gamma : (z, \tau) \mapsto ((c\tau + d)^{-1}z, \gamma\tau)$$

of $\text{SL}_2(\mathbb{Z})$.

The universal elliptic curve $\bar{\mathcal{E}}^{\text{an}} \rightarrow \bar{\mathcal{M}}_{1,1}^{\text{an}}$ is obtained by glueing in the Tate curve as described in [26, § 5]. The restriction to the q -disk \mathbb{D} of $\bar{\mathcal{E}}^{\text{an}}$ minus the double point of \bar{E}_0 is the quotient of $\mathbb{C}^* \times \mathbb{D}$ by the \mathbb{Z} -action

$$n : (w, q) \mapsto \begin{cases} (q^n w, q) & q \neq 0 \\ (w, 0) & q = 0. \end{cases}$$

To relate this to the algebraic construction, note that $\mathcal{M}_{1,2+1}^{\text{an}}$ is the analytic variety $\Gamma \backslash (\mathbb{C}^* \times \mathbb{C} \times \mathfrak{h})$, where $\gamma(\xi, z, \tau) = ((c\tau + d)\xi, \gamma(z, \tau))$. The function

$$(\xi, z, \tau) \mapsto (\xi^{-4}g_2(\tau), \xi^{-6}g_3(\tau), [(2\pi i)^{-2}\wp_\tau(z), (2\pi i)^{-3}\wp'_\tau(z), 1])$$

from $\mathbb{C}^* \times \mathbb{C} \times \mathfrak{h}$ to $\mathbb{C}^2 - D^{-1}(0) \times \mathbb{P}^2$ induces a biholomorphism

$$\Gamma \backslash (\mathbb{C}^* \times \mathbb{C} \times \mathfrak{h}) \rightarrow \mathcal{M}_{1,2+\bar{1}}^{\text{an}}.$$

It is invariant with respect to the \mathbb{C}^* -action $\lambda \cdot (\xi, z, \tau) = (\lambda\xi, z, \tau)$ on $\Gamma \backslash (\mathbb{C}^* \times \mathbb{C} \times \mathfrak{h})$ and the \mathbb{C}^* -action on $\mathcal{M}_{1,2+\bar{1}}^{\text{an}}$ that multiplies the abelian differential by λ .

4.5.2. Orbifold fundamental groups. Since $\mathcal{M}_{1,1}^{\text{an}}$ is the orbifold quotient of \mathfrak{h} by $\text{SL}_2(\mathbb{Z})$, there is a natural isomorphism

$$\pi_1^{\text{top}}(\mathcal{M}_{1,1}, p) \cong \text{Aut}(\mathfrak{h} \rightarrow \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}) \cong \text{SL}_2(\mathbb{Z}),$$

where p denotes the projection $\mathfrak{h} \rightarrow \mathcal{M}_{1,1}^{\text{an}}$. The inclusion of the imaginary axis in \mathfrak{h} induces a canonical isomorphism

$$\pi_1^{\text{top}}(\mathcal{M}_{1,1}, \bar{\mathfrak{t}}) \cong \pi_1^{\text{top}}(\mathcal{M}_{1,1}, p) \cong \text{SL}_2(\mathbb{Z}). \tag{4.3}$$

The orbifold fundamental group of $\mathcal{M}_{1,1}^{\text{an}}$ is a central extension of $\text{SL}_2(\mathbb{Z})$ by \mathbb{Z} . There are natural isomorphisms

$$\pi_1^{\text{top}}(\mathcal{M}_{1,\bar{1}}, \vec{v}_o) \cong B_3 \cong \widetilde{\text{SL}}_2(\mathbb{Z}),$$

where B_3 denotes the braid group on three strings and $\widetilde{\text{SL}}_2(\mathbb{Z})$ denotes the inverse image of $\text{SL}_2(\mathbb{Z})$ in the universal covering group $\widetilde{\text{SL}}_2(\mathbb{R})$ of $\text{SL}_2(\mathbb{R})$. Again, this is well known; details can be found in [26, § 8].

Similarly, there are natural isomorphisms

$$\pi_1^{\text{top}}(\mathcal{E}, \vec{v}_o) \cong \pi_1^{\text{top}}(\mathcal{E}, p') \cong \text{Aut}(\mathbb{C} \times \mathfrak{h} \rightarrow \mathcal{E}^{\text{an}}) \cong \Gamma \cong \text{SL}_2(\mathbb{Z}) \times \mathbb{Z}^2,$$

where $p' : \mathbb{C} \times \mathfrak{h} \rightarrow \mathcal{E}^{\text{an}}$ is the projection.

5. The local system \mathbb{H}

Roughly speaking, the local system \mathbb{H} over $\mathcal{M}_{1,1}$ is the ‘motivic local system’ $R^1\pi_*\mathbb{Q}$ associated to the universal elliptic curve $\pi : \mathcal{E} \rightarrow \mathcal{M}_{1,1}$. While we could work in Voevodsky’s category of motivic sheaves [60], we will instead define \mathbb{H} to be a set of compatible realizations: Betti, \mathbb{Q} -de Rham, Hodge, and ℓ -adic étale. These are described below in detail.

In subsequent sections, we will abuse notation and denote the pullback of \mathbb{H} to $\mathcal{M}_{1,n+\bar{r}}$ by \mathbb{H} for all $r, n \geq 0$ with $r + n > 0$.

5.1. Betti realization

The Betti realization of \mathbb{H} is the orbifold local system

$$\mathbb{H}^B := R^1\pi_*^{\text{an}}\mathbb{Q}$$

over $\mathcal{M}_{1,1}^{\text{an}}$ associated to the universal elliptic curve $\pi^{\text{an}} : \mathcal{E}^{\text{an}} \rightarrow \mathcal{M}_{1,1}^{\text{an}}$. Since Poincaré duality induces an isomorphism $H^1(E) \cong H_1(E)$ for all elliptic curves E , \mathbb{H}^B is also isomorphic to the local system whose fiber over $[E] \in \mathcal{M}_{1,1}$ is $H_1(E, \mathbb{Q})$.

Since \mathfrak{h} is contractible, there is a natural isomorphism

$$H_1(\mathcal{E}_{\mathfrak{h}}, \mathbb{Z}) \cong H_1(E_{\tau}, \mathbb{Z}) \cong \mathbb{Z}\mathbf{a} \oplus \mathbb{Z}\mathbf{b}$$

for each $\tau \in \mathfrak{h}$. The sections \mathbf{a} and \mathbf{b} thus trivialize the pullback of \mathbb{H}^B to \mathfrak{h} . The induced action of $\mathrm{SL}_2(\mathbb{Z})$ on $H_1(\mathcal{E}_{\mathfrak{h}}, \mathbb{Z})$ is given by left multiplication:

$$\gamma : \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix} \mapsto \gamma \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}.$$

It corresponds to a right action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{b}$.

Denote the dual basis of $H^1(\mathcal{E}_{\mathfrak{h}}, \mathbb{Z})$ by $\check{\mathbf{a}}, \check{\mathbf{b}}$. The action of $\mathrm{SL}_2(\mathbb{Z})$ on this frame is given by $\gamma : (\check{\mathbf{b}} \ -\check{\mathbf{a}}) \mapsto (\check{\mathbf{b}} \ -\check{\mathbf{a}})\gamma$. The basis $\check{\mathbf{a}}, \check{\mathbf{b}}$ corresponds to the basis $-\mathbf{b}, \mathbf{a}$ of $H_1(E_{\tau})$ under Poincaré duality. The corresponding action of $\mathrm{SL}_2(\mathbb{Z})$ on this frame is:

$$\gamma : (\mathbf{a} \ -\mathbf{b}) \mapsto (\mathbf{a} \ -\mathbf{b})\gamma. \tag{5.1}$$

This defines a left action of $\mathrm{SL}_2(\mathbb{Z})$ on $H^B \times \mathfrak{h}$, where $H^B = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{b}$ should be thought of as the fiber of \mathbb{H}^B over p , and also over $\check{\mathbf{t}}$.

5.2. The flat vector bundle $\mathcal{H}^{\mathrm{an}}$ and its canonical extension

Denote the flat connection on the holomorphic vector bundle

$$\mathcal{H}^{\mathrm{an}} = \mathbb{H}^B \otimes \mathcal{O}_{\mathcal{M}_{1,1}^{\mathrm{an}}}$$

by ∇_0 . The pullback $\mathcal{H}_{\mathfrak{h}}^{\mathrm{an}}$ of $\mathcal{H}^{\mathrm{an}}$ to \mathfrak{h} is the vector bundle $\mathcal{O}_{\mathcal{M}_{1,1}^{\mathrm{an}}}\mathbf{a} \oplus \mathcal{O}_{\mathcal{M}_{1,1}^{\mathrm{an}}}\mathbf{b}$. The sections \mathbf{a} and \mathbf{b} are flat.

Define a holomorphic section \mathbf{w} of $\mathcal{H}_{\mathfrak{h}}^{\mathrm{an}}$ by

$$\mathbf{w}(\tau) = 2\pi i \omega_{\tau} \in H^1(E_{\tau}, \mathbb{C}),$$

where ω_{τ} denotes the class of the holomorphic 1-form dz in $H^1(E_{\tau})$.

This bundle has a Hodge filtration

$$\mathcal{H}^{\mathrm{an}} = F^0\mathcal{H}^{\mathrm{an}} \supset F^1\mathcal{H}^{\mathrm{an}} \supset F^2\mathcal{H}^{\mathrm{an}} = 0.$$

The section \mathbf{w} trivializes $F^1\mathcal{H}_{\mathfrak{h}}^{\mathrm{an}}$.

The sections \mathbf{a} and \mathbf{w} descend to give a framing

$$\mathcal{H}_{\mathbb{D}^*}^{\mathrm{an}} = \mathcal{O}_{\mathbb{D}^*}\mathbf{a} \oplus \mathcal{O}_{\mathbb{D}^*}\mathbf{w}$$

of the pullback of $\mathcal{H}^{\mathrm{an}}$ to the punctured q -disk. Since $\log q = 2\pi i\tau$,

$$\nabla_0\mathbf{w} = \nabla_0(-2\pi i\mathbf{b} + \log q \mathbf{a}) = \mathbf{a} \frac{dq}{q}.$$

Since $\nabla_0\mathbf{a} = 0$, the connection on $\mathcal{H}_{\mathfrak{h}}$ is given by

$$\nabla_0 = d + \mathbf{a} \frac{\partial}{\partial \mathbf{w}} \frac{dq}{q}. \tag{5.2}$$

To define an extension $\overline{\mathcal{H}}^{\text{an}}$ of \mathcal{H}^{an} to a vector bundle over $\overline{\mathcal{M}}_{1,1}^{\text{an}}$, it suffices to extend $\mathcal{H}_{\mathbb{D}^*}^{\text{an}}$ to a vector bundle over \mathbb{D} . We do this by defining

$$\overline{\mathcal{H}}_{\mathbb{D}}^{\text{an}} = \mathcal{O}_{\mathbb{D}}\mathbf{a} \oplus \mathcal{O}_{\mathbb{D}}\mathbf{w}.$$

The formula (5.2) implies that the connection ∇_0 extends to a meromorphic connection on $\overline{\mathcal{H}}^{\text{an}}$ with a regular singular point at the cusp e_o . The residue of the connection at the cusp is the nilpotent operator $\mathbf{a}\partial/\partial\mathbf{w}$. This implies:

Proposition 5.1. *The flat vector bundle $(\overline{\mathcal{H}}^{\text{an}}, \nabla_0)$ is Deligne’s canonical extension of \mathcal{H}^{an} to $\overline{\mathcal{M}}_{1,1}^{\text{an}}$. The holomorphic subbundle $F^1\mathcal{H}^{\text{an}}$ extends to the holomorphic subbundle $F^1\overline{\mathcal{H}}^{\text{an}}$ that is spanned (locally) by \mathbf{w} .*

5.3. Algebraic de Rham realization

A vector bundle on $\mathcal{M}_{1,1/\mathbb{Q}}$ with a connection is a vector bundle on $\mathbb{A}_{\mathbb{Q}}^2 - \{0\}$ with a \mathbb{G}_m -action, endowed with a \mathbb{G}_m -invariant connection that is trivial on each \mathbb{G}_m -orbit. Define

$$\overline{\mathcal{H}}^{\text{DR}} = \mathcal{O}_{\mathbb{A}_{\mathbb{Q}}^2}S \oplus \mathcal{O}_{\mathbb{A}_{\mathbb{Q}}^2}T.$$

Extend the action $\lambda : (u, v) \mapsto (\lambda^{-4}u, \lambda^{-6}v)$ of \mathbb{G}_m on \mathbb{A}^2 to this bundle by defining

$$\lambda \cdot S = \lambda^{-1}S \quad \text{and} \quad \lambda \cdot T = \lambda T.$$

Define a connection ∇_0 on $\overline{\mathcal{H}}^{\text{DR}}$ by

$$\nabla_0 = d + \left(-\frac{1}{12} \frac{dD}{D} \otimes T + \frac{3}{2} \frac{\alpha}{D} \otimes S \right) \frac{\partial}{\partial T} + \left(-\frac{u}{8} \frac{\alpha}{D} \otimes T + \frac{1}{12} \frac{dD}{D} \otimes S \right) \frac{\partial}{\partial S},$$

where $\alpha = 2udv - 3vdu$ and $D = u^3 - 27v^2$.

The connection is \mathbb{G}_m -invariant, trivial on each orbit, defined over \mathbb{Q} , has regular singularities along the discriminant divisor $D = u^3 - 27v^2 = 0$ and is holomorphic on its complement. It therefore descends to a rational connection over $\overline{\mathcal{M}}_{1,1/\mathbb{Q}}$, with a regular singular point at the cusp e_o .

Define a Hodge filtration

$$\overline{\mathcal{H}}^{\text{DR}} = F^0\overline{\mathcal{H}}^{\text{DR}} \supset F^1\overline{\mathcal{H}}^{\text{DR}} \supset F^2\overline{\mathcal{H}}^{\text{DR}} = 0$$

on $\overline{\mathcal{H}}^{\text{DR}}$ by setting $F^1\overline{\mathcal{H}}^{\text{DR}} = \mathcal{O}_{\mathbb{A}_{\mathbb{Q}}^2}T$. This is a \mathbb{G}_m -invariant subbundle, and thus defined over $\mathcal{M}_{1,1/\mathbb{Q}}$.

The following statement follows from [27, Propositions 19.6, 19.7], and Proposition 4.1.⁷

Proposition 5.2. *There is a natural isomorphism*

$$(\overline{\mathcal{H}}^{\text{an}}, \nabla_0) \cong (\overline{\mathcal{H}}^{\text{DR}}, \nabla_0) \otimes_{\mathcal{O}_{\overline{\mathcal{M}}_{1,1/\mathbb{Q}}}} \mathcal{O}_{\overline{\mathcal{M}}_{1,1}^{\text{an}}}$$

that respects the Hodge filtration. When pulled back to $\mathcal{M}_{1,1/\mathbb{Q}} = \mathbb{A}^2 - \{0\}$, the section T corresponds to dx/y and the section S to $x dx/y$. After pulling back to the q -disk along the slice $q \mapsto (g_2(q), g_3(q)) \in \mathbb{A}^2$, we have $T = \mathbf{w}$ and $S = \mathbf{a} - 2G_2(q)\mathbf{w}$.

⁷The normalizations in [27] differ from those here. They are related by $T = 2\pi i \hat{T}$, $S = \hat{S}/2\pi i$, and $A = 2\pi i \mathbf{a}$.

5.4. Hodge realization

The Hodge realization of \mathbb{H} is the polarized variation of \mathbb{Q} -Hodge structure (PVHS) of weight 1 over $\mathcal{M}_{1,1}^{\text{an}}$ whose underlying local system is the Betti incarnation \mathbb{H}^B of \mathbb{H} and whose associated flat vector bundle, with its Hodge filtration, is the flat vector bundle $(\overline{\mathcal{H}}^{\text{DR}}, \nabla_0, F^\bullet) \otimes \mathcal{O}_{\mathcal{M}_{1,1}^{\text{an}}}$. It is naturally polarized by the cup product. Since it is of weight 1, its weight filtration is

$$0 = W_0\mathbb{H} \subseteq W_1\mathbb{H} = \mathbb{H}.$$

We compute the limit mixed Hodge structure on the fiber $H := H_{\vec{\mathfrak{t}}}$ of \mathbb{H} over $\vec{\mathfrak{t}} = \partial/\partial q$.

The fiber H^{DR} of $\overline{\mathcal{H}}^{\text{DR}}$ over the cusp e_o has \mathbb{Q} -basis S and T . Proposition 5.2 implies that $T = \mathbf{w}$ and that, when $q = 0$,

$$S = \mathbf{a} - G_2(0)\mathbf{w}/2 = \mathbf{a} + \mathbf{w}/12.$$

Consequently,

$$H^{\text{DR}} = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{w}.$$

The residue of the connection at e_o is the nilpotent operator

$$N := \mathbf{a} \frac{\partial}{\partial \mathbf{w}} \in \text{End } H^{\text{DR}}.$$

The associated monodromy weight filtration of H^{DR} (centered at the weight, 1, of \mathbb{H}) is

$$0 = M_{-1}H^{\text{DR}} \subset M_0H^{\text{DR}} = M_1H^{\text{DR}} \subset M_2H^{\text{DR}} = H^{\text{DR}}.$$

It remains to determine the Betti structure H^B on H associated to the tangent vector $\vec{\mathfrak{t}} = \partial/\partial q$. This is computed according to Schmid’s prescription [50]. The complex vector space underlying H is $H^{\text{DR}} \otimes \mathbb{C}$. Its integral lattice is

$$H_{\mathbb{Z}}^B := H^0(\mathfrak{h}, \mathbb{H}^B) = \mathbb{Z}\mathbf{a} \oplus \mathbb{Z}\mathbf{b}.$$

The comparison isomorphism $H^B \rightarrow H^{\text{DR}} \otimes \mathbb{C}$ associated to $\vec{\mathfrak{t}}$ takes $\mathbf{v} \in H^B$ to

$$\lim_{q \rightarrow 0} q^N \mathbf{v} = \lim_{q \rightarrow 0} (\text{id} + \log q \mathbf{a} \partial/\partial \mathbf{w}) \mathbf{v}.$$

Since $\mathbf{b} = ((\log q)\mathbf{a} - \mathbf{w})/(2\pi i)$, the comparison isomorphism takes $\mathbf{a} \in H^B$ to $\mathbf{a} \in H^{\text{DR}}$ and $\mathbf{b} \in H^B$ to

$$\lim_{q \rightarrow 0} (\text{id} + \log q \mathbf{a} \partial/\partial \mathbf{w}) \mathbf{b} = -(2\pi i)^{-1} \mathbf{w}.$$

Since \mathbf{w} spans $F^1 H^{\text{DR}}$, it follows that the limit MHS is isomorphic to $\mathbb{Z}(0) \oplus \mathbb{Z}(-1)$. The copy of $\mathbb{Z}(0)$ is spanned by \mathbf{a} and the copy of $\mathbb{Z}(-1)$ is spanned by $\mathbf{b} = -(2\pi i)^{-1} \mathbf{w}$.

The image of the positive generator of $\pi_1^{\text{top}}(\mathbb{D}^*)$ in $\pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{\mathfrak{t}}) = \text{SL}_2(\mathbb{Z})$ is

$$\sigma_o := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Formula (5.1) implies that in $\text{End } H$

$$\log \sigma_o = \mathbf{a} \frac{\partial}{\partial \mathbf{b}} = -2\pi i \mathbf{a} \frac{\partial}{\partial \mathbf{w}}.$$

Remark 5.3. For the uninitiated, it may seem strange that the pure Hodge structure on $H^1(E_\tau)$ becomes a mixed Hodge structure in the limit. One way to come to terms with this is to think of the limit MHS $H = H_{\vec{t}}$ as being the mixed Hodge structure on $H^1(E_{\vec{t}})$, where $E_{\vec{t}}$ denotes a smoothing of the nodal cubic E_0 in the direction of \vec{t} .⁸ There is a continuous retraction $E_{\vec{t}} \rightarrow E_0$ whose composition with the inclusion $E_0^* \hookrightarrow E_{\vec{t}}$ is the inclusion $E_0^* \hookrightarrow E_0$, where E_0^* denotes the non-singular locus of E_0 , which is isomorphic to \mathbb{C}^* . This sequence induces an exact sequence of MHS

$$0 \rightarrow H^1(E_0) \rightarrow H^1(E_{\vec{t}}) \rightarrow H^1(E_0^*) \rightarrow 0,$$

which exhibits $H^1(E_{\vec{t}})$ as an extension of $H^1(\mathbb{C}^*) = \mathbb{Z}(-1)$ by $H^1(E_0) = \mathbb{Z}(0)$. This MHS is not always split – if one replaces \vec{t} by $\lambda\vec{t}$, then the MHS on $H^1(E_{\lambda\vec{t}})$ is the extension of $\mathbb{Z}(-1)$ by $\mathbb{Z}(0)$ corresponding to $\lambda \in \mathbb{C}^* \cong \text{Ext}^1(\mathbb{Z}(-1), \mathbb{Z}(0))$. To prove this, one replaces q by q/λ in the above computations of the limit.

5.5. Étale realization

The ℓ -adic realization of \mathbb{H} is the lisse sheaf

$$\mathbb{H}_\ell := R^1\pi_*\mathbb{Q}_\ell$$

over the stack $\mathcal{M}_{1,1/\mathbb{Q}}$. Its fiber over the moduli point of an elliptic curve $(E, 0)$ over a number field K in $\overline{\mathbb{Q}}$ is $\pi_1(E, 0) \otimes_{\mathbb{Q}_\ell}(-1)$, endowed with the natural action of $G_K := \text{Gal}(\overline{\mathbb{Q}}/K)$. As in the Hodge case, its fiber $H_\ell = H^1_{\text{ét}}(E_{\vec{t}/\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ over $\vec{t} = \text{Spec } \overline{\mathbb{Q}}((q^{1/n} : n \geq 1))$ is a split extension of $\mathbb{Q}_\ell(-1)$ by $\mathbb{Q}_\ell(0)$. Details can be found in Nakamura [45].

The local system \mathbb{H}_ℓ is determined by its monodromy representation

$$\rho_{H,\ell} : \pi_1(\mathcal{M}_{1,1/\mathbb{Q}}, \vec{t}) \rightarrow \text{Aut } H_\ell.$$

Its restriction

$$\pi_1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, \vec{t}) \rightarrow \text{Aut } H_\ell = \text{Aut}(\mathbb{Q}_\ell(0) \oplus \mathbb{Q}_\ell(-1))$$

to the geometric fundamental group is $G_{\mathbb{Q}}$ -equivariant and corresponds to the action of $\pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{t})$ on $H^B \otimes \mathbb{Q}_\ell$ under the comparison isomorphisms

$$H_\ell \cong H^B \otimes \mathbb{Q}_\ell \quad \text{and} \quad \pi_1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, \vec{t})^{\text{op}} \cong \pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{t})^\wedge,$$

where $()^\wedge$ denotes profinite completion and op denotes opposite group. (Cf. § 2.4.)

5.6. Summary

The key point of the previous discussion is that the fibers over \vec{t} of the local system \mathbb{H}^B , \mathbb{H}_ℓ , $\overline{\mathcal{H}}^{\text{DR}}$ are naturally isomorphic to the realizations of the object $H = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)$ of MTM. In this sense, the fibers over \vec{t} can be lifted canonically to an object H of MTM. The local systems \mathbb{H}^B , $\overline{\mathcal{H}}^{\text{DR}}$ and \mathbb{H}_ℓ are determined by the mixed Tate motive H and the action of $\text{SL}_2(\mathbb{Z})$ on its Betti realization. More precisely:

⁸The notion of the fiber $E_{\vec{v}}$ of \mathcal{E} over a non-zero tangent vector \vec{v} of the origin of the q -disk can be made precise. See, for example, [27, Appendix C].

- (i) There is an object $H = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)$ of **MTM**, endowed with two weight filtrations: M_\bullet , its weight filtration in **MTM**, and the second weight filtration

$$0 = W_0H \subset W_1H = H.$$

Its Betti realization $H^B = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{b}$ is the fiber of \mathbb{H}^B over $\vec{\mathfrak{t}}$; its de Rham realization is the fiber $H^{\text{DR}} = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{w}$ of $\overline{\mathcal{H}}^{\text{DR}}$ over the cusp. The comparison isomorphism takes \mathbf{a} to \mathbf{a} and \mathbf{b} to $-(2\pi i)^{-1}\mathbf{w}$.

- (ii) There is an action of $\pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{\mathfrak{t}}) \cong \text{SL}_2(\mathbb{Z})$ on H^B , where $\gamma \in \text{SL}_2(\mathbb{Z})$ acts via the formula (5.1). This determines the \mathbb{Q} -local system \mathbb{H}^B over $\mathcal{M}_{1,1}^{\text{an}}$.
- (iii) There is a filtered bundle with connection $(\overline{\mathcal{H}}^{\text{DR}}, F^\bullet, \nabla_0)$ over $\overline{\mathcal{M}}_{1,1}/\mathbb{Q}$ whose complexification is isomorphic to the canonical extension of $\mathcal{H} := \mathbb{H}^B \otimes \mathcal{O}_{\mathcal{M}_{1,1}^{\text{an}}}$ to $\overline{\mathcal{M}}_{1,1}^{\text{an}}$ and where F^\bullet is the usual Hodge filtration. Together these give \mathbb{H}^B the structure of a polarized variation of Hodge structure over $\mathcal{M}_{1,1}^{\text{an}}$.
- (iv) The fiber of $\overline{\mathcal{H}}^{\text{DR}}$ over the cusp e_o of $\overline{\mathcal{M}}_{1,1}/\mathbb{Q}$ is naturally isomorphic to H^{DR} . The filtration M_\bullet is the monodromy weight filtration of the residue $N = \mathbf{a}\partial/\partial\mathbf{w} \in \text{End } H^{\text{DR}}$ of ∇ at e_o . The limit MHS on the fiber of the polarized variation \mathbb{H}^B over $\vec{\mathfrak{t}}$ is the Hodge realization of H .
- (v) For each prime number ℓ , the action $\pi_1(\mathcal{M}_{1,1}/\overline{\mathbb{Q}}, \vec{\mathfrak{t}}) \rightarrow \text{Aut } H_\ell$ induced by the action of $\text{SL}_2(\mathbb{Z})$ on H^B via the comparison isomorphism is $G_{\mathbb{Q}}$ -equivariant.

This is the basic universal elliptic motive. Other universal elliptic motives will include $S^n \mathbb{H}(r) := (S^n \mathbb{H}) \otimes \mathbb{Q}(r)$.

6. Universal mixed elliptic motives

Suppose that $* \in \{1, \vec{1}, 2\}$. Denote the natural tangential base point of $\mathcal{M}_{1,*}/\mathbb{Q}$ constructed in Paragraph 4.2 by \vec{v}_o . It induces a section $\vec{v}_{o*} : G_{\mathbb{Q}} \rightarrow \pi_1(\mathcal{M}_{1,*}/\mathbb{Q}, \vec{v}_o)$ of the natural homomorphism $\pi_1(\mathcal{M}_{1,*}/\mathbb{Q}, \vec{v}_o) \rightarrow G_{\mathbb{Q}}$ and therefore an action of $G_{\mathbb{Q}}$ on $\pi_1(\mathcal{M}_{1,*}/\overline{\mathbb{Q}}, \vec{v}_o)$.

Let \mathbb{H} be the structure described in the summary in § 5.6, pulled back to $\mathcal{M}_{1,*}$.

Definition 6.1 (Mixed elliptic motives). A *universal mixed elliptic motive* \mathbb{V} over \mathbb{Z} of type $*$ consists of:

- (i) an object V of **MTM** (which is called the *fiber of \mathbb{V} over \vec{v}_o*) whose weight filtration is denoted by M_\bullet ;
- (ii) an increasing filtration W_\bullet of V in **MTM** that satisfies

$$V = \bigcup_m W_m V \quad \text{and} \quad \bigcap_m W_m V = 0;$$

- (iii) a bifiltered vector bundle $(\mathcal{V}, W_\bullet, F^\bullet)$ over $\overline{\mathcal{M}}_{1,*}/\mathbb{Q}$ whose fiber over e_o is the \mathbb{Q} -de Rham realization of (V, W_\bullet) ;
- (iv) an integrable flat connection

$$\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega_{\overline{\mathcal{M}}_{1,*}/\mathbb{Q}}^1(\log \Delta)$$

defined over \mathbb{Q} with nilpotent residue along each component of the boundary divisor Δ which preserves W_\bullet and satisfies Griffiths transversality:

$$\nabla : F^p \mathcal{V} \rightarrow F^{p-1} \mathcal{V} \otimes \Omega_{\overline{\mathcal{M}}_{1,*}/\mathbb{Q}}^1(\log \Delta);$$

- (v) a homomorphism $\rho_V : \pi_1^{\text{top}}(\mathcal{M}_{1,*}^{\text{an}}, \vec{v}_o) \rightarrow \text{Aut}(V^B, W_\bullet)$.

Denote by $(\mathbb{V}^B, W_\bullet)$ the filtered \mathbb{Q} -local system over $\mathcal{M}_{1,*}^{\text{an}}$ whose fiber over \vec{v}_o is (V^B, W_\bullet) and whose monodromy representation is ρ_V ;

- (vi) an isomorphism $(\mathcal{V}, W_\bullet, \nabla) \otimes_{\mathcal{O}} \mathcal{O}_{\overline{\mathcal{M}}_{1,*}^{\text{an}}}$ with the canonical extension of the filtered flat bundle $(\mathbb{V}^B, W_\bullet) \otimes_{\mathcal{O}_{\mathcal{M}_{1,*}^{\text{an}}}}$ to $\overline{\mathcal{M}}_{1,*}^{\text{an}}$ that induces the comparison isomorphism of $(V^{\text{DR}}, W_\bullet) \otimes \mathbb{C}$ with $(V^B, W_\bullet) \otimes \mathbb{C}$ on the fiber over \vec{v}_o .

These are required to satisfy:

- (a) for each prime number ℓ , the homomorphism

$$\rho_{V,\ell} : \pi_1(\mathcal{M}_{1,*}/\overline{\mathbb{Q}}, \vec{v}_o) \rightarrow \text{Aut } V_\ell$$

induced by ρ_V via the comparison isomorphism $V_\ell \cong V^B \otimes \mathbb{Q}_\ell$ is $G_{\mathbb{Q}}$ -equivariant;

- (b) each weight graded quotient $\text{Gr}_m^W \mathbb{V}$ of \mathbb{V} is isomorphic to a direct sum of copies (with multiplicities) of $S^{m+2r} \mathbb{H}(r)$.

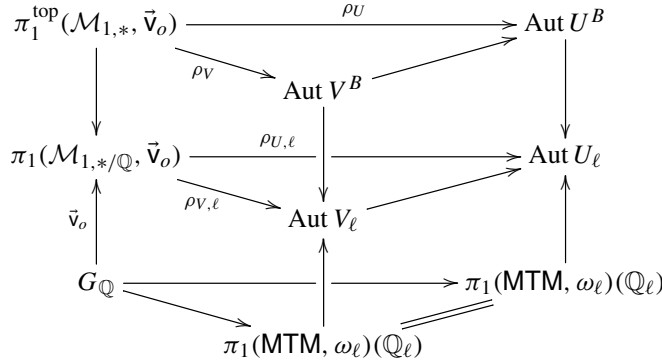
Remark 6.2. The last condition implies that M_\bullet is the relative weight filtration associated to the nilpotent endomorphism $\log \sigma_o$ (the positive generator of the fundamental group of the punctured q -disk) of the filtered vector space (V^B, W_\bullet) . The isomorphism of the canonical extension of $\mathbb{V}^B \otimes \mathcal{O}_{\mathcal{M}_{1,*}^{\text{an}}}$ with the complexification of $(\mathcal{V}, W_\bullet, F^\bullet)$ defines an admissible variation of MHS over $\mathcal{M}_{1,*}^{\text{an}}$ whose limit MHS over \vec{v}_o is naturally isomorphic to the Hodge realization of $(V, M_\bullet, W_\bullet)$.

In addition, since V is an object of MTM, the associated Galois representation $G_{\mathbb{Q}} \rightarrow \text{Aut}(V \otimes \mathbb{Q}_\ell)$ is unramified at all primes $p \neq \ell$ and is crystalline at ℓ .

Remark 6.3. Lemma 4.3 of [29] implies that the filtration W_\bullet of a universal mixed elliptic motive V is uniquely determined by the filtration M_\bullet and the $\pi_1^{\text{top}}(\mathcal{M}_{1,*}, \vec{v}_o)$ action on V^B .

Definition 6.4. A morphism $\phi : \mathbb{V} \rightarrow \mathbb{U}$ of universal mixed elliptic motives consists of a morphism $\phi^{\text{MTM}} : V \rightarrow U$ in MTM and a morphism $\phi^{\text{DR}} : (\mathcal{V}, F^\bullet, \nabla) \rightarrow (\mathcal{U}, F^\bullet, \nabla)$ of their de Rham realizations. These are required to be compatible with all additional structures in the sense that

- (i) the morphisms $V^{\text{DR}} \rightarrow U^{\text{DR}}$ induced by ϕ^{MTM} and ϕ^{DR} are equal;
- (ii) for all prime numbers ℓ , the diagram



commutes, where the homomorphisms $\text{Aut } V^B \rightarrow \text{Aut } V_\ell$ and $\text{Aut } U^B \rightarrow \text{Aut } U_\ell$ are induced by the comparison maps;

- (iii) the induced map $\mathbb{V}^B \rightarrow \mathbb{U}^B$ of local systems induces a morphism of variations of MHS over $\mathcal{M}_{1,*}^{\text{an}}$.

Denote the category of mixed elliptic motives of type $* \in \{1, \bar{1}, 2\}$ by MEM_* . There is a functor $\vec{v}_o^* : \text{MEM}_* \rightarrow \text{MTM}$ that takes a mixed elliptic motive \mathbb{V} to its fiber (V, M_\bullet) over the base point \vec{v}_o .

Example 6.5 (Geometrically constant mixed elliptic motives). Suppose that $* \in \{1, \bar{1}, 2\}$. Objects \mathbb{V} of MEM_* for which the representation ρ_V is trivial will be called *geometrically constant*. These have the property that the two weight filtrations W_\bullet and M_\bullet coincide on their fiber over \vec{v}_o and are characterized by this property when $* \neq \bar{1}$. (Cf. Proposition 10.6.) One can think of the geometrically constant objects of MEM_* as pullbacks of objects of MTM along the structure morphism $\mathcal{M}_{1,*} \rightarrow \text{Spec } \mathbb{Z}$.

Example 6.6 (Simple universal mixed elliptic motives). These are the Tate twists $S^m \mathbb{H}(r)$ of symmetric powers of \mathbb{H} . That they are universal elliptic motives follows from the discussion in the previous section that was summarized in § 5.6. The mixed Tate motive underlying $S^m \mathbb{H}(r)$ is

$$S^m H(r) = \mathbb{Q}(r) \oplus \mathbb{Q}(r - 1) \oplus \cdots \oplus \mathbb{Q}(r - m).$$

Definition 6.7. An object V of MEM_* is *W-pure of weight r* if $\text{Gr}_j^W V = 0$ when $j \neq r$. It is *M-pure of weight m* if $\text{Gr}_j^M V = 0$ when $j \neq m$.

The simple object $S^m \mathbb{H}(r)$ of MEM_* is *W-pure* of weight $m - 2r$. It is *M-pure* if and only if $m = 0$. Not all objects of MEM_* are *W-pure* or *geometrically constant*. The simplest non-trivial examples are extensions of \mathbb{Q} by $S^{2n} \mathbb{H}(2n + 1)$ over $\mathcal{M}_{1,2}$ for each $n \geq 1$. These are the elliptic polylogarithms of Beilinson and Levin [7].

Example 6.8. An important and non-trivial example of a pro-object of MEM_* is provided by the local system over $\mathcal{M}_{1,2}$ whose fiber over $[E, x]$ is the Lie algebra $\mathfrak{p}(E', x)$ of the unipotent completion⁹ of $\pi_1(E', x)$. Its restriction to $\mathcal{M}_{1,\bar{1}}$ is the local system whose fiber over $[E, \bar{V}]$ is the Lie algebra of the unipotent completion of $\pi_1(E', \bar{V})$.

Choose a parameter w on the fiber \bar{E}_0 over $q = 0$ of the Tate curve that takes the value 1 at the identity and defines an isomorphism of the smooth points E_0 of \bar{E}_0 with \mathbb{G}_m . It is unique up to the involution $w \mapsto 1/w$ of \bar{E}_0 . The tangent vector

$$\vec{w}_o : \text{Spec } \mathbb{Z}((w)) \rightarrow E_0$$

is integrally defined and is non-zero mod p for all prime numbers p . It will be used as a base point for both E'_0 and also for E'_\dagger via the inclusion $E_0 \rightarrow E'_\dagger$. Note that E'_0 is isomorphic to $\mathbb{P}^1 - \{0, 1, \infty\}$.

Theorem 6.9 [30]. *The Lie algebra $\mathfrak{p}(E'_\dagger, \vec{w}_o)$ is a pro-object of MTM. The inclusion $E_0 \rightarrow E'_\dagger$ induces a morphism*

$$\mathfrak{p}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{w}_o) \cong \mathfrak{p}(E'_0, \vec{w}_o) \rightarrow \mathfrak{p}(E'_\dagger, \vec{w}_o).$$

Corollary 6.10. *There is an object \mathfrak{p} of MEM_2 whose fiber over \vec{v}_o is $\mathfrak{p}(E'_\dagger, \vec{w}_o)$.*

6.1. Variants: higher levels and more decorations

When $r + n > 0$, one can make a similar definition of mixed elliptic motives over the moduli stack $\mathcal{M}_{1,n+\bar{r}/\mathcal{O}_N}[N]$ of decorated genus 1 curves with a level $N \geq 1$ structure, where $\mathcal{O}_N = \mathbb{Z}[\mu_N, 1/N]$. One first has to choose an integrally defined tangent vector \vec{v}_o at a cusp with everywhere good reduction. (That is, a tangent vector defined over \mathcal{O}_N that is non-zero mod \mathfrak{p} for all prime ideals \mathfrak{p} of \mathcal{O}_N .) When $N > 1$ or $r + n > 2$, there are several possible choices of tangent vector \vec{v}_0 . All give equivalent categories of mixed elliptic motives. This will be proved in [30]. The coordinate ring of the tannakian fundamental group of level N universal mixed elliptic motives should be an ind-object of $\text{MTM}(\mathcal{O}_N)$ and there might be an interesting connection to Deligne’s work [16].

6.2. A ‘theorem of the fixed part’

We conclude this section by proving a mild generalization of the Theorem of the Fixed Part. For this, we need the following result, which follows from GAGA.

Lemma 6.11. *Suppose that k is a subfield of \mathbb{C} and that X is a smooth variety defined over k with an action by a k -algebraic group G . If \mathcal{V} is a G -invariant vector bundle over X with a rational, G -invariant connection $\nabla : \mathcal{V} \rightarrow \mathcal{V} \otimes \Omega^1_X \otimes_{\mathcal{O}_X} k(X)$, then for all*

⁹Unipotent completion is briefly reviewed in § 10.3.

$P \in X(k)$ the square

$$\begin{CD} H^0(X, (\mathcal{V}, \nabla))^G @>>> V_P \\ @VVV @VVV \\ H^0(X^{\text{an}}, (\mathcal{V}^{\text{an}}, \nabla))^G @>>> V_P \otimes_k \mathbb{C} \end{CD}$$

is cartesian. Here V_P denotes the fiber of \mathcal{V} over P . □

We also need the stack version. When the connection is trivial on G -orbits, there is an isomorphism

$$H^0(X, (\mathcal{V}, \nabla))^G \cong H^0(G \backslash X, (\mathcal{V}, \nabla)).$$

This implies that the square

$$\begin{CD} H^0(G \backslash X, (\mathcal{V}, \nabla)) @>>> V_P \\ @VVV @VVV \\ H^0(G \backslash X^{\text{an}}, (\mathcal{V}^{\text{an}}, \nabla))^G @>>> V_P \otimes_k \mathbb{C} \end{CD}$$

is also cartesian.

The next result is a version of the Theorem of the Fixed Part for universal mixed elliptic motives.

Proposition 6.12. *For all objects \mathbb{V} of MEM_* there is an object $H^0(\mathcal{M}_{1,*}, \mathbb{V})$ of MTM whose Hodge realization is the canonical mixed Hodge structure on the invariants $H^0(\mathcal{M}_{1,*}^{\text{an}}, \mathbb{V}^B)$ of \mathbb{V}^B . It is a subobject of the fiber V of \mathbb{V} over \tilde{v}_o .*

Proof. The Theorem of the Fixed Part implies that $H^0(\mathcal{M}_{1,*}, \mathbb{V}^B)$ has a natural MHS and that this is a sub-MHS of the Hodge realization of V with $M_\bullet = W_\bullet$. Since the Hodge realization functor on MTM is fully faithful, $H^0(\mathcal{M}_{1,*}, \mathbb{V}^B)$ is the Hodge realization of an object of MTM . Denote it by $H^0(\mathcal{M}_{1,*}, \mathbb{V})$. Compatibility with the \mathbb{Q} -de Rham realization follows from Lemma 6.11. □

Lemma 6.11 and Proposition 6.12 also imply that the geometrically constant object of MEM_* that restricts to $H^0(\mathcal{M}_{1,*}, \mathbb{V})$ over \tilde{v}_o is a sub-MEM of \mathbb{V} .

Corollary 6.13. *Every W -pure object of MEM_* is semi-simple.*

Proof. By standard arguments, it suffices to show that every extension

$$0 \rightarrow \mathbb{V} \rightarrow \mathbb{E} \rightarrow \mathbb{Q}(0) \rightarrow 0$$

in MEM_* , where \mathbb{E} is W -pure of weight 0, splits. The definition of universal mixed elliptic motives implies that, since \mathbb{E} is W -pure, as an $\text{SL}_2(\mathbb{Z})$ -module, E^B is isomorphic to a direct sum of copies of symmetric powers of H^B . It follows that

$$H^0(\mathcal{M}_{1,*}, \mathbb{E}^B) \rightarrow H^0(\mathcal{M}_{1,*}, \mathbb{Q})$$

is a surjective morphism of objects of MTM of W -weight 0. Since these are trivial $\text{SL}_2(\mathbb{Z})$ -modules, the two weight filtrations M and W are equal, so they are both also M -pure of weight 0. The morphism therefore splits. The choice of a splitting of $E \rightarrow \mathbb{Q}(0)$ in MTM induces a splitting in MEM_* . □

7. Tannakian considerations

The category MEM_* , $*$ $\in \{1, \bar{1}, 2\}$, is a rigid abelian tensor category over \mathbb{Q} . The functor that takes an object of MEM_* to its fiber (V, M_\bullet) over \vec{v}_o defines a functor $\vec{v}_o^* : \text{MEM}_* \rightarrow \text{MTM}$. It is faithful, exact and preserves tensor products and unit objects. Consequently, each fiber functor $\omega : \text{MTM} \rightarrow \text{Vec}_F$ gives rise to a fiber functor $\omega \circ \vec{v}_o^* : \text{MEM}_* \rightarrow \text{Vec}_F$ that we shall also denote by ω .

Proposition 7.1. *The category MEM_* is a neutral tannakian category over \mathbb{Q} . □*

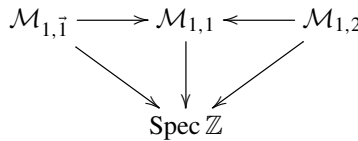
The standard fiber functors for both MTM and MEM_* are:

$$\omega^B : \text{MEM}_* \rightarrow \text{Vec}_{\mathbb{Q}}, \quad \omega^{\text{DR}} : \text{MEM}_* \rightarrow \text{Vec}_{\mathbb{Q}}, \quad \text{and} \quad \omega_\ell : \text{MEM}_* \rightarrow \text{Vec}_{\mathbb{Q}_\ell}.$$

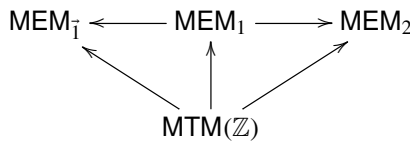
Other useful fiber functors include $\text{Gr}_\bullet^W \omega^B$, $\text{Gr}_\bullet^W \omega^{\text{DR}}$ and $\text{Gr}_\bullet^W \omega_\ell$.

Tannaka duality implies that, when $\omega = \omega^B$ and $\omega = \omega^{\text{DR}}$, the category MEM_* is equivalent to the category $\text{Rep}(\pi_1(\text{MEM}_*, \omega))$ of finite dimensional representations of $\pi_1(\text{MEM}_*, \omega)$.

The morphisms



induce functors



In § 7.2 we show that there is a ‘restriction functor’ $\text{MEM}_2 \rightarrow \text{MEM}_{\bar{1}}$ and that $\text{MEM}_1 \rightarrow \text{MEM}_{\bar{1}}$ factors through it: $\text{MEM}_1 \rightarrow \text{MEM}_2 \rightarrow \text{MEM}_{\bar{1}}$.

7.1. Generalities

The functor $c : \text{MTM} \rightarrow \text{MEM}_*$ that takes a mixed Tate motive (V, M_\bullet) to the geometrically constant mixed elliptic motive with fiber (V, M_\bullet) over \vec{v}_o is fully faithful and induces a surjective homomorphism

$$\pi_1(\text{MEM}_*, \omega) \rightarrow \pi_1(\text{MTM}, \omega)$$

for each fiber functor $\omega : \text{MTM} \rightarrow \text{Vec}_F$. It is split by the homomorphism \vec{v}_o^* .

Set

$$\pi_1^{\text{geom}}(\text{MEM}_*, \omega) = \ker\{\pi_1(\text{MEM}_*, \omega) \rightarrow \pi_1(\text{MTM}, \omega)\}.$$

One thus has a split extension

$$1 \rightarrow \pi_1^{\text{geom}}(\text{MEM}_*, \omega) \rightarrow \pi_1(\text{MEM}_*, \omega) \rightarrow \pi_1(\text{MTM}, \omega) \rightarrow 1$$

which is split by \vec{v}_o^* .

One also has the group $\pi_1(\text{MEM}_*, \vec{v}_o^*)$. It is an affine group scheme in MTM . More precisely:

Proposition 7.2. *The coordinate ring of $\pi_1(\text{MEM}_*, \vec{V}_o^*)$ is a Hopf algebra in the category of ind-objects of MTM. The Betti, de Rham and ℓ -adic realizations of $\mathcal{O}(\pi_1(\text{MEM}_*, \vec{V}_o^*))$ are the coordinate rings of*

$$\pi_1^{\text{geom}}(\text{MEM}_*, \omega^B), \quad \pi_1^{\text{geom}}(\text{MEM}_*, \omega^{\text{DR}}) \quad \text{and} \quad \pi_1^{\text{geom}}(\text{MEM}_*, \omega_\ell),$$

respectively. □

A universal mixed elliptic motive V is *semi-simple* if it is a direct sum of simple universal elliptic motives:

$$V = \bigoplus_{j=1}^N S^{mj} \mathbb{H}(r_j).$$

The category MEM_*^{ss} of semi-simple universal elliptic motives is the full tannakian subcategory of MEM_* generated by the $S^m \mathbb{H}(r)$. For each of the standard fiber functors ω , we have

$$\pi_1(\text{MEM}_*^{\text{ss}}, \omega) \cong \text{GL}(H_\omega)$$

where V_ω denotes $\omega(V)$ for all $V \in \text{MEM}_*$. The simple object \mathbb{H} corresponds to the defining representation of $\text{GL}(H_\omega)$, and the simple object $\mathbb{Q}(-1)$ is $\Lambda^2 \mathbb{H}$, and thus corresponds to the one-dimensional representation $\det : \text{GL}(H_\omega) \rightarrow \mathbb{G}_m$. Combining these, we see that $S^m \mathbb{H}(r)$ corresponds to the m th symmetric power of the defining representation, twisted by $\det^{\otimes(-r)}$. Note that if \mathbb{V} is a W -pure object V of MEM_* of weight m , then the scalar matrix λid_H in $\text{GL}(H_\omega)$ acts on V_ω as $\lambda^m \text{id}_V$.

Proposition 7.3. *For each of the standard fiber functors, the surjection*

$$\pi_1(\text{MEM}_*, \omega) \rightarrow \text{GL}(H_\omega) \tag{7.1}$$

is split and has prounipotent kernel.

Proof. Corollary 6.13 implies that the W -graded quotients $\text{Gr}_m^W \mathbb{V}$ of an $\text{MEM} \mathbb{V}$ are semi-simple. That is, we have a functor $\text{Gr}_\bullet^W : \text{MEM}_* \rightarrow \text{MEM}_*^{\text{ss}}$. It is exact and thus induces a homomorphism

$$\text{GL}(H_\omega) \rightarrow \pi_1(\text{MEM}_*, \text{Gr}_\bullet^W \omega). \tag{7.2}$$

Each choice of an F -rational point ($F = \mathbb{Q}, \mathbb{Q}_\ell$) of the ‘path torsor’ $\text{Isom}^\otimes(\omega, \text{Gr}_\bullet^W \omega)$ gives an isomorphism $\pi_1(\text{MEM}_*, \text{Gr}_\bullet^W \omega) \rightarrow \pi_1(\text{MEM}_*, \omega)$. Composing it with (7.2) gives a splitting of (7.1).

The fact that the kernel is prounipotent is a consequence of the general fact that if \mathcal{C} is a neutral tannakian category whose category of semi-simple objects is \mathcal{C}^{ss} , then the kernel of the homomorphism $\pi_1(\mathcal{C}, \omega) \rightarrow \pi_1(\mathcal{C}^{\text{ss}}, \omega)$ induced by the inclusion $\mathcal{C}^{\text{ss}} \rightarrow \mathcal{C}$ has prounipotent kernel. □

Remark 7.4. Later we will need to know that the functor $\mathbb{V} \mapsto \text{Gr}_\bullet^W \text{Gr}_\bullet^M V$ that takes a mixed elliptic motive \mathbb{V} to the associated bigraded of its fiber over \vec{V}_o is exact. This is proved in Appendix B.

7.2. The restriction functor $\text{MEM}_2 \rightarrow \text{MEM}_1$

Set $\mathcal{N} = \mathcal{L}^{-1}$. Recall that $\mathcal{M}_{1,2} = \mathcal{E}'$. Since \mathcal{N}^{an} is a covering space of \mathcal{E}^{an} , there are neighborhoods U of the zero section of \mathcal{N}^{an} and V of the zero section of \mathcal{E}^{an} such that V gets mapped biholomorphically onto U by the projection $\mathcal{N} \rightarrow \mathcal{E}$ and such that U is a deformation retract of \mathcal{N} . Denote the complement of the zero section in U and V by U' and V' . Then the sequence of maps

$$\mathcal{M}_{1,1}^{\text{an}} \xleftarrow{\cong} U' \xrightarrow{\cong} V' \hookrightarrow \mathcal{M}_{1,2}^{\text{an}}$$

induces a homomorphism $\pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{v}_o) \rightarrow \pi_1^{\text{top}}(\mathcal{M}_{1,2}, \vec{v}_o)$. This induces a $G_{\mathbb{Q}}$ -equivariant homomorphism $\pi_1(\mathcal{M}_{1,1}/\overline{\mathbb{Q}}, \vec{v}_o) \rightarrow \pi_1(\mathcal{M}_{1,2}/\overline{\mathbb{Q}}, \vec{v}_o)$.

Proposition 7.5. *There is a natural restriction functor $\text{MEM}_2 \rightarrow \text{MEM}_1$ that is the identity on the fiber over the base point \vec{v}_o and takes the object \mathbb{V} with monodromy representation $\rho_{\mathbb{V}} : \pi_1^{\text{top}}(\mathcal{M}_{1,2}, \vec{v}_o) \rightarrow \text{Aut } V^B$ to an object of MEM_1 with monodromy representation the composite*

$$\pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{v}_o) \rightarrow \pi_1^{\text{top}}(\mathcal{M}_{1,2}, \vec{v}_o) \rightarrow \text{Aut } V^B.$$

Proof. The only task is to prove that if \mathbb{V} is an object of MEM_2 , then its de Rham realization $(\mathcal{V}^{\text{DR}}, \nabla)$, a vector bundle with connection over $\mathcal{M}_{1,2}/\overline{\mathbb{Q}}$, pulls back to a vector bundle with connection over $\mathcal{M}_{1,1}/\overline{\mathbb{Q}}$ satisfying the required compatibilities with the other realizations.

Denote the zero section of $\mathcal{E}/\overline{\mathbb{Q}}$ by Z and its ideal sheaf in $\mathcal{O}_{\mathcal{E}}$ by \mathfrak{m}_Z . Then

$$\mathcal{O}_{\mathcal{N}} = \text{Gr}^{\bullet} \mathcal{O}_{\mathcal{E}} := \bigoplus_{n \geq 0} \mathfrak{m}_Z^n / \mathfrak{m}_Z^{n+1}.$$

There is a natural isomorphism $\mathcal{O}_{\mathcal{E}} \cong \mathcal{O}_Z \oplus \mathfrak{m}_Z$ that is induced by the inclusion $Z \hookrightarrow \mathcal{E}$ and the projection $\mathcal{E} \rightarrow \mathcal{M}_{1,1} \cong Z$. This induces the eigenspace decomposition

$$\mathcal{O}_{\mathcal{E}} / \mathfrak{m}_Z^2 = \mathcal{O}_Z \oplus \mathfrak{m}_Z / \mathfrak{m}_Z^2 = \mathcal{O}_Z \oplus \mathcal{N}^{\vee}$$

under the natural \mathbb{G}_m -action, where $(\)^{\vee}$ denotes dual. Note that $\mathcal{O}_{\mathcal{N}}$ is the graded \mathcal{O}_Z algebra

$$\mathcal{O}_{\mathcal{N}} = \text{Sym}_{\mathcal{O}_Z} \mathcal{N} = \mathcal{O}_Z \oplus \mathcal{N}^{\vee} \oplus S^2 \mathcal{N}^{\vee} \oplus \dots$$

The pullback of \mathcal{V}^{DR} from \mathcal{E} to \mathcal{N} is defined to be the graded $\mathcal{O}_{\mathcal{N}}$ -module

$$\text{Gr}^{\bullet} \mathcal{V}^{\text{DR}} := (\mathcal{V}^{\text{DR}} \otimes_{\mathcal{O}_{\mathcal{E}}} \mathcal{O}_{\mathcal{E}} / \mathfrak{m}_Z^2) \otimes_{\mathcal{O}_{\mathcal{E}} / \mathfrak{m}_Z^2} \mathcal{O}_{\mathcal{N}}.$$

The connection ∇ on \mathcal{V}^{DR} induces a connection¹⁰

$$\overline{\nabla} : \text{Gr}^{\bullet} \mathcal{V}^{\text{DR}} \rightarrow (\text{Gr}^{\bullet} \mathcal{V}^{\text{DR}}) \otimes \Omega_{\mathcal{N}}^1(\log Z)$$

which is characterized by the properties:

¹⁰Examples of how this works in practice can be found in [27, § 13].

- (i) it has regular singular points along Z ,
- (ii) it is invariant under the \mathbb{G}_m -action on \mathcal{N} ,
- (iii) the residues of ∇ and $\bar{\nabla}$ along Z , which lie in $H^0(Z, \text{End } \mathcal{V}^{\text{DR}}|_Z)$, are equal.

To complete the proof, we sketch a proof of the compatibility of the monodromy representations with the Betti realizations. More precisely, we explain why the diagram

$$\begin{array}{ccc}
 \pi_1(\mathcal{M}_{1,1}^{\text{an}}, \bar{\mathbf{v}}) & \longrightarrow & \pi_1(\mathcal{M}_{1,2}^{\text{an}}, \bar{\mathbf{v}}) \\
 & \searrow \rho_{\bar{\nabla}} & \downarrow \rho_{\nabla} \\
 & & \text{Aut } V_P
 \end{array}$$

commutes, where $P \in Z$, $\bar{\mathbf{v}}$ is a non-zero element of $T_P \mathcal{M}_{1,2}^{\text{an}}$ (so $\bar{\mathbf{v}} \in \mathcal{M}_{1,1}^{\text{an}}$), and where ρ_{∇} and $\rho_{\bar{\nabla}}$ are the monodromy representations of ∇ and $\bar{\nabla}$. It suffices to consider the case where $\bar{\mathbf{v}} = \partial/\partial \xi$. Let $\tau_o \in \mathfrak{h}$ be a point that lies above $P \in Z$.

By standard ODE theory (see, for example, [61]), since ∇ has nilpotent residue at each point along Z , there is a polynomial

$$p(\tau, \xi, T) \in \text{Aut } V_P \otimes_{\mathbb{C}} \mathcal{O}(\mathfrak{h} \times \mathbb{C})[T]$$

whose regularized value $p(\tau_o, 1, 0)$ at $\bar{\mathbf{v}} := \partial/\partial \xi \in T_P \mathcal{M}_{1,2}$ is the identity of V_P and with the property that all flat sections of \mathcal{V} are of the form $v p(\tau, \xi, \log \xi)$ for some $v \in V_P$. The monodromy of ∇ about an element $\gamma \in \pi_1(\mathcal{M}_{1,2}^{\text{an}}, \bar{\mathbf{v}})$ is obtained by taking the analytic continuation \tilde{p} of p along γ and then taking its regularized value at $\bar{\mathbf{v}}$:

$$\rho_{\nabla}(\gamma) = \tilde{p}(\tau_o, 1, 0).$$

On the other hand, ODE theory implies that the flat sections of $\bar{\nabla}$ over $\mathcal{M}_{1,\bar{1}}$ are of the form $p(\tau, 1, \log \xi)$. The regularized monodromy representation of $\bar{\nabla}$ on $\gamma \in \pi_1(\mathcal{M}_{1,\bar{1}}^{\text{an}}, \bar{\mathbf{v}})$ is computed from \tilde{p} . Since we may assume that γ lies in the neighborhood U' of Z , we have the claimed compatibility of monodromy representations. □

8. Hodge theoretic considerations

For $F = \mathbb{Q}$ or \mathbb{R} , define $\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})$ to be the category of admissible variations of F -MHS over $\mathcal{M}_{1,*}^{\text{an}}$ whose weight graded quotients are sums of polarized variations of Hodge structure of the form $S^m \mathbb{H} \otimes A$, where A is a Hodge structure.¹¹ It is neutral tannakian. Denote the forgetful functor that takes a variation to the F -vector space underlying its fiber over $\bar{\mathbf{v}}_o$ by ω_o . When $F = \mathbb{Q}$ we will omit the subscript.

Theorem 8.1. *The forgetful functor $\text{MEM}_* \rightarrow \text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$ that takes a universal mixed elliptic motive \mathbb{V} to the associated variation of MHS \mathbb{V}^{an} over $\mathcal{M}_{1,*}^{\text{an}}$ is fully faithful. Consequently,*

$$\pi_1(\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H}), \omega_o) \rightarrow \pi_1(\text{MEM}_*, \omega^B)$$

is surjective.

¹¹It is important to note that A is not necessarily of type (p, p) .

Proof. Since $\text{Hom}_{\mathcal{C}}(\mathbb{A}, \mathbb{B})$ is naturally isomorphic to $\text{Hom}_{\mathcal{C}}(\mathbb{Q}(0), \text{Hom}_{\mathbb{Q}}(\mathbb{A}, \mathbb{B}))$ for all objects \mathbb{A} and \mathbb{B} of MEM_* and $\mathcal{C} = \text{MEM}_*$ and $\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$, it suffices to show that

$$\text{Hom}_{\text{MEM}_*}(\mathbb{Q}(0), \mathbb{V}) \rightarrow \text{Hom}_{\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})}(\mathbb{Q}(0), \mathbb{V}^{\text{an}})$$

is an isomorphism for all objects \mathbb{V} of MEM_* .

Injectivity is easily proved and is left to the reader. We will prove surjectivity. Suppose that $\mathbb{V} = (V, \mathbb{V}^{\text{DR}}, \rho_V)$ is an object of MEM_* and that $\phi^{\text{an}} : \mathbb{Q}(0)^{\text{an}} \rightarrow \mathbb{V}^{\text{an}}$ is a morphism of variations of MHS. It induces a morphism of MHS

$$v : \mathbb{Q}(0) \rightarrow V^{\text{MHS}} := (V^B, W_{\bullet}, F^{\bullet})$$

of limit MHS over \vec{V}_o . Since the Hodge realization functor $\text{MTM} \rightarrow \text{MHS}$ is fully faithful [17], this map is the Hodge realization of a morphism $\mathbb{Q}(0) \rightarrow V$ in MTM .

Identify the morphism $v : \mathbb{Q}(0) \rightarrow V$ with the set of the realizations $(v^B \in V^B, v^{\text{DR}} \in V^{\text{DR}}, v_{\ell} \in V_{\ell})$ of the image of $1 \in \mathbb{Q}(0)$. For each prime ℓ

$$v_{\ell} \in H^0(\mathcal{M}_{1,*/\mathbb{Q}}, V_{\ell})$$

as v_{ℓ} is fixed by both $G_{\mathbb{Q}}$ and $\pi_1(\mathcal{M}_{1,*/\overline{\mathbb{Q}}}, \vec{V}_o)$.

To complete the construction of a morphism $\phi : \mathbb{Q}(0) \rightarrow \mathbb{V}$ in MEM_* that lifts ϕ^{an} , we need to construct a morphism $\mathbb{Q}(0)_{\mathcal{M}_{1,*/\mathbb{Q}}}^{\text{DR}} \rightarrow \mathbb{V}^{\text{DR}}$. Such a map corresponds to an element of $H^0(\mathcal{M}_{1,*/\mathbb{Q}}, (\mathcal{V}, \nabla))$, where $\mathbb{V}^{\text{DR}} = (\mathcal{V}, \nabla)$. The vector $v^B \in V^B$ lies in the image of the restriction map

$$H^0(\overline{\mathcal{M}}_{1,*}^{\text{an}}, (\mathcal{V}^{\text{an}}, \nabla)) \rightarrow V^B \otimes \mathbb{C},$$

where \mathcal{V}^{an} denotes Deligne’s canonical extension of $\mathbb{V}^B \otimes \mathcal{O}_{\mathcal{M}_{1,*}}^{\text{an}}$ to $\overline{\mathcal{M}}_{1,*}^{\text{an}}$. Lemma 6.11 implies that v^{DR} lies in the image of the corresponding homomorphism

$$H^0(\overline{\mathcal{M}}_{1,*/\mathbb{Q}}, (\mathcal{V}, \nabla)) \rightarrow V^{\text{DR}}. \quad \square$$

Part 2. Simple extensions in MEM_*

In this section we compute the groups $\text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^m \mathbb{H}(r))$ for all m and r . These computations are made possible by the fact that the Hodge realization is fully faithful, which means that an extension in MEM_* is non-trivial if and only if its Hodge realization is non-trivial. We also prove the corresponding statement for ℓ -adic Galois realizations. This part concludes with a discussion of $\text{Ext}_{\text{MEM}_*}^2(\mathbb{Q}, S^m \mathbb{H}(r))$ and its relation to standard conjectures in number theory.

9. Cohomology of $\mathcal{M}_{1,*}$

Here we recall the basic facts we need. Full details in the Hodge case can be found in [28]. References in the ℓ -adic case are given below.

9.1. The cohomology of $\mathcal{M}_{1,1}^{\text{an}}$

The first basic observation is that, since we are regarding $\mathcal{M}_{1,1}^{\text{an}}$ as the orbifold $\text{SL}_2(\mathbb{Z}) \backslash \mathfrak{h}$, there is a natural isomorphism

$$H^\bullet(\mathcal{M}_{1,1}^{\text{an}}, \mathbb{V}) \cong H^\bullet(\text{SL}_2(\mathbb{Z}), V)$$

for all $\text{SL}_2(\mathbb{Z})$ -modules V , where \mathbb{V} denotes the local system over $\mathcal{M}_{1,1}^{\text{an}}$ that corresponds to V . Since $\text{SL}_2(\mathbb{Z})$ is virtually free, this implies that $H^j(\mathcal{M}_{1,1}^{\text{an}}, \mathbb{V})$ vanishes for all $j \geq 2$ whenever \mathbb{V} is a local system of \mathbb{Q} -modules. Since $-I \in \text{SL}_2(\mathbb{Z})$ acts as $(-1)^m$ on $H^\bullet(\text{SL}_2(\mathbb{Z}), S^m H)$, it follows that $H^\bullet(\mathcal{M}_{1,1}^{\text{an}}, S^m \mathbb{H})$ vanishes when m is odd. When $m = 0$, $H^1(\mathcal{M}_{1,1}^{\text{an}}, \mathbb{Q}) = 0$ as $H_1(\text{SL}_2(\mathbb{Z}); \mathbb{Z}) = \mathbb{Z}/12$.

Well-known results of Shimura, Manin, Drinfeld and Zucker (cf. [40, 62]) imply that the cohomology groups $H^\bullet(\mathcal{M}_{1,1}^{\text{an}}, S^n \mathbb{H})$ and their mixed Hodge structures can be expressed in terms of modular forms. Denote the space of (holomorphic) modular forms of $\text{SL}_2(\mathbb{Z})$ of weight w by \mathfrak{M}_w and the subspace of cusp forms by \mathfrak{M}_w^c . Recall that these are trivial when w is odd.

In this section we regard \mathbb{H} as a polarized variation of Hodge structure over $\mathcal{M}_{1,1}^{\text{an}}$ of weight 1. As explained in § 4, its fiber over the tangent vector $\bar{\mathfrak{t}}$ is

$$H^B = \mathbb{Q}\mathfrak{a} \oplus \mathbb{Q}\mathfrak{b} \quad \text{and} \quad H^{\text{DR}} = F^0 H^{\text{DR}} = \mathbb{Q}\mathfrak{a} \oplus \mathbb{Q}\mathfrak{w},$$

where the comparison isomorphism $H^B \otimes \mathbb{C} \rightarrow H^{\text{DR}} \otimes \mathbb{C}$ takes \mathfrak{a} to \mathfrak{a} and identifies \mathfrak{w} with $-2\pi i \mathfrak{b}$. The Hodge filtration $F^1 H^{\text{DR}}$ is spanned by \mathfrak{w} .

For $f \in \mathfrak{M}_{2n+2}$ define

$$\omega_f = 2\pi i f(\tau) \mathfrak{w}^{2n} d\tau = (2\pi i)^{2n+1} f(\tau) (\mathfrak{b} - \tau \mathfrak{a})^{2n} d\tau \in E^1(\mathfrak{h}) \otimes S^{2n} H.$$

This is a holomorphic 1-form on \mathfrak{h} with values in $S^{2n} \mathcal{H}_{\mathfrak{h}}$. Since it is $\text{SL}_2(\mathbb{Z})$ invariant and since the section \mathfrak{w} spans $F^1 \mathcal{H}_{\mathfrak{h}}$,

$$\omega_f \in H^0(\mathcal{M}_{1,1}^{\text{an}}, \Omega^1 \otimes F^{2n} S^{2n} \mathcal{H}).$$

Since this form is holomorphic, it defines a class in $H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n} \mathbb{H})$. When f is the Eisenstein series G_{2n} , we denote ω_f by ψ_{2n} .

Remark 9.1. The particular scalings have been chosen to have the property that, if $f \in \mathfrak{M}_{2n+2}$ has rational Fourier coefficients, then ω_f is an element of the \mathbb{Q} de Rham cohomology $H_{\text{DR}}^1(\mathcal{M}_{1,1/\mathbb{Q}}, S^{2n} \mathcal{H})$. (Cf. [27, § 21].) In particular, the class of ψ_{2n+2} under the residue map

$$H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n} \mathbb{H}) \rightarrow \mathbb{Q}(-2n - 1)$$

is a rational generator of $\mathbb{Q}(-2n - 1)^{\text{DR}}$.

The following statement is a composite of well-known results. (Cf. [7, 62].)

Theorem 9.2 (Shimura, Zucker, Manin–Drinfeld). *For each $n > 0$, the group $H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H})$ has a natural MHS with Hodge numbers $(2n + 1, 0)$, $(0, 2n + 1)$ and $(2n + 1, 2n + 1)$. The map $\mathfrak{M}_{2n+2} \rightarrow H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{C}})$ that takes f to the class of ω_f is injective and has image $F^{2n+1}H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H})$. The image of \mathfrak{M}_{2n+2}^o is*

$$H^{2n+1,0}(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}) := F^{2n+1}W_{2n+1}H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}).$$

The MHS $H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H})$ splits over \mathbb{Q} . The copy of $\mathbb{Q}(-2n - 1)^{\text{DR}}$ is spanned by ψ_{2n+2} .

9.2. The Hodge structures associated to a Hecke eigenform

It is useful to decompose the real MHS on $H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H})$ under the action of the Hecke correspondences. Denote the set of normalized Hecke eigen cusp forms of $\text{SL}_2(\mathbb{Z})$ of weight w by \mathfrak{B}_w . It is a basis of \mathfrak{M}_w^o .

For $f \in \mathfrak{B}_{2n+2}$, let K_f be the subfield of \mathbb{C} generated by the Fourier coefficients of f . It is a totally real field. For each prime number p , $T_p(f) = a_p f$ where a_p is the p th Fourier coefficient of f . Let

$$V_f := \bigcap_p \ker\{T_p - a_p \text{id} : H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}}) \rightarrow H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}})\}.$$

This is a two-dimensional \mathbb{R} Hodge substructure of $H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H})$. Its complexification $V_{f,\mathbb{C}}$ is spanned by ω_f and its complex conjugate. As a real Hodge structure, $\mathcal{M}_{1,1}^{\text{an}}$ decomposes

$$H^1(\mathcal{M}_{1,1}^{\text{an}}, S^m\mathbb{H}_{\mathbb{R}}) \cong \mathbb{R}(-2n - 1) \oplus \bigoplus_{f \in \mathfrak{B}_{2n+2}} V_f. \tag{9.1}$$

The copy of $\mathbb{R}(-2n - 1)$ is spanned by the class ψ_{2n+2} .

There is a similar decomposition of $H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H})$ as a \mathbb{Q} -MHS. Suppose that $f \in \mathfrak{B}_{2n+2}$. As above, let a_p be the p th Fourier coefficient of f . Let $m_p(x)$ be the minimal polynomial of a_p over \mathbb{Q} . Then

$$M_f = \bigcap_p \ker\{m_p(T_p) : H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{Q}}) \rightarrow H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{Q}})\}$$

is a simple \mathbb{Q} Hodge substructure of $H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{Q}})$ with the property that

$$M_f \otimes \mathbb{R} = \bigoplus_g V_g,$$

where g ranges over the Hecke eigen cusp forms whose Fourier expansions are Galois conjugate to that of f . It has dimension $2 \dim_{\mathbb{Q}} K_f$. There is a decomposition

$$H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{Q}}) \cong \mathbb{Q}(-2n - 1) \oplus \bigoplus_f M_f.$$

where f ranges over the Galois equivalence classes of Hecke eigen cusp forms of weight $2n + 2$.

9.3. Cohomology of $\mathcal{M}_{1,1}^{\text{an}}$ and $\mathcal{M}_{1,2}^{\text{an}}$

These are easily deduced from the cohomology of $\mathcal{M}_{1,1}^{\text{an}}$ using the Leray spectral sequences of the projections $\mathcal{M}_{1,\bar{1}} \rightarrow \mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2} \rightarrow \mathcal{M}_{1,1}$. We state the results and leave the proof to the reader.

Proposition 9.3. *There is an isomorphism*

$$H^1(\mathcal{M}_{1,\bar{1}}^{\text{an}}, \mathbb{Q}) \cong \mathbb{Q}(-1).$$

This group is generated by the class of the form (4.2) corresponding to the Eisenstein series G_2 .¹² If $m > 0$, then the projection to $\mathcal{M}_{1,1}^{\text{an}}$ induces an isomorphism

$$H^1(\mathcal{M}_{1,\bar{1}}^{\text{an}}, S^m \mathbb{H}) \cong H^1(\mathcal{M}_{1,1}^{\text{an}}, S^m \mathbb{H})$$

and the cup product $H^1(\mathcal{M}_{1,\bar{1}}^{\text{an}}, S^m \mathbb{H}) \otimes H^1(\mathcal{M}_{1,\bar{1}}^{\text{an}}, \mathbb{Q}) \rightarrow H^2(\mathcal{M}_{1,\bar{1}}^{\text{an}}, S^m \mathbb{H})$ is an isomorphism of MHS, so that

$$H^2(\mathcal{M}_{1,\bar{1}}^{\text{an}}, S^m \mathbb{H}) \cong H^1(\mathcal{M}_{1,1}^{\text{an}}, S^m \mathbb{H})(-1).$$

In the case of $\mathcal{M}_{1,2}^{\text{an}}$, we have:

Proposition 9.4. *There are natural isomorphisms of MHS*

$$H^1(\mathcal{M}_{1,2}^{\text{an}}, \mathbb{H}) \cong \mathbb{Q}(-1) \quad \text{and} \quad H^2(\mathcal{M}_{1,2}^{\text{an}}, \mathbb{H}) = 0.$$

If $m > 1$, then the projection induces an isomorphism

$$H^1(\mathcal{M}_{1,2}^{\text{an}}, S^m \mathbb{H}) \cong H^1(\mathcal{M}_{1,1}^{\text{an}}, S^m \mathbb{H})$$

in degree 1, and in degree 2, there is an isomorphism of MHS

$$H^2(\mathcal{M}_{1,2}^{\text{an}}, S^m \mathbb{H}) \cong H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{m+1} \mathbb{H}) \oplus H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{m-1} \mathbb{H})(-1).$$

In particular, $H^2(\mathcal{M}_{1,2}, S^m \mathbb{H})$ is non-trivial only when m is odd. □

This implies that

$$\begin{aligned} &H^2(\mathcal{M}_{1,2}, \mathbb{H} \otimes S^{2n} \mathbb{H}) \\ &\cong H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n+2} \mathbb{H}) \oplus H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n} \mathbb{H})^{\oplus 2}(-1) \oplus H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n-2} \mathbb{H})(-2). \end{aligned}$$

One can show that the cup product

$$H^1(\mathcal{M}_{1,2}^{\text{an}}, \mathbb{H}) \otimes H^1(\mathcal{M}_{1,2}^{\text{an}}, S^{2n} \mathbb{H}) \rightarrow H^2(\mathcal{M}_{1,2}, \mathbb{H} \otimes S^{2n} \mathbb{H})$$

is injective and has image in the middle factor $H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n+2} \mathbb{H})^{\oplus 2}(-1)$.

¹²It is natural to extend the definition of the ψ_{2n} ($n > 1$) to the case $n = 1$ by defining $\psi_2 = 2\pi i G_2(\tau) d\tau - (1/2)d\xi/\xi$. Then ψ_2 is a generator of $H_{\text{DR}}^1(\mathcal{M}_{1,\bar{1}}/\mathbb{Q})$ and $H^1(\mathcal{M}_{1,\bar{1}}^{\text{an}}, \mathbb{Q}(1))$. Note also that $\psi_2 = -\frac{1}{24} \frac{dD}{D}$, where $D = u^3 - 27v^2$ denotes the discriminant function on $\mathcal{M}_{1,\bar{1}}$.

10. Relative unipotent completion

The next task is to describe the tannakian fundamental group of $\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$. Computing it will allow us to bound the size of $\pi_1(\text{MEM}_{*, \omega^B})$. To do this, we need to review some basic facts about relative completion of discrete groups and facts about the relative completion of modular groups and their relation to extensions of variations of MHS. References for this material include [23, 28, 29].

10.1. Relative unipotent completion

Suppose that Γ is a discrete group and that R is a reductive algebraic group over a field F of characteristic zero. Suppose that $\rho : \Gamma \rightarrow R(F)$ is a Zariski dense representation. Consider the full subcategory $\mathcal{R}(\Gamma, \rho)$ of $\text{Rep}_F(\Gamma)$ consisting of those Γ -modules M that admit a filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_n = M$$

in $\text{Rep}_F(\Gamma)$, where the action of Γ on each graded quotient M_j/M_{j-1} factors through a rational representation $R \rightarrow \text{Aut}(M_j/M_{j-1})$ via ρ . This category is neutral tannakian over F . The completion of Γ relative to ρ is the affine group scheme $\pi_1(\mathcal{R}(\Gamma, \rho), \omega)$ over F , where ω takes a module M to its underlying vector space. It is an extension

$$1 \rightarrow \mathcal{U}(\Gamma, \rho) \rightarrow \pi_1(\mathcal{R}(\Gamma, \rho), \omega) \rightarrow R \rightarrow 1,$$

where $\mathcal{U}(\Gamma, \rho)$ is pronipotent. There is a natural homomorphism

$$\tilde{\rho} : \Gamma \rightarrow \pi_1(\mathcal{R}(\Gamma, \rho), \omega)(F)$$

whose composition with the canonical quotient map $\pi_1(\mathcal{R}(\Gamma, \rho), \omega) \rightarrow R$ is ρ . It is Zariski dense. The coordinate ring of $\pi_1(\mathcal{R}(\Gamma, \rho), \omega)$ is the Hopf algebra of matrix entries of the objects of $\mathcal{R}(\Gamma, \rho)$.

10.2. Cohomological properties

Set $\mathcal{G} = \pi_1(\mathcal{R}(\Gamma, \rho), \omega)$ and $\mathcal{U} = \mathcal{U}(\Gamma, \rho)$. Denote the Lie algebra of \mathcal{U} by \mathfrak{u} . It is pronilpotent. The action of \mathcal{G} on \mathfrak{u} induced by the inner action of \mathcal{G} on \mathcal{U} induces an action of \mathcal{G} on $H^\bullet(\mathfrak{u})$. Standard arguments imply that it factors through the quotient mapping $\mathcal{G} \rightarrow R$, so that $H^\bullet(\mathfrak{u})$ is an R -module. The homomorphism $\Gamma \rightarrow \mathcal{G}(F)$ induces a homomorphism $H^\bullet(\mathcal{G}, V) \rightarrow H^\bullet(\Gamma, V)$. There is a natural isomorphism

$$H^\bullet(\mathcal{G}, V) \cong [H^\bullet(\mathcal{U}, V)]^R.$$

The following basic property of relative completion is easily proved by adapting the proof of [31, Theorem 4.6] or [32, Theorem 8.1]. A proof in the de Rham case is given in [23].

Proposition 10.1. *For all R -modules V , there are natural isomorphisms*

$$H^1(\mathcal{G}, V) \cong [H^1(\mathfrak{u}) \otimes V]^R \cong H^1(\Gamma, V)$$

and an injection

$$H^2(\mathcal{G}, V) \cong [H^2(\mathfrak{u}) \otimes V]^R \hookrightarrow H^2(\Gamma, V).$$

Using the fact (cf. [29]) that, for all \mathcal{G} -modules V , there are natural isomorphisms

$$\mathrm{Ext}_{\mathrm{Rep}(\mathcal{G})}^{\bullet}(A, B) \cong H^{\bullet}(\mathcal{G}, \mathrm{Hom}_F(A, B)) \cong H^{\bullet}(\mathfrak{u}, \mathrm{Hom}_F(A, B))^R,$$

one can restate the previous result as follows. For all \mathcal{G} -modules A and B , the natural map

$$\mathrm{Ext}_{\mathrm{Rep}(\mathcal{G})}^j(A, B) \rightarrow \mathrm{Ext}_{\mathrm{Rep}_F(\Gamma)}^j(A, B)$$

is an isomorphism in degree 1 and injective in degree 2.

10.3. Unipotent completion

Unipotent completion is the special case where R is the trivial group. In this case, $\mathcal{R}(\Gamma, \rho)$ is the category of unipotent representations of Γ . The unipotent completion of Γ over F will be denoted by $\Gamma_{/F}^{\mathrm{un}}$.

Suppose that $(E, 0)$ is an elliptic curve over \mathbb{C} . Set $E' = E - \{0\}$. Let b be a base point of E' . (So b can be a point of E' or a non-zero tangent vector $\vec{v} \in T_0E$.) Denote the unipotent completion of $\pi_1(E', b)$ by $\mathcal{P}(E, b)$ and its Lie algebra by $\mathfrak{p}(E', b)$. For all such E and b , the coordinate ring $\mathcal{O}(\mathcal{P}(E, b))$ and Lie algebra $\mathfrak{p}(E', b)$ of $\mathcal{P}(E, b)$ have natural mixed Hodge structures that are compatible with their algebraic structures. The weight filtration W_{\bullet} of $\mathfrak{p}(E', b)$ is (essentially) its lower central series:

$$W_{-n}\mathfrak{p}(E', b) = L^n\mathfrak{p}(E', b).$$

There is a canonical isomorphism $\mathrm{Gr}_{\bullet}^W \mathfrak{p}(E', b) \cong \mathbb{L}(H_1(E))$ of the associated weight graded of $\mathfrak{p}(E', b)$ with the free Lie algebra generated by $H_1(E)$. These form a variation of MHS over $\mathcal{M}_{1,2}^{\mathrm{an}}$ when b is a point of E' , and over $\mathcal{M}_{1,1}^{\mathrm{an}}$ when b is a tangential base point at 0.

There is also a canonical MHS on the unipotent completion of $\pi_1(E'_1, \vec{v}_o)$. It is the limit MHS associated to the tangent vector \vec{v}_o of $\overline{\mathcal{M}}_{1,2}$ at the identity of the nodal cubic. It has a weight filtration W_{\bullet} as above, and a relative weight filtration M_{\bullet} . This limit MHS is computed explicitly in [27, § 18].

Denote the commutator subalgebra of $\mathfrak{p} = \mathfrak{p}(E', b)$ by \mathfrak{p}' and its commutator subalgebra by \mathfrak{p}'' . Then there is a short exact sequence

$$0 \rightarrow H_1(\mathfrak{p}') \rightarrow \mathfrak{p}/\mathfrak{p}'' \rightarrow H_1(E) \rightarrow 0$$

in the category of Lie algebras with mixed Hodge structure. There is a canonical isomorphism of mixed Hodge structures

$$H_1(\mathfrak{p}') \cong [\mathrm{Sym} H_1(E)](1) = \bigoplus_{m \geq 0} S^m H^1(E)(m + 1).$$

Consequently, these variations yield extensions

$$0 \rightarrow S^m \mathbb{H}(m + 1) \rightarrow \mathbb{V}_m \rightarrow \mathbb{H}(1) \rightarrow 0$$

of variations of MHS over $\mathcal{M}_{1,2}^{\mathrm{an}}$ and $\mathcal{M}_{1,1}^{\mathrm{an}}$ for each $m \geq 0$. But since the log of monodromy about the cusp of E' lies in \mathfrak{p}' and spans the copy of $\mathbb{Q}(1)$ in $H_1(\mathfrak{p}')$, the monodromy of

\mathbb{V}_m about any loop in a fiber of $\mathcal{M}_{1,1}^{\text{an}} \rightarrow \mathcal{M}_{1,1}^{\text{an}}$ is trivial. Consequently, the restriction of this extension to $\mathcal{M}_{1,1}^{\text{an}}$ descends to a variation over $\mathcal{M}_{1,1}^{\text{an}}$.

By standard manipulations, each of these variations is equivalent to a variation

$$0 \rightarrow S^{m+1}\mathbb{H}(m+1) \oplus S^{m-1}\mathbb{H}(m) \rightarrow \mathbb{E}_m \rightarrow \mathbb{Q} \rightarrow 0. \tag{10.1}$$

Similar constructions can be made in the ℓ -adic case.

The following result follows from Proposition 9.3 when $*$ = $\vec{1}$ and the vanishing of $H^1(\mathcal{M}_{1,*}^{\text{an}}, \mathbb{Q})$ when $*$ \neq $\vec{1}$.

Proposition 10.2. *The unipotent completion of $\pi_1^{\text{un}}(\mathcal{M}_{1,*}^{\text{an}}, x)$ is trivial when $*$ \in $\{1, 2\}$ and is isomorphic to \mathbb{G}_a when $*$ = $\vec{1}$.*

10.4. Relative completion of $\pi_1^{\text{top}}(\mathcal{M}_{1,*}, \vec{v}_o)$

Denote by $\mathcal{G}_*^{\text{rel}}(x)$ the completion of $\pi_1^{\text{top}}(\mathcal{M}_{1,*}, x)$ with respect to the natural homomorphism $\pi_1^{\text{top}}(\mathcal{M}_{1,*}, x) \rightarrow \text{SL}(H_x^B)$. It is an extension

$$1 \rightarrow \mathcal{U}_*^{\text{rel}}(x) \rightarrow \mathcal{G}_*^{\text{rel}}(x) \rightarrow \text{SL}(H_x) \rightarrow 1,$$

where $\mathcal{U}_*^{\text{rel}}(x)$ is prounipotent and $\text{SL}(H_x)$ denotes $\text{SL}(H_x^B)$.

Theorem 10.3 [23, 28]. *For each choice of base point x of $\mathcal{M}_{1,*}^{\text{an}}$, the coordinate ring of $\mathcal{G}_*^{\text{rel}}(x)$ has a natural mixed Hodge structure. The coordinate rings $\{\mathcal{O}(\mathcal{G}_*^{\text{rel}}(x))\}_x$ form an ind-object of $\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$. The MHS on the coordinate ring of $\mathcal{G}_*^{\text{rel}}(\vec{v}_o)$ is the limit MHS of this variation of MHS associated to the tangent vector \vec{v}_o . In this case, the weight filtration is the relative weight filtration M_\bullet .*

Denote the Lie algebra of $\mathcal{G}_*^{\text{rel}}(x)$ by $\mathfrak{g}_*^{\text{rel}}(x)$ and the Lie algebra of $\mathcal{U}_*^{\text{rel}}(x)$ by $\mathfrak{u}_*^{\text{rel}}(x)$. The Lie algebra $\mathfrak{g}_*^{\text{rel}}(x)$ is dual to the cotangent space $\mathfrak{m}/\mathfrak{m}^2$, where \mathfrak{m} denotes the ideal of $\mathcal{O}(\mathcal{G}_*^{\text{rel}}(x))$ of functions that vanish at the identity. Its bracket is induced by the skew symmetrized coproduct of $\mathcal{O}(\mathcal{G}_*^{\text{rel}}(x))$. The theorem implies that $\mathfrak{g}_*^{\text{rel}}(x)$ is an inverse limit of Lie algebras with mixed Hodge structure and that $\mathfrak{u}_*^{\text{rel}}(x)$ is a pronilpotent Lie algebra in the category of mixed Hodge structures. These mixed Hodge structures have the property that

$$\mathfrak{g}_*^{\text{rel}}(x) = W_0\mathfrak{g}_*^{\text{rel}}(x) \quad \text{and} \quad \mathfrak{u}_*^{\text{rel}}(x) = W_{-1}\mathfrak{g}_*^{\text{rel}}(x).$$

We now fix the base point of $\mathcal{M}_{1,*}^{\text{an}}$ to be \vec{v}_o and write $\mathcal{G}_*^{\text{rel}}$ for $\mathcal{G}_*(\vec{v}_o)$, $\mathcal{U}_*^{\text{rel}}$ for $\mathcal{U}_*(\vec{v}_o)$ and similarly for their Lie algebras. The coordinate ring of $\mathcal{G}_*^{\text{rel}}$ and each of the Lie algebras has two weight filtrations, W_\bullet and M_\bullet . Both are compatible with the algebraic operations.

Since $\pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{1}) \cong \text{SL}_2(\mathbb{Z})$ has virtual cohomological dimension 1, Proposition 10.1 implies that $H^2(\mathfrak{u}_1) = 0$. This implies that \mathfrak{u}_1 is topologically freely generated by $H_1(\mathfrak{u})$. (See the discussion in § 18.) Proposition 10.1 also implies that there are natural $\text{SL}(H)$ -module isomorphisms

$$H^1(\mathfrak{u}_1^{\text{rel}}) \cong \bigoplus_{n \geq 1} H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}) \otimes (S^{2n}H)^\vee \cong \bigoplus_{n \geq 1} H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}) \otimes (S^{2n}H)(2n).$$

Both are isomorphisms of MHS.

Recall from [28, § 14] that the exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{v}_o) \rightarrow \pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{t}) \rightarrow 1$$

induces an exact sequence

$$0 \rightarrow \mathbb{G}_a \rightarrow \mathcal{G}_1^{\text{rel}} \rightarrow \mathcal{G}_1^{\text{rel}} \rightarrow 1$$

and that this sequence splits, so that there is a natural isomorphism

$$\mathcal{G}_1^{\text{rel}} \cong \mathcal{G}_1^{\text{rel}} \times \mathbb{G}_a.$$

The projection onto \mathbb{G}_a is induced by the Hurewicz homomorphism

$$\pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{v}_o) \rightarrow H_1(\mathcal{M}_{1,1}^{\text{an}}, \mathbb{Q}) \cong \mathbb{Q},$$

and is obtained by integrating the 1-form (4.2) along loops.

Applying completion to the extension

$$1 \rightarrow \pi_1(E'_1, \vec{w}_o) \rightarrow \pi_1^{\text{top}}(\mathcal{M}_{1,2}, \vec{v}_o) \rightarrow \pi_1^{\text{top}}(\mathcal{M}_{1,1}, \vec{t}) \rightarrow 1$$

gives an extension

$$1 \rightarrow \mathcal{P} \rightarrow \mathcal{G}_2^{\text{rel}} \rightarrow \mathcal{G}_1^{\text{rel}} \rightarrow 1$$

where \mathcal{P} is the unipotent completion of $\pi_1(E'_1, \vec{w}_o)$. The corresponding sequence of Lie algebras

$$0 \rightarrow \mathfrak{p} \rightarrow \mathfrak{g}_2^{\text{rel}} \rightarrow \mathfrak{g}_1^{\text{rel}} \rightarrow 0$$

is exact in the category of pro-objects of $\text{MHS}_{\mathbb{Q}}$.

Proposition 10.4. *There are natural $\text{SL}(H)$ -equivariant isomorphisms of MHS*

$$H^1(u_1) \cong \bigoplus_{n \geq 1} (H_{\text{cusp}}^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H})(2n) \oplus \mathbb{Q}(-1)) \otimes S^{2n}H,$$

$$H^1(u_1) \cong H^1(u_1) \oplus \mathbb{Q}(-1) \quad \text{and} \quad H^1(u_2) \cong H^1(u_1) \oplus H. \quad \square$$

10.5. Variations of MHS over $\mathcal{M}_{1,*}^{\text{an}}$

The category MHS of \mathbb{Q} -MHS is neutral tannakian. Let ω^B be the fiber functor that takes a MHS to its underlying \mathbb{Q} -vector space. Set $\pi_1(\text{MHS}) = \pi_1(\text{MHS}, \omega^B)$.

Suppose that \mathcal{G} is an affine group scheme over \mathbb{Q} whose coordinate ring is a Hopf algebra in the category of ind-objects of MHS . A *Hodge representation* of \mathcal{G} is a \mathcal{G} -module V , endowed with a MHS for which the corresponding coaction

$$V \rightarrow V \otimes \mathcal{O}(\mathcal{G})$$

is a morphism of MHS. The action of $\pi_1(\text{MHS})$ on $\mathcal{O}(\mathcal{G})$ respects its Hopf algebra structure. We can thus form the semi-direct product $\pi_1(\text{MHS}) \ltimes \mathcal{G}$. The Hodge representations of \mathcal{G} are precisely the representations of $\pi_1(\text{MHS}) \ltimes \mathcal{G}$.

Recall the definition of $\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$ from the beginning of § 8. A proof of the following result is sketched in [29]. A more detailed proof of a more general result will appear in a joint paper with Gregory Pearlstein.

Theorem 10.5. *Taking the fiber at the base point x defines an equivalence of categories from $\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$ to the category of Hodge representations of $\mathcal{G}_*^{\text{rel}}(x)$. Equivalently, for all base points, there is a natural isomorphism*

$$\pi_1(\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H}), \omega_x^B) \cong \pi_1(\text{MHS}) \ltimes \mathcal{G}_*^{\text{rel}}(x),$$

where ω_x^B denotes the fiber functor that takes a variation to the \mathbb{Q} -vector space underlying its fiber over x .

This result applies to tangential base points as well. This follows as the weight filtration W_\bullet can be recovered from the relative weight filtration M_\bullet and the monodromy logarithm N . (Cf. [29, Lemma 4.3].) It implies that a Hodge representation of $\mathcal{G}_*^{\text{rel}}$ must also respect the weight filtration W_\bullet .

Proposition 10.6. *If \mathbb{V} is an admissible variation of mixed Hodge structure over $\mathcal{M}_{1,*}^{\text{an}}$ whose monodromy representation factors through the relative completion of $\pi_1(\mathcal{M}_{1,*}^{\text{an}}, \vec{v}_o)$, then the weight filtration W_\bullet and the relative weight filtration M_\bullet on the fiber over the tangential base point \vec{v}_o are equal if and only if \mathbb{V} has unipotent monodromy. If $* \in \{1, 2\}$, every admissible variation with unipotent monodromy is constant. If $* = \vec{1}$, every admissible unipotent variation is pulled back from an admissible unipotent variation of MHS over \mathbb{C}^* along the discriminant map $\mathcal{M}_{1,\vec{1}}^{\text{an}} \rightarrow \mathbb{C}^*$.*

Proof. Suppose that \mathbb{V} is an admissible variation of MHS over $\mathcal{M}_{1,*}^{\text{an}}$ whose monodromy factors through the relative completion of $\pi_1(\mathcal{M}_{1,*}^{\text{an}}, \vec{v}_o)$. Denote its fiber over \vec{v}_o by V . Let $N \in \text{End } V$ be the local monodromy logarithm. The definition of the relative weight filtration implies that $M_\bullet V = W_\bullet V$ if and only if N acts trivially on $\text{Gr}_\bullet^W V$. But since each W graded quotient of V is a sum of $S^m H(r)$, and since N acts on H as $\mathbf{ad}/\partial \mathbf{b}$, it follows that $M_\bullet = W_\bullet$ if and only if $\text{Gr}_\bullet^W \mathbb{V}$ is a trivial local system – that is, if and only if \mathbb{V} is a unipotent local system.

If \mathbb{V} is a unipotent local system, then its monodromy representation factors through the unipotent completion of $\pi_1(\mathcal{M}_{1,*}^{\text{an}}, x)$. By Proposition 10.2, this is trivial when $* = 1, 2$. So \mathbb{V} is a trivial local in this case, and therefore trivial as a variation of MHS as well. When $* = \vec{1}$, the discriminant map $\mathcal{M}_{1,\vec{1}}^{\text{an}} \rightarrow \mathbb{C}^*$ induces an isomorphism on unipotent fundamental groups. So the last statement follows from the classification of unipotent variations of MHS in [34]. □

10.6. Density results

The forgetful functor

$$\text{MEM}_* \rightarrow \mathcal{R}(\pi_1^{\text{top}}(\mathcal{M}_{1,*}, \vec{v}_o), \rho)$$

induces a homomorphism $\mathcal{G}_* \rightarrow \pi_1(\text{MEM}_*, \omega^B)$. There is thus a natural homomorphism

$$\pi_1^{\text{top}}(\mathcal{M}_{1,*}, \vec{v}_o) \rightarrow \pi_1(\text{MEM}_*, \omega^B)(\mathbb{Q}).$$

Theorem 10.7. *The image of the natural homomorphism $\mathcal{G}_*^{\text{rel}} \rightarrow \pi_1(\text{MEM}_*, \omega^B)$ is $\pi_1^{\text{geom}}(\text{MEM}_*, \omega^B)$. Consequently, $\pi_1^{\text{geom}}(\text{MEM}_*, \omega^B)$ is the Zariski closure of the image of the natural homomorphism $\pi_1^{\text{top}}(\mathcal{M}_{1,*}, \vec{v}_o) \rightarrow \pi_1(\text{MEM}_*, \omega^B)(\mathbb{Q})$.*

Proof. Since $\text{MEM}_* \rightarrow \text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$ takes geometrically constant MEMs to constant variations of MHS, the surjection $\pi_1(\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})) \rightarrow \pi_1(\text{MEM}_*, \omega^B)$ commutes with their projections to $\pi_1(\text{MHS})$ and to $\pi_1(\text{MTM}, \omega^B)$. So one has the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathcal{G}_*^{\text{rel}} & \longrightarrow & \pi_1(\text{MHS}) \times \mathcal{G}_*^{\text{rel}} & \longrightarrow & \pi_1(\text{MHS}) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \pi_1^{\text{geom}}(\text{MEM}_*, \omega^B) & \longrightarrow & \pi_1(\text{MEM}_*, \omega^B) & \longrightarrow & \pi_1(\text{MTM}, \omega^B) \longrightarrow 1
 \end{array}$$

The surjectivity of the middle map, which follows from Theorem 8.1, and the splitting $\pi_1(\text{MEM}_*, \omega^B) \cong \pi_1(\text{MTM}_*, \omega^B) \times \pi_1^{\text{geom}}(\text{MEM}_*, \omega^B)$, imply the surjectivity of the left-hand map. The last assertion follows as the canonical homomorphism $\pi_1^{\text{top}}(\mathcal{M}_{1,*}, \vec{v}_o) \rightarrow \mathcal{G}_*^{\text{rel}}(\mathbb{Q})$ is Zariski dense. \square

Recall that $\mathcal{K} = \mathcal{K}^B$ is the kernel of the natural homomorphism $\pi_1(\text{MTM}, \omega^B) \rightarrow \mathbb{G}_m$. It is prounipotent. Denote the kernel of $\pi_1(\text{MEM}_*, \omega^B) \rightarrow \pi_1(\text{MTM}, \omega^B)$ by $\mathcal{U}_*^{\text{MEM}}$. Since the commutative diagram (in which all fiber functors are ω^B)

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathcal{U}_*^{\text{geom}} & \longrightarrow & \mathcal{U}_*^{\text{MEM}} & \longrightarrow & \mathcal{K} \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \pi_1^{\text{geom}}(\text{MEM}_*) & \longrightarrow & \pi_1(\text{MEM}_*) & \longrightarrow & \pi_1(\text{MTM}) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \text{SL}(H) & \longrightarrow & \text{GL}(H) & \longrightarrow & \mathbb{G}_m \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

has exact rows and columns, we have:

Corollary 10.8. *The kernel of the natural homomorphism $\pi_1(\text{MEM}_*, \omega) \rightarrow \text{GL}(H_\omega)$ is prounipotent. It is a split extension*

$$1 \rightarrow \mathcal{U}_*^{\text{geom}} \rightarrow \mathcal{U}_*^{\text{MEM}} \rightarrow \mathcal{K} \rightarrow 1$$

of \mathcal{K} by $\mathcal{U}_*^{\text{geom}} := \mathcal{U}_*^{\text{MEM}} \times_{\pi_1(\text{MEM}_*, \omega)} \pi_1^{\text{geom}}(\text{MEM}_*, \omega)$. \square

The splitting is induced by the base point \vec{v}_o .

11. Extensions of variations of mixed hodge structure over $\mathcal{M}_{1,*}^{\text{an}}$

In this section we give a brief summary of results from [29] in the case of $\mathcal{M}_{1,*}^{\text{an}}$ and apply these to compute the extension groups in $\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$. It will be natural to work with real variations of MHS as well as \mathbb{Q} -variations. So in this section, F will be \mathbb{Q} or \mathbb{R} .

11.1. Computation of the Ext groups

By the results of the previous section, the category $\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})$ is the category of representations of

$$\widehat{\mathcal{G}}_* := \pi_1(\text{MHS}_F) \ltimes \mathcal{G}_*^{\text{rel}}.$$

Consequently, for an object \mathbb{V} of $\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})$ with fiber V over \vec{v}_o , there is a natural isomorphism

$$\text{Ext}_{\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})}^\bullet(F, \mathbb{V}) \cong H^\bullet(\widehat{\mathcal{G}}_*, V)$$

that is compatible with products. The conjugation action of $\widehat{\mathcal{G}}_*$ on $\mathcal{G}_*^{\text{rel}}$ induces an action of $\pi_1(\text{MHS}_F)$ on $H^\bullet(\mathcal{G}_*^{\text{rel}}, V)$. So $H^\bullet(\mathcal{G}_*^{\text{rel}}, V)$ has a natural F -MHS. An explicit construction of this MHS is given in [29].

The spectral sequence of the extension

$$1 \rightarrow \mathcal{G}_*^{\text{rel}} \rightarrow \widehat{\mathcal{G}}_* \rightarrow \pi_1(\text{MHS}_F) \rightarrow 1$$

and the fact that $\text{Ext}_{\text{MHS}_F}^j(A, B)$ vanishes for all MHS A and B when $j > 1$, implies that there is an exact sequence

$$0 \rightarrow \text{Ext}_{\text{MHS}}^1(F, H^{j-1}(\mathcal{G}_*^{\text{rel}}, V)) \rightarrow \text{Ext}_{\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})}^j(F, \mathbb{V}) \rightarrow \Gamma H^j(\mathcal{G}_*^{\text{rel}}, V) \rightarrow 0, \tag{11.1}$$

where Γ denotes the functor $\text{Hom}_{\text{MHS}}(F, _)$. For all objects \mathbb{V} of $\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})$, there is a natural homomorphism [29, Thm. 7.2]

$$\text{Ext}_{\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})}^\bullet(F, \mathbb{V}) \rightarrow H_{\mathcal{D}}^\bullet(\mathcal{M}_{1,*}^{\text{an}}, \mathbb{V}). \tag{11.2}$$

In the case of modular curves, we get the best possible result.

Theorem 11.1 [29, Prop. 6.2, Cor. 7.6]. *The natural homomorphism $H^\bullet(\mathcal{G}_*^{\text{rel}}, V) \rightarrow H^\bullet(\mathcal{M}_{1,*}^{\text{an}}, \mathbb{V})$ is an isomorphism of MHS. Consequently, the homomorphism (11.2) is an isomorphism.*

Plugging this into the exact sequence (11.1) yields the following useful computational tool.

Corollary 11.2. *For all $j \geq 0$, there is an exact sequence*

$$0 \rightarrow \text{Ext}_{\text{MHS}_F}^1(F, H^{j-1}(\mathcal{M}_{1,*}^{\text{an}}, \mathbb{V})) \rightarrow \text{Ext}_{\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})}^j(F, \mathbb{V}) \rightarrow \Gamma H^j(\mathcal{M}_{1,*}^{\text{an}}, \mathbb{V}) \rightarrow 0.$$

This and the cohomology computations in § 9 imply:

Proposition 11.3. *The extension groups $\text{Ext}_{\text{MHS}_F(\mathcal{M}_{1,*}, \mathbb{H})}^j(F, \mathbb{V})$ vanish when $j > 2$ and $* \in \{1, \bar{1}\}$, and when $j > 3$ if $* = 2$.*

First the 1-extensions. The cohomology computations in § 9 imply that:

Theorem 11.4. *When $\ast \in \{1, \bar{1}, 2\}$, there is an exact sequence*

$$0 \rightarrow \text{Ext}_{\text{MHS}}^1(F, F(r)) \rightarrow \text{Ext}_{\text{MHS}(\mathcal{M}_{1,\ast,\mathbb{H}})}^1(F, F(r)) \rightarrow \Gamma H^1(\mathcal{M}_{1,\ast}^{\text{an}}, F(r)) \rightarrow 0.$$

The right-hand group is trivial except when $\ast = \bar{1}$ and $r = 1$. In this case, it is spanned by the class ψ_2 associated to the Eisenstein series G_2 . When $\ast \in \{1, \bar{1}\}$, and $m > 0$, we have

$$\text{Ext}_{\text{MHS}(\mathcal{M}_{1,\ast,\mathbb{H}})}^1(F, S^m \mathbb{H}(r)) = \begin{cases} F & m = 2n > 0 \text{ and } r = 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

When $\ast = 2$, we have

$$\text{Ext}_{\text{MHS}(\mathcal{M}_{1,2,\mathbb{H}})}^1(F, S^m \mathbb{H}(r)) = \begin{cases} F & m = 1 \text{ and } r = 1, \\ F & m = 2n > 0 \text{ and } r = 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

When $m = 2n > 0$, the extension corresponds to the Eisenstein series G_{2n+2} .¹³ □

Remark 11.5. The generator of $\text{Ext}_{\text{MHS}(\mathcal{M}_{1,2,\mathbb{H}})}^1(\mathbb{Q}, \mathbb{H}(1))$ is the extension whose fiber over $(E, 0, x)$

$$0 \rightarrow H_1(E, \mathbb{Q}) \rightarrow H_1(E, \{0, x\}; \mathbb{Q}) \rightarrow \tilde{H}_0(\{0, x\}; \mathbb{Q}) \rightarrow 0.$$

In degree 2, we consider only extensions of real variations of MHS, and so, only real Deligne–Beilinson cohomology. The reason for doing this should become apparent in § 21. We will compute the answer only for $\ast \in \{1, \bar{1}\}$ as we shall not need the case $\ast = 2$.

Theorem 11.6. *For all integers $m \geq 0$ and r we have*

$$\text{Ext}_{\text{MHS}(\mathcal{M}_{1,\ast,\mathbb{H}})}^2(\mathbb{R}, S^m \mathbb{H}(r)) = \begin{cases} \mathbb{R} & \ast = \bar{1}, m = 0 \text{ and } r \geq 2, \\ \mathbb{R} \oplus \bigoplus_{f \in \mathfrak{B}_{2n+2}} V_f & m = 2n > 0, r \geq 2n + 2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $H^2(\mathcal{M}_{1,\ast}^{\text{an}}, S^m \mathbb{H})$ vanishes for all $m \geq 0$ when $\ast \in \{1, \bar{1}\}$, we have

$$H_{\mathcal{D}}^2(\mathcal{M}_{1,\ast}, S^m \mathbb{H}_{\mathbb{R}}(r)) = \text{Ext}_{\text{MHS}}^1(\mathbb{R}, H^1(\mathcal{M}_{1,\ast}, S^m \mathbb{H}(r))).$$

This vanishes when m is odd. So suppose that $m = 2n$. The decomposition (9.1) implies that if $r \geq 2n + 2$, where $n > 0$ when $\ast = 1$ and $n \geq 0$ when $\ast = \bar{1}$, we have

$$\begin{aligned} H_{\mathcal{D}}^2(\mathcal{M}_{1,\ast}, S^{2n} \mathbb{H}_{\mathbb{R}}(r)) &= \text{Ext}_{\text{MHS}}^1(\mathbb{R}, H^1(\mathcal{M}_{1,\ast}, S^{2n} \mathbb{H}(r))) \\ &\cong \text{Ext}_{\text{MHS}}^1(\mathbb{R}, \mathbb{R}(r - 2n - 1)) \oplus \bigoplus_{f \in \mathfrak{B}_{2n+2}} V_f(r) \end{aligned}$$

and that this ext group vanishes in the remaining cases.

¹³These extensions correspond to the elliptic polylogarithms of Beilinson–Levin [7]. Their restriction to $\mathcal{M}_{1,1}^{\text{an}}$ is described in [28, § 13.3].

To complete the proof, recall that if A is a real mixed Hodge structure, then

$$\text{Ext}_{\text{MHS}}^1(\mathbb{R}, A(r)) \cong A_{\mathbb{C}} / (i^r A_{\mathbb{R}} + F^r A).$$

Since V_f has Hodge type $(2n + 1, 0)$ and $(0, 2n + 1)$,

$$V_{f, \mathbb{C}} = i^r V_f + F^r V_f$$

when $r < 2n + 2$ and $F^0 V_f = 0$ when $r \geq 2n + 2$. This implies $\text{Ext}_{\text{MHS}}^1(\mathbb{R}, V_f(r))$ vanishes when $r < 2n + 2$ and is isomorphic to V_f when $r \geq 2n + 2$. Similarly, $\text{Ext}_{\text{MHS}}^1(\mathbb{R}, \mathbb{R}(r - 2n - 1))$ vanishes when $r - 2n - 1 \leq 0$ and is isomorphic to \mathbb{R} when $r \geq 2n - 2$. □

11.2. The $\overline{\mathcal{F}}_{\infty}$ action

In this section, we take $* \in \{1, \bar{1}\}$. Since $\mathcal{M}_{1,*}$ and the universal elliptic curve over it are defined over \mathbb{Z} , complex conjugation (aka ‘real Frobenius’) $\mathcal{F}_{\infty} \in \text{Gal}(\mathbb{C}/\mathbb{R})$ acts on $\mathcal{E}^{\text{an}} \rightarrow \mathcal{M}_{1,*}^{\text{an}}$. This implies that \mathcal{F}_{∞} acts on $\mathbb{H}_{\mathbb{R}}$, and thus on $H^{\bullet}(\mathcal{M}_{1,*}^{\text{an}}, S^{2n} \mathbb{H}_{\mathbb{R}})$ as well. It extends to a \mathbb{C} -linear involution of $H^{\bullet}(\mathcal{M}_{1,*}^{\text{an}}, S^{2n} \mathbb{H}_{\mathbb{C}})$. As is well known, this can be computed in terms of modular symbols. (Cf. [28, Lemma 17.7].)

For each $f \in \mathfrak{B}_{2n+2}$, V_f is preserved by \mathcal{F}_{∞} . Write

$$V_f = V_f^+ \oplus V_f^-$$

where \mathcal{F}_{∞} acts at $+1$ on V_f^+ and -1 on V_f^- . The action on the class of the Eisenstein series is given by:

Lemma 11.7. *The real Frobenius operator \mathcal{F}_{∞} multiplies the class of ψ_{2n+2} in $H^1(\mathcal{M}_{1,*}^{\text{an}}, S^{2n} \mathbb{H}_{\mathbb{C}})$ by -1 .*

Proof. Since \mathcal{F}_{∞} commutes with the action of the Hecke operators, \mathcal{F}_{∞} multiplies the eigenform ψ_{2n+2} by ± 1 . To determine this multiple we use the residue map

$$\text{Res} : H^1(\mathcal{M}_{1,*}^{\text{an}}, S^{2n} \mathbb{H}_{\mathbb{C}}) \rightarrow S^{2n} H / \text{im } N \cong \mathbb{R} \mathbf{b}^{2n},$$

where N denotes the monodromy logarithm $-\mathbf{a}\partial/\partial\mathbf{b}$. It anti-commutes with \mathcal{F}_{∞} as \mathcal{F}_{∞} reverses the orientation of the q -disk. The result follows as \mathbf{b}^{2n} is invariant and $\text{Res}_{q=0} \psi_{2n+2} = (2\pi i \mathbf{b})^{2n} G_{2n+2}(0)$. □

Let

$$\overline{\mathcal{F}}_{\infty} : H^{\bullet}(\mathcal{M}_{1,*}^{\text{an}}, S^{2n} \mathbb{H}_{\mathbb{C}}) \rightarrow H^{\bullet}(\mathcal{M}_{1,*}^{\text{an}}, S^{2n} \mathbb{H}_{\mathbb{C}})$$

be the composition of \mathcal{F}_{∞} with complex conjugation on the coefficients. This is \mathbb{C} -antilinear. It corresponds to complex conjugation on $H_{\text{DR}}^{\bullet}(\mathcal{M}_{1,*}/\mathbb{R}, S^{2n} \mathcal{H}_{\mathbb{R}}) \otimes \mathbb{C}$ under the comparison isomorphism. It therefore preserves the Hodge filtration and the real structure. It lifts to an involution of the real Deligne–Beilinson cohomology

$$H_{\mathcal{D}}^{\bullet}(\mathcal{M}_{1,*}^{\text{an}}, S^{2n} \mathbb{H}(r)) \cong \text{Ext}_{\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})}^{\bullet}(\mathbb{R}, S^{2n} \mathbb{H}(r)).$$

The (Betti) copy of $\mathbb{Q}(r - 2n - 1)$ in $H^1(\mathcal{M}_{1,*}^{\text{an}}, S^{2n} \mathbb{H}_{\mathbb{Q}}(r))$ is spanned by

$$(2\pi i)^{r-2n-1} \psi_{2n+2}.$$

(Cf. Remark 9.1.) Lemma 11.7 implies that $\overline{\mathcal{F}}_{\infty}$ multiplies it by $(-1)^r$. These observations yield the following refinement of Theorems 11.4 and 11.6.

Proposition 11.8. *If $*$ $\in \{1, \bar{1}, 2\}$, then*

$$H_{\mathcal{D}}^1(\mathcal{M}_{1,*}^{\text{an}}, \mathbb{R}(r))^{\bar{\mathcal{F}}_{\infty}} \cong \begin{cases} \mathbb{R} & * = \bar{1}, r = 1, \\ \mathbb{R} & r \geq 3 \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

If $$ $\in \{1, \bar{1}\}$, then for all integers $m > 0$ and r*

$$H_{\mathcal{D}}^1(\mathcal{M}_{1,*}^{\text{an}}, S^m \mathbb{H}_{\mathbb{R}}(r)) = H_{\mathcal{D}}^1(\mathcal{M}_{1,*}^{\text{an}}, S^m \mathbb{H}_{\mathbb{R}}(r))^{\bar{\mathcal{F}}_{\infty}}.$$

It is non-zero if and only if $m = 2n$, $r = 2n + 1$ and $m \geq 0$ when $$ $= \bar{1}$ and $m > 0$ when $*$ $= 1$. In degree 2*

$$H_{\mathcal{D}}^2(\mathcal{M}_{1,*}^{\text{an}}, S^m \mathbb{H}_{\mathbb{R}}(r))^{\bar{\mathcal{F}}_{\infty}} = \begin{cases} \mathbb{R} & * = \bar{1}, m = 0 \text{ and } r \geq 2 \text{ even}, \\ \mathbb{R} \oplus \bigoplus_{f \in \mathcal{B}_{2n+2}} V_f^- & m = 2n > 0, r \geq 2n + 2 \text{ even}, \\ \bigoplus_{f \in \mathcal{B}_{2n+2}} V_f^+ & m = 2n > 0, r > 2n + 2 \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

12. Weighted and crystalline completion

In this section we review the definitions of the ℓ -adic weighted and crystalline completion functors and make some initial observations about the crystalline completion of $\pi_1(\mathcal{M}_{1,*/\mathbb{Q}}, \bar{\mathcal{V}}_o)$. More details can be found in [31, 32].

Fix a prime number ℓ . Suppose that Γ is a profinite group and that R is a reductive algebraic group over \mathbb{Q}_{ℓ} , endowed with a central cocharacter $c : \mathbb{G}_m \rightarrow R$. Suppose that $\rho : \Gamma \rightarrow R(\mathbb{Q}_{\ell})$ is a continuous, Zariski dense representation. Say that a representation V of R has weight m if its pullback to \mathbb{G}_m via c is of weight m . Since c is central, Schur’s lemma implies that every irreducible representation of R has a weight.

Consider the full subcategory $\mathcal{R}_{\mathbb{Q}_{\ell}}^{\text{cts}}(\Gamma, \rho, c)$ of the category of continuous Γ -modules $\text{Rep}_{\mathbb{Q}_{\ell}}^{\text{cts}}(\Gamma)$ consisting of those Γ -modules M that are finite dimensional and admit a (necessarily unique) filtration

$$0 = W_n M \subseteq W_{n+1} M \subseteq \dots \subseteq W_{N-1} M \subseteq W_N M = M,$$

where the action of Γ on each graded quotient $\text{Gr}_m^W M$ factors through a rational representation $R \rightarrow \text{Aut}(\text{Gr}_m^W M)$ of weight m via ρ . Note that every irreducible R -module is an object of $\mathcal{R}_{\mathbb{Q}_{\ell}}^{\text{cts}}(\Gamma, \rho, c)$.

This category is neutral tannakian over \mathbb{Q}_{ℓ} . The weighted completion of Γ relative to ρ is the affine group scheme $\pi_1(\mathcal{R}_{\mathbb{Q}_{\ell}}^{\text{cts}}(\Gamma, \rho, c), \omega)$ over \mathbb{Q}_{ℓ} , where ω is the fiber functor that takes a module M to its underlying vector space. It is an extension

$$1 \rightarrow \mathcal{U}(\Gamma, \rho, c) \rightarrow \pi_1(\mathcal{R}_{\mathbb{Q}_{\ell}}^{\text{cts}}(\Gamma, \rho, c), \omega) \rightarrow R \rightarrow 1$$

where $\mathcal{U}(\Gamma, \rho, c)$ is prounipotent. There is a natural homomorphism

$$\tilde{\rho} : \Gamma \rightarrow \pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c), \omega)(\mathbb{Q}_\ell)$$

whose composition with the canonical quotient map $\pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c), \omega) \rightarrow R$ is ρ . It is Zariski dense.

Levi’s theorem implies that the central cocharacter $c : \mathbb{G}_m \rightarrow R$ can be lifted to a cocharacter $\tilde{c} : \mathbb{G}_m \rightarrow \pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c), \omega)$. So every object V of $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c)$ is a \mathbb{G}_m -module via \tilde{c} . Denote the subspace of V on which \mathbb{G}_m acts via the n th power of the standard character by V_n . The inclusion $V_n \hookrightarrow V$ induces an isomorphism $V_n \cong \text{Gr}_n^W V$.

Proposition 12.1 [31, Theorem 3.12]. *The isomorphism $V \cong \bigoplus_{n \in \mathbb{Z}} V_n$ defines a splitting*

$$V \cong \bigoplus_{n \in \mathbb{Z}} \text{Gr}_\bullet^W V$$

of the weight filtration that is natural with respect to morphisms of $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c)$. It is compatible with duals and tensor products.

Although this isomorphism is natural, it is *not* canonical as it depends on the choice of lift \tilde{c} of the central cocharacter c .

Corollary 12.2. *For each $n \in \mathbb{Z}$, the functor $\text{Gr}_n^W : \mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c) \rightarrow \text{Rep}_F(R)$ is exact.*

Example 12.3. Here we recall from [31] the setup for the weighted completion of $\pi_1(\text{Spec } \mathbb{Z}[1/\ell])$. Let $\Gamma = \pi_1(\text{Spec } \mathbb{Z}[1/\ell], \overline{\mathbb{Q}})$, $R = \mathbb{G}_m$ and $\rho : \Gamma \rightarrow \mathbb{G}_m(\mathbb{Q}_\ell)$ be the ℓ -adic cyclotomic character $\chi_\ell : G_{\mathbb{Q}} \rightarrow \mathbb{Z}_\ell^\times \subset \mathbb{Q}_\ell^\times$. The defining representation of \mathbb{G}_m thus corresponds to $\mathbb{Q}_\ell(1)$. Since this has weight -2 , we define the central cocharacter to be the homomorphism $c : \mathbb{G}_m \rightarrow \mathbb{G}_m$ defined by $z \mapsto z^{-2}$. The weighted completion of Γ with respect to χ_ℓ and c is

$$\mathcal{A}_\ell^{\text{wtd}} = \pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\pi_1(\text{Spec } \mathbb{Z}[1/\ell], \overline{\mathbb{Q}}), \rho, c), \omega).$$

12.1. Crystalline completion

Suppose that $K \subset \overline{\mathbb{Q}}$ is a number field. Set $G_K = \text{Gal}(\overline{\mathbb{Q}}/K)$. We continue with the notation of the previous paragraph. Suppose in addition that there is a surjection $\Gamma \rightarrow G_K$ with a distinguished splitting $s : G_K \rightarrow \Gamma$. Define an object V of $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c)$ to be *crystalline* (with respect to s) if the representation $\rho \circ s : G_K \rightarrow \text{GL}(V)$ is unramified at all primes that do not lie over ℓ and that it is crystalline at all primes that lie over ℓ .¹⁴

Let $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(\Gamma, \rho, c)$ be the full subcategory of $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c)$ whose objects are the crystalline objects. It is tannakian. The ℓ -adic crystalline completion of (Γ, s) is defined to be

$$\pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(\Gamma, \rho, c), \omega).$$

It is a quotient of the weighted completion $\pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c), \omega)$.

¹⁴In some applications it is natural to relax this definition and allow crystalline representations to be ramified at a set S of primes. We will not need to do that here as $\mathcal{M}_{1,*/\mathbb{Z}}$ has good reduction at all primes.

As in the case of weighted completion, every object of $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c)$ has a natural weight filtration W_\bullet , the functors Gr_n^W are exact. Each lift \tilde{c} of the central cocharacter c to $\pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c), \omega)$ defines a natural splitting of the weight filtration of each object of $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\Gamma, \rho, c)$.

12.2. Crystalline completion of $G_{\mathbb{Q}}$

The setup is similar to that in Example 12.3. Let $\Gamma = G_{\mathbb{Q}}$, $R = \mathbb{G}_m$ and $\rho : G_{\mathbb{Q}} \rightarrow \mathbb{G}_m(\mathbb{Q}_\ell)$ the ℓ -adic cyclotomic character χ_ℓ . The central cocharacter $c : \mathbb{G}_m \rightarrow \mathbb{G}_m$ is defined by $z \mapsto z^{-2}$. We also take $K = \mathbb{Q}$ and the homomorphism $\Gamma \rightarrow G_{\mathbb{Q}}$ to be the identity. Define

$$\mathcal{A}_\ell^{\text{cris}} = \pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(G_{\mathbb{Q}}, \chi_\ell, c), \omega).$$

There is a canonical surjection $\mathcal{A}_\ell^{\text{wtd}} \rightarrow \mathcal{A}_\ell^{\text{cris}}$, where $\mathcal{A}_\ell^{\text{wtd}}$ denotes the weighted completion of $\pi_1(\text{Spec } \mathbb{Z}[1/\ell], \overline{\mathbb{Q}})$ defined in Example 12.3.

The ℓ -adic realization functor $\text{MTM} \rightarrow \mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(G_{\mathbb{Q}}, \chi_\ell, c)$ induces a homomorphism $\mathcal{A}_\ell^{\text{cris}} \rightarrow \pi_1(\text{MTM}, \omega_\ell)$.

Theorem 12.4 [31]. *For all ℓ , the natural homomorphism $\mathcal{A}_\ell^{\text{cris}} \rightarrow \pi_1(\text{MTM}, \omega_\ell)$ is an isomorphism.*

12.3. Weighted and crystalline completion of $\pi_1(\mathcal{M}_{1,*}, \vec{v}_o)$

The monodromy of the local system \mathbb{H}_ℓ over $\mathcal{M}_{1,*}/\mathbb{Z}$ induces a homomorphism

$$\rho_\ell : \pi_1(\mathcal{M}_{1,*}/\mathbb{Z}[1/\ell], \vec{v}_o) \rightarrow \text{GL}(H_\ell).$$

Define $c : \mathbb{G}_m \rightarrow \text{GL}(H)$ to be the cocharacter $\lambda \mapsto \lambda \text{id}_H$. It is central and assigns W -weight 1 to H_ℓ . Recall from §5.5 that, as a $G_{\mathbb{Q}}$ -module, H_ℓ is isomorphic to $\mathbb{Q}_\ell(0) \oplus \mathbb{Q}_\ell(-1)$. Define

$$\mathcal{G}_*^{\text{wtd},\ell} = \pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\pi_1(\mathcal{M}_{1,*}/\mathbb{Z}[1/\ell], \vec{v}_o), \rho_\ell, c), \omega).$$

To define the ℓ -adic crystalline completion of $\pi_1(\mathcal{M}_{1,*}/\mathbb{Q}, \vec{v}_o)$, we take s to be the section of $\pi_1(\mathcal{M}_{1,*}/\mathbb{Q}, \vec{v}_o) \rightarrow G_{\mathbb{Q}}$ induced by \vec{v}_o . Define

$$\mathcal{G}_*^{\text{cris},\ell} = \pi_1(\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(\pi_1(\mathcal{M}_{1,*}/\mathbb{Q}, \vec{v}_o), \rho_\ell, c), \omega).$$

Both groups are extensions of $\text{GL}(H_\ell)$ by a pronipotent group:

$$\begin{aligned} 1 &\rightarrow \mathcal{U}_*^{\text{wtd},\ell} \rightarrow \mathcal{G}_*^{\text{wtd},\ell} \rightarrow \text{GL}(H_\ell) \rightarrow 1 \\ 1 &\rightarrow \mathcal{U}_*^{\text{cris},\ell} \rightarrow \mathcal{G}_*^{\text{cris},\ell} \rightarrow \text{GL}(H_\ell) \rightarrow 1. \end{aligned}$$

Proposition 12.5. *There are homomorphisms*

$$\mathcal{G}_*^{\text{rel}} \otimes \mathbb{Q}_\ell \longrightarrow \mathcal{G}_*^{\text{wtd},\ell} \longrightarrow \mathcal{G}_*^{\text{cris},\ell} \longrightarrow \mathcal{A}_\ell^{\text{cris}},$$

the last two of which are surjective, such that the diagram

$$\begin{array}{ccccccc}
 \pi_1^{\text{top}}(\mathcal{M}_{1,*}, \vec{v}_o) & \longrightarrow & \pi_1(\mathcal{M}_{1,*/\mathbb{Q}}, \vec{v}_o) & \longrightarrow & \pi_1(\mathcal{M}_{1,*/\mathbb{Z}[1/\ell]}, \vec{v}_o) & \longrightarrow & \pi_1(\text{Spec } \mathbb{Z}[1/\ell]) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \mathcal{G}_*^{\text{rel}}(\mathbb{Q}_\ell) & \longrightarrow & \mathcal{G}_*^{\text{wtd}, \ell}(\mathbb{Q}_\ell) & \longrightarrow & \mathcal{G}_*^{\text{cris}, \ell}(\mathbb{Q}_\ell) & \longrightarrow & \mathcal{A}_\ell^{\text{cris}}
 \end{array}$$

commutes. The sequence $\mathcal{G}_*^{\text{rel}} \otimes \mathbb{Q}_\ell \rightarrow \mathcal{G}_*^{\text{cris}, \ell} \rightarrow \mathcal{A}_\ell^{\text{cris}} \rightarrow 1$ is exact and the morphism $\mathcal{G}_*^{\text{cris}, \ell} \rightarrow \mathcal{A}_\ell^{\text{cris}}$ is split.

Proof. The first homomorphism is induced by the forgetful functor

$$\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}(\pi_1(\mathcal{M}_{1,*/\mathbb{Z}[1/\ell]}, \vec{v}_o), \rho, c) \rightarrow \mathcal{R}_{\mathbb{Q}_\ell}(\pi_1^{\text{top}}(\mathcal{M}_{1,*}, \vec{v}_o), \rho).$$

The existence of the middle surjection follows from the fact that

$$\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}} := \mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(\pi_1(\mathcal{M}_{1,*/\mathbb{Q}}, \vec{v}_o), \rho_\ell, c)$$

is a full subcategory of $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cts}}$. The existence of the right-hand square follows from the commutativity of the diagram

$$\begin{array}{ccc}
 \pi_1(\mathcal{M}_{1,*/\mathbb{Q}}, \vec{v}_o) & \longrightarrow & G_{\mathbb{Q}} \\
 \downarrow & & \downarrow \chi_\ell \\
 \text{GL}(H_\ell) & \xrightarrow{\det^{-1}} & G_m(\mathbb{Q}_\ell) \\
 \swarrow c=(\)^1 \text{id}_H & & \nwarrow c=(\)^{-2} \\
 & G_m(\mathbb{Q}_\ell) &
 \end{array}$$

The last statement follows from the right exactness of completion and from the fact that one has functors $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(G_{\mathbb{Q}}, \chi_\ell, c) \rightarrow \mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}} \rightarrow \mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(G_{\mathbb{Q}}, \chi_\ell, c)$ whose composite is the identity. The first is the inclusion of the geometrically constant objects, the second is restriction to the fiber over \vec{v}_o . □

13. Extensions of ℓ -adic sheaves over $\mathcal{M}_{1,*/\mathbb{Q}}$

Our goal in this section is to prove the ℓ -adic analogues of the results in § 11. The work [48] of Olsson, which we use as a ‘black box’, is key. These results are not needed in the sequel. They are included to show how the computations in this paper are related to certain fundamental questions in number theory.

The ℓ -adic analogues of the groups $\text{Ext}_{\text{MHS}(\mathcal{M}_{1,*/\mathbb{H}})}^\bullet(\mathbb{R}, S^m \mathbb{H}(r))$ are the groups $H^\bullet(\mathcal{G}_*^{\text{cris}, \ell}, S^m H_\ell(r))$. In this section, we compute these cohomology groups in degree 1 and bound them in degree 2.

For convenience, we denote the category $\mathcal{R}_{\mathbb{Q}_\ell}^{\text{cris}}(\pi_1(\mathcal{M}_{1,*/\mathbb{Z}[1/\ell]}, \vec{v}_o), \rho_\ell, c)$ by MEM_*^ℓ , so that MEM_*^ℓ is equivalent to $\text{Rep}(\mathcal{G}_*^{\text{cris}, \ell})$ and

$$H^\bullet(\mathcal{G}_*^{\text{cris}, \ell}, S^m H_\ell(r)) = \text{Ext}_{\text{MEM}_*^\ell}^\bullet(\mathbb{Q}_\ell, S^m \mathbb{H}_\ell(r)).$$

In this and subsequent sections, we will use the following notation. Denote the category of $G_{\mathbb{Q}}$ modules that are unramified at all $p \neq \ell$ and crystalline at ℓ by \mathcal{C}_{ℓ} . For an object V of \mathcal{C}_{ℓ} , define

$$H_{\text{sm}}^{\bullet}(G_{\mathbb{Q}}, V) := \text{Ext}_{\mathcal{C}_{\ell}}^{\bullet}(\mathbb{Q}_{\ell}, V).$$

This group is typically denoted by $H_f^{\bullet}(G_{\mathbb{Q}}, V)$. We have chosen a non-standard notation to avoid a notational conflict as f will often denote a cusp form of $\text{SL}_2(\mathbb{Z})$.

13.1. Computation of $H^1(\mathcal{G}_*^{\text{cris}, \ell}, S^m H_{\ell}(r))$

The following result is the étale analogue of Theorem 11.4.

Proposition 13.1. *When $* \in \{1, \bar{1}\}$, we have*

$$H^1(\mathcal{G}_*^{\text{cris}, \ell}, S^m H_{\ell}(r)) = \begin{cases} \mathbb{Q}_{\ell} & * = 1, m = 0 \text{ and } r \geq 3 \text{ odd,} \\ \mathbb{Q}_{\ell} & * = \bar{1}, m = 0 \text{ and } r \geq 1 \text{ odd,} \\ \mathbb{Q}_{\ell} & m = 2n > 0 \text{ and } r = 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

When $* = 2$, we have

$$H^1(\mathcal{G}_*^{\text{cris}, \ell}, S^m H_{\ell}(r)) = \begin{cases} \mathbb{Q}_{\ell} & m = 0 \text{ and } r \geq 3 \text{ odd,} \\ \mathbb{Q}_{\ell} & m = 1 \text{ and } r = 1, \\ \mathbb{Q}_{\ell} & m = 2n > 0 \text{ and } r = 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will prove the result when $* = 1$. The remaining cases are left to the reader. The proof uses weighted completion, which is easier to compute than crystalline completion. Since the natural homomorphism $\mathcal{G}_1^{\text{wtd}, \ell} \rightarrow \mathcal{G}_1^{\text{cris}, \ell}$ is surjective (Proposition 12.5), it induces an injection

$$H^1(\mathcal{G}_1^{\text{cris}, \ell}, S^m H_{\ell}(r)) \hookrightarrow H^1(\mathcal{G}_1^{\text{wtd}, \ell}, S^m H_{\ell}(r)). \tag{13.1}$$

By [31, Theorem 4.6], there is a natural isomorphism

$$H^1(\mathcal{G}_1^{\text{wtd}, \ell}, S^m H_{\ell}(r)) \cong H^1(\pi_1(\mathcal{M}_{1,1/\mathbb{Z}[1/\ell]}, \vec{\mathbf{v}}_o))$$

Set $G_{\ell} = \pi_1(\text{Spec } \mathbb{Z}[1/\ell], \bar{\mathbb{Q}})$. The Hochschild–Serre spectral sequence of the extension

$$1 \rightarrow \pi_1(\mathcal{M}_{1,1/\bar{\mathbb{Q}}}, \vec{\mathbf{v}}_o) \rightarrow \pi_1(\mathcal{M}_{1,1/\mathbb{Z}[1/\ell]}, \vec{\mathbf{v}}_o) \rightarrow G_{\ell} \rightarrow 1$$

degenerates at E_2 as the kernel has virtual cohomological dimension 1 and because the extension is split. This implies that

$$H^1(\mathcal{G}_1^{\text{wtd}, \ell}, S^m H_{\ell}(r)) \cong \begin{cases} H^1(G_{\ell}, \mathbb{Q}_{\ell}(r)) & m = 0, \\ H^0(G_{\ell}, H_{\text{ét}}^1(\mathcal{M}_{1,1/\bar{\mathbb{Q}}}, S^m \mathbb{H}_{\ell}(r))) & m > 0. \end{cases}$$

When $m = 0$, the image of (13.1) equals the image of

$$H_{\text{sm}}^1(G_{\mathbb{Q}}, \mathbb{Q}_{\ell}(r)) \hookrightarrow H^1(G_{\ell}, \mathbb{Q}_{\ell}(r)) \cong H^1(\mathcal{G}_1^{\text{wtd}, \ell}, \mathbb{Q}_{\ell}(r)).$$

The $m = 0$ case now follows from Soulé’s computation [55].

Now suppose that $m > 0$. Then we have an injection

$$H^1(\mathcal{G}_1^{\text{cris}, \ell}, S^m H_{\ell}(r)) \hookrightarrow H^0(G_{\ell}, H_{\text{ét}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^m \mathbb{H}_{\ell}(r))).$$

The comparison theorem and Theorem 9.2 imply that

$$H^0(G_{\ell}, H_{\text{ét}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^m \mathbb{H}_{\ell}(r))) \cong \mathbb{Q}_{\ell}$$

when $m = 2n > 0$ and $r = 2n + 1$, and is zero otherwise.

To complete the proof, we have to show that (13.1) is surjective when $n > 0$. To do this, consider the extension \mathbb{E}'_{2n+1} of $\pi_1(\mathcal{M}_{1,1/\mathbb{Q}}, \vec{\mathfrak{t}})$ -modules corresponding to the subextension of the extension (10.1) over $\mathcal{M}_{1,1}^{\text{an}}$ with $m = 2n + 1$ that is an extension of \mathbb{Q}_{ℓ} by $S^{2n} \mathbb{H}_{\ell}(2n + 1)$. The restriction of $S^{2n} \mathbb{H}_{\ell}(2n + 1)$ to the base point $\vec{\mathfrak{t}}$ splits as a sum

$$S^{2n} H_{\ell}(2n + 1) = \mathbb{Q}_{\ell}(1) \oplus \mathbb{Q}_{\ell}(2) \oplus \cdots \oplus \mathbb{Q}_{\ell}(2n + 1).$$

So the fiber of \mathbb{E}'_{2n+1} over $\vec{\mathfrak{t}}$ splits as a sum of extensions of \mathbb{Q}_{ℓ} by $\mathbb{Q}_{\ell}(j)$, with $1 \leq j \leq 2n + 1$. Nakamura [45, Theorems 3.3 and 3.5] shows that all but the last of these is trivial, and the last one is a non-zero multiple of the generator of $H_{\text{sm}}^1(G_{\mathbb{Q}}, \mathbb{Q}_{\ell}(2n + 1))$.¹⁵ This implies that the restriction of the \mathbb{E}'_{2n+1} to $\vec{\mathfrak{t}}$ is crystalline, so that (13.1) is surjective. \square

The following technical computation will be needed in the next section.

Corollary 13.2. *Suppose that $* \in \{1, \vec{1}\}$. If $e \geq 0$, then*

$$\text{Ext}_{\text{MEM}_*}^1(S^{2n} H_{\ell}(2n + 1), S^{2m} H_{\ell}(2m - e)) = 0.$$

Proof. In order for the ext group to be non-zero, the weight of $S^{2n} H_{\ell}(2n + 1)$ has to be greater than the weight of $S^{2m} H_{\ell}(2m - e)$. That is, we have $-2n - 2 > -2m + 2e$ or, equivalently, $0 \leq n < m - e - 1$. In particular, $n < m$. Since $n < m$

$$\begin{aligned} \text{Ext}_{\text{MEM}_*}^1(S^{2n} H_{\ell}(2n + 1), S^{2m} H_{\ell}(2m - e)) &\cong \text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}_{\ell}, S^{2n} H_{\ell} \otimes S^{2m} H_{\ell}(2m - e - 1)) \\ &\cong \bigoplus_{t=0}^{2n} \text{Ext}^1(\mathbb{Q}_{\ell}, S^{2n+2m-2t} H_{\ell}(2m - e - t - 1)). \end{aligned}$$

Since $t \leq 2n$ and $e \geq 0$, we have $2m - e - t - 1 < 2n + 2m - 2t$. So none of the summands on the right-hand side can be non-zero. The result follows. \square

13.2. A bound on $H^2(\mathcal{G}_*^{\text{cris}, \ell}, S^m H_{\ell}(r))$

In this section we prove the following analogue of the second statement of Proposition 11.8.

¹⁵See also Remarks 14.2 and 15.2.

Recall from [52, Theorem 1.2.4] that $H_{\text{ét}}^1(\mathcal{M}_{1,*}/\overline{\mathbb{Q}}, S^m \mathbb{H}_\ell(r))$ is a crystalline $G_{\mathbb{Q}}$ -module.

Theorem 13.3. *Fix a prime number ℓ . If $* \in \{1, \bar{1}\}$ and $m \geq 0$ and r are integers, then $H^2(\mathcal{G}_*^{\text{cris},\ell}, S^m H_\ell(r))$ vanishes when m is odd and when $r < m + 2$. When $m = 2n$, there is an injection*

$$H^2(\mathcal{G}_*^{\text{cris},\ell}, S^{2n} H_\ell(r)) \hookrightarrow H_{\text{sm}}^1(G_{\mathbb{Q}}, H^1(\mathcal{M}_{1,*}/\overline{\mathbb{Q}}, S^{2n} H_\ell(r))).$$

More precisely, when $r \geq 2n + 2$, there is an injection

$$H^2(\mathcal{G}_*^{\text{cris},\ell}, S^{2n} H_\ell(r)) \hookrightarrow \begin{cases} \mathbb{Q}_\ell & * = \bar{1}, n = 0, r \geq 2 \text{ even,} \\ \mathbb{Q}_\ell \oplus H_{\text{sm}}^1(G_{\mathbb{Q}}, H_{\text{cusp}}^1(\mathcal{M}_{1,*}/\overline{\mathbb{Q}}, S^{2n} \mathbb{H}_\ell(r))) & n > 0, r \text{ even,} \\ H_{\text{sm}}^1(G_{\mathbb{Q}}, H_{\text{cusp}}^1(\mathcal{M}_{1,*}/\overline{\mathbb{Q}}, S^{2n} \mathbb{H}_\ell(r))) & n > 0, r \text{ odd.} \end{cases}$$

Proof. We will prove the theorem when $* = 1$. The proof of the case $* = \bar{1}$ is similar and is left to the reader.

We begin by establishing the vanishing of $H^2(\mathcal{G}_1^{\text{cris},\ell}, S^m H_\ell(r))$ when $r < m + 2$ and when m is odd. There are at least two ways to do this. Both exploit the vanishing of $H^1(\mathcal{G}_1^{\text{cris},\ell}, S^m H_\ell(r))$ when $r < m + 1$ or m is odd. We will use Lie algebra cohomology. The other approach uses Yoneda extensions.

Denote the Lie algebra of $\mathcal{U}_*^{\text{cris},\ell}$ by $\mathfrak{u}_*^{\text{cris},\ell}$. Since

$$\text{Ext}_{\text{MEM}_1}^j(\mathbb{Q}_\ell, S^m \mathbb{H}_\ell(r)) \cong H^j(\mathfrak{u}_1^{\text{cris},\ell}, S^m H_\ell(r))^{\text{GL}(H)}$$

and since there is a natural isomorphism

$$\text{Gr}_\bullet^W H^j(\mathfrak{u}_1^{\text{cris},\ell}, S^m H_\ell(r)) \cong H^j(\text{Gr}_\bullet^W \mathfrak{u}_1^{\text{cris},\ell}, S^m H_\ell(r)),$$

it follows that

$$\text{Gr}_\bullet^W H_1(\mathfrak{u}_1^{\text{cris},\ell}) \cong \bigoplus_{n>0} S^{2n} H_\ell(2n + 1).$$

Since $\text{Gr}_\bullet^W \mathfrak{u}_1^{\text{cris},\ell}$ is generated as a Lie algebra in the category of $\text{GL}(H)$ -modules by the image of any $\text{GL}(H)$ -invariant section of $\text{Gr}_\bullet^W \mathfrak{u}_1^{\text{cris},\ell} \rightarrow \text{Gr}_\bullet^W H_1(\mathfrak{u}_1^{\text{cris},\ell})$, and since

$$S^a H \otimes S^b H \cong \bigoplus_{t \geq 0} S^{a+b-2t} H(-t)$$

it follows that if $S^m H_\ell(r)$ appears in $\text{Gr}_\bullet^W \mathfrak{u}_1^{\text{cris},\ell}$, then m is even and $r \geq m + 1$. This implies that if $S^m H_\ell(r)$ occurs in $\Lambda^2 \text{Gr}_\bullet^W \mathfrak{u}_1^{\text{cris},\ell}$, then m is even and $r \geq m + 2$. Consequently, the group of $\text{GL}(H)$ -invariant 2-cochains

$$\text{Hom}_{\text{GL}(H)}(\Lambda^2 \text{Gr}_\bullet^W \mathfrak{u}_1^{\text{cris},\ell}, S^m H_\ell(r))$$

is non-zero only when m is even and $r \geq m + 2$. It follows that $H^2(\mathfrak{u}_1^{\text{cris},\ell}, S^m H_\ell(r))$ vanishes when m is odd and when $r < 2m + 2$.

There are natural homomorphisms

$$H^2(\mathcal{G}_1^{\text{cris},\ell}, S^{2n} H_\ell(r)) \rightarrow H^2(\mathcal{G}_1^{\text{wt},\ell}, S^{2n} H_\ell(r)) \rightarrow H_{\text{ét}}^2(\mathcal{M}_{1,1/\mathbb{Z}[1/\ell]}, S^m \mathbb{H}_\ell(r))$$

General properties of weighted completion (cf. [31, Theorem 4.6]) imply that the right-hand map is injective. Our next task is to show that the second mapping is injective. We do this using Yoneda extensions.

Suppose that the 2-extension

$$0 \rightarrow S^{2n} H_\ell(r) \rightarrow E \rightarrow F \rightarrow \mathbb{Q}_\ell \rightarrow 0$$

in MEM_1^ℓ represents a class in the kernel of

$$H^2(\mathcal{G}_1^{\text{cris},\ell}, S^{2n} H_\ell(r)) \rightarrow H^2(\mathcal{G}_1^{\text{wt},\ell}, S^{2n} H_\ell(r))$$

By Yoneda’s criterion, there is a $\mathcal{G}_1^{\text{wt},\ell}$ -module V and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & S^{2n} H_\ell(r) & \longrightarrow & V & \longrightarrow & F \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{id}_V & & \downarrow \\ 0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & \mathbb{Q}_\ell \longrightarrow 0 \end{array}$$

with exact rows in $\text{Rep}(\mathcal{G}_1^{\text{wt},\ell})$. We view V as a $G_{\mathbb{Q}}$ -module via the homomorphism

$$G_{\mathbb{Q}} \rightarrow \pi_1(\mathcal{M}_{1,1/\mathbb{Q}}, \vec{\mathfrak{t}}) \rightarrow \mathcal{G}_1^{\text{wt}}(\mathbb{Q}_\ell) \rightarrow \text{Aut } V$$

where the first homomorphism is the section induced by the base point $\vec{\mathfrak{t}}$. To prove injectivity, it suffices to show that V is a crystalline $G_{\mathbb{Q}}$ -module as the fact that it is a $\mathcal{G}_\ell^{\text{wt}}$ -module implies that it is unramified away from ℓ .

The action of $\mathcal{G}_\ell^{\text{wt}}$ on V determines compatible weight filtrations M_\bullet on E, F and V .

Let B be the kernel of $F \rightarrow \mathbb{Q}_\ell$. By the Pruning Lemma below (Lemma 13.4), we may assume that each W_\bullet graded quotient of B is a sum of copies of $S^m H_\ell(r)$ where $r > m$ or, equivalently, that $B = M_{-2}B$. In this case B is an extension

$$0 \rightarrow M_{-4}B \rightarrow B \rightarrow \mathbb{Q}_\ell(1)^d \rightarrow 0$$

for some $d \geq 0$. This implies that E is also an extension

$$0 \rightarrow M_{-4}E \rightarrow E \rightarrow \mathbb{Q}_\ell(1)^d \rightarrow 0.$$

The pushout of the extension

$$0 \rightarrow E \rightarrow V \rightarrow \mathbb{Q}_\ell \rightarrow 0$$

along $E \rightarrow \mathbb{Q}_\ell(1)^d$ is the extension

$$0 \rightarrow \mathbb{Q}_\ell(1)^d \rightarrow F/M_{-4}F \rightarrow \mathbb{Q}_\ell \rightarrow 0.$$

Since F is crystalline, this is an extension of crystalline $G_{\mathbb{Q}}$ -modules. Proposition 11.3 of [33] implies that V is also a crystalline $G_{\mathbb{Q}}$ -module and therefore a $\mathcal{G}_1^{\text{cris},\ell}$ module. This completes the proof of injectivity.

The final task is to show that the image of

$$H^2(\mathcal{G}_1^{\text{cris},\ell}, S^{2n} H_\ell(r)) \rightarrow H^1(G_\ell, H_{\text{ét}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^{2n} \mathbb{H}_\ell(r))) \tag{13.2}$$

is contained in $H_{\text{sm}}^1(G_\mathbb{Q}, H_{\text{ét}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^{2n} \mathbb{H}_\ell(r)))$. When $n = 0$, this is clear.

To prove this when $n > 0$, we use the fact that $\mathcal{G}_1^{\text{rel}} \otimes \mathbb{Q}_\ell$ is a crystalline representation of $G_\mathbb{Q}$, or more accurately, that its coordinate ring $\mathcal{O}(\mathcal{G}_1^{\text{rel}}) \otimes \mathbb{Q}_\ell$ is a crystalline $G_\mathbb{Q}$ -module. (That it is unramified away from ℓ is proved in [33], the full statement is proved in [48].)

Let $\mathcal{A}_\ell^{\text{Mod}}$ be the Zariski closure of the image of $G_\mathbb{Q}$ in $\text{Aut}(\mathcal{G}_{1/\mathbb{Q}_\ell}^{\text{rel}})$.¹⁶ Representations of $G_\mathbb{Q}$ that factor through $G_\mathbb{Q} \rightarrow \mathcal{A}_\ell^{\text{Mod}}(\mathbb{Q}_\ell)$ are crystalline. Denote by $\mathcal{C}_1^{\text{cris},\ell}$ the tannakian category of finite dimensional representations of $G_\mathbb{Q} \times \mathcal{G}_{1/\mathbb{Q}_\ell}^{\text{rel}}$. Its tannakian fundamental group (with respect to the underlying \mathbb{Q}_ℓ vector space) is a split extension

$$1 \rightarrow \mathcal{G}_{1/\mathbb{Q}_\ell}^{\text{rel}} \rightarrow \pi_1(\mathcal{C}_1^{\text{cris}}) \rightarrow \mathcal{A}_\ell^{\text{Mod}} \rightarrow 1.$$

The splitting is induced by the base point \vec{t} . There is a commutative diagram

$$\begin{array}{ccc} G_\mathbb{Q} \times \mathcal{G}_{1/\mathbb{Q}_\ell}^{\text{rel}} & \longrightarrow & \mathcal{A}_\ell^{\text{Mod}} \times \mathcal{G}_{1/\mathbb{Q}_\ell}^{\text{rel}} \\ \downarrow & & \downarrow \\ \mathcal{G}_1^{\text{wtd},\ell} & \longrightarrow & \mathcal{G}_1^{\text{cris},\ell} \end{array} \tag{13.3}$$

Now suppose that $n > 0$. Since $H^2(\mathcal{G}_1^{\text{rel}}, S^{2n} H(r))$ vanishes for all n and r , since $H^0(\mathcal{G}_{1/\mathbb{Q}_\ell}^{\text{rel}}, S^{2n} H_\ell(r)) = 0$ when $n > 0$, and since there are $G_\mathbb{Q}$ -module isomorphisms

$$H^1(\mathcal{G}_1^{\text{rel}}, S^{2n} H_\ell(r)) \cong [H^1(\mathfrak{u}_1^{\text{rel}}) \otimes S^{2n} H_\ell(r)]^{\text{SL}(H)} \cong H_{\text{ét}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^{2n} \mathbb{H}_\ell(r))$$

there are isomorphisms

$$H^2(G_\mathbb{Q} \times \mathcal{G}_{1/\mathbb{Q}_\ell}^{\text{rel}}, S^{2n} H_\ell(r)) \cong H^1(G_\mathbb{Q}, H_{\text{ét}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^{2n} \mathbb{H}_\ell(r)))$$

and

$$\begin{aligned} H^2(\mathcal{G}_{1/\mathbb{Q}_\ell}^{\text{rel}}, S^{2n} H_\ell(r)) &\cong H_{\text{sm}}^1(G_\mathbb{Q}, H_{\text{ét}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^{2n} \mathbb{H}_\ell(r))) \\ &\cong H_{\text{sm}}^1(\mathcal{M}_{1,1/\mathbb{Q}}, S^{2n} \mathbb{H}_\ell(r)). \end{aligned}$$

Applying H^2 to the diagram (13.3) implies that the image of (13.2) is contained in $H_{\text{sm}}^1(G_\mathbb{Q}, H_{\text{ét}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^{2n} \mathbb{H}_\ell(r)))$. □

Our final task is to prove the Pruning Lemma.

Lemma 13.4 (Pruning Lemma). *Suppose that $* \in \{1, \vec{1}\}$. Let \mathcal{G} be either $\mathcal{G}_*^{\text{wtd},\ell}$ or $\mathcal{G}_*^{\text{cris},\ell}$. If*

$$0 \rightarrow B \rightarrow E \rightarrow \mathbb{Q}_\ell \rightarrow 0 \tag{13.4}$$

¹⁶This group should be a quotient of the \mathbb{Q}_ℓ form of the fundamental group of Brown’s category of mixed modular motives over \mathbb{Z} , perhaps even isomorphic to it.

is an extension of \mathcal{G} -modules, then there is a \mathcal{G} -submodule A of B such that the extension (13.4) is pulled back from an extension

$$0 \rightarrow A \rightarrow F \rightarrow \mathbb{Q}_\ell \rightarrow 0$$

and $A = M_{-2}A$.

Proof. All modules in this proof will be \mathcal{G} -modules and all extension groups will be extensions of \mathcal{G} -modules. First recall that if C is a submodule of B and if $\text{Ext}^1(\mathbb{Q}_\ell, B/C) = 0$, then $\text{Ext}^1(\mathbb{Q}_\ell, C) \rightarrow \text{Ext}^1(\mathbb{Q}_\ell, B)$ is surjective.

Note that $S^m H_\ell(r) = M_{-2} S^m H_\ell(r)$ if and only if $r > m$. So the condition that $A = M_{-2}A$ is equivalent to the condition that each copy of $S^m H_\ell(r)$ in $\text{Gr}_\bullet^W A$ satisfies $r > m$.

If $M_{-2}B \neq B$, choose the largest integer w such that $\text{Gr}_w^W B$ contains a copy of $S^m H_\ell(r)$ with $r \leq m$. Then decompose $\text{Gr}_w^W B$

$$\text{Gr}_w^W B = D^+ \oplus D^-,$$

where D^+ is the sum of factors $S^m H_\ell(r)$ with $r > m$ and D^- is the sum of the factors with $r \leq m$.

Corollary 13.2 implies that the extension

$$0 \rightarrow D^- \rightarrow B/W_{w-1} \rightarrow C \rightarrow 0$$

splits. There is therefore a projection $B \rightarrow D^-$ and an extension

$$0 \rightarrow K \rightarrow B \rightarrow D^- \rightarrow 0.$$

Corollary 13.2 implies that $\text{Ext}^1(\mathbb{Q}_\ell, D^-)$ vanishes, which implies that $\text{Ext}^1(\mathbb{Q}_\ell, K) \rightarrow \text{Ext}^1(\mathbb{Q}_\ell, B)$ is surjective. Repeat this procedure until $M_{-2}K = K$. \square

14. The elliptic polylogarithm

Recall from Corollary 6.10 that there is an object \mathfrak{p} of MEM_2 whose corresponding local system has fiber the Lie algebra $\mathfrak{p}(E', x)$ of $\pi_1^{\text{un}}(E', x)$ over the point $[E, x]$ of $\mathcal{M}_{1,2}^{\text{an}}$. Denote by \mathfrak{p}' the sublocal system whose fiber over $[E, x]$ is the commutator subalgebra $[\mathfrak{p}(E', x), \mathfrak{p}(E', x)]$. It is an object of MEM_2 .

The elliptic polylogarithm of Beilinson and Levin is the object $\text{Pol}_2^{\text{ell}} := \mathfrak{p}/[\mathfrak{p}', \mathfrak{p}']$ of MEM_2 , [7, Proposition 1.4.3]. It is an extension

$$0 \rightarrow (\text{Sym } \mathbb{H}(1))(1) \rightarrow \text{Pol}_2^{\text{ell}} \rightarrow \mathbb{H}(1) \rightarrow 0 \tag{14.1}$$

and therefore an element of

$$\begin{aligned} & \bigoplus_{m \geq 0} \text{Ext}_{\text{MEM}_2}^1(\mathbb{H}(1), S^m \mathbb{H}(m+1)) \\ & \cong \bigoplus_{m \geq 0} \text{Ext}_{\text{MEM}_2}^1(\mathbb{Q}, \mathbb{H} \otimes S^m \mathbb{H}(m+1)) \\ & \cong \bigoplus_{m \geq 1} \text{Ext}_{\text{MEM}_2}^1(\mathbb{Q}, S^{m-1} \mathbb{H}(m)) \oplus \bigoplus_{m \geq 0} \text{Ext}_{\text{MEM}_2}^1(\mathbb{Q}, S^{m+1} \mathbb{H}(m+1)). \end{aligned} \tag{14.2}$$

By Proposition 7.5, \mathfrak{p} , \mathfrak{p}' and $\mathbf{Pol}_2^{\text{ell}}$ restrict to objects of $\mathbf{MEM}_{\bar{1}}$, which we will also denote by $\mathfrak{p}_{\bar{1}}$, $\mathfrak{p}'_{\bar{1}}$ and $\mathbf{Pol}_1^{\text{ell}}$. The fiber of $\mathfrak{p}_{\bar{1}}$ over the point $[E, \vec{v}]$ of $\mathcal{M}_{1,1}^{\text{an}}$ is $\mathfrak{p}(E', \vec{v})$.

The monodromy of the local system $\mathbf{Pol}_1^{\text{ell}}$ over $\mathcal{M}_{1,\bar{1}}$ about the fiber of the \mathbb{G}_m -bundle $\mathcal{M}_{1,\bar{1}} \rightarrow \mathcal{M}_{1,1}$ is the unipotent automorphism of $\mathfrak{p}(E', \vec{v})$ induced by the element of $\pi_1(E', \vec{v})$ that rotates the tangent vector once about the identity. Its logarithm spans a copy of $\mathbb{Q}(1)$ in $\text{Der } \mathfrak{p}(E', \vec{v})$ and induces a map

$$\mathbf{Pol}_1^{\text{ell}}(1) \rightarrow \mathbf{Pol}_1^{\text{ell}}/(\text{Sym } \mathbb{H}(1))(2) \xrightarrow{\sim} \mathbb{H}(2) \hookrightarrow (\text{Sym } \mathbb{H}(1))(1).$$

The coinvariants $\mathbf{Pol}_1^{\text{ell}}/\mathbb{H}(2)$ of this action descend to a local system on $\mathcal{M}_{1,1}$, which we denote by $\mathbf{Pol}_1^{\text{ell}}$. It is an object of \mathbf{MEM}_1 .

The computations in the previous sections imply that some of the extensions in the decomposition (14.2) are trivial in $\text{MHS}(\mathcal{M}_{1,2}, \mathbb{H})$. The next result shows that all extensions in $\mathbf{Pol}_*^{\text{ell}}$ that can be non-zero are indeed non-trivial.

Proposition 14.1. *The elements of $\text{Ext}_{\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})}^1(\mathbb{Q}, S^m \mathbb{H}(m+1))$ that occur in the decomposition (14.2) and its analogues for $\mathbf{Pol}_1^{\text{ell}}$ and $\mathbf{Pol}_{\bar{1}}^{\text{ell}}$, are non-trivial when*

- (i) $m = 2n > 0$, when $* = 1, 2$,
- (ii) $m = 2n \geq 0$, when $* = \bar{1}$.

In addition, the extension in $\text{Ext}_{\text{MHS}(\mathcal{M}_{1,2}, \mathbb{H})}^1(\mathbb{Q}, \mathbb{H}(1))$ is non-trivial.

Proof. The vanishing follows from Theorem 11.4. There are several ways to prove the non-vanishing. Here, we appeal to the result [7, Lemma 1.5.3] of Beilinson–Levin, which implies that the fiber of the kernel of the monodromy logarithm

$$\log \sigma_o : \mathbf{Pol}_2^{\text{ell}} \rightarrow \mathbf{Pol}_2^{\text{ell}}$$

over the tangent vector \vec{v}_o , where σ_o is the generator of the fundamental group of the punctured q -disk, is the fiber over $\vec{w}_o = \bar{1}0$ of the classical polylogarithm local system $\mathbf{Pol}^{\text{class}}$ over $\mathbb{P}^1 - \{0, 1, \infty\}$. This is the subextension

$$0 \rightarrow \bigoplus_{m \geq 1} \mathbb{Q}(m) \rightarrow \mathbf{Pol}_{\vec{w}_o}^{\text{class}} \rightarrow \mathbb{Q}(1) \rightarrow 0$$

of the extension (14.1) obtained by replacing each copy of $S^m H(m+1)$ by the space $\mathbb{Q}(m+1)$ of its lowest weight vectors. This is an extension of $\mathbb{Q}(1)$ by $\mathbb{Q}(m+1)$. It is well known that the fiber of the polylog variation of MHS over \vec{v}_o is given by the zeta values

$$\zeta(m) \in \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, \mathbb{Q}(m)) \cong \text{Ext}_{\text{MHS}}^1(\mathbb{Q}(1), \mathbb{Q}(m+1)).$$

This has infinite order when $m \geq 3$ is odd, which implies that the $(2n+1)$ st subextension of $\mathbf{Pol}_2^{\text{ell}}$ (the ‘ $(2n+1)$ st elliptic polylog’) is non-trivial in

$$0 \rightarrow S^{2n+1} \mathbb{H}(2n+2) \rightarrow \mathbb{E}_{2n+1} \rightarrow \mathbb{H}(1) \rightarrow 0$$

is non-trivial in

$$\text{Ext}^1(\mathbb{H}(1), S^{2n+1} \mathbb{H}(2n+2)) \cong \text{Ext}^1(\mathbb{Q}, S^{2n+2} \mathbb{H}(2n+2)) \oplus \text{Ext}^1(\mathbb{Q}, S^{2n} \mathbb{H}(2n+1)).$$

Since the first summand is trivial, this implies that the second component of \mathbb{E}_{2n+1} is non-trivial. □

Remark 14.2. The non-triviality of these extensions also follows from the explicit computation [27, Proposition 18.3] of the limit MHS on $\mathfrak{p} = \mathfrak{p}(E'_1, \vec{w}_o)$. Alternatively, one can compute the period of the limit MHS on the fiber of $\mathbf{Pol}_1^{\text{ell}}$ over \vec{v}_o directly. This was done by Brown in [11, Lemma 7.1].

Finally, one can prove the non-triviality using ℓ -adic methods. Since \mathfrak{p} is a mixed elliptic motive, one can check the non-triviality of an extension in any realization. Nakamura [45, Theorems 3.3 and 3.5] computed the ℓ -adic Galois action on \mathfrak{p} and showed that the extension above is given by a non-zero rational multiple of the Soulé character.

15. Simple extensions in MEM*

We can now assemble the results of this part to give a computation of the extension groups of \mathbb{Q} by $S^m\mathbb{H}(r)$ in MEM_* .

Theorem 15.1. *When $* \in \{1, \bar{1}\}$, we have*

$$\text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^m\mathbb{H}(r)) = \begin{cases} \mathbb{Q} & * = 1, m = 0 \text{ and } r \geq 3 \text{ odd,} \\ \mathbb{Q} & * = \bar{1}, m = 0 \text{ and } r \geq 1 \text{ odd,} \\ \mathbb{Q} & m = 2n > 0 \text{ and } r = 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

When $* = 2$, we have

$$\text{Ext}_{\text{MEM}_2}^1(\mathbb{Q}, S^m\mathbb{H}(r)) = \begin{cases} \mathbb{Q} & m = 0 \text{ and } r \geq 3 \text{ odd,} \\ \mathbb{Q} & m = 1 \text{ and } r = 1, \\ \mathbb{Q} & m = 2n > 0 \text{ and } r = 2n + 1, \\ 0 & \text{otherwise.} \end{cases}$$

The natural maps

$$\text{Ext}_{\text{MEM}_1}^1(\mathbb{Q}, S^m\mathbb{H}(r)) \rightarrow \text{Ext}_{\text{MEM}_{\bar{1}}}^1(\mathbb{Q}, S^m\mathbb{H}(r)) \rightarrow \text{Ext}_{\text{MEM}_2}^1(\mathbb{Q}, S^m\mathbb{H}(r))$$

are both injective for all m and r .

When $m = 2n$ and $r = 2n + 1$, the group $\text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^m\mathbb{H}(r))$ is generated by the $2n$ th elliptic polylogarithmic extension, which corresponds to the Eisenstein series of weight $2n + 2$. When $m = 0$ and $r \geq 3$ is odd, it is generated by the r th zeta element; when $* = \bar{1}$ and $r = 1$, it is generated by an invertible function associated to the Eisenstein series of weight 2, which can be regarded as the discriminant function on $\mathcal{M}_{1,\bar{1}}$. When $* = 2$ and $m = 1$, it is generated by the extension that corresponds to the tautological section of the universal elliptic curve over $\mathcal{M}_{1,2}$.

Proof. When $m = 0$ and $r \geq 3$, the result follows from the fact that the restriction mapping $\text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, \mathbb{Q}(r)) \rightarrow \text{Ext}_{\text{MTM}_*}^1(\mathbb{Q}, \mathbb{Q}(r))$ is an isomorphism. When $m = 0$ and

$r = 1$, the result follows from the fact that $\mathcal{O}(\mathcal{M}_{1,*})^\times$ is trivial when $* = 1, 2$ and generated by the discriminant when $* = \bar{1}$.

Suppose that $m > 0$. Since the functor $\text{MEM}_* \rightarrow \text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$ is fully faithful (Theorem 8.1), the induced homomorphisms

$$\text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^m \mathbb{H}(r)) \rightarrow \text{Ext}_{\text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})}^1(\mathbb{Q}, S^m \mathbb{H}(r)) \tag{15.1}$$

are injective. So Theorem 11.4 shows that the extension groups are no bigger than claimed in the statement.

When $m > 0$, surjectivity follows from the existence of the elliptic polylogarithms in MEM_* . (Proposition 14.1.) The \mathbb{Q} -de Rham realization of the generator of

$$\text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^{2n} \mathbb{H}(2n + 1)), \quad n > 0$$

can be written down explicitly using Proposition 5.2 and Remark 9.1, or deduced from [27, Theorem 20.2]. □

Remark 15.2. Since the fiber H of \mathbb{H} over \bar{t} is $\mathbb{Q}(0) \oplus \mathbb{Q}(-1)$, the fiber V over \bar{t} of a generator of $\text{Ext}_{\text{MHS}(\mathcal{M}_{1,1})}^1(\mathbb{Q}, S^{2n} \mathbb{H}(2n + 1))$ is an extension of \mathbb{Q} by

$$\mathbb{Q}(1) \oplus \mathbb{Q}(2) \oplus \cdots \oplus \mathbb{Q}(2n + 1)$$

and thus determines elements

$$\lambda_j \in \text{Ext}_{\text{MHS}}^1(\mathbb{Q}, \mathbb{Q}(j)), \quad 1 \leq j \leq 2n + 1.$$

It is easy to see that $\lambda_j = 0$ when $j < 2n + 1$; the proof is in the next paragraph. The proof of the theorem implies that the class of λ_{2n+1} is a non-zero rational multiple of the class of $\zeta(2n + 1)$. And indeed, this is exactly what one sees in the computations of Brown [11] and Nakamura [45, Theorems 3.3 and 3.5].

We know that $N := \log \sigma_o$ acts on the limit V as a morphism of type $(-1, -1)$. Since the residue of the form ψ_{2n} at $q = 0$ is non-zero, the map

$$N : \text{Gr}_0^W V \rightarrow \text{Gr}_{-1}^W V$$

is non-zero and therefore injective. Suppose that $1 \leq j \leq 2n$. Since $j < 2n + 1$, $N : S^{2n} H \rightarrow S^{2n} H$ maps $\mathbb{Q}(j)$ onto $\mathbb{Q}(j + 1)$. It thus maps the extension λ_j of \mathbb{Q} by $\mathbb{Q}(j)$ isomorphically into an extension of $\mathbb{Q}(1)$ by $\mathbb{Q}(j + 1)$ that lies inside $S^{2n} H(2n + 1)$. But all such extensions are split, so $\lambda_j = 0$.

16. Structure theorems

The results of the preceding sections allow us to determine how the fundamental groups of the various categories MEM_* are related.

Let ω be any fiber functor that is the composition of the fiber functor $\text{MEM}_* \rightarrow \text{MTM}$ with any fiber functor $\text{MTM} \rightarrow \text{Vec}_F$. Denote the unipotent completion of $\pi_1(E_{\bar{t}}', \bar{\mathbf{w}}_o)$ by \mathcal{P} . The inclusion $E_{\bar{t}}' \hookrightarrow \mathcal{M}_{1,2}$ induces a restriction functor $\text{MEM}_2 \rightarrow \text{Rep}(\mathcal{P})$ and therefore a homomorphism $\mathcal{P} \rightarrow \pi_1(\text{MEM}_2, \omega)$.

The inclusion of a formal punctured disk $\mathbb{D}' \hookrightarrow E'_1$ induces a homomorphism $\mathbb{Q}(1) = \pi_1^{\text{un}}(\mathbb{D}', \vec{w}_o) \rightarrow \mathcal{P}$. We will denote this subgroup of \mathcal{P} by $\mathbb{G}_a(1)$. It is also the unipotent fundamental group of the fiber of $\mathcal{M}_{1,1} \rightarrow \mathcal{M}_{1,1}$. As above, restriction to the fiber induces a homomorphism $\mathbb{G}_a(1) \rightarrow \pi_1(\text{MEM}_1, \omega)$ such that the diagram

$$\begin{CD} \mathbb{G}_a(1) @>>> \pi_1(\text{MEM}_1, \omega) \\ @VVV @VVV \\ \mathcal{P} @>>> \pi_1(\text{MEM}_2, \omega) \end{CD}$$

commutes.

Theorem 16.1. *There are exact sequences*

$$1 \rightarrow \mathcal{P} \rightarrow \pi_1(\text{MEM}_2, \omega) \rightarrow \pi_1(\text{MEM}_1, \omega) \rightarrow 1$$

and

$$1 \rightarrow \mathbb{G}_a(1) \rightarrow \pi_1(\text{MEM}_1, \omega) \rightarrow \pi_1(\text{MEM}_1, \omega) \rightarrow 1.$$

Proof. It suffices to prove the result when $\omega = \omega^B$. We first show that \mathcal{P} is a normal subgroup of $\pi_1(\text{MEM}_2, \omega^B)$. Since the Lie algebra of \mathcal{P} is an object of MEM_2 , there is a natural action of $\pi_1(\text{MEM}_2, \omega^B)$ on \mathcal{P} . The composite

$$\mathcal{P} \rightarrow \pi_1(\text{MEM}_2, \omega^B) \rightarrow \text{Aut } \mathcal{P}$$

is the inner action. It is injective as \mathcal{P} has trivial center, which shows that \mathcal{P} is a subgroup of $\pi_1(\text{MEM}_2, \omega^B)$.

To see that it is normal, consider the diagram

$$\begin{CD} \mathcal{P} @>>> \widehat{\mathcal{G}}_2 \\ @| @VVV \\ \mathcal{P} @>>> \pi_1(\text{MEM}_2, \omega^B) \end{CD}$$

Since the right-hand vertical map is surjective, and since \mathcal{P} is a normal subgroup of $\widehat{\mathcal{G}}_2$, it follows that \mathcal{P} is a normal subgroup of $\pi_1(\text{MEM}_2, \omega^B)$.

We can thus consider the quotient group $\pi_1(\text{MEM}_2, \omega^B)/\mathcal{P}$. Its representations are objects of MEM_2 that have trivial monodromy on each fiber of $\mathcal{M}_{1,2} \rightarrow \mathcal{M}_{1,1}$, and are thus constant on each fiber. This implies that its representations are precisely the pullbacks of objects of MEM_1 along the projection. This implies that $\pi_1(\text{MEM}_2, \omega^B)/\mathcal{P}$ is isomorphic to $\pi_1(\text{MEM}_1, \omega^B)$.

That $\mathbb{G}_a(1)$ is a normal subgroup of $\pi_1(\text{MEM}_1, \omega)$ follows from the commutativity of the diagram

$$\begin{CD} \mathbb{G}_a(1) @>>> \widehat{\mathcal{G}}_1 @>>> \pi_1(\text{MEM}_1, \omega^B) \\ @VVV @VVV @VVV \\ \mathcal{P} @>>> \widehat{\mathcal{G}}_2 @>>> \pi_1(\text{MEM}_2, \omega^B) \end{CD}$$

As in the $* = 2$, case, $\pi_1(\text{MEM}_1, \omega^B)/\mathbb{G}_a(1) \cong \pi_1(\text{MEM}_1, \omega^B)$. □

Corollary 16.2. *There are natural isomorphisms*

$$\pi_1(\text{MEM}_{\bar{1}}, \omega) \cong \pi_1(\text{MEM}_1, \omega) \times \mathbb{G}_a(1)$$

and

$$\pi_1(\text{MEM}_2, \omega) \cong \pi_1(\text{MEM}_{\bar{1}}, \omega) \times_{\mathbb{G}_a(1)} \mathcal{P}.$$

Proof. Since $\text{Ext}_{\text{MTM}}^1(\mathbb{Q}, \mathbb{Q}(1)) = 0$, the isomorphism $\text{Ext}_{\text{MEM}_{\bar{1}}}^1(\mathbb{Q}, \mathbb{Q}(1)) \cong \mathbb{Q}$ of Theorem 15.1 restricts to an isomorphism

$$H^1(\pi_1^{\text{geom}}(\text{MEM}_{\bar{1}}, \omega), \mathbb{Q}(1)) \cong \mathbb{Q}.$$

The composition of the corresponding homomorphism $\pi_1^{\text{geom}}(\text{MEM}_{\bar{1}}, \omega) \rightarrow \mathbb{Q}(1)$ with the inclusion $\mathbb{G}_a(1) \rightarrow \pi_1^{\text{geom}}(\text{MEM}_{\bar{1}}, \omega)$ is easily seen to be an isomorphism. There is therefore an isomorphism

$$\pi_1^{\text{geom}}(\text{MEM}_{\bar{1}}, \omega) \cong [\pi_1^{\text{geom}}(\text{MEM}_{\bar{1}}, \omega)/\mathbb{G}_a(1)] \times \mathbb{G}_a(1) \cong \pi_1^{\text{geom}}(\text{MEM}_1, \omega) \times \mathbb{G}_a(1).$$

Since all of these isomorphisms are respected by the $\pi_1(\text{MTM}, \omega)$ action, there is a natural isomorphism

$$\pi_1(\text{MEM}_{\bar{1}}, \omega) \cong \pi_1(\text{MTM}, \omega) \times (\pi_1^{\text{geom}}(\text{MEM}_1, \omega) \times \mathbb{G}_a(1)) \cong \pi_1(\text{MEM}_1) \times \mathbb{G}_a(1).$$

The second isomorphism follows from Theorem 16.1 and the first assertion. The splitting is induced by the homomorphism $\pi_1(\text{MEM}_{\bar{1}}, \omega) \rightarrow \pi_1(\text{MEM}_2, \omega)$:

$$\begin{array}{ccccccc}
 & & & & \pi_1(\text{MEM}_{\bar{1}}, \omega) & & \\
 & & & & \swarrow & \downarrow & \\
 1 & \longrightarrow & \mathcal{P} & \longrightarrow & \pi_1(\text{MEM}_2, \omega) & \longrightarrow & \pi_1(\text{MEM}_1, \omega) \longrightarrow 1.
 \end{array}$$

□

17. Motivic remarks

Suppose that $* \in \{1, \bar{1}\}$. We expect that the ext groups of the category MEM_* to be faithfully represented in the various categories of realizations. To be more precise, set $\text{MHS}_* := \text{MHS}(\mathcal{M}_{1,*}, \mathbb{H})$ and $\text{MEM}_*^\ell = \text{Rep}(\mathcal{G}_*^{\text{cris}, \ell})$. The realization functors

$$\text{real}_{\text{MHS}} : \text{MEM}_* \rightarrow \text{MHS}_* \quad \text{and} \quad \text{real}_\ell : \text{MEM}_* \rightarrow \text{MEM}_*^\ell$$

induce ‘regulator mappings’

$$\text{reg}_{\mathbb{Q}} : \text{Ext}_{\text{MEM}_*}^\bullet(\mathbb{Q}, S^m \mathbb{H}(r)) \rightarrow \text{Ext}_{\text{MHS}_*}^\bullet(\mathbb{Q}, S^m \mathbb{H}(r))^{\overline{\mathcal{F}}_\infty} \cong H_{\mathcal{D}}^\bullet(\mathcal{M}_{1,*}^{\text{an}}, S^m \mathbb{H}_{\mathbb{Q}}(r))^{\overline{\mathcal{F}}_\infty}$$

and

$$\text{reg}_\ell : \text{Ext}_{\text{MEM}_*}^\bullet(\mathbb{Q}, S^m \mathbb{H}(r)) \rightarrow \text{Ext}_{\text{MEM}_*^\ell}^\bullet(\mathbb{Q}_\ell, S^m \mathbb{H}_\ell(r)) \cong H^\bullet(\mathcal{G}_*^{\text{cris}, \ell}, S^m H_\ell(r))$$

that are compatible with products. We have already seen that, in degree 1, the first is an isomorphism and the second is an isomorphism after tensoring with \mathbb{Q}_ℓ . We can also map to real Deligne cohomology

$$\text{reg}_{\mathbb{R}} : \text{Ext}_{\text{MEM}_*}^\bullet(\mathbb{Q}, S^m \mathbb{H}(r)) \rightarrow \text{Ext}_{\text{MHS}_*}^\bullet(\mathbb{R}, S^m \mathbb{H}_{\mathbb{R}}(r))^{\overline{\mathcal{F}}_\infty} \cong H_{\mathcal{D}}^\bullet(\mathcal{M}_{1,*}^{\text{an}}, S^m \mathbb{H}_{\mathbb{R}}(r))^{\overline{\mathcal{F}}_\infty}.$$

In degree 1, this map is an isomorphism after tensoring with \mathbb{R} . The story is more interesting in degree 2, where we have regulator mappings

$$\begin{aligned} \text{reg}_{\mathbb{R}} : \text{Ext}_{\text{MEM}_*}^2(\mathbb{Q}, S^m \mathbb{H}(r)) &\rightarrow H_{\mathcal{D}}^2(\mathcal{M}_{1,*}^{\text{an}}, S^m \mathbb{H}_{\mathbb{R}}(r))^{\overline{\mathcal{F}}_{\infty}} \\ &\cong \text{Ext}_{\text{MHS}}^1(\mathbb{R}, H^1(\mathcal{M}_{1,*}^{\text{an}}, S^m \mathbb{H}(r)))^{\overline{\mathcal{F}}_{\infty}} \end{aligned} \tag{17.1}$$

and

$$\begin{aligned} \text{reg}_{\ell} : \text{Ext}_{\text{MEM}_*}^2(\mathbb{Q}, S^m \mathbb{H}(r)) &\rightarrow H^2(\mathcal{G}_*^{\text{cris}, \ell}, S^m H_{\ell}(r)) \\ &\cong H_{\text{sm}}^1(G_{\mathbb{Q}}, H_{\text{ét}}^1(\mathcal{M}_{1,*}/\overline{\mathbb{Q}}, S^m \mathbb{H}_{\ell}(r))). \end{aligned} \tag{17.2}$$

Conjecture 17.1. For all $m \geq 0$ and all $r \in \mathbb{Z}$:

- (i) The degree 2 regulator mapping (17.1) is an isomorphism after tensoring $\text{Ext}_{\text{MEM}_*}^2(\mathbb{Q}, S^m \mathbb{H}(r))$ with \mathbb{R} .
- (ii) For each prime number ℓ , the degree 2 regulator mapping (17.2) is an isomorphism after tensoring $\text{Ext}_{\text{MEM}_*}^2(\mathbb{Q}, S^m \mathbb{H}(r))$ with \mathbb{Q}_{ℓ} .

The first conjecture is an analogue of Beilinson’s conjecture [6, Conjecture 8.4.1] and his result for weight 2 modular forms [5]; the second is an analogue of a version of the Bloch–Kato conjecture (cf. [20, 4.2.2]).

In §25 we show that (i) holds if and only if the cup product

$$\begin{aligned} \sum_{a+b=n} \text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^{2a} \mathbb{H}(2a+1)) \otimes \text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^{2b} \mathbb{H}(2b+1)) \\ \rightarrow \bigoplus_{r=0}^b \text{Ext}_{\text{MEM}_*}^2(\mathbb{Q}, S^{2n-2r} \mathbb{H}(2n+2-r)) \end{aligned}$$

is surjective.

Part 3. Toward a presentation of $\pi_1(\text{MEM}_*)$

The goal of this part is to find a presentation of $\pi_1(\text{MEM}_*, \omega^{\text{DR}})$, or more accurately, a presentation of the Lie algebra $\mathfrak{u}_*^{\text{MEM}}$ of its pronipotent radical. While we do not achieve this goal, we are able to come close, thanks to computations of Brown [11] and Pollack [49]. Specifically, we are able to determine the ‘quadratic heads’ of all relations in $\mathfrak{u}_*^{\text{MEM}}$. If the variant Conjecture 17.1(i) of standard conjecture holds, then the quadratic head of every non-trivial minimal relation is non-trivial.

In §23, we show that the de Rham realization of every object of MEM_* has a canonical bigrading which splits the Hodge filtration and both weight filtrations. This is an important tool. It means that $\mathfrak{u}_*^{\text{MEM}}$ is the completion of a bigraded Lie algebra in the category of $\mathfrak{sl}(H)$ -modules.

18. Presentations of graded Lie algebras

Before attempting the problem of finding a presentation of $\pi_1(\text{MEM}_*, \omega)$, we first consider a more abstract setting. Suppose that \mathcal{G} is an affine group scheme over a field of characteristic zero that is an extension

$$1 \rightarrow \mathcal{U} \rightarrow \mathcal{G} \rightarrow S \rightarrow 1$$

of a connected, reductive group S by a prounipotent group \mathcal{U} . Suppose that the Lie algebra \mathfrak{g} of \mathcal{G} is graded:

$$\mathfrak{g} = \prod_{n \leq 0} \mathfrak{g}_n$$

where \mathfrak{g}_0 is isomorphic to the Lie algebra \mathfrak{s} of S and

$$\mathfrak{u} = \prod_{n < 0} \mathfrak{g}_n$$

is the Lie algebra of \mathcal{U} . For simplicity, we will suppose that each \mathfrak{g}_n is finite dimensional. The splitting $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{u}$ gives a Levi decomposition $\mathcal{G} \cong S \ltimes \mathcal{U}$ of \mathcal{G} . To give a presentation of \mathcal{G} , it suffices to give a presentation of the associated graded Lie algebra \mathfrak{u}_\bullet of \mathfrak{u} in the category of S -modules.

The first step in finding a minimal such presentation of \mathfrak{u}_\bullet is to choose an S -invariant splitting of the graded surjection

$$\phi : \mathfrak{u}_\bullet \rightarrow H_1(\mathfrak{u}_\bullet).$$

This induces a graded S -equivariant surjection $\mathbb{L}(H_1(\mathfrak{u}_\bullet)) \rightarrow \mathfrak{u}_\bullet$ that induces the identity on H_1 .

Set $\mathfrak{f} = \mathbb{L}(H_1(\mathfrak{u}_\bullet))$. The ideal of relations $\mathfrak{r} := \ker \phi$ is a graded submodule of the commutator subalgebra $[\mathfrak{f}, \mathfrak{f}]$. The Lie algebra version of Hopf’s theorem implies that, since ϕ induces an isomorphism on H_1 , there is a natural isomorphism

$$H_2(\mathfrak{u}_\bullet) \cong \mathfrak{r}/[\mathfrak{r}, \mathfrak{f}]$$

of S -modules. The image of any S -invariant section $\psi : H_2(\mathfrak{u}_\bullet) \hookrightarrow \mathfrak{r}$ of the canonical surjection $\mathfrak{r} \rightarrow H_2(\mathfrak{u}_\bullet)$ will be a minimal set of generators of \mathfrak{r} . That is, \mathfrak{r} is the ideal $(\text{im } \psi)$ generated by $\text{im } \psi$ and there are isomorphisms

$$\text{im } \psi \xrightarrow{\cong} \mathfrak{r}/[\mathfrak{r}, \mathfrak{f}] = H_2(\mathfrak{u}_\bullet).$$

So, loosely speaking, every pronilpotent Lie algebra in $\text{Rep}_F(R)$ has a minimal presentation of the form

$$\mathfrak{u} \cong \mathbb{L}(H_1(\mathfrak{u}))^\wedge / (H_2(\mathfrak{u})).$$

Of course, one has to determine the mapping ψ . Its quadratic part is easy to describe.

Denote the lower central series of a Lie algebra \mathfrak{h} by

$$\mathfrak{h} = L^1\mathfrak{h} \supseteq L^2\mathfrak{h} \supseteq L^3\mathfrak{h} \supseteq \dots$$

The bracket $\mathfrak{f} \otimes \mathfrak{f} \rightarrow \mathfrak{f}$ induces an isomorphism

$$\Lambda^2 H_1(\mathfrak{u}_\bullet) \xrightarrow{\cong} L^2\mathfrak{f}/L^3\mathfrak{f}.$$

Examining the Chevalley–Eilenberg cochain complex of \mathfrak{u}_\bullet , one sees immediately that:

Lemma 18.1. *The composition*

$$H_2(\mathfrak{u}_\bullet) \xrightarrow{\psi} L^2\mathfrak{f} \longrightarrow L^2\mathfrak{f}/L^3\mathfrak{f} \xrightarrow{\simeq} \Lambda^2 H_1(\mathfrak{u}_\bullet)$$

is the dual of the cup product $\Lambda^2 H^1(\mathfrak{u}_\bullet) \rightarrow H^2(\mathfrak{u}_\bullet)$. Equivalently, there is an exact sequence

$$H_2(\mathfrak{u}_\bullet) \xrightarrow{\Delta} \Lambda^2 H_1(\mathfrak{u}_\bullet) \xrightarrow{[\cdot, \cdot]} L^2\mathfrak{u}_\bullet/L^3\mathfrak{u}_\bullet \longrightarrow 0$$

where Δ denotes the coproduct; i.e., the graded dual of the cup product.

Definition 18.2. The quadratic leading term (or part) of a relation $r \in \mathfrak{r}$ is its image in $L^2\mathfrak{u}_\bullet/L^3\mathfrak{u}_\bullet$.

The final observation is that the canonical isomorphism

$$H^\bullet(\mathcal{G}, V) \cong H^\bullet(\mathfrak{u}, V)^S$$

implies that (provided that every irreducible S -module is absolutely irreducible) there is a natural isomorphism

$$H_\bullet(\mathfrak{u}_\bullet) \cong \bigoplus_{\alpha \in \check{S}} H^\bullet(\mathcal{G}, V_\alpha)^\vee \otimes V_\alpha, \tag{18.1}$$

where \check{S} denotes the set of isomorphism classes of irreducible S -modules, $(\)^\vee$ denotes dual, and $\{V_\alpha\}$ an S -module in the class α .

If we fix a Cartan subalgebra \mathfrak{h} of \mathfrak{s} and a set of positive roots, we can decompose \mathfrak{s} as

$$\mathfrak{s} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-,$$

where \mathfrak{n}_+ (respectively \mathfrak{n}_-) is the nilpotent subalgebra consisting of positive (respectively negative) root vectors. So we can think of \mathfrak{u}_\bullet as being generated as an \mathfrak{n}_- module by its highest weight vectors. Applying this to the isomorphism (18.1) gives the following result.

Proposition 18.3. *The space of highest weight vectors of the generating space of a minimal presentation of \mathfrak{u}_\bullet projects isomorphically to*

$$H_0(\mathfrak{n}_-, H_1(\mathfrak{u}_\bullet)) \cong H_1(\mathfrak{u}_\bullet)/\mathfrak{n}_- H_1(\mathfrak{u}_\bullet) \cong \bigoplus_{\alpha \in \check{S}} H^1(\mathcal{G}, V_\alpha)^\vee.$$

The space of highest weight vectors in a minimal space $\mathfrak{r}/[\mathfrak{r}, \mathfrak{f}]$ of relations projects isomorphically to

$$H_0(\mathfrak{n}_-, \mathfrak{r}/[\mathfrak{r}, \mathfrak{f}]) \cong H_0(\mathfrak{n}_-, H_2(\mathfrak{u})) \cong H_2(\mathfrak{u})/\mathfrak{n}_- H_2(\mathfrak{u}_\bullet) \cong \bigoplus_{\alpha \in \check{S}} H^2(\mathcal{G}, V_\alpha)^\vee.$$

Combined with Lemma 18.1, we see that the space $\mathfrak{r}/(\mathfrak{r} \cap L^3\mathfrak{f})$ of quadratic leading terms of the space of minimal relations is dual to the cup product.

Corollary 18.4. *The diagram*

$$\begin{array}{ccc}
 H_0(\mathfrak{n}_-, \tau/(\tau \cap L^3\mathfrak{f}))^{\subset} & \xrightarrow{\quad} & \Lambda^2 H_0(\mathfrak{n}_-, H_1(\mathfrak{u}_\bullet)) \\
 \downarrow & & \downarrow \\
 \bigoplus_{\alpha \in \check{S}} H^2(\mathcal{G}, V_\alpha)^\vee & \xrightarrow{\Delta} & \bigoplus_{\beta, \gamma \in \check{S}} H^1(\mathcal{G}, V_\beta)^\vee \otimes H^1(\mathcal{G}, V_\gamma)^\vee
 \end{array}$$

commutes, where Δ denotes the dual of the cup product.

19. Splittings

A fundamental fact about motives is that (say) the complex vector space underlying a mixed Hodge structure is naturally (though not canonically) isomorphic to the direct sum of its weight graded quotients. In [Appendix B](#) we show that we can simultaneously split the weight filtrations M_\bullet and W_\bullet on the various realizations of all objects of MEM_* , as well as all $\widehat{\mathcal{G}}_*$ and $\mathcal{G}_*^{\text{cris}, \ell}$ modules. This will simplify the problem of writing down the presentations as it allows us to work in the category of bigraded Lie algebras with $\mathfrak{sl}(H)$ -action, instead of in the much more slippery category of pronilpotent Lie algebras.

Specifically, let \mathcal{G} be any of the groups $\pi_1(\text{MEM}_*, \omega)$, where $* \in \{1, \bar{1}, 2\}$ and ω is ω^{DR} or ω^B . Then every \mathcal{G} -module V has a natural bigrading

$$V = \bigoplus_{m, n \in \mathbb{Z}} V_{m, n}$$

in the category of \mathbb{Q} -vector spaces with the property that

$$M_m V = \bigoplus_{\substack{r \leq m \\ n \in \mathbb{Z}}} V_{r, n} \quad \text{and} \quad W_n V = \bigoplus_{\substack{r \leq n \\ m \in \mathbb{Z}}} V_{m, r}.$$

It is natural in the sense that it is preserved by \mathcal{G} -module homomorphisms. It is compatible with \otimes and Hom . One can also choose the bigradings to be compatible with the functors $\text{MEM}_2 \rightarrow \text{MEM}_{\bar{1}} \rightarrow \text{MEM}_1$.

Remark 19.1. One consequence of the proof is that the functors $\text{Gr}_\bullet^M \text{Gr}_\bullet^W$ are exact. If one knew this in advance, one could use the fiber functor $\text{Gr}_\bullet^M \text{Gr}_\bullet^W$ to prove the result. Analogous results hold for $\widehat{\mathcal{G}}_*$ -modules and $\mathcal{G}_*^{\text{cris}, \ell}$ -modules. One can arrange for their splittings to be compatible with those of $\pi_1(\text{MEM}_*, \omega)$ via the homomorphisms $\widehat{\mathcal{G}}_* \rightarrow \pi_1(\text{MEM}_*, \omega^B)$ and $\mathcal{G}_*^{\text{cris}, \ell} \rightarrow \pi_1(\text{MEM}_*, \omega^\ell) \otimes \mathbb{Q}_\ell$.

Remark 19.2. In [§ 23](#), we will construct a canonical \mathbb{Q} -de Rham bigrading of each object of MEM_* which splits both weight filtrations and also the Hodge filtration. This bigrading generalizes, and reduces to, the canonical grading of the \mathbb{Q} -DR realization of a mixed Tate motive which splits the weight filtration and the Hodge filtration.

19.1. Bases

In the subsequent sections, it will be important to distinguish between $H = H^1(E_{\check{\tau}})$ and $\check{H} := H(1) = H_1(E_{\check{\tau}})$. The intersection pairing induces an isomorphism

$$H^B \rightarrow \text{Hom}(H^B, \mathbb{Q}) = \check{H}^B; \quad x \mapsto (x \cdot \quad)$$

of their Betti realizations. Both are isomorphic to $H^B = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{b} \cong \check{H}^B$. Their de Rham realizations are different:

$$H^{\text{DR}} = \mathbb{Q}\mathbf{a} \oplus \mathbb{Q}\mathbf{w}$$

where $\mathbf{w} = -2\pi i\mathbf{b}$. Note that \mathbf{w} spans a copy of $\mathbb{Q}(-1)$ in $H(1)$. The de Rham realization of $H(1)$ is

$$\mathbb{Q}A \oplus \mathbb{Q}T$$

where $\mathbf{a} = 2\pi iA$ and $T = -\mathbf{b}$. This is the \mathbb{Q} -de Rham basis used in [27]. In this part, we will be mainly working in homotopy, and thus with $H(1) = H_1(E_{\check{\tau}})$.

Note that $\text{SL}(H)$ and $\text{SL}(\check{H})$ are naturally isomorphic, as are their Lie algebras. We will identify them.

Below we will take ω to be either ω^B or ω^{DR} . When we use ω^B , we will use the \mathbb{Q} Betti basis \mathbf{a}, \mathbf{b} of \check{H} ; when we use ω^{DR} , we will use the \mathbb{Q} -DR basis A, T of \check{H} .

19.2. Splittings of $\mathfrak{g}_*^{\text{MEM}}$

Fix $* \in \{1, \bar{1}, 2\}$. Set $\mathfrak{g} = \mathfrak{g}_*^{\text{MEM}}$ and $\mathfrak{u} = \mathfrak{u}_*^{\text{MEM}}$. By the results of the previous section, the Lie algebras \mathfrak{g} and \mathfrak{u} are isomorphic to the degree completion of their associated bigraded Lie algebras

$$\mathfrak{g} \cong \prod_{\substack{n \leq 0 \\ m \in \mathbb{Z}}} \mathfrak{g}_{m,n} \quad \text{and} \quad \mathfrak{u} \cong \prod_{\substack{n < 0 \\ m \in \mathbb{Z}}} \mathfrak{u}_{m,n}.$$

Since $\mathfrak{gl}(H) \cong \text{Gr}_0^W \mathfrak{g}$ and $\mathfrak{u}_{m,n} = \mathfrak{g}_{m,n}$ when $n < 0$, we have the decomposition

$$\mathfrak{g} \cong \mathfrak{gl}(H) \times \prod_{\substack{n < 0 \\ m \in \mathbb{Z}}} \mathfrak{u}_{m,n} \cong \mathfrak{gl}(H) \times \mathfrak{u}.$$

This decomposition corresponds to a Levi decomposition

$$\pi_1(\text{MEM}_*, \omega) \cong \text{GL}(H) \times \mathcal{U}_*^{\text{MEM}}.$$

Since a prounipotent group is determined by its Lie algebra, to give a presentation of $\pi_1(\text{MEM}_*, \omega)$, it suffices to give a presentation of $\mathfrak{u} = \mathfrak{u}_*^{\text{MEM}}$ in the category of $\text{GL}(H)$ -modules. To do this, it suffices to give a presentation of the bigraded Lie algebra

$$\text{Gr } \mathfrak{u} := \bigoplus_{m,n} \mathfrak{u}_{m,n}$$

as a bigraded $\mathfrak{sl}(H)$ -module, where $\mathfrak{sl}(H)$ is the bigraded Lie algebra

$$\mathfrak{sl}(H) = \mathfrak{s}_{-2,0} \oplus \mathfrak{s}_{0,0} \oplus \mathfrak{s}_{2,0} \cong \mathbb{Q}(1) \oplus \mathbb{Q}(0) \oplus \mathbb{Q}(-1).$$

The elements

$$\mathbf{e}_0^{\text{DR}} = -A \frac{\partial}{\partial T} \quad \text{and} \quad \mathbf{e}_0^B = 2\pi i \mathbf{e}_0^{\text{DR}} = \mathbf{a} \frac{\partial}{\partial \mathbf{b}} \tag{19.1}$$

span $\mathfrak{s}_{-2,0}^{\text{DR}}$ and $\mathfrak{s}_{-2,0}^B$, respectively. The subalgebra $\mathfrak{s}_{0,0}$ is a Cartan subalgebra and is diagonal with respect to the basis A, T of H^{DR} . It is natural to assign the $\mathfrak{sl}(H)$ weights 1 to T and -1 to A . With this convention, $A\partial/\partial T$ has $\mathfrak{sl}(H)$ weight -2 .

Each weight graded quotient $\text{Gr}_n^W V$ of a \mathcal{G} -module V is an $\text{SL}(H)$ -module. The subspace $V_{m,n}$ of $\text{Gr}_n^W V$ is the set of vectors of $\mathfrak{sl}(H)$ weight $m - n$. In other words, the three notions of weight are related by the formula

$$M\text{-weight} = \mathfrak{sl}(H)\text{-weight} + W\text{-weight}. \tag{19.2}$$

Convention 19.3. Most formulas will hold for both the Betti and de Rham realizations. For this reason, we will often omit the B or the DR and write, for example, \mathbf{e}_0 , which can be interpreted as \mathbf{e}_0^B in the Betti realization and \mathbf{e}_0^{DR} in the de Rham realization.

20. Generators of u_*^{MEM}

In this and subsequent sections, $\text{Gr } V$, where V is in MEM_* , will denote $\text{Gr}_\bullet^M \text{Gr}_\bullet^W V$. Since there is a natural isomorphism $\text{Gr } u_*^{\text{MEM}} \cong u_*^{\text{MEM}}$, to find a presentation of u_*^{MEM} , it suffices to find a presentation of its associated bigraded $\text{Gr } u_*^{\text{MEM}}$.

The abelianization of $\text{Gr } u_*^{\text{MEM}}$ is easily computed by plugging the computations of Theorem 15.1 into isomorphism (18.1) and Proposition 18.3.

Proposition 20.1. *For $* \in \{1, \bar{1}, 2\}$, we have*

$$H_1(\text{Gr } u_*^{\text{MEM}}) \cong \bigoplus_{\substack{m \geq 0 \\ r \in \mathbb{Z}}} \text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^m \mathbb{H}(r))^\vee \otimes S^m H(r)$$

$$\cong \begin{cases} \bigoplus_{m>0} \mathbb{Q}(2m+1) \oplus \bigoplus_{n>0} S^{2n} H(2n+1) & * = 1, \\ \bigoplus_{m>0} \mathbb{Q}(2m+1) \oplus \bigoplus_{n \geq 0} S^{2n} H(2n+1) & * = \bar{1}, \\ \bigoplus_{m>0} \mathbb{Q}(2m+1) \oplus H(1) \oplus \bigoplus_{n>0} S^{2n} H(2n+1) & * = 2. \end{cases}$$

Since the surjection $\pi_1(\text{MEM}_*, \omega) \rightarrow \pi_1(\text{MTM}, \omega)$ is split, it follows that the sequence

$$0 \rightarrow H_1(\text{Gr } u_*^{\text{geom}}) \rightarrow H_1(\text{Gr } u_*^{\text{MEM}}) \rightarrow H_1(\text{Gr } \mathfrak{k}) \rightarrow 0$$

is exact. Since

$$H_1(\text{Gr } \mathfrak{k}) = \bigoplus_{m>0} \mathbb{Q}(2m+1),$$

is a trivial $\text{SL}(H)$ -module and since $H_1(u_*^{\text{geom}})^{\text{SL}(H)} = 0$, we obtain the following result.

Corollary 20.2. *There is a natural $\text{GL}(H)$ -module isomorphism*

$$H_1(\text{Gr } u_*^{\text{MEM}}) \cong H_1(\text{Gr } \mathfrak{k}) \oplus H_1(\text{Gr } u_*^{\text{geom}}).$$

Following the procedure in § 18, we choose a graded $GL(H)$ -invariant (and therefore bigraded) section of $\text{Gr} u_*^{\text{MEM}} \rightarrow H_1(\text{Gr} u_*^{\text{MEM}})$. This induces a surjection

$$\mathbb{L}(H_1(\text{Gr} u_*^{\text{MEM}})) \rightarrow \text{Gr} u_*^{\text{MEM}}.$$

We may assume that these maps are compatible with the projections $u_1^{\text{MEM}} \rightarrow u_2^{\text{MEM}} \rightarrow u_1^{\text{MEM}}$. We now fix a basis of $H_1(\text{Gr} u_*^{\text{MEM}})$.¹⁷ Its image under the section will be a generating set of $\text{Gr} u_*^{\text{MEM}}$. These generators will not be unique as they will depend on the choice of the section. More will be said about this in Remark 20.3.

For the rest of this section $* \in \{1, \bar{1}\}$. Recall that $\mathbb{Q}(n)$ is the one-dimensional \mathbb{G}_m -module \mathbb{Q} on which \mathbb{G}_m acts by the n th power of the standard character. Let

$$\sigma_{2m+1} \in \text{Ext}_{\text{MTM}}^1(\mathbb{Q}, \mathbb{Q}(2m+1))^\vee \otimes \mathbb{Q}(2m+1) \subseteq H_1(\text{Gr} \mathfrak{k})$$

be the basis vector on which the motivic zeta value $\zeta^m(2m+1)$ defined in [10] takes the value 1. Let

$$\mathbf{z}_{2m+1} \in \text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^0 H(2m+1))^\vee \otimes \mathbb{Q}(2m+1) \subseteq H_1(\text{Gr} u_*^{\text{MEM}})$$

be the corresponding element of $H_1(\text{Gr} u_*^{\text{MEM}})$. The projection $u_*^{\text{MEM}} \rightarrow \mathfrak{k}$ takes \mathbf{z}_{2m+1} to σ_{2m+1} . Let

$$\mathbf{e}_{2n} \in \text{Ext}_{\text{MEM}_*}^1(\mathbb{Q}, S^{2n-2} \mathbb{H}(2n-1))^\vee \otimes S^{2n-2} H(2n-1)$$

be the element whose value on the class ψ_{2n} of the normalized Eisenstein series is the highest weight vector $2\pi i \mathbf{b}^{2n-2} \in \mathbb{Q}(1)$ of $S^{2n-2} H(2n-1)$.

Remark 20.3. Just as in the case of the generators σ_{2m+1} of $\text{Gr} \mathfrak{k}$, it is important to note that, even though the \mathbf{z}_{2m+1} 's are canonical in $H_1(\text{Gr} u_*^{\text{MEM}})$, they are not canonical in $\text{Gr} u_*^{\text{MEM}}$ as:

- (i) The lift of \mathbf{z}_{2m+1} from $H_1(u_*^{\text{MEM}})$ to $\text{Gr} u_*^{\text{MEM}}$ is not unique. For example, \mathbf{z}_{11} can be replaced by any element of $\mathbf{z}_{11} + \mathbb{Q}[\mathbf{z}_3, [\mathbf{z}_3, \mathbf{z}_5]]$.
- (ii) Even if we fix the lift of \mathbf{z}_{2m+1} to $\text{Gr} u_*^{\text{MEM}}$, we can adjust it by any $SL(H)$ invariant element of $\text{Gr}_{-4m-2}^M \text{Gr}_{-4m-2}^W u_*^{\text{geom}}$.

Note, however, that the geometric generators \mathbf{e}_{2n} are unique as $\text{Gr}_{-2}^M \text{Gr}_{-2n}^W u_*^{\text{MEM}}$ is one-dimensional for each $n \geq 1$.

Remark 20.4. The results of § 29 imply that there is a canonical \mathbb{Q} -de Rham choice of the homomorphism

$$\bigoplus_{m \geq 1} \text{Gr}_{-4m-2}^M \mathfrak{k} \rightarrow \bigoplus_{m \geq 1} \text{Gr}_{-4m-2}^M \text{Gr}_{-4m-2}^W u_1^{\text{MEM}}.$$

Namely, it is the homomorphism that takes $\mathbf{z} \in \text{Gr}_{-4m-2}^M \mathfrak{k}$ to its component (denoted by $\mathbf{z}^{(4m+2)}$ in § 29) of the image of $\vec{\mathbf{v}}_{o^*}(\mathbf{z})$ in the $\mathfrak{sl}(H)$ invariants of $\text{Gr}_{\bullet}^W u_1^{\text{MEM}}$, where u_1^{MEM} is identified with its W -graded quotient using the canonical \mathbb{Q} -DR splitting of W constructed in § 23.

¹⁷We shall see shortly that there is a natural choice.

Notation 20.5. Denote the adjoint action of the enveloping algebra of $\mathfrak{g}_*^{\text{MEM}}$ on $\mathfrak{u}_*^{\text{MEM}}$ by $f \cdot u$. In particular, if $x \in \mathfrak{g}_*^{\text{MEM}}$ and $u \in \mathfrak{u}_*^{\text{MEM}}$, then $x^j \cdot u = \text{ad}_x^j(u)$.

With this notation,

$$S^{2n-2}H(2n-1) = \text{span}\{\mathbf{e}_0^j \cdot \mathbf{e}_{2n} : 0 \leq j \leq 2n-2\}.$$

One also has the relation $\mathbf{e}_0^{2n-1} \cdot \mathbf{e}_{2n} = 0$. Note that

$$\mathbf{z}_{2m+1} \in \text{Gr}_{-4m-2}^M \text{Gr}_{-4m-2}^W \mathfrak{u}_*^{\text{MEM}} \quad \text{and} \quad \mathbf{e}_0^j \cdot \mathbf{e}_{2n} \in \text{Gr}_{-2-2j}^M \text{Gr}_{-2n}^W \mathfrak{u}_*^{\text{MEM}}.$$

Their images under the chosen section of $\mathbb{L}(H_1(\text{Gr} \mathfrak{u}_*^{\text{MEM}})) \rightarrow \text{Gr} \mathfrak{u}_*^{\text{MEM}}$ will be a generating set, so that, for example,

$$\text{Gr} \mathfrak{u}_*^{\text{MEM}} = \mathfrak{f}_*/(\text{relations}),$$

where

$$\mathfrak{f}_1 := \mathbb{L}(\mathbf{e}_0^j \cdot \mathbf{e}_{2n+2}, \mathbf{z}_{2m+1} : n \geq 0, m \geq 1, 0 \leq j \leq 2n)$$

and $\mathfrak{f}_1 := \mathfrak{f}_1/(\mathbf{e}_2)$. This is free. Denote the Lie subalgebra of \mathfrak{f}_* generated by the $\mathbf{e}_0^j \cdot \mathbf{e}_{2n+2}$, $j, n \geq 0$, by $\mathfrak{f}_*^{\text{geom}}$. The Lie algebra $\mathfrak{u}_*^{\text{geom}}$ is a quotient of $\mathfrak{f}_*^{\text{geom}}$.

We will refer to the \mathbf{e}_{2n} with $n \geq 0$ as *geometric generators* as they generate $\text{Gr} \mathfrak{u}_*^{\text{geom}}$. The \mathbf{z}_{2m+1} 's will be called *arithmetic generators* as they come from lifts of elements of \mathfrak{k} .

Remark 20.6. Each element of $\text{Gr} \mathfrak{u}_*^{\text{geom}}$ can be expressed as a Lie word in the \mathbf{e}_{2n} 's, where $n \geq 0$. Since each \mathbf{e}_{2n} is in $\text{Gr}_{-2}^M \mathfrak{u}_*^{\text{MEM}}$, each element of Gr_{-2d}^M can be expressed as a Lie word of degree d in the generators \mathbf{e}_{2n} . We will therefore refer to $\text{Gr}_{-2d}^M \mathfrak{u}_*^{\text{MEM}}$ as the *degree d elements of $\text{Gr} \mathfrak{u}_*^{\text{MEM}}$* .

21. General comments about the relations in $\mathfrak{u}_*^{\text{MEM}}$

Before embarking on the problem of determining the relations between the generators of $\mathfrak{u}_*^{\text{MEM}}$, it is useful to ponder their shape and structure. As above, $* \in \{1, 1\}$. In this and subsequent sections, we denote $\pi_1(\text{MEM}_*, \omega^B)$ by $\mathcal{G}_*^{\text{MEM}}$ and $\pi_1(\text{MEM}_*, \omega^B)^{\text{geom}}$ by $\mathcal{G}_*^{\text{geom}}$.

Relations between the \mathbf{e}_{2n} 's will be called *geometric relations*. Since \mathfrak{k} is free, there are no relations between the σ_{2m+1} 's and hence no relations between the \mathbf{z}_{2m+1} 's modulo the \mathbf{e}_{2n} 's. Since $\mathfrak{u}_*^{\text{geom}}$ is an ideal in $\mathfrak{u}_*^{\text{MEM}}$, there are relations of the form

$$[\mathbf{z}_{2m+1}, \mathbf{e}_{2n+2}] = \text{Lie word in the geometric generators } \mathbf{e}_{2j}, j \geq 0. \tag{21.1}$$

These will be called *arithmetic relations*. They describe the action of \mathfrak{k} on $\mathfrak{u}_*^{\text{geom}}$ and will be discussed in greater detail in joint work of the first author with Brown.

The bigraded ideals of relations are

$$\mathfrak{r}_* = \ker\{\mathfrak{f}_* \rightarrow \text{Gr} \mathfrak{u}_*^{\text{MEM}}\} \quad \text{and} \quad \mathfrak{r}_*^{\text{geom}} := \ker\{\mathfrak{f}_*^{\text{geom}} \rightarrow \text{Gr} \mathfrak{u}_*^{\text{geom}}\} = \mathfrak{r}_* \cap \mathfrak{f}_*^{\text{geom}}.$$

These are $\mathfrak{gl}(H)$ -modules.

A minimal space of relations of $\mathfrak{u}_*^{\text{MEM}}$ is a minimal bigraded subspace of the space of $\mathfrak{sl}(H)$ highest weight vectors of \mathfrak{r}_* that generate it as an ideal. A minimal set of relations of $\mathfrak{u}_*^{\text{MEM}}$ is a bigraded basis of a minimal subspace of generators of \mathfrak{r}_* .

Proposition 18.3 implies that each minimal subspace of relations in $\mathfrak{u}_*^{\text{MEM}}$ is isomorphic to

$$H_0(\mathfrak{n}_-, \mathfrak{r}_*/[\mathfrak{r}_*, \mathfrak{f}_*]),$$

where \mathfrak{n}_- is the Lie subalgebra $\mathbb{Q}\mathfrak{e}_0$ of $\mathfrak{sl}(H)$. It also implies that the space of minimal relations of $\mathfrak{u}_*^{\text{MEM}}$ of $\mathfrak{sl}(H)$ -weight $2n$ and M -weight $-2d$, and thus W -weight $-2n - d$, is dual to

$$H^2(\mathcal{G}_*^{\text{MEM}}, S^{2n} H(2n + d)).$$

Remark 20.6 implies that the geometric relations of M -weight $-2d$ have degree d in the $\{\mathfrak{e}_{2j} : j \geq 0\}$. It also implies that the right-hand side of the arithmetic relation (21.1) has degree $2m + 2$ in the $\{\mathfrak{e}_{2j} : j \geq 0\}$ as it has M -weight $-4m - 4$.

A minimal space of geometric relations is a subspace of the $\mathfrak{sl}(H)$ highest weight vectors in $\mathfrak{r}_*^{\text{geom}}$ that injects into $H_0(\mathfrak{n}_-, \mathfrak{r}_*/[\mathfrak{r}_*, \mathfrak{f}_*])$ under the canonical projection

$$\mathfrak{r}_*^{\text{geom}} \rightarrow \mathfrak{r}_*/[\mathfrak{r}_*, \mathfrak{f}_*] \rightarrow H_0(\mathfrak{n}_-, \mathfrak{r}_*/[\mathfrak{r}_*, \mathfrak{f}_*]).$$

Since \mathfrak{k} is free, $[\mathfrak{r}_*, \mathfrak{f}_*] \subseteq \mathfrak{r}_*^{\text{geom}}$, which implies that any minimal space of geometric relations is isomorphic to

$$H_0(\mathfrak{n}_-, \mathfrak{r}_*^{\text{geom}}/[\mathfrak{r}_*, \mathfrak{f}_*]).$$

A minimal set of geometric relations is a basis of a minimal space of geometric relations. Those of $\mathfrak{sl}(H)$ -weight $2n$ and M -weight $-2d$ are dual to the image of the restriction mapping

$$H^2(\mathcal{G}_*^{\text{MEM}}, S^{2n} H(2n + d)) \rightarrow H^2(\mathcal{G}_*^{\text{geom}}, S^{2n} H(2n + d)).$$

As remarked above, these have degree d in the \mathfrak{e}_{2m} 's and W -weight $-2n - 2d$. They occur as highest weight vectors of copies of $S^{2n} H(2n + d)$ in the relations.

21.1. Conjectural size and shape of the relations

In this section we explain how standard conjectures in number theory, if true, would constrain the size and form of a minimal set of relations of $\text{Gr}\mathfrak{u}_*^{\text{MEM}}$. In § 25 we will give an *unconditional* proof that there is a set of minimal relations is of this form and give a partial computation of it.

Corollary 18.4 implies that the highest weight vectors in the space of leading quadratic terms

$$\mathfrak{r}_*/([\mathfrak{r}_*, \mathfrak{f}_*] + \mathfrak{r}_* \cap L^3 \mathfrak{f}_*) = \mathfrak{r}_*/(\mathfrak{r}_* \cap L^3 \mathfrak{f}_*)$$

of a set of minimal relations of $\mathfrak{u}_*^{\text{MEM}}$ of M -weight d and $\mathfrak{sl}(H)$ -weight $2n$ is dual to the image of the map

$$\begin{aligned} & \bigoplus_{\substack{j+k=n+d-2 \\ j,k \geq 0}} H^1(\mathcal{G}_*^{\text{MEM}}, S^{2j} H(2j + 1)) \otimes H^1(\mathcal{G}_*^{\text{MEM}}, S^{2k} H(2k + 1)) \\ & \rightarrow H^2(\mathcal{G}_*^{\text{MEM}}, S^{2n} H(2n + d)) \end{aligned} \tag{21.2}$$

obtained by composing the cup product with a $GL(H)$ -invariant projection

$$S^{2j}H(2j + 1) \otimes S^{2k}H(2k + 1) \rightarrow S^{2n}H(2n + d). \tag{21.3}$$

Since \mathfrak{k} is free and acts trivially on $H_1(u^{\text{geom}})$, we have $[\mathfrak{k}, \mathfrak{r}_*^{\text{geom}}] \subseteq L^3\mathfrak{f}_*^{\text{geom}}$. It follows that

$$\mathfrak{r}_*^{\text{geom}} / (\mathfrak{r}_*^{\text{geom}} \cap L^3\mathfrak{f}_*^{\text{geom}}) \rightarrow \mathfrak{r}_* / (\mathfrak{r}_* \cap L^3\mathfrak{f}_*)$$

is injective. The image is the set of quadratic leading terms (Definition 18.2) of the geometric relations. The quadratic parts of geometric relations of M -weight $-2d$ have degree d as expressions in the $\{\mathbf{e}_{2m} : m \geq 0\}$, but are quadratic in \mathfrak{f}_* .¹⁸

To better understand the relations we restrict to the case $* = 1$. Since $u_1^{\text{MEM}} = u_1^{\text{MEM}} \oplus \mathbb{Q}\mathbf{e}_2$, where \mathbf{e}_2 is central, there is no loss of generality.¹⁹ The generator \mathbf{e}_{2m+2} of \mathfrak{f}_1 (Betti or de Rham, according to taste) is dual to a generator of $H^1(\mathcal{G}_*^{\text{MEM}}, S^{2m}\mathbb{H}(2m + 1))$ and is a highest weight vector of the unique copy of $S^{2m}H$ in $H_1(\mathfrak{f}_1)$. When $m = 0$, the generator \mathbf{z}_{2m+1} of \mathfrak{f}_1 is dual to a generator of $H^1(\mathcal{G}_1^{\text{MEM}}, \mathbb{Q}(2m + 1))$. The projection (21.3) is dual to an inclusion

$$S^{2n}H(2n + d) \hookrightarrow H_1(\mathfrak{f}_1) \otimes H_1(\mathfrak{f}_1)$$

and thus corresponds to a line of highest weight vectors in $H_1(\mathfrak{f}_1)^{\otimes 2}$ of $\mathfrak{sl}(H)$ -weight $2n$. The line lies in Gr_{-2d}^M as the highest weight vector of $S^{2n}H(2n + d)$ has M -weight $-2d$. Since the cup product is anti-commutative, the image of $S^{2n}H(2n + d)$ lies in $\Lambda^2 H_1(\mathfrak{f}_1) \cong L^2\mathfrak{f}_1/L^3$.

To get an idea of the shape of the relations and to see how they might be computed, we assume Conjecture 17.1(i) for the rest of this section. This means that we identify $H^\bullet(\mathcal{G}_*^{\text{MEM}}, S^{2m}H(r)) \otimes \mathbb{R}$ with $H_{\mathcal{D}}^\bullet(\mathcal{M}_{1,*}^{\text{an}}, S^{2m}\mathbb{H}_{\mathbb{R}}(r))^{\overline{\mathcal{F}}_\infty}$.

Recall that V_f denotes the two-dimensional real Hodge structure associated to a Hecke eigen cusp form f and that \mathfrak{B}_{2n+2} denotes the set of normalized Hecke eigen cusp forms of weight $2n + 2$. Each V_f decomposes as the sum of two one-dimensional eigenspaces V_f^\pm of the de Rham Frobenius $\overline{\mathcal{F}}_\infty$. The cup product

$$H_{\text{cusp}}^1(\text{SL}_2(\mathbb{Z}), S^{2n}H) \otimes H_{\text{cusp}}^1(\text{SL}_2(\mathbb{Z}), S^{2n}H) \rightarrow H_{\text{cusp}}^2(\text{SL}_2(\mathbb{Z}), \mathbb{Q})$$

induces an isomorphism $\text{Hom}(V_f^\pm, \mathbb{R}) \cong V_f^\mp$. Conjecture 17.1(i) and Proposition 11.8 imply that there is an isomorphism

$$H^2(\mathcal{G}_1^{\text{MEM}}, S^{2n}H(2n + d))^\vee \otimes \mathbb{R} \cong \begin{cases} \mathbb{R} \oplus \bigoplus_{f \in \mathfrak{B}_{2n+2}} V_f^+ & n > 0, d \geq 2 \text{ even,} \\ \bigoplus_{f \in \mathfrak{B}_{2n+2}} V_f^- & n > 0, d > 2 \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

The copy of \mathbb{R} when d is even corresponds to the Eisenstein series G_{2n+2} .

¹⁸For example, $[\mathbf{e}_0^j \cdot \mathbf{e}_{2a}, \mathbf{e}_0^k \cdot \mathbf{e}_{2b}]$, where $a, b > 1$ and $j, k \geq 0$, is a quadratic element of \mathfrak{f}_1 , but has degree $d = j + k + 2$ in the \mathbf{e}_m 's.

¹⁹To understand the case $* = 2$, by Corollary 16.2, it suffices to understand the action of u_1^{MEM} on \mathfrak{p} . The action of the geometric generators is described in §22.

Proposition 21.1. *When $\ast = 1$, Conjecture 17.1(i) implies that the cup product (21.2) is surjective for all n and d .*

Proof. The interpretation of Brown’s period computations [11] in terms of Deligne–Beilinson cohomology [29, § 7] implies that the image of the cup product (21.2) under the regulator $\text{reg}_{\mathbb{R}}$, after tensoring with \mathbb{R} , is isomorphic to $H_{\mathcal{D}}^2(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}(2n + d))$. \square

Corollary 21.2. *If we assume the Conjecture 17.1(i), then each minimal relation is determined by its leading quadratic term. That is, the projections $\mathfrak{r}_1/[\mathfrak{r}_1, \mathfrak{f}_1] \rightarrow \mathfrak{r}_1/(\mathfrak{r}_1 \cap L^3\mathfrak{f}_1)$ and*

$$H_0(\mathfrak{n}_-, \mathfrak{r}_1/[\mathfrak{r}_1, \mathfrak{f}_1]) \rightarrow H_0(\mathfrak{n}_-, \mathfrak{r}_1/(\mathfrak{r}_1 \cap L^3\mathfrak{f}_1))$$

are isomorphisms.

So, granted Conjecture 17.1(i), for each $f \in \mathfrak{B}_{2n+2}$ and each $d \geq 2$, there is a map

$$V_f^\epsilon \rightarrow \text{Gr}_{-2d}^M \text{Gr}_{-2n-2d}^W H_0(\mathfrak{n}_-, \mathfrak{r}_1/[\mathfrak{r}_1, \mathfrak{f}_1]) \tag{21.4}$$

where ϵ is $+$ when d is even and $-$ when d is odd.²⁰ The map

$$V_f^\epsilon \hookrightarrow \text{Gr}_{-2d}^M H_0(\mathfrak{n}_-, \mathfrak{r}_1/(\mathfrak{r}_1 \cap L^3\mathfrak{f}_1))$$

obtained by composing (21.4) with the projection is dual to the cup product. It is non-zero and therefore injective. The last statement of Proposition 21.1 implies that the image of this last map lies in the geometric part

$$\text{Gr}_{-2d}^M H_0(\mathfrak{n}_-, \mathfrak{r}_1^{\text{geom}}/(\mathfrak{r}_1^{\text{geom}} \cap L^3\mathfrak{f}_1^{\text{geom}})).$$

That is, the leading quadratic terms of each cuspidal relation are geometric. More is true:

Proposition 21.3. *Under the assumption that Conjecture 17.1(i) holds, the image of (21.4) lies in the subspace $\text{Gr}_{-2d}^M \text{Gr}_{-2n-2d}^W H_0(\mathfrak{n}_-, \mathfrak{r}_1^{\text{geom}}/[\mathfrak{r}_1, \mathfrak{f}_1])$ of the space of minimal relations.*

Proof. As pointed out above, there are no relations between the \mathbf{z}_{2m-1} ’s modulo the ideal generated by the \mathbf{e}_{2n} ’s. Since $\mathfrak{u}_1^{\text{geom}}$ is an ideal, each relation $[\mathbf{z}_{2m-1}, \mathbf{e}_{2n}]$ lies in $\mathfrak{f}_1^{\text{geom}}$. So we can replace every occurrence of \mathbf{z}_{2m-1} in a cuspidal relation by an element of $\mathfrak{f}_1^{\text{geom}}$. So every cuspidal relation lies in $\mathfrak{f}_1^{\text{geom}}$. \square

When $d \geq 2$ is even, write $d = 2m + 2$, where $m \geq 0$. The dual

$$\mathbb{R} \hookrightarrow \text{Gr}_{-2d}^M \text{Gr}_{-2n-2d}^W H_0(\mathfrak{n}_-, \mathfrak{r}_1/(\mathfrak{r}_1 \cap L^3\mathfrak{f}_1))$$

of the composition of the cup product with the projection associated to the Eisenstein series G_{2n+2} will contain the leading quadratic terms of the arithmetic relation (21.1) associated to the Eisenstein series G_{2m+2} . Determining this map is a joint project with Brown.

²⁰This map is dual to the composite of the cup product with the projection onto $V_f^{-\epsilon}$.

22. The monodromy representation

Any relation that holds in $\mathfrak{g}_1^{\text{MEM}}$ will hold in any homomorphic image. Relations that hold between the images of the \mathbf{e}_{2n} in a homomorphic image give an upper bound on the relations in $\mathfrak{u}_1^{\text{MEM}}$. In this section we determine how the generators \mathbf{e}_{2n} of $\text{Gr} \mathfrak{u}_1^{\text{MEM}}$ act on the unipotent fundamental group of E'_1 .

As in previous sections, \mathfrak{p} denotes the Lie algebra of the unipotent completion of $\pi_1(E'_1, \vec{\mathbf{w}})$. Recall that \check{H} denotes $H(1)$. It can be thought of as the first homology group of the first order Tate curve E'_1 . The splittings given by Proposition B.1 give natural isomorphisms

$$\text{Gr}_\bullet^W \mathfrak{p}^B \cong \mathbb{L}(\check{H}^B) = \mathbb{L}(\mathbf{a}, \mathbf{b}) \quad \text{and} \quad \text{Gr}_\bullet^W \mathfrak{p}^{\text{DR}} \cong \mathbb{L}(\check{H}^{\text{DR}}) = \mathbb{L}(A, T)$$

which extend to the natural bigraded isomorphism obtained by splitting each W -graded quotient into its $\mathfrak{sl}(H)$ weight spaces.

The natural monodromy action $\mathcal{G}_1^{\text{MEM}} \rightarrow \text{Aut } \mathfrak{p}$ induces a Lie algebra homomorphism $\mathfrak{u}_1^{\text{MEM}} \rightarrow \text{Der } \mathfrak{p}$. This induces a $\mathfrak{gl}(H)$ -invariant bigraded monodromy homomorphism

$$\text{Gr} \mathfrak{u}_1^{\text{MEM}} \rightarrow \text{Gr} \text{Der } \mathfrak{p} \cong \text{Der} \text{Gr } \mathfrak{p} \cong \text{Der } \mathbb{L}(\check{H}). \tag{22.1}$$

Set $\theta = [\mathbf{a}, \mathbf{b}] \in \text{Gr}_{-2}^M \text{Gr}_{-2}^W \mathbb{L}(\check{H})$.

Proposition 22.1. *The image of the graded monodromy action (22.1) lies in*

$$\text{Der}^0 \mathbb{L}(\check{H}) := \{\delta \in \text{Der } \mathbb{L}(\check{H}) : \delta(\theta) = 0\}.$$

Proof. Set $E = E'_1$. Let γ_o be the element of $\pi_1(E', \vec{\mathbf{w}})$ obtained by rotating the tangent vector once about the identity. Observe that $\log \gamma_o \in W_{-2}\mathfrak{p}$ and that its image in Gr_{-2}^W is $[\mathbf{a}, \mathbf{b}]$. Each element of the group $\pi_1(\mathcal{M}_{1,1}^{\text{an}}, \vec{\mathbf{v}}_o)$ is represented by an element of $\text{Diff}^+(E, 0)$ that acts trivially on T_0E , the tangent space of E at the identity. This implies that the image of $\text{Aut } \mathfrak{p}$ in $\pi_1(\mathcal{M}_{1,1}^{\text{an}}, \vec{\mathbf{v}}_o)$ is contained in the subgroup of elements that fix $\log \gamma_o \in \mathfrak{p}$ and therefore act trivially on $\text{Gr}_{-2}^W \mathfrak{p}$. This implies that the image of $\text{Gr} \mathfrak{u}_1^{\text{geom}} \rightarrow \text{Der} \text{Gr } \mathbb{L}(H)$ lies in $\text{Der}^0 \mathbb{L}(H)$.

To complete the proof, recall that, using the splitting $\mathfrak{k} \rightarrow \mathfrak{u}_1^{\text{MEM}}$ induced by the tangent vector $\vec{\mathbf{t}}$, we can identify $\mathfrak{u}_1^{\text{MEM}}$ with $\mathfrak{k} \times \mathfrak{u}_1^{\text{geom}}$. Since $\log \gamma_o$ spans a copy of $\mathbb{Q}(1)$ in \mathfrak{p} , the image of $\mathfrak{k} \rightarrow \mathfrak{u}_1^{\text{MEM}} \rightarrow \text{Der } \mathfrak{p}$ acts trivially on it. It follows that $\text{Gr } \mathfrak{k}$ acts trivially on θ , which completes the proof. \square

Since, by Corollary 16.2, $\mathfrak{u}_1^{\text{MEM}} \cong \mathfrak{u}_1^{\text{MEM}} \oplus \mathbb{Q}\mathbf{e}_2$ where \mathbf{e}_2 is central, there are, by restriction, natural monodromy representations

$$\mathfrak{u}_1^{\text{MEM}} \rightarrow \text{Der } \mathfrak{p} \quad \text{and} \quad \text{Gr} \mathfrak{u}_1^{\text{MEM}} \rightarrow \text{Der}^0 \mathbb{L}(\check{H}). \tag{22.2}$$

Restricting to this smaller algebra will simplify the computations in subsequent sections.

Remark 22.2. Another way to see that there is a natural monodromy representation $\text{Gr} \mathfrak{u}_1^{\text{MEM}} \rightarrow \text{Der}^0 \mathbb{L}(\check{H})$ is to observe that, since the centralizer of a non-zero element x in

a free Lie algebra is spanned by x ,

$$\text{Der}^0 \mathbb{L}(\check{H}) \cap \text{Inn} \mathbb{L}(\check{H}) = \mathbb{Q} \text{ ad } \theta.$$

This implies that $W_{-3} \text{Der}^0 \mathbb{L}(\check{H}) \rightarrow W_{-3} \text{OutDer} \mathbb{L}(\check{H})$ is an isomorphism. Since every generator of $\text{Gr} u_1^{\text{MEM}}$ has weight ≤ -4 , the homomorphism $u_1^{\text{MEM}} \rightarrow \text{OutDer} \mathbb{L}(\check{H})$ lifts to $\text{Der}^0 \mathbb{L}(\check{H})$.

For a basis $\mathbf{v}_1, \mathbf{v}_2$ of \check{H} and each $n \geq 0$, define the derivation $\epsilon_{2n}(\mathbf{v}_1, \mathbf{v}_2)$ of $\mathbb{L}(\check{H})$ by

$$\epsilon_{2n}(\mathbf{v}_1, \mathbf{v}_2) := \begin{cases} -\mathbf{v}_2 \frac{\partial}{\partial \mathbf{v}_1} & n = 0; \\ \text{ad}_{\mathbf{v}_1}^{2n-1}(\mathbf{v}_2) - \sum_{\substack{j+k=2n-1 \\ j>k>0}} (-1)^j [\text{ad}_{\mathbf{v}_1}^j(\mathbf{v}_2), \text{ad}_{\mathbf{v}_1}^k(\mathbf{v}_2)] \frac{\partial}{\partial \mathbf{v}_2} & n > 0. \end{cases} \quad (22.3)$$

Here we are identifying $\mathbb{L}(\check{H})$ with its image in $\text{Der} \mathbb{L}(\check{H})$ under the inclusion $\text{ad} : \mathbb{L}(\check{H}) \hookrightarrow \text{Der} \mathbb{L}(\check{H})$. Each $\epsilon_{2n}(\mathbf{v}_1, \mathbf{v}_2)$ annihilates θ and is thus in $\text{Der}^0 \mathbb{L}(\check{H})$, [27, Proposition 21.2].²¹ It is useful to note that if $c_1, c_2 \in \mathbb{C}$, then

$$\epsilon_{2n}(c_1 \mathbf{v}_1, c_2 \mathbf{v}_2) = c_1^{2n-1} c_2 \epsilon_{2n}(\mathbf{v}_1, \mathbf{v}_2).$$

Recall the notation of § 19.1. Set

$$\epsilon_{2n} = \epsilon_{2n}^{\text{DR}} := \epsilon_{2n}(T, A) \quad \text{and} \quad \epsilon_{2n}^B = 2\pi i \epsilon_{2n}^{\text{DR}} = \epsilon_{2n}(-\mathbf{b}, \mathbf{a}).$$

These lie in $\text{Gr}_{-2}^M \text{Gr}_{-2n}^W \text{Der}^0 \mathbb{L}(\check{H})$ and are $\mathfrak{sl}(H)$ -highest weight vectors of weight $2n - 2$.²² When $n = 0$, this agrees with the definition (19.1) of \mathbf{e}_0 given earlier.

Pollack [49] found relations between the ϵ_{2n} .²³ The relevance of Pollack’s computations to bounding the relations in $\text{Gr} u_{\bullet}^{\text{MEM}}$ comes from the following result.

Theorem 22.3 [28, Theorem 15.7]. *The monodromy representation (22.2) takes \mathbf{e}_0 to ϵ_0 and $\mathbf{e}_{2n} \in \text{Gr} u_1^{\text{MEM}}$ to $2\epsilon_{2n}/(2n - 2)!$ when $n > 0$.*

Remark 22.4. One consequence of this result is that a presentation of u_1^{geom} determines a presentation of u_2^{geom} as $\text{Gr} u_2^{\text{geom}} \cong \text{Gr} u_1 \rtimes \mathbb{L}(H)$ where the action is given by the \mathbf{e}_{2n} .

23. The canonical de Rham splitting

The \mathbb{Q} -vector space V^{DR} that underlies a universal mixed elliptic motive \mathbb{V} in MEM_* is naturally isomorphic to its associated graded module

$$V^{\text{DR}} \cong \text{Gr}_{\bullet}^M V^{\text{DR}}$$

²¹These derivations occur in the work [59] of Tsunogai on the action of the absolute Galois group on the fundamental group of a once punctured elliptic curve. They also occur in the paper of Calaque et al [13, § 3.1].

²²One can show that there is a unique copy of $S^{2n-2}H(2n - 1)$ in $\text{Gr}_{-2n}^W \text{Der}^0 \mathbb{L}(H)$. It has highest weight vector ϵ_{2n} .

²³Note that in [49], ϵ_{2n} denotes our $\epsilon_{2n}(\mathbf{a}, -\mathbf{b})$.

with respect to the weight filtration M_\bullet . The canonical way to realize this isomorphism is via the identification

$$F^m V^{\text{DR}} \cap M_{2m} V^{\text{DR}} \cong \text{Gr}_{2m}^M V^{\text{DR}}$$

induced by the inclusion $F^m V^{\text{DR}} \cap M_{2m} V^{\text{DR}} \hookrightarrow M_{2m} V^{\text{DR}}$. This grading splits both the Hodge filtration and the weight filtration M_\bullet . As shown in Appendix B, this isomorphism can be lifted to a natural isomorphism

$$V^{\text{DR}} \cong \text{Gr}_\bullet^W \text{Gr}_\bullet^M V^{\text{DR}}$$

of V^{DR} with its associated bigraded module which splits the Hodge filtration and both weight filtrations. The purpose of this section is to show that this lift is canonical.

Theorem 23.1. *For every object \mathbb{V} of MEM_* , there is a unique isomorphism*

$$V^{\text{DR}} \cong \bigoplus_{m,n} \text{Gr}_n^W \text{Gr}_{2m}^M V^{\text{DR}}$$

of rational vector spaces, where V^{DR} is the \mathbb{Q} -de Rham realization of the fiber $V \in \text{MTM}$ of \mathbb{V} over \bar{V}_o with the following properties:

- (i) *The isomorphism is natural with respect to morphisms in MEM_* and compatible with tensor products and duals.*
- (ii) *This bigrading of V^{DR} refines the standard grading $V^{\text{DR}} \cong \bigoplus_p F^p M_{2p} V^{\text{DR}}$ of the de Rham realization of a mixed Tate motive. That is, for each $p \in \mathbb{Z}$, this isomorphism restricts to an isomorphism*

$$F^p V^{\text{DR}} \cap M_{2p} V^{\text{DR}} \cong \bigoplus_n \text{Gr}_n^W \text{Gr}_{2p}^M V^{\text{DR}}.$$

Proof. Denote $\pi_1(\text{MEM}_*, \omega^{\text{DR}})$ by \mathcal{G}_* , its pronipotent radical by \mathcal{U}_* and its Lie algebra by \mathfrak{g}_* . Results from § 7 imply that there are natural homomorphisms $\mathcal{G}_1 \rightarrow \mathcal{G}_2 \rightarrow \mathcal{G}_1$. This means that every object of MEM_* can be viewed as an object of $\text{MEM}_{\bar{1}}$. So it suffices to prove the result for $* = \bar{1}$. Applying Proposition B.1 to $\mathcal{G}_{\bar{1}}$ implies that such compatible bigradings which split the Hodge filtration and both weight filtrations exist.

To prove uniqueness, observe that such bigradings correspond to a lift of the central cocharacter $\mathbb{G}_m \rightarrow \text{GL}(H)$ that gives the \mathbb{Q} -DR splitting of H to a cocharacter $\mathbb{G}_m \rightarrow \mathcal{G}_{\bar{1}}$. This cocharacter lies in $F^0 W_0 M_0 \mathcal{G}_{\bar{1}}$ and is unique up to conjugation by an element of $F^0 W_0 M_0 \mathcal{U}_{\bar{1}}$. But since $\text{Gr} \mathfrak{u}_{\bar{1}}$ is generated the $\mathbf{e}_0^j \cdot \mathbf{e}_{2n}$ and the \mathbf{z}_{2m+1} with $n, m > 0$, all of which lie in M_{-2} , and since $F^0 \cap M_{-2} = 0$, $F^0 W_0 M_0 \mathfrak{u}_{\bar{1}} = 0$. Uniqueness follows. \square

Denote the \mathbb{Q} -DR realization of the Lie algebra of the unipotent fundamental group of $(E_{\bar{1}}', \bar{W}_o)$ by \mathfrak{p} . The isomorphism $H_1(\mathfrak{p}) \cong \mathbb{Q}A \oplus \mathbb{Q}T$ given in § 19.1 induces an isomorphism $\text{Gr} \mathfrak{p} \cong \mathbb{L}(A, T)$ of bigraded Lie algebras. The canonical \mathbb{Q} -DR bigrading induces an isomorphism

$$\psi : \mathfrak{p} \rightarrow \mathbb{L}(A, T)^\wedge$$

which is compatible with the bracket and satisfies

$$T \in \text{Gr}_{-1}^W \text{Gr}_0^M \mathbb{L}(A, T), \quad A \in \text{Gr}_{-1}^W \text{Gr}_{-2}^M \mathbb{L}(A, T). \tag{23.1}$$

The KZB connection [13, 43] is the \mathbb{Q} -DR realization of connection associated to the local system of unipotent fundamental groups of punctured elliptic curves over $\mathcal{M}_{1,2}$. This was suggested in [43] and proved in [27, 44]. It gives an isomorphism

$$\phi : \mathfrak{p} \otimes \mathbb{C} \rightarrow \mathbb{L}(A, T)^\wedge \otimes \mathbb{C}$$

of $\mathfrak{p} \otimes \mathbb{C}$ with the completion of the bigraded Lie algebra $\mathbb{L}(A, T)$ which splits the Hodge filtration and both weight filtrations. (Cf. [27, §15].) In order to exploit the formulas derived in [27] using the elliptic KZB connection, we need to show that the KZB bigrading agrees with the canonical one constructed above.

Proposition 23.2. *The isomorphism*

$$\phi \circ \psi^{-1} : \mathbb{L}(A, T)^\wedge \otimes \mathbb{C} \rightarrow \mathbb{L}(A, T)^\wedge \otimes \mathbb{C}$$

is the identity, so that the canonical bigrading of \mathfrak{p} constructed above and the bigrading given by the KZB connection agree.

Proof. The isomorphism $\phi : \mathfrak{p} \otimes \mathbb{C} \rightarrow \mathbb{L}(A, T)^\wedge \otimes \mathbb{C}$ constructed from the KZB connection induces the isomorphism

$$H_1(\mathfrak{p}) \otimes \mathbb{C} \cong \mathbb{C}T \oplus \mathbb{C}A$$

of §19.1 which respects the canonical bigrading. It follows that the map induced by $\phi \circ \psi^{-1}$ on H_1 is the identity.

Since $\phi \circ \psi^{-1}$ corresponds to a morphism of mixed Tate structures, it induces an isomorphism

$$F^p \cap M_{2p} \mathbb{L}(A, T)^\wedge \otimes \mathbb{C} \rightarrow F^p \cap M_{2p} \mathbb{L}(A, T)^\wedge \otimes \mathbb{C}$$

for all $p \in \mathbb{Z}$. Since $F^0 \mathbb{L}(A, T)^\wedge \otimes \mathbb{C} = \mathbb{C}T$, it follows that $\phi \circ \psi^{-1}(T) = T$. Since

$$F^{-1} \cap M_{-2} \mathbb{L}(A, T)^\wedge = \prod_{n \geq 0} \mathbb{C}T^n \cdot A,$$

it follows that $\phi \circ \psi^{-1} : A \mapsto A + \sum_{n=1}^\infty c_n T^n \cdot A$ where each $c_n \in \mathbb{C}$.

As in the previous section, we denote by γ_o the element of $\pi_1^{\text{un}}(E'_1, \vec{w}_o)$ that rotates the tangent vector \vec{w}_o in $T_0 E'_1$ once about 0. Since $\log \gamma_o$ spans a copy of $\mathbb{Q}(1)$ in \mathfrak{p} , its image under ψ is $2\pi i [T, A]$. But since, by [27, §12], the residue of the KZB connection at the identity of E'_1 is $[T, A]$, the image of $\log \gamma_o$ under ϕ is also $2\pi i [T, A]$. So $\phi \circ \psi^{-1}([T, A]) = [T, A]$. Therefore

$$[T, A] = \phi \circ \psi^{-1}([T, A]) = [T, A] + \sum_{n=1}^\infty c_n T^{n+1} \cdot A.$$

It follows that all c_n vanish, so that $\phi \circ \psi^{-1}(A) = A$ and $\phi \circ \psi^{-1}$ is the identity. □

By identifying $\mathfrak{g}_1^{\text{MEM}}$ with its associated bigraded Lie algebra via the de Rham splitting, we can regard each \mathbf{e}_{2n} as an element of $\mathfrak{g}_1^{\text{MEM}}$. The following result is a consequence of the previous result and Theorem 22.3.

Corollary 23.3. *The homomorphism $\mathfrak{g}_*^{\text{MEM}} \rightarrow \text{Der } \mathbb{L}(A, T)^\wedge$ respects the bigrading when $*$ = $\bar{1}, 2$. It takes \mathbf{e}_0 to ϵ_0 and \mathbf{e}_{2n} to $2\epsilon_{2n}/(2n - 2)!$ when $n > 0$.*

Remark 23.4. The theorem implies that the Hodge realization of an object of MEM_1 is an Eisenstein variation of MHS over $\mathcal{M}_{1,1}^{\text{an}}$. (See [28, Remark 16.5] for a definition.) This implies, in particular, that the weight filtration W_\bullet of the vector bundle \mathcal{V} over $\overline{\mathcal{M}}_{1,1}/\mathbb{Q}$ that underlies the canonical extension of \mathbb{V} to $\overline{\mathcal{M}}_{1,1}$ is isomorphic to its weight graded quotient:

$$\mathcal{V} \cong \bigoplus_n \text{Gr}_n^W \mathcal{V}$$

and that, in the notation of [28], the connection on \mathcal{V} is of the form $\nabla_0 + \Omega$, where ∇_0 denotes the connection on $\text{Gr}_n^W \mathcal{V}$ that is described in § 5.2 and

$$\Omega = \sum_n \psi_{2n}(\varphi_{2n}) \in \Omega^1_{\overline{\mathcal{M}}_{1,1}/\mathbb{Q}} \otimes \text{End}\left(\bigoplus_n \text{Gr}_n^W \mathcal{V}\right),$$

where $\varphi_{2n} \in \text{Gr}_{-2}^M \text{Gr}_{-2n}^W \text{End } V^{\text{DR}}$ is the image of \mathbf{e}_{2n} under the monodromy homomorphism $\text{Gr } \mathfrak{u}_1^{\text{DR}} \rightarrow \text{End } \text{Gr } V^{\text{DR}}$.

24. Pollack’s relations

In this section we recall Pollack’s relations [49], which give an upper bound on the leading quadratic terms of these relations. If the cup product (21.2) is surjective, as would follow from Conjecture 17.1(i), this will give an upper bound on the space of minimal relations in $\mathfrak{u}_1^{\text{MEM}}$.

Pollack found all highest weight relations of degree $d = 2$ that hold between the ϵ_{2n} and all highest weight relations between the ϵ_{2n} of higher degree $d > 2$ that hold modulo a certain filtration that we now define.

Suppose that \mathfrak{n} is a Lie algebra. Denote its commutator subalgebra by \mathfrak{n}' . Define the filtration P^\bullet of \mathfrak{n} by

$$P^0 = \mathfrak{n}, \quad P^m \mathfrak{n} = L^m \mathfrak{n}' \quad m > 0.$$

where L^m denotes the m th terms of the lower central series. The filtration P^\bullet of $\mathbb{L}(\check{H})$ induces a filtration²⁴ on $\text{Der } \mathbb{L}(\check{H})$ by

$$P^m \text{Der}^0 \mathbb{L}(\check{H}) = \{\delta \in \text{Der}^0 \mathbb{L}(\check{H}) : \delta P^j \mathbb{L}(\check{H}) \subseteq P^{j+m} \mathbb{L}(\check{H})\}.$$

It has the property that $[P^j \mathfrak{n}, P^k \mathfrak{n}] \subseteq P^{j+k} \mathfrak{n}$.

Lemma 24.1. *The image of $L^m \text{Gr } \mathfrak{u}_1^{\text{MEM}}$ in $\text{Der}^0 \mathbb{L}(\check{H})$ under the monodromy homomorphism $\text{Gr } \mathfrak{u}_1^{\text{MEM}} \rightarrow \text{Der}^0 \mathbb{L}(\check{H})$ is contained in $P^m \text{Der}^0 \mathbb{L}(\check{H})$.*

Proof. A derivation of $\mathbb{L}(\check{H})$ lies in $P^1 \text{Der}^0 \mathbb{L}(\check{H})$ if and only if it acts trivially on both $H_1(\mathbb{L}(\check{H}))$ and $H_1(\mathbb{L}(\check{H})')$. The map $(\text{Sym } \check{H})(1) \rightarrow H_1(\mathbb{L}(\check{H})')$ defined by

$$x_1 \dots x_n \mapsto \sum_{\sigma \in \Sigma_n} (x_{\sigma(1)} \dots x_{\sigma(n)}) \cdot \theta$$

²⁴In § 28, we will see that this is the filtration induced by the natural *elliptic depth filtration* of $\text{Der } \mathfrak{p}$.

is an isomorphism. If $\delta \in \text{Der}^0 \mathbb{L}(\check{H})$, then

$$\delta((x_1 x_2 \dots x_n) \cdot \theta) \equiv \sum_{j=1}^n (\delta(x_j) x_1 x_2 \dots x_{j-1} x_{j+1} \dots x_n) \cdot \theta \pmod{L^2 \mathbb{L}(\check{H})'}$$

So if δ acts trivially on $H_1(\mathbb{L}(\check{H}))$, then it acts trivially on $H_1(\mathbb{L}(\check{H})')$.

Since the image of u_1^{MEM} in $\mathfrak{sl}(H)$ is trivial, its image in $\text{Der}^0 \mathbb{L}(\check{H})$ acts trivially on $H_1(\mathbb{L}(\check{H}))$ and therefore lies in $P^1 \text{Der}^0 \mathbb{L}(\check{H})$. It follows that $L^m u_1^{\text{MEM}}$ is mapped into $P^m \text{Der}^0 \mathbb{L}(\check{H})$. □

Regard $S^{2m} H(2m + 1)$ as the copy of $S^{2m} H$ in $\text{Gr} u_1^{\text{MEM}}$ generated by e_{2m+2} . If $a \geq b > 0$, then the image of the bracket in \mathfrak{f}_1 is

$$[S^{2a} H(2a + 1), S^{2b} H(2b + 1)] \cong \begin{cases} \bigoplus_{r=0}^{2b} S^{2a+2b-2r} H(2a + 2b + 2 - r) & a > b, \\ \bigoplus_{r=1}^{2a} S^{4a-4r+2} H(4a - 2r + 3) & a = b. \end{cases}$$

The determination of the highest weight vectors in $L^2 \mathfrak{f}_1 / L^3$ of degree d is an exercise in the representation theory of \mathfrak{sl}_2 .

Proposition 24.2. *If a, b and d are positive integers satisfying $0 \leq d - 2 \leq 2 \min(a, b)$, then*

$$w_{a,b}^d := \frac{1}{4} \sum_{\substack{i+j=d-2 \\ i \geq 0, j \geq 0}} (-1)^i \binom{d-2}{i} (2a-i)!(2b-j)! [e_0^i \cdot e_{2a+2}, e_0^j \cdot e_{2b+2}]$$

spans the space of $\mathfrak{sl}(H)$ highest weight vectors in the subspace

$$\text{Gr}_{-2d}^M [S^{2a} H(2a + 1), S^{2b} H(2b + 1)]$$

of u_1^{MEM} . Its image in $\text{Gr}_{-2d}^M \text{Der}^0 \mathbb{L}(\check{H})$ is

$$w_{a,b}^d := \sum_{\substack{i+j=d-2 \\ i \geq 0, j \geq 0}} (-1)^i \binom{d-2}{i} \frac{(2a-i)!(2b-j)!}{(2a)!(2b)!} [\epsilon_0^i \cdot \epsilon_{2a+2}, \epsilon_0^j \cdot \epsilon_{2b+2}],$$

which is an $\mathfrak{sl}(H)$ highest weight vector in $\text{Gr}_{-2d}^M \text{Der}^0 \mathbb{L}(\check{H})$.

Both $w_{a,b}^d$ and $w_{a,b}^d$ have W -weight $-2a - 2b - 4$ and $\mathfrak{sl}(H)$ -weight $2a + 2b - 2d + 4$. Note that $w_{a,b}^d$ vanishes when either a or b is zero as e_2 and ϵ_2 are central. Since $w_{a,b}^d = (-1)^{d+1} w_{b,a}^d$, we can, and will, assume that the coefficients in an expression

$$\sum_{a+b=n} c_a w_{a,b}^d$$

satisfy $c_a + (-1)^d c_b = 0$.

Before stating Pollack’s result we need to recall a few basic facts about cuspidal cocycles. This material is standard, but conventions are not. We will use those given in [28, § 17]. The left action of $SL_2(\mathbb{Z})$ on $H^1(E)$ is given by $(\mathbf{a}, -\mathbf{b}) \mapsto (\mathbf{a}, -\mathbf{b})\gamma$. (Cf. [28, Lemma 9.2].)

Denote the standard cochain complex of $SL_2(\mathbb{Z})$ with coefficients in the left-module V by $C^\bullet(SL_2(\mathbb{Z}), V)$ and its differential by δ . Set

$$Z_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^m H) = \{r \in C^1(SL_2(\mathbb{Z}), S^m H) : \delta r = 0, r(T) = 0\},$$

where $T = \begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$. These groups vanish when m is odd. Cuspidal cocycles are determined by their value on $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and are characterized by the functional equations

$$(I + S)r = (I + U + U^2)r = 0,$$

where $U = ST$. We will identify a cuspidal cocycle with its value on S .

The real Frobenius operator \mathcal{F}_∞ acts on cuspidal cocycles via the formula

$$\mathcal{F}_\infty : \mathbf{a} \rightarrow \mathbf{a}, \quad \mathcal{F}_\infty : \mathbf{b} \rightarrow -\mathbf{b}.$$

Let

$$Z_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^m H) = Z_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^m H)^+ \oplus Z_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^m H)^-$$

be the eigenspace decomposition. The $+$ (respectively $-$) eigenspace consists of polynomials with even (respectively odd) degree in \mathbf{a} .

The map $Z_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{2n} H) \rightarrow H_{\text{cusp}}^1(\mathcal{M}_{1,1}, S^{2n} \mathbb{H})$ that takes a cuspidal cocycle to its cohomology class has kernel spanned by the unique cuspidal coboundary $\delta(\mathbf{a}^{2n}) = \mathbf{b}^{2n} - \mathbf{a}^{2n}$. There is thus a short exact sequence

$$0 \rightarrow \mathbb{Q}\delta(\mathbf{a}^{2n}) \rightarrow Z_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{2n} H) \rightarrow H_{\text{cusp}}^1(\mathcal{M}_{1,1}, S^{2n} \mathbb{H}) \rightarrow 0. \tag{24.1}$$

It is equivariant with respect to the \mathcal{F}_∞ action. The class map has a section given by modular symbols.

Recall that the modular symbol of a cusp form f of $SL_2(\mathbb{Z})$ of weight $2n + 2$ is the homogeneous polynomial

$$r_f(\mathbf{a}, \mathbf{b}) = \sum_{j=0}^{2n} a_f(j) \mathbf{a}^j \mathbf{b}^{2n-j} := (2\pi i)^{2n+1} \int_0^{i\infty} f(\tau) (\mathbf{b} - \tau \mathbf{a})^{2n} d\tau \in S^{2n} H$$

of degree $2n$. This decomposes $r_f = r_f^+ + r_f^-$, where $r_f^\pm \in Z_{\text{cusp}}^1(SL_2(\mathbb{Z}), S^{2n} H)^\pm \otimes \mathbb{C}$.

Theorem 24.3 (Pollack). *If $n > 0$, then*

$$\sum_{a+b=n} c_a [\epsilon_{2a+2}, \epsilon_{2b+2}] = 0$$

if and only if there is a cusp form f of $SL_2(\mathbb{Z})$ of weight $2n + 2$ with $r_f^+(\mathbf{a}, \mathbf{b}) = \sum c_a \mathbf{a}^{2a} \mathbf{b}^{2n-2a}$. If $n \geq d \geq 2$, then

$$\sum_{a+b=n} c_a w_{a,b}^d \equiv 0 \pmod{P^3 \text{Der}^0 \mathbb{L}(\check{H})}$$

if and only if

$$\sum_{\substack{a+b=n \\ 2a, 2b \geq d-2}} c_a \mathbf{a}^{2a-d+2} \mathbf{b}^{2b-d+2} \in Z^1_{\text{cusp}}(\text{SL}_2(\mathbb{Z}), S^{2n-2d+4}H)^\epsilon,$$

where ϵ is $+$ when d is even and $-$ when d is odd.

Here f is arbitrary, in that it can have complex Fourier coefficients. The relations with arithmetic significance correspond to normalized Hecke eigenforms.

Remark 24.4. The second statement can be rewritten to focus on the cocycle:

$$\sum_{\substack{A+B=2N \\ A, B \equiv d \pmod{2}}} c_A \mathbf{a}^A \mathbf{b}^B \in Z^1_{\text{cusp}}(\text{SL}_2(\mathbb{Z}), S^{2N}H)^\epsilon$$

if and only if

$$\sum_{\substack{A+B=2N \\ A, B \equiv d \pmod{2}}} c_A w_{a,b}^d \equiv 0 \pmod{P^3 \text{Der}^0 \mathbb{L}(\check{H})},$$

where $a = (A + d - 2)/2$ and $b = (B + d - 2)/2$.

Remark 24.5. The trivial cuspidal cocycle $\mathbf{b}^{2N} - \mathbf{a}^{2N}$ in $Z^1_{\text{cusp}}(\text{SL}_2(\mathbb{Z}), S^{2N}H)$ corresponds to the even degree relation

$$w_{k-1, N+k-1}^{2k} - w_{N+k-1, k-1}^{2k} = 2w_{k-1, N+k-1}^{2k} \in \text{Gr}_{-4k}^M \text{Gr}_{-2N-4k}^W \text{Der}^0 \mathbb{L}(\check{H})/P^3.$$

This has $\mathfrak{sl}(H)$ -weight $2N$. There is one non-trivial such relation for each $k > 1$.

25. Pollack’s relations are motivic

In this section we use Brown’s period computations [11] to prove that there are relations in $\text{Gr} u_1^{\text{MEM}}$ that project to Pollack’s relations in $(\text{Der} \mathbb{L}(H))/P^3$. These have also been proved in [11, §§ 16, 20] using tannakian methods. Both proofs use the same general approach, which was suggested in [25]. Namely, that relations in u_1^{MEM} correspond to certain non-trivial extensions of MHS in the coordinate ring $\mathcal{O}(\mathcal{G}_1^{\text{rel}})$ of the relative completion of $\text{SL}_2(\mathbb{Z})$. The period computations in [11], which are key, imply the non-triviality of these extensions and thus the existence of the lifted relations.

One can ask if the lifted Pollack relations generate all relations in u_1^{MEM} . We show that if we assume Conjecture 17.1, then the lifts of the Pollack relations generate all relations in u_1^{MEM} . We continue with the notation of the previous four sections.

Theorem 25.1. *Pollack’s relations that correspond to period polynomials of cusp forms are motivic. That is, for each cusp form f of $\text{SL}_2(\mathbb{Z})$ of weight $2n + 2$ and each $d \geq 2$, there is an element $\mathbf{r}_{f,d}$ of $\text{Gr}_{-2d}^M v_1^{\text{geom}} \otimes \mathbb{C}$ with*

$$\mathbf{r}_{f,d} \equiv \sum_{a+b=d-2} c_a \mathbf{w}_{a,b}^d \pmod{L^3 f_1^{\text{geom}}},$$

where

$$r_f^\epsilon(\mathbf{a}, \mathbf{b}) = \sum_{\substack{a+b=n \\ 2a, 2b \geq d-2}} c_a \mathbf{a}^{2a-d+2} \mathbf{b}^{2b-d+2} \in Z_{\text{cusp}}^1(\text{SL}_2(\mathbb{Z}), S^{2n-2d+4}H)^\epsilon$$

is the even or odd part of the modular symbol of f , where ϵ is the sign of $(-1)^d$. The Pollack relation corresponding to the trivial cuspidal cocycle lifts to the arithmetic relation

$$\begin{aligned} & [\mathbf{z}_{2m-1}, \mathbf{e}_{2n+2}] \\ & \equiv \frac{(2m-2)!}{(2n+2m)!} \binom{2n+2}{2} \frac{B_{2n+2m}}{B_{2n+2}} \sum_{\substack{i+j=2m-2 \\ i, j \geq 0}} (-1)^i \frac{(2n+i)!}{i!} [\mathbf{e}_0^i \cdot \mathbf{e}_{2m}, \mathbf{e}_0^j \cdot \mathbf{e}_{2n+2m}] \end{aligned}$$

mod $L^3 \text{Gr}_{-4m}^M \mathfrak{f}_1^{\text{geom}}$.

The ‘tail’ of $\mathbf{r}_{f,d}$ is well defined only modulo the ideal generated by the $\mathbf{r}_{g,e}$ where g has lower weight and $e < d$. Note also that if the Fourier coefficients of an eigenform f are rational, then the coefficients of the quadratic part of $\mathbf{r}_{f,d}$ a rational multiplies of a fixed complex number.

The arithmetic relations look a little nicer if we set

$$\epsilon_{2k} = \frac{(2k-2)!}{2} \mathbf{e}_{2k}.$$

The image of ϵ_{2k} in the derivation algebra is ϵ_{2k} . With this normalization, the arithmetic relations become

$$[\mathbf{z}_{2m-1}, \epsilon_{2n+2}] \equiv \frac{(2n+2)!}{(2n+2m)!} \frac{B_{2n+2m}}{B_{2n+2}} \mathbf{w}_{m-1, n+m-1}^{2m} \text{ mod } L^3 \text{Gr}_{-8}^M \mathfrak{f}_1^{\text{geom}}.$$

When $m = 2$, these become

$$[\mathbf{z}_3, \epsilon_{2n}] \equiv \frac{1}{2} \frac{1}{\binom{2n+2}{2}} \frac{B_{2n+2}}{B_{2n}} \mathbf{w}_{1,n}^4 \text{ mod } L^3 \text{Gr}_{-8}^M \mathfrak{f}_1^{\text{geom}}.$$

This agrees with Pollack’s computations of $[\tilde{z}_3, \epsilon_{2n}]$ for $n \leq 6$ in $\text{Der}^0 \mathbb{L}(H)$ if one takes \mathbf{z}_3 to $1/12$ times his derivation \tilde{z}_3 , as $\mathbf{w}_{1,n}^4$ goes to his E_{2n}^4 in the derivation algebra.

Baumard and Schneps [3] used combinatorial methods to prove that Pollack’s cubic relations between the ϵ_{2n} in $(\text{Der}^0 \mathbb{L}(H))/P^3$ lift to relations in $\text{Der}^0 \mathbb{L}(H)$. Brown gives a stronger version of the second statement in [12]. Since we have proved that Pollack’s relations lift to relations in $\mathfrak{u}^{\text{MEM}}$, it follows that all of his relations lift to relations in the derivation algebra.

Corollary 25.2. *Pollack’s relations in $\text{Der}^0 \mathbb{L}(\check{H})/P^3$ lift to relations in $\text{Der}^0 \mathbb{L}(\check{H})$ for all $d \geq 3$. The Lie algebra \mathfrak{k} acts trivially on $\text{Der}^0 \mathbb{L}(\check{H})/P^3$.*

Proof of Theorem 25.1. The idea behind the proof is to bound the quadratic heads of the relations in $\mathfrak{u}_1^{\text{MEM}}$ from above using Pollack’s relations and from below using the cup product in the Deligne cohomology of $\mathcal{G}_1^{\text{rel}}$.

For $* \in \{1, \bar{1}\}$, denote the image of $\pi_1(\mathcal{G}_*^{\text{geom}}) \rightarrow \text{Aut } \mathfrak{p}$ by $\mathcal{S}_*^{\text{geom}}$. Corollary 6.10 implies that the coordinate ring of the image is an object of MTM. Set

$$\mathcal{S}_* = \pi_1(\text{MTM}) \times \mathcal{S}_*^{\text{geom}}.$$

Corollary 16.2 implies that there is a natural isomorphism $\mathcal{S}_1^{\text{geom}} \cong \mathcal{S}_1^{\text{geom}} \times \mathbb{G}_a(1)$. So it suffices to prove the case $* = 1$.

Denote by P^\bullet the filtration of $\text{Aut } \mathfrak{p}$ induced by the filtration P^\bullet of \mathfrak{p} . It is obtained by exponentiating the filtration P^\bullet of $\text{Der } \mathfrak{p}$ defined above. It restricts to a filtration P^\bullet of \mathcal{S}_* . One has the following factorizations

$$\begin{array}{ccccccc} \widehat{\mathcal{G}}_* & \longrightarrow & \mathcal{G}_*^{\text{MEM}} & \longrightarrow & \mathcal{S}_* & \longrightarrow & \mathcal{S}_*/P^3 \\ & & \searrow & & \downarrow & & \downarrow \\ & & & & \text{Aut } \mathfrak{p} & \longrightarrow & (\text{Aut } \mathfrak{p})/P^3 \end{array}$$

of the monodromy representation. There are therefore maps

$$H^\bullet(\mathcal{S}_*/P^3, S^m H(r)) \rightarrow H^\bullet(\mathcal{G}_*^{\text{MEM}}, S^m H(r)) \rightarrow H^\bullet(\widehat{\mathcal{G}}_*, S^m H(r)) \cong H^\bullet_{\mathcal{D}}(\mathcal{G}_*^{\text{rel}}, S^m H(r))$$

that are compatible with cup products. Proposition 11.8 and Theorems 15.1 and 22.3 imply that each of these maps is an isomorphism in degree 1 for all m and r .

In the rest of the proof, $\epsilon_d \in \{+, -\}$ is the sign of $(-1)^d$ and ϵ'_d is the sign of $(-1)^{d+1}$.

Recall from [11] that when $n > 0$, the Haberlund–Petersson inner product induces a non-singular pairing

$$Z^1_{\text{cusp}}(\text{SL}_2(\mathbb{Z}), S^{2n} H)^\pm \otimes H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n} \mathbb{H})^\mp \rightarrow \mathbb{Q}.$$

under which the sequence (24.1) is dual to the sequence

$$0 \rightarrow H^1_{\text{cusp}}(\mathcal{M}_{1,1}^{\text{an}}, S^{2n} \mathbb{H}) \rightarrow H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n} \mathbb{H}) \rightarrow S^{2n} H / \text{im } \mathbf{e}_0 \rightarrow 0.$$

The duality (Lemma 18.1) between cup products and quadratic heads of the relations implies that the Pollack relations of degree d with SL_2 highest weight $2n$ correspond to a surjection

$$r_{\mathcal{S}} : H^2(\mathcal{S}_1/P^3, S^{2n} H(2n+d)) \rightarrow [Z^1_{\text{cusp}}(\text{SL}_2(\mathbb{Z}), S^{2n} H)^{\epsilon_d}]^\vee \cong H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n} \mathbb{H})^{\epsilon'_d},$$

where $[\]^\vee$ denotes dual, and that the restriction of $r_{\mathcal{S}}$ to the image of the cup product

$$\begin{aligned} & \bigoplus_{j+k=n+d-2} H^1(\mathcal{S}_1/P^3, S^{2j} H(2j+1)) \otimes H^1(\mathcal{S}_1/P^3, S^{2k} H(2k+1)) \\ & \rightarrow H^2(\mathcal{S}_1/P^3, S^{2n} H(2n+d)) \end{aligned}$$

is an isomorphism.

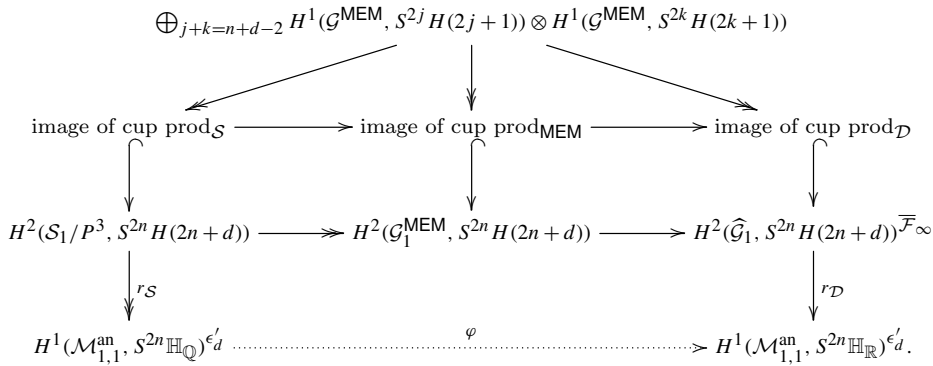
By the results of § 11, there is an isomorphism

$$H^\bullet(\widehat{\mathcal{G}}_1, S^{2n} H(2n+d)) \cong H^\bullet_{\mathcal{D}}(\mathcal{M}_{1,1}^{\text{an}}, S^{2n} H(2n+d)).$$

Since the class of the Eisenstein series in $H^1_{\mathcal{D}}(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}(2n+1))$ is $\overline{\mathcal{F}}_{\infty}$ -invariant, the image of the cup product lies in $H^2_{\mathcal{D}}(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}H_{\mathbb{R}}(2n+d))^{\overline{\mathcal{F}}_{\infty}}$. Define $r_{\mathcal{D}}$ to be the projection

$$\begin{aligned} H^2_{\mathcal{D}}(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}H_{\mathbb{R}}(2n+d))^{\overline{\mathcal{F}}_{\infty}} &\cong \text{Ext}_{\text{MHS}}^1(\mathbb{R}, H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}H_{\mathbb{R}}(2n+d))^{\overline{\mathcal{F}}_{\infty}}) \\ &\cong H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}})^{\epsilon'_d} \\ &\rightarrow H^1_{\text{cusp}}(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}})^{\epsilon'_d}. \end{aligned}$$

Since the H^1 's of \mathcal{S}_1/P^3 , $\mathcal{G}_1^{\text{MEM}}$ and $\widehat{\mathcal{G}}_1$ are all isomorphic, we have the following commutative diagram:



Brown’s period computations [11, Corollary 11.2] and the computation of the cup product in [29, §§ 8, 10] imply that there is a homomorphism φ that makes the diagram commute and is a multiple of the adjoint of the Haberlund–Petersson pairing. It is therefore an isomorphism after tensoring with \mathbb{R} . Since the restriction of $r_{\mathcal{S}}$ to the image of $\text{cup prod}_{\mathcal{S}}$ is an isomorphism,

$$\text{image of cup prod}_{\mathcal{S}} \rightarrow \text{image of cup prod}_{\text{MEM}}$$

is injective. This implies that Pollack’s relations lift to $\mathfrak{u}_1^{\text{MEM}}$.

The precise form of the arithmetic relation follows from [29, Theorem 10.5] and the duality (Corollary 18.4) between cup products and the quadratic terms of the relations. \square

As a corollary of the proof we obtain the following statement which relates surjectivity of the cup product with standard conjectures. The point of this result is that the degree 1 cohomology is motivic, so the image of the cup product should be as well.

Theorem 25.3. *The image of the composition*

$$\begin{aligned} &\bigoplus_{j+k=n+d-2} H^1(\mathcal{G}_1^{\text{MEM}}, S^{2j}H(2j+1)) \otimes H^1(\mathcal{G}_1^{\text{MEM}}, S^{2k}H(2k+1)) \\ &\rightarrow H^2(\mathcal{G}_1^{\text{MEM}}, S^{2n}H(2n+d)) \rightarrow H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}})^{\epsilon'} \end{aligned}$$

of the cup product with the projection to the $\overline{\mathcal{F}}_{\infty}$ invariant part of the real Deligne cohomology of $\mathcal{M}_{1,1}^{\text{an}}$ is a \mathbb{Q} -form of $H^2_{\mathcal{D}}(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}}(2n+d))^{\overline{\mathcal{F}}_{\infty}}$. Here, $\epsilon' \in \{+, -\}$ is the sign of $(-1)^{d+1}$.

The corresponding result also holds for $* = \bar{1}, 2$. The cohomology classes dual to the relations account for the degree 2 real Deligne cohomology of $\mathcal{M}_{1,*}^{\text{an}}$ that is in the image of the cup product. For example, when $* = \bar{1}$, the relation $[\mathbf{e}_2, \mathbf{e}_{2n}] = 0$ is dual to the copy of \mathbb{R} in $H_{\mathcal{D}}^2(\mathcal{M}_{1,1}^{\text{an}}, \mathbb{R}(2n))$ when $n \geq 2$ given in Proposition 11.8. Similarly, when $* = 2$, the relations that give the action of the \mathbf{e}_{2n} on the generators A, T (i.e., the formula for ϵ_{2n} of $\mathbb{L}(\check{H})$) correspond to copies of \mathbb{R} in $H_{\mathcal{D}}^2(\mathcal{M}_{1,2}^{\text{an}}, S^{2m+1}\mathbb{H}_{\mathbb{R}}(2r))^{\overline{\mathcal{F}}^\infty}$.

From the discussion in § 18, we know that the map

$$\mathbf{r}_*/[\mathbf{r}_*, \mathbf{f}_*] \rightarrow \mathbf{r}_*/(\mathbf{r}_* \cap L^3 \mathbf{f}_*), \tag{25.1}$$

which takes a minimal relation to its quadratic head, is injective if and only if the cup product

$$\begin{aligned} & \bigoplus_{j+k=n+d-2} H^1(\mathcal{G}_*^{\text{MEM}}, S^{2j}H(2j+1)) \otimes H^1(\mathcal{G}_*^{\text{MEM}}, S^{2k}H(2k+1)) \\ & \rightarrow H^2(\mathcal{G}_*^{\text{MEM}}, S^{2n}H(2n+d)) \end{aligned} \tag{25.2}$$

is surjective for all n and d . Since, by Theorem 25.3, the image of the cup product under

$$\text{reg}_{\mathbb{Q}} : H^2(\mathcal{G}_1^{\text{MEM}}, S^{2n}H(2n+d)) \rightarrow H_{\mathcal{D}}^2(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}}(2n+d))^{\overline{\mathcal{F}}^\infty}$$

is a \mathbb{Q} -form of

$$H_{\mathcal{D}}^2(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}}(2n+d))^{\overline{\mathcal{F}}^\infty} \cong H^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}_{\mathbb{R}})^{\epsilon'}$$

Conjecture 17.1(i) is equivalent to the surjectivity of (25.1).

Corollary 25.4. *Suppose that $* = 1$. The following statements are equivalent:*

- (i) *Conjecture 17.1(i) is true;*
- (ii) *every non-trivial minimal relation in $\mathbf{u}_*^{\text{MEM}}$ has a non-trivial quadratic head – that is, equation (25.1) is injective;*
- (iii) *the cup product (25.2) is surjective for all n and d .*

26. Problems, questions and conjectures

26.1. ℓ -adic analogues

The existence of the lifts of the Pollack relations to $\mathbf{u}_1^{\text{MEM}}$ was proved using Hodge theory. One can ask whether one can also establish their existence using ℓ -adic methods and whether the ℓ -adic analogues of Theorem 25.3 and Corollary 25.4 hold.

Conjecture 26.1. For all prime numbers ℓ and all $n, d > 0$, the cup product

$$\begin{aligned} & \bigoplus_{j+k=n+d-2} H^1(\mathcal{G}_1^{\text{cris}, \ell}, S^{2j}H_{\mathbb{Q}_\ell}(2j+1)) \otimes H^1(\mathcal{G}_1^{\text{cris}, \ell}, S^{2k}H_{\mathbb{Q}_\ell}(2k+1)) \\ & \rightarrow H^2(\mathcal{G}_1^{\text{cris}, \ell}, S^{2n}H_{\mathbb{Q}_\ell}(2n+d)) \\ & \xrightarrow{\text{projn}} H_{\text{sm}}^1(G_{\mathbb{Q}}, H_{\text{cusp}}^1(\mathcal{M}_{1,1/\overline{\mathbb{Q}}}, S^{2n}\mathbb{H}_{\mathbb{Q}_\ell}(2n+d))) \end{aligned}$$

is surjective and the natural homomorphism

$$H^2(\mathcal{G}_1^{\text{MEM}}, S^m H(r)) \otimes \mathbb{Q}_\ell \rightarrow H^2(\mathcal{G}_1^{\text{cris}, \ell}, S^m H_{\mathbb{Q}_\ell}(r))$$

is an isomorphism.

26.2. Does \mathfrak{p} generate MEM_1 ?

Another approach to understanding $\mathcal{G}_1^{\text{MEM}}$ is to ask whether it is faithfully represented in $\text{Aut } \mathfrak{p}$.

Question 26.2. Is the homomorphism $\mathcal{G}_1^{\text{MEM}} \rightarrow \text{Der } \mathfrak{p}$ injective? Equivalently, does \mathfrak{p} generate MEM_1 as a tannakian category?

If true, this would be an elliptic analogue of Brown’s theorem [10]. The proof [57] of the Oda Conjecture implies the kernel of $\mathcal{G}_1^{\text{MEM}} \rightarrow \text{Aut } \mathfrak{p}$ lies in $\mathcal{G}_1^{\text{geom}}$, so the conjectured statement is equivalent to injectivity of $\mathcal{G}_1^{\text{geom}} \rightarrow \text{Aut } \mathfrak{p}$.

26.3. Eisenstein quotients

One can ask how faithfully MEM_1 is represented in Hodge theory. To explain this, recall from [28, §16] that $\mathcal{G}_1^{\text{eis}}$ denotes the maximal Tate quotient of $\mathcal{G}_1^{\text{rel}}$ in the category of affine groups with MHS. Since the Hodge realization of $\mathcal{G}_1^{\text{geom}}$ is a mixed Hodge–Tate structure, the quotient mapping $\mathcal{G}_1^{\text{rel}} \rightarrow \mathcal{G}_1^{\text{geom}}$ factors through a surjective homomorphism $\mathcal{G}_1^{\text{eis}} \rightarrow \mathcal{G}_1^{\text{geom}}$.

Conjecture 26.3. The natural surjection $\mathcal{G}_1^{\text{eis}} \rightarrow \mathcal{G}_1^{\text{geom}}$ is an isomorphism.

If true, this implies, via Corollary 16.2, that $\mathcal{G}_*^{\text{MEM}} \cong \pi_1(\text{MTM}) \times \mathcal{G}_*^{\text{eis}}$ for $* \in \{1, \bar{1}, 2\}$.

Denote the Lie algebra of the prounipotent radical of $\mathcal{G}_1^{\text{eis}}$ by $\mathfrak{u}_1^{\text{eis}}$. Brown’s period relations imply that Pollack’s geometric (i.e., cuspidal) relations lift to relations in $\mathfrak{u}_1^{\text{eis}}$. If conjecture 17.1(i) is true, then $\mathcal{G}_1^{\text{eis}} \rightarrow \mathcal{G}_1^{\text{geom}}$ is an isomorphism.

26.4. Massey products

The higher order terms of the relations $\mathbf{r}_{f,d}$ correspond to non-vanishing matrix Massey products²⁵ in $H_{\mathcal{D}}^2(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}(2n+d))$ of the classes

$$G_{2m+2} \in H_{\mathcal{D}}^1(\mathcal{M}_{1,1}^{\text{an}}, S^{2m}\mathbb{H}(2m+1))$$

of the normalized Eisenstein series G_{2m+2} . One approach to determining the relations in $\mathcal{G}_1^{\text{MEM}}$ is to compute these.

Pollack [49] found explicit lifts of the first two cubic relations to $\text{Der}^0 \mathbb{L}(H)$. The additional terms imply that there are non-trivial matrix Massey triple products of degree

²⁵See [46] for the definition of matrix Massey products. The point for us is that the cohomology of a prounipotent Lie algebra is generated from its degree 1 cohomology by matrix Massey products.

2 in the cohomology of $\mathcal{G}_1^{\text{MEM}}$. Specifically, as Pollack points out, the cubic relation

$$80[\epsilon_{12}, [\epsilon_4, \epsilon_0]] + 16[\epsilon_4, [\epsilon_{12}, \epsilon_0]] - 250[\epsilon_{10}, [\epsilon_6, \epsilon_0]] - 125[\epsilon_6, [\epsilon_{10}, \epsilon_0]] + 280[\epsilon_8, [\epsilon_8, \epsilon_0]] - 462[\epsilon_4, [\epsilon_4, \epsilon_8]] - 1725[\epsilon_6, [\epsilon_6, \epsilon_4]] = 0$$

that corresponds to the normalized cusp form Δ of weight 12, can be rewritten (after dividing by -40) as

$$4(\omega_{1,5}^3 + \omega_{5,1}^3) - 25(\omega_{2,4}^3 + \omega_{4,2}^3) + 42\omega_{3,3}^3 + \frac{231}{20}[\epsilon_4, [\epsilon_4, \epsilon_8]] + \frac{345}{8}[\epsilon_6, [\epsilon_6, \epsilon_4]] = 0.$$

The first three terms comprise the quadratic head of the relation. It corresponds to the multiple $4(x^9y + xy^9) - 25(x^7y^3 + x^5y^7) + 42x^5y^5$ of \mathbf{r}_Δ^- . The terms of the cubic tail imply that the projection of the Massey triple products²⁶

$$\frac{231}{20} \langle \mathbb{G}_4, \mathbb{G}_4, \mathbb{G}_8 \rangle \quad \text{and} \quad \frac{345}{8} \langle \mathbb{G}_6, \mathbb{G}_6, \mathbb{G}_4 \rangle$$

to the cuspidal summand of $H_D^2(\mathcal{M}_{1,1}^{\text{an}}, S^{10}\mathbb{H}(13))$ under the multiplication maps

$$S^2\mathbb{H}(3) \otimes S^2\mathbb{H}(3) \otimes S^6\mathbb{H}(7) \rightarrow S^{10}\mathbb{H}(13) \quad \text{and} \quad S^4\mathbb{H}(5) \otimes S^4\mathbb{H}(5) \otimes S^2\mathbb{H}(3) \rightarrow S^{10}\mathbb{H}(13)$$

is equal. It also implies that (with the correct normalization) these equal the images of

$$4\mathbb{G}_{12} \smile \mathbb{G}_4 = -25\mathbb{G}_{10} \smile \mathbb{G}_6 = 42\mathbb{G}_8 \smile \mathbb{G}_8$$

in $H_D^2(\mathcal{M}_{1,1}^{\text{an}}, S^{10}\mathbb{H}(13))$.

Problem 1. Compute (matrix) Massey products in $H_D^2(\mathcal{M}_{1,1}^{\text{an}}, S^{2n}\mathbb{H}(2n + d))$ of the classes of Eisenstein series. Use them to compute higher order terms of the relations $\mathbf{r}_{f,d}$.

The indeterminacy in the tails of the relations $\mathbf{r}_{f,d}$ and the indeterminacy in the matrix Massey products correspond.

26.5. Motivic sheaves

Ayoub [2], using the work of Voevodsky [60] has constructed a category of motivic sheaves. Arapura [1], using the work of Nori, has also constructed a category of motivic sheaves. These constructions are not known to be equivalent.

Question 26.4. Are universal elliptic motives the realization of motivic sheaves over $\mathcal{M}_{1,*/\mathbb{Z}}$?

This makes sense when $* = \bar{1}$ as $\mathcal{M}_{1,\bar{1}/\mathbb{Z}}$ is a scheme. When $* = 1$, define a motivic sheaf over $\mathcal{M}_{1,1/\mathbb{Z}}$ to be a \mathbb{G}_m -invariant motivic sheaf on $\mathcal{M}_{1,\bar{1}/\mathbb{Z}}$ that is trivial on \mathbb{G}_m orbits. Similarly when $* = 2$.

An affirmative answer to this question will imply that the specialization of a universal mixed elliptic motive (as a set of compatible realizations) to a point $[E] \in \mathcal{M}_{1,1}(F)$ corresponding to an elliptic curve over F will actually be the set of realizations of a motive over F .

²⁶Since there are no cusp forms of weight < 12 , these Massey products have no indeterminacy.

Part 4. Relation to MTM and the genus 0 story

Universal mixed elliptic motives are related to the study of the unipotent fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$ via degeneration to the nodal cubic \bar{E}_0 . This is because the nodal cubic can be identified with $\mathbb{P}^1_{/\mathbb{Z}}$ with 0 and ∞ identified. The corresponding group is $E_0 = \mathbb{G}_{m/\mathbb{Z}}$ and the associated punctured elliptic curve E'_0 is $\mathbb{G}_m - \{\text{id}\} = \mathbb{P}^1 - \{0, 1, \infty\}$. Unless otherwise stated, in this part we will work with \mathbb{Q} -de Rham realizations and its canonical bigrading constructed in §23.

27. Degeneration to the nodal cubic

In this section we recall some basic formulas from [27, Part 3] which were deduced from the elliptic KZB connection [13, 43]. Denote the natural parameter on $\mathbb{P}^1_{/\mathbb{Z}}$ by w , so that

$$\mathbb{P}^1_{/\mathbb{Z}} - \{0, 1, \infty\} = \text{Spec } \mathbb{Z}[w, 1/w, 1/(w - 1)].$$

Denote the tangent vector $\partial/\partial w$ at $w = 1$ by \vec{w}_o . We can identify the nodal cubic E_0 (the fiber of the universal elliptic curve over $q = 0$) with \mathbb{P}^1 with 0 and ∞ identified.

The morphism $(\mathbb{P}^1_{/\mathbb{Z}} - \{0, 1, \infty\}, \vec{w}_o) \rightarrow (E'_{\mathbb{Z}/[[q]]}, \vec{w}_o)$ induces a homomorphism (cf. [27, §18])

$$\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{w}_o) \rightarrow \pi_1^{\text{un}}(E'_t, \vec{w}_o) \tag{27.1}$$

on unipotent fundamental groups. The induced Lie algebra homomorphism

$$\text{Lie } \pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{w}_o) \rightarrow \text{Lie } \pi_1^{\text{un}}(E'_t, \vec{w}_o)$$

is *not* a morphism in MEM_1 , but it is a morphism of MTM and is thus $\pi_1(\text{MTM})$ -equivariant.

To write down formulas, we now identify the de Rham realizations of \mathfrak{k} and $\text{Lie } \pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{w}_o)$ with the completions of their M_\bullet graded quotients

$$\mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \sigma_9, \dots) \quad \text{and} \quad \mathbb{L}(X_0, X_1) \cong \mathbb{L}(X_0, X_1, X_\infty)/(X_0 + X_1 + X_\infty),$$

respectively via the canonical \mathbb{Q} -de Rham splitting of mixed Tate motives. Likewise, we identify $\mathfrak{p} := \text{Lie } \pi_1^{\text{un}}(E'_t, \vec{w}_o)^{\text{DR}}$ with the completion of its associated (M_\bullet, W_\bullet) -bigraded module via the \mathbb{Q} -de Rham splitting of mixed elliptic motives constructed in §23. To fix notation, recall from §19.1 that

$$\check{H}^{\text{DR}} = H_1(E'_t)^{\text{DR}} = \mathbb{Q}A \oplus \mathbb{Q}T$$

where T spans the copy of $\mathbb{Q}(0)^{\text{DR}}$ and A spans the copy of $\mathbb{Q}(1)^{\text{DR}}$. The associated bigraded of \mathfrak{p} is naturally isomorphic to $\mathbb{L}(\check{H}^{\text{DR}}) = \mathbb{L}(A, T)$.

In [27, §18] it is shown that, after identification with associated graded Lie algebras as above, the map

$$\mathbb{L}(X_0, X_1, X_\infty)^\wedge/(X_0 + X_1 + X_\infty) \rightarrow \mathbb{L}(A, T)^\wedge \tag{27.2}$$

induced by (27.1) is given by

$$\begin{aligned} X_0 &\mapsto R_0 = \left(\frac{T}{e^T - 1}\right) \cdot A, \\ X_1 &\mapsto R_1 = [T, A], \\ X_\infty &\mapsto R_\infty = \left(\frac{T}{e^{-T} - 1}\right) \cdot A, \end{aligned} \tag{27.3}$$

where \cdot denotes the adjoint action of $\mathbb{Q}\langle\langle A, T \rangle\rangle = U\mathbb{L}(A, T)^\wedge$ on $\mathbb{L}(A, T)^\wedge$. This homomorphism is injective and \mathfrak{k} -equivariant.²⁷

Recall from § 5 that σ_o denotes the positive integral generator of the fundamental group $\pi_1(\mathbb{D}^*, \vec{1})$ of the q -disk, which is isomorphic to $\mathbb{Q}(1)$. It acts on $\pi_1^{\text{un}}(E'_t, \vec{w}_o)$ and thus on \mathfrak{p} . The image of σ_o in $\text{Aut } \mathfrak{p}$ lies in the pronipotent subgroup $W_{-2} \text{Aut } \mathfrak{p}$ and thus has a logarithm. Set

$$N = N^{\text{DR}} := \frac{1}{2\pi i} \log \sigma_o \in \text{Der } \mathfrak{p}.$$

In [27, § 13] it is shown that, after identification with the associated bigraded module,

$$N = \sum_{m \geq 0} (2m - 1) \frac{B_{2m}}{(2m)!} \epsilon_{2m} \in W_0 \text{Gr}_{-2}^M \text{Der}^0 \mathbb{L}(A, T)^\wedge, \tag{27.4}$$

where B_{2m} denotes the $2m$ th Bernoulli number. It commutes with the action of \mathfrak{k} as it spans a copy of $\mathbb{Q}(1)$.

28. Depth filtrations

There is a natural depth filtration in the elliptic case which generalizes the depth filtration in classical case, $\mathbb{P}^1 - \{0, 1, \infty\}$. The classical and elliptic depth filtrations are quite closely related via the weight filtration W_\bullet . Unfortunately, the link between them is not as straightforward as one might hope, as we shall see.

In this section we set $E = E'_t$ and $U = \mathbb{P}^1 - \{0, 1, \infty\}$. We will denote the Lie algebra of $\pi_1^{\text{un}}(X, \vec{w}_o)$ by $\mathfrak{p}(X)$, when $X = U, \mathbb{G}_m, E$, and E' .

Recall that the depth filtration D^\bullet of the Lie algebra $\pi_1^{\text{un}}(U, \vec{w}_o)$ is defined by

$$D^d \mathfrak{p}(U) = \begin{cases} \mathfrak{p}(U) & d = 0, \\ \ker\{\mathfrak{p}(U) \rightarrow \mathfrak{p}(\mathbb{G}_m)\} & d = 1, \\ L^d D^1 \mathfrak{p}(U) & d > 1, \end{cases}$$

where L^d denotes the d th term of the lower central series. In the elliptic case, define²⁸

$$D^d \mathfrak{p}(E') = \begin{cases} \mathfrak{p}(E') & d = 0, \\ \ker\{\mathfrak{p}(E') \rightarrow \mathfrak{p}(E)\} & d = 1, \\ L^d D^1 \mathfrak{p}(E') & d > 1. \end{cases}$$

²⁷Similar and related formulas appear in [13, Proposition 4.9] and [19, Proposition 3.8].

²⁸This is the filtration P^\bullet defined in § 24.

These filtrations are motivic and thus preserved by \mathfrak{k} . The first is a filtration of $\mathfrak{p}(U)$ in MTM and the second is a filtration of $\mathfrak{p}(E')$ in MEM_T . They can thus be described on the associated graded modules: $D^d\mathbb{L}(X_0, X_1)$ is the ideal spanned by the Lie monomials whose degree in X_1 is $\geq d$; and $D^d\mathbb{L}(A, T)$ is the ideal spanned by the Lie monomials in which $\theta := [T, A]$ occurs at least d times.

Remark 28.1. Note that $\text{Gr}_\bullet^M \mathfrak{p}(E')$ is naturally isomorphic to the T -adic completion of $\mathbb{L}(A, T)$.

The morphisms

$$\begin{array}{ccc} U & \hookrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow \\ E' & \hookrightarrow & E \end{array}$$

preserve the base point \bar{W}_o and thus induce homomorphisms

$$\begin{array}{ccc} \mathfrak{p}(U) & \longrightarrow & \mathfrak{p}(\mathbb{G}_m) \\ \downarrow & & \downarrow \\ \mathfrak{p}(E') & \longrightarrow & \mathfrak{p}(E) \end{array}$$

That the vertical homomorphisms are injective follows from the fact that a Lie subalgebra of a free Lie algebra is free. It follows that $\mathfrak{p}(U) \rightarrow \mathfrak{p}(E')$ preserves D^\bullet .

Proposition 28.2. *The inclusion $\mathfrak{p}(U) \hookrightarrow \mathfrak{p}(E')$ is strictly compatible with the depth filtrations. That is,*

$$D^d \mathfrak{p}(U) = \mathfrak{p}(U) \cap D^d \mathfrak{p}(E').$$

Proof. Since Gr_\bullet^W and Gr_\bullet^M are exact, we can work with the associated graded. The Lie algebra $D^1\mathbb{L}(X_0, X_1)$ is freely generated by the set $\{X_0^m \cdot X_1 : m \geq 0\}$. Since it is free, it follows that for all $m \geq 1$, $D^m\mathbb{L}(X_0, X_1) = L^m D^1\mathbb{L}(X_0, X_1)$. This remains true after completion when we replace ‘generated’ by ‘topologically generated’. The Lie algebra $D^1\mathbb{L}(A, T)$ is also free. Its abelianization is naturally isomorphic to the free $\text{Sym } \check{H}$ -module generated by θ . The map (27.2) induces the inclusion

$$H_1(D^1\mathbb{L}(X_0, X_1)^\wedge) \rightarrow H_1(D^1\mathbb{L}(A, T)^\wedge)$$

that takes $X_0^m \cdot X_1$ to $T^m \cdot \theta$ and is therefore injective. It follows that

$$L^m D^1\mathbb{L}(X_0, X_1)^\wedge = \mathbb{L}(X_0, X_1)^\wedge \cap L^m D^1\mathbb{L}(A, T)^\wedge,$$

which implies the result. □

The weight filtration W_\bullet of $\mathfrak{p}(E')$ restricts to a filtration of $\mathfrak{p}(U)$:

$$W_{-m} \mathfrak{p}(U) := \mathfrak{p}(U) \cap W_{-m} \mathfrak{p}(E').$$

The three filtrations D^\bullet , M_\bullet and W_\bullet are related by the ‘convolution formula’.²⁹

²⁹The convolution of two filtrations F^\bullet and G^\bullet of a vector space V is the filtration $F * G$ defined by $(F * G)^n V := \sum_{j+k=n} F^j V \cap G^k V$. There is a natural isomorphism $\text{Gr}_{F * G}^n V \cong \bigoplus_{j+k=n} \text{Gr}_F^j \text{Gr}_G^k V$.

Proposition 28.3. For all $m \geq 0$, we have

$$W_{-m}\mathfrak{p}(U) = \sum_{n+d \geq m} M_{-2n}\mathfrak{p}(U) \cap D^d\mathfrak{p}(U),$$

so that

$$\text{Gr}_{-m}^W \mathfrak{p}(U) \cong \bigoplus_{n+d=m} \text{Gr}_{-2n}^M \text{Gr}_D^d \mathfrak{p}(U).$$

Proof. Identify $\mathfrak{p}(U)$ with $\mathbb{L}(X_0, X_1)^\wedge$ and $\mathfrak{p}(E')$ with $\mathbb{L}(A, T)^\wedge$. We first show that if $m = n + d$, then

$$M_{-2n}\mathfrak{p}(U) \cap D^d\mathfrak{p}(U) \subseteq W_{-m}\mathfrak{p}(E').$$

Suppose that w is a homogeneous element of $\mathbb{L}(X_0, X_1)$ of bidegree (a, b) in X_0 and X_1 . Then $w \in W_{-2n} \cap D^d$ if and only if $b \geq d$ and $a + b \geq n$. These conditions imply that $a + 2b \geq m$. On the other hand, the corresponding word in $\mathbb{L}(A, T)^\wedge$ lies in $W_{-a-2b}\mathbb{L}(A, T)^\wedge$, which is contained in W_{-m} when $a + 2b \geq m$. It follows that

$$W_{-m}\mathfrak{p}(U) \supseteq \sum_{n+d \geq m} M_{-2n}\mathfrak{p}(U) \cap D^d\mathfrak{p}(U).$$

To prove the reverse inclusion, define a filtration G_\bullet on $\mathbb{L}(X_0, X_1)^\wedge$ by

$$G_{-m}\mathbb{L}(X_0, X_1)^\wedge = \sum_{n+d \geq m} M_{-2n}\mathbb{L} \cap D^d\mathbb{L}(X_0, X_1)^\wedge.$$

The first part implies that the identity $(\mathbb{L}(X_0, X_1)^\wedge, G_\bullet) \rightarrow (\mathbb{L}(X_0, X_1)^\wedge, W_\bullet)$ is filtration preserving. To prove $G_\bullet = W_\bullet$, it suffices to show that the identity induces an isomorphism on associated gradeds. An easy linear algebra argument implies that

$$\text{Gr}_{-m}^G \mathbb{L}(X_0, X_1) \cong \bigoplus_{n+d=m} \text{Gr}_{-2n}^M \text{Gr}_D^d \mathbb{L}(X_0, X_1).$$

This implies that $\text{Gr}_{-m}^G \mathbb{L}(X_0, X_1)$ is spanned by the monomials w in X_0 and X_1 of bidegree (a, b) with $a + 2b = m$. Since these also span $\text{Gr}_{-m}^W \mathbb{L}(X_0, X_1)$, the identity induces an isomorphism on associated gradeds. □

Remark 28.4. One might suspect that one can invert this formula to get a formula for D^\bullet in terms of M_\bullet and W_\bullet . However, while it is true that

$$\text{Gr}_D^d \mathbb{L}(X_0, X_1) = \sum_{m-n=d} \text{Gr}_{-m}^W \text{Gr}_{-2n}^M \mathbb{L}(X_0, X_1),$$

it is *not* true that $D^d\mathbb{L}(X_0, X_1) = \bigoplus_{m-n=d} W_{-m}\mathbb{L}(X_0, X_1) \cap M_{-2n}\mathbb{L}(X_0, X_1)$. The point being that the right-hand side is *not* the convolution of W_\bullet and M_\bullet .

Remark 28.5. The proposition does not hold if U is replaced by E' . For example,

$$A^{n-1}T^{m-n-1} \cdot \theta \in \text{Gr}_{-2n}^M \text{Gr}_{-m}^W \mathbb{L}(\check{H}).$$

On the other hand, it lies in $D^1\mathbb{L}(\check{H})$ and projects to a non-trivial element of $\text{Gr}_D^1 \mathbb{L}(\check{H})$. So the relation $n + d = m$ between depth and the weights holds if and only if the element is of the form $A^{n-1} \cdot \theta$.

The depth filtrations on $\mathfrak{p}(E')$ and $\mathfrak{p}(U)$ induce depth filtrations on their derivation algebras in the standard way. These are motivic as the depth filtrations on $\mathfrak{p}(E')$ and $\mathfrak{p}(U)$ are.

Define the Lie algebra of *nodal derivations* $\text{Der}^N \mathfrak{p}(E')$ of $\mathfrak{p}(E')$ to be

$$\{\delta \in \text{Der } \mathfrak{p}(E') : [\delta, N] = 0, \delta(\log \sigma_o) = 0 \text{ and } \delta(\mathfrak{p}(U)) \subset \mathfrak{p}(U)\}.$$

This is a pro-object of **MTM**. Define the *extendable derivations* $\text{Der}^E \mathfrak{p}(U)$ to be the image of the restriction map

$$\text{Der}^N \mathfrak{p}(E') \rightarrow \text{Der } \mathfrak{p}(U).$$

It is also a pro-object of **MTM**. The image of the natural action of \mathfrak{k} on $\mathfrak{p}(E')$ lands in $\text{Der}^N \mathfrak{p}(E')$, so that we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{k} & \xrightarrow{\phi_E} & \text{Der}^N \mathfrak{p}(E') \\ & \searrow \phi_P & \downarrow \text{res} \\ & & \text{Der}^E \mathfrak{p}(U). \end{array}$$

The kernel of the restriction map res is spanned by N . Since this has weight 0, the restriction map is an isomorphism after applying W_{-1} . The depth filtrations of $\text{Der } \mathfrak{p}(E')$ and $\text{Der } \mathfrak{p}(U)$ induce depth filtrations on $\text{Der}^N \mathfrak{p}(E')$ and $\text{Der}^E \mathfrak{p}(U)$, respectively.

The following useful result was proved by Pollack [49, 4.5].

Lemma 28.6 (Pollack). *If $\delta \in \text{Der}^0 \mathbb{L}(A, T)$, then $\delta(A) \in D^d \mathbb{L}(A, T)$ if and only if $\delta(T) \in D^d \mathbb{L}(A, T)$. \square*

Note that it is important that $\delta(\theta) = 0$. The derivation $\delta = \text{ad}_A$ is not in $D^1 \text{Der } \mathbb{L}(A, T)$, even though $\delta(A), \delta(T) \in D^1 \mathbb{L}(A, T)$.

Lemma 28.7. *If $\delta \in \text{Der}^0 \mathbb{L}(A, T)$, then*

$$\begin{aligned} \delta \in D^d \text{Der}^0 \mathbb{L}(A, T) &\iff \delta(A), \delta(T) \in D^d \mathbb{L}(A, T) \\ &\iff \delta(A) \in D^d \mathbb{L}(A, T) \text{ or } \delta(T) \in D^d \mathbb{L}(A, T). \end{aligned}$$

Proof. The left to right implications are clear. Pollack’s lemma 28.6 implies that the right-hand statement implies the middle one. To complete the proof, we have to show that the middle statement implies the left-hand statement. To do this, it suffices to prove that if $\delta(A)$ and $\delta(T)$ are in $D^d \mathbb{L}(A, T)$ and if $u \in D^1 \mathbb{L}(A, T)$, then $\delta(u) \in D^{d+1} \mathbb{L}(A, T)$. Since $D^1 \mathbb{L}(A, T)$ is free, and since its abelianization is the rank 1 free $\text{Sym } \check{H}$ module generated by the class of θ , $D^1 \mathbb{L}(A, T)$ is generated by elements of the form $f(A, T) \cdot \theta$, where $f(A, T) \in \mathbb{Q}(A, T)$. The assumptions imply that $\delta f(A, T) \in D^d \mathbb{Q}(A, T)$. So, since $\delta(\theta) = 0$,

$$\delta(f(A, T) \cdot \theta) = \delta(f(A, T)) \cdot \theta \in D^{d+1} \mathbb{L}(A, T)$$

from which it follows that δ takes $D^1 \mathbb{L}(A, T)$ into $D^{d+1} \mathbb{L}(A, T)$. \square

We can now prove that the elliptic and classical depth filtrations agree.

Proposition 28.8. *The restriction mapping $\text{Der}^N \mathfrak{p}(E') \rightarrow \text{Der}^E \mathfrak{p}(U)$ preserves the depth filtration and is strict with respect to it.*

Proof. Identify X_0 and X_1 with their images in $\mathbb{L}(A, T)^\wedge$. If $\delta \in \text{Der}^0 \mathbb{L}(A, T)$, then

$$\delta(X_0) = \delta \left(\frac{T}{e^T - 1} \cdot A \right) = \delta(A) + \sum_{m=1}^\infty \frac{B_{2m}}{(2m)!} \delta(T^{2m-1}) \cdot \theta. \tag{28.1}$$

Observe that if $\delta \in D^d \text{Der}^0 \mathbb{L}(A, T)$ and $n \geq 1$, then

$$\delta(T^n \cdot A) = \delta(T^{n-1} \cdot \theta) = \delta(T^{n-1}) \cdot \theta \in D^{d+1} \mathbb{L}(A, T).$$

So if $\delta \in D^d \text{Der}^N \mathbb{L}(A, T)$, then $\delta(X_0) \in D^d \mathbb{L}(A, T)^\wedge$. So, by Proposition 28.2,

$$\delta(X_0) \in \mathbb{L}(X_0, X_1)^\wedge \cap D^d \mathbb{L}(A, T) \in D^d \mathbb{L}(X_0, X_1)^\wedge.$$

This implies that the restriction mapping $\text{Der}^N \mathfrak{p}(E') \rightarrow \text{Der}^E \mathfrak{p}(U)$ respects the depth filtrations.

It remains to show that the restriction mapping is strictly compatible with the depth filtrations. Suppose that the restriction of $\delta \in \text{Der}^N \mathbb{L}(A, T)$ to $\mathbb{L}(X_0, X_1)^\wedge$ is in $D^d \text{Der} \mathbb{L}(X_0, X_1)$. That is, that $\delta(X_0) \in D^d \mathbb{L}(X_0, X_1)^\wedge$. Then equation (28.1) implies that $\delta(A) \in D^d \mathbb{L}(A, T)$. Applying Lemma 28.7, we see that $\delta \in D^d \text{Der}^0 \mathbb{L}(A, T)$, as required. \square

So the depth filtration on \mathfrak{k} can be computed by pulling back the depth filtration either from $\text{Der} \mathfrak{p}(U)$ (the ‘classical’ case) or from $\text{Der} \mathfrak{p}(E')$ (the ‘elliptic’ case). Next, we use this to show that the depth filtration on \mathfrak{k} is closely related to the elliptic weight filtration W_\bullet .

Theorem 28.9. *For all $m \geq 0$, we have*

$$W_{-m} \mathfrak{k} = \sum_{n+d \geq m} M_{-2n} \mathfrak{k} \cap D^d \mathfrak{k},$$

so that

$$\text{Gr}_{-m}^W \mathfrak{k} \cong \bigoplus_{n+d=m} \text{Gr}_{-2n}^M \text{Gr}_D^d \mathfrak{k}.$$

Proof. Let γ_1 be the class of the canonical loop in $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{\nu}_o)$ about 1 in \mathcal{U} . Its logarithm $\log \gamma_1$ spans a copy of $\mathbb{Q}(1)$ in $\mathfrak{p}(U)$. Define

$$\text{Der}^0 \mathfrak{p}(U) = \{\delta \in \text{Der} \mathfrak{p}(U) : \delta(\log \gamma_1) = 0\}.$$

This is motivic as it is the kernel of the morphism $\text{Der} \mathfrak{p}(U) \rightarrow \mathfrak{p}(U)$ that takes the derivation δ to $\delta(\log \gamma_1)$. So it is a pro-object of MTM that is filtered by W_\bullet . The image of the canonical homomorphism $\mathfrak{k} \rightarrow \text{Der} \mathfrak{p}(U)$ is contained in $\text{Der}^0 \mathfrak{p}(U)$.

Choose any element γ_0 of $\pi_1(U, \vec{\nu}_o)$ that, together with γ_1 , generates $\pi_1(U, \vec{\nu}_o)$. Define a linear map $\phi : \mathfrak{p}(U) \rightarrow \text{Der}^0 \mathfrak{p}(U)$ by $f \mapsto \delta_f$, where $\delta_f(\log \gamma_0) = [\log \gamma_0, f]$. This has kernel $\mathbb{Q} \log \gamma_0$. Note that ϕ is *not* motivic, or even a morphism of MHS. However, its restriction to $M_{-4} \mathfrak{p}(U)$ is strict with respect to the depth filtration and both weight filtrations as we now show.

Since $\log \gamma_0$ is not in $M_{-4}\mathfrak{p}(U)$ and since M_{-2n} is the n th term of the lower central series of $\mathfrak{p}(U)$, the restriction

$$M_{-4}\mathfrak{p}(U) \rightarrow \text{Der}^0 \mathfrak{p}(U) \tag{28.2}$$

of ϕ to $M_{-4}\mathfrak{p}(U)$ is injective and strictly compatible with M_\bullet .

The inclusion (28.2) is also strict with respect to the depth filtration D^\bullet because, for $f \in M_{-4}\mathfrak{p}(U)$,

$$\delta_f \in D^d \text{Der} \mathfrak{p}(U) \iff \delta_f(\log \gamma_0) \in D^d \mathfrak{p}(U) \iff f \in D^d \mathfrak{p}(U).$$

Similarly, when $m \geq 3$, $\delta_f \in W_{-m} \text{Der} \mathfrak{p}(U)$ if and only if $f \in W_{-m}\mathfrak{p}(U)$. In other words, (28.2) is strict with respect to W_\bullet as well.

It is well known that the image of $\mathfrak{k} \rightarrow M_{-4} \text{Der}^0 \mathfrak{p}(U)$ is contained in the image of ϕ . Since $M_{-4}\mathfrak{p}(U) \rightarrow \text{Der}^0 \mathfrak{p}(U)$ is injective, this lifts to a (non-motivic) map $\mathfrak{k} \rightarrow \mathfrak{p}(U)$. The result now follows from Proposition 28.3 and the fact that $\mathfrak{p}(U) \rightarrow \text{Der}^0 \mathfrak{p}(U)$ is strict with respect to the depth filtration and the two weight filtrations. \square

Although we cannot ‘invert’ the convolution formula, we nonetheless are able to express the depth graded quotients of \mathfrak{k} in terms of the classical and elliptic weight filtrations.

Corollary 28.10. *The depth and elliptic weight filtrations on $\text{Gr}_{-2n}^M \mathfrak{k}$ are related by*

$$W_{-m} \text{Gr}_{-2n}^M \mathfrak{k} = D^{m-n} \text{Gr}_{-2n}^M \mathfrak{k}.$$

Consequently, there is a natural isomorphism $\text{Gr}_{-m}^W \text{Gr}_{-2n}^M \mathfrak{k} \cong \text{Gr}_{-2n}^M \text{Gr}_D^{m-n} \mathfrak{k}$. \square

29. The infinitesimal Galois action and Ihara–Takao congruences

The Ihara–Takao congruence (Theorem 29.6 below) was proved numerically by Ihara and Takao in [36] and proved again using modular symbols by Schneps [51]. In this section, we use Pollack’s relations to give a more conceptual explanation of the Ihara–Tako congruence, which makes it clearer why cusp forms impose relations in the depth graded quotients of \mathfrak{k} . We begin with a discussion of the infinitesimal Galois action $\phi_E : \mathfrak{k} \rightarrow \text{Der} \mathfrak{p}(E')$. We continue with the notation of the previous section. In particular, U denotes $\mathbb{P}^1 - \{0, 1, \infty\}$.

The derivations ϵ_{2n} are highest weight vectors. As we will see below, to ‘first order’, the images of the \mathbf{z}_{2m+1} in $\text{Der}^0 \mathbb{L}(\dot{H})$ are lowest weight vectors. For each $n \geq 0$, set³⁰

$$\check{\epsilon}_{2n} = \check{\epsilon}_{2n}^{\text{DR}} := \epsilon_{2n}(A, T) = \frac{1}{(2n-2)!} \epsilon_0^{2n-2} \cdot \epsilon_{2n}$$

Its Betti counterpart is

$$\check{\epsilon}_{2n}^B := \epsilon_{2n}(\mathbf{a}, -\mathbf{b}) = \frac{1}{(2n-2)!} (\epsilon_0^B)^{2n-2} \cdot \epsilon_{2n}^B = (2\pi i)^{2n-1} \check{\epsilon}_{2n}^{\text{DR}}.$$

These are lowest weight vectors for $\mathfrak{sl}(H)$.³¹

³⁰Recall the definition of $\epsilon_{2n}(\mathbf{v}_1, \mathbf{v}_2)$, equation (22.3).

³¹The equality $\check{\epsilon}_{2n}^B = (2\pi i)^{2n-1} \check{\epsilon}_{2n}^{\text{DR}}$ reflects the fact that $\check{\epsilon}_{2n}$ spans a copy of $\mathbb{Q}(2n-1)$ in $\text{Gr}_{-2n+2}^W \text{Der} \mathfrak{p}(E')$.

We will identify \mathfrak{k} with the completion of $\text{Gr}_\bullet^M \mathfrak{k}$ using the \mathbb{Q} -de Rham splitting of M_\bullet . Identify $\mathfrak{p}(E')$ with the completion of its associated (M_\bullet, W_\bullet) bigraded quotient via the \mathbb{Q} -de Rham splitting. This gives an isomorphism of $\text{Gr}_\bullet^M \mathfrak{p}(E')$ with the T -adic completion of $\mathbb{L}(A, T)$. Choose generators of \mathfrak{z}_{2m+1} of $\text{Gr}_\bullet^M \mathfrak{k}$ that project to the natural generators (see § 20) of $H_1(\mathfrak{k})$.

The infinitesimal Galois action $\phi_E : \mathfrak{k} \rightarrow \text{Der } \mathfrak{p}(E')$ is a morphism in MTM. So it can be identified with the mapping

$$\phi_E : \mathbb{L}(\sigma_3, \sigma_5, \sigma_7, \dots) \rightarrow \text{Der}^0 \mathbb{L}(A, T)^\wedge$$

induced by applying Gr^M to ϕ_E . Since ϕ_E is not a morphism in MEM_1 , it does not preserve W_\bullet . For $\sigma \in \text{Gr}_{-4m-2}^M \mathfrak{k}$, let $\sigma^{(r)}$ be the component of $\phi_E(\sigma)$ that lies in $\text{Gr}_{-r}^W \text{Gr}_{-4m-2}^M \text{Der}^0 \mathbb{L}(A, T)$.

Call an element of $\text{Der } \mathfrak{p}(E')$ *geometric* if it lies in the image of $\mathfrak{g}_1^{\text{geom}} \rightarrow \text{Der } \mathfrak{p}(E')$. Theorem 22.3 implies that, after identifying with the associated bigraded, the Lie algebra of geometric derivations is the subalgebra of $\text{Der}^0 \mathbb{L}(A, T)$ generated by $\check{\epsilon}_0$ and the $\epsilon_0^j \cdot \epsilon_{2n}$ with $j \geq 0$ and $n \geq 0$. Observe that

$$\sigma_{2m+1}^{(4m+2)} \equiv \mathfrak{z}_{2m+1} \text{ mod geometric derivations}$$

as $\mathfrak{z}_{2m+1} \mapsto \sigma_{2m+1}$ under $u_1^{\text{MEM}} \rightarrow \mathfrak{k}$.

Proposition 29.1. *If $r \neq 4m + 2$, then $\sigma_{2m+1}^{(r)}$ is geometric. The remaining term $\sigma_{2m+1}^{(4m+2)}$ is well-defined modulo the image of $L^3 \text{Gr } u_1^{\text{geom}}$.*

Proof. Denote the normalizer of the image of a Lie algebra homomorphism $\mathfrak{a} \rightarrow \text{Der } \mathfrak{p}(E')$ by $N(\mathfrak{a})$. Since $\mathfrak{g}_1^{\text{MEM}}/\mathfrak{g}_1^{\text{geom}} \cong \mathfrak{k}$, and since there is a monodromy homomorphism $\mathfrak{g}^{\text{MEM}} \rightarrow \text{Der } \mathfrak{p}(E')$, the image of ϕ_E lies in $N(\mathfrak{g}_1^{\text{geom}})$. Since u_1^{geom} is an ideal of $\mathfrak{g}_1^{\text{geom}}$, since $N(u_1^{\text{geom}})/\text{im } u_1^{\text{geom}}$ is a $\mathfrak{g}_1^{\text{geom}}/u_1^{\text{geom}}$ -module, and since $\mathfrak{g}_1^{\text{geom}}/u_1^{\text{geom}} \cong \mathfrak{sl}(H)$, it follows that

$$N(\mathfrak{g}_1^{\text{geom}})/\text{im } \mathfrak{g}_1^{\text{geom}} \subseteq [N(u_1^{\text{geom}})/\text{im } u_1^{\text{geom}}]^{\mathfrak{sl}(H)}.$$

After identification with associated bigradeds, we see that, modulo geometric derivations, the image of \mathfrak{k} is contained in $[\text{Der}^0 \mathbb{L}(A, T)]^{\mathfrak{sl}(H)}$.

Since the $\mathfrak{sl}(H)$ -invariants have the property that their M - and W -weights are equal by (19.2), $\sigma_{2m+1}^{(r)}$ must be geometric when $r \neq 4m + 2$.

The second assertion follows immediately from the fact that

$$H_1(u_1^{\text{geom}})^{\text{SL}(H)} = \Lambda^2 H_1(u_1^{\text{geom}})^{\text{SL}(H)} = 0.$$

Since $\text{Gr}_r^W u_1^{\text{geom}} = \text{Gr}_r^W u_1^{\text{geom}}$ for all $r \neq -2$, this implies that all $\mathfrak{sl}(H)$ -invariants in $\text{Gr}_{-4m-2}^W u_1^{\text{geom}}$ lie in $L^3 \text{Gr } u_1^{\text{geom}}$. □

The next result is an immediate consequence of the fact that the odd W -graded quotients of u_1^{geom} vanish.

Corollary 29.2. *If $\sigma_{2m+1}^{(r)} \neq 0$, then r is even and $\geq 2m + 2$, so that*

$$\phi_E(\sigma_{2m+1}) = \sum_{k \geq m+1} \sigma_{2m+1}^{(2k)}.$$

Remark 29.3. Brown’s proof [10] of the injectivity of $\phi_{\mathbb{P}} : \mathfrak{k} \rightarrow \text{Der } \mathfrak{p}(U)$ and the Oda Conjecture, whose proof was completed by Takao [57], imply that the homomorphism $\mathfrak{k} \rightarrow N(\mathfrak{g}_1^{\text{geom}})/\text{im } \mathfrak{g}_1^{\text{geom}}$ is injective.

The fact that the image of \mathfrak{k} lies in the centralizer of N gives finer information about the $\mathfrak{z}_{2m+1}^{(r)}$.

Proposition 29.4. *For all $m \geq 1$, we have $\phi_E(\sigma_{2m+1}) \equiv \check{\epsilon}_{2m+2} \pmod{D^2}$.*

Proof. Recall the formula for the monodromy logarithm N from (27.4). Since $\phi_E(\sigma_{2m+1})$ commutes with N , the $\sigma_{2m+1}^{(2k)}$ satisfy the system of equations

$$[\epsilon_0, \sigma_{2m+1}^{(2k)}] = c_4[\epsilon_4, \sigma_{2m+1}^{(2k-4)}] + c_6[\epsilon_6, \sigma_{2m+1}^{(2k-6)}] + \dots + c_{2k-2m-2}[\epsilon_{2k-2m-2}, \sigma_{2m+1}^{(2m+2)}],$$

where $c_{2k} = (2k - 1)B_{2k}/(2k)!$. This equation determines $\sigma_{2m+1}^{(2k)}$ up to an element of $\ker \epsilon_0$ and implies that $\sigma_{2m+1}^{(2m+2)}$ is a lowest weight vector of $\text{Der}^0 \mathbb{L}(A, T)$. Since

$$\text{Gr}_{-4m-2}^M \ker\{\epsilon_0 : H_1(u_1^{\text{geom}}) \rightarrow H_1(u_1^{\text{geom}})\} = \mathbb{Q}\check{\epsilon}_{2m+2},$$

it follows that $\sigma_{2m+1}^{(2m+2)}$ is a multiple (possibly zero) of $\check{\epsilon}_{2m+2}$. It also implies that

$$W_{-2m-3} \ker\{\epsilon_0 : \text{Der}_{<0}^{\text{geom}} \mathbb{L}(A, T) \rightarrow \text{Der}_{<0}^{\text{geom}} \mathbb{L}(A, T)\} \subset L^2 \text{Der}_{<0}^{\text{geom}} \mathbb{L}(A, T),$$

where $\text{Der}_{<0}^{\text{geom}} \mathbb{L}(A, T)$ denotes the geometric derivations of $\mathbb{L}(A, T)$ of negative W -weight. Combined with Proposition 29.1, this and the equation above imply that $\sigma_{2m+1}^{(2k)} \in L^2 \text{Der}_{<0}^{\text{geom}} \mathbb{L}(A, T)$ when $2k > 2m + 2$. Since D^\bullet is a central filtration of $\text{Der} \mathbb{L}(A, T)$, $D^2 \text{Der}_{<0}^{\text{geom}} \mathbb{L}(A, T) \supseteq L^2 \text{Der}_{<0}^{\text{geom}} \mathbb{L}(A, T)$. It follows that

$$\phi_E(\sigma_{2m+1}) \equiv \sigma_{2m+1}^{(2m+2)} \pmod{D^2}.$$

It remains to show that $\sigma_{2m+1}^{(2m+2)} = \check{\epsilon}_{2m+2}$. This can be proved either by computing the cocycle of the étale realization or the period of a Hodge realization. Nakamura computed the cocycle in [45, Theorems 3.3 and 3.5]. We deduce it from Brown’s computation [11, Lemma 7.1] of the period of the Eisenstein series G_{2m+2} . His result implies that

$$\phi_E(\sigma_{2m+1}) \equiv \frac{(2m)!}{2} \text{im } \check{\epsilon}_{2m+2} \pmod{D^2},$$

where $\check{\epsilon}_{2m+2} := \mathbf{e}_0^{2m} \cdot \mathbf{e}_{2m+2}/(2m)!$ and im means its image in $\text{Der}^0 \mathbb{L}(A, T)$. Thus, by Theorem 22.3, is $\check{\epsilon}_{2m+2}$. □

Brown [12, Theorem 1.2] has computed $\phi_E(\sigma_{2m+1}) \pmod{W_{-2m-5}}$ for a certain choice of σ_{2m+1} . This gives the second term in the expansion in Corollary 29.2.

The $\check{\epsilon}_{2n}$ satisfy Pollack’s relations as well. The elliptic analogue of the Ihara–Takao congruence is now an immediate consequence of Proposition 29.4 and Pollack’s relations.

Theorem 29.5. *If $n > 0$, then*

$$\sum_{a+b=n} c_a[\phi_E(\sigma_{2a+1}), \phi_E(\sigma_{2b+1})] \equiv 0 \pmod{D^3 \text{Der } \mathfrak{p}(E')}$$

if and only if there is a cusp form f of $\text{SL}_2(\mathbb{Z})$ of weight $2n + 2$ with $r_f^+(x, y) = \sum c_a x^{2a} y^{2n-2a}$.

The classical case of the Ihara–Takao congruences now follows from the elliptic case using the formulas (27.3).

Theorem 29.6 (Ihara–Takao, Goncharov, Schneps). *If $n > 0$, then*

$$\sum_{a+b=n} c_a[\phi_{\mathbb{P}}(\sigma_{2a+1}), \phi_{\mathbb{P}}(\sigma_{2b+1})] \equiv 0 \pmod{D^3 \text{Der } \mathfrak{p}(U)}$$

if and only if there is a cusp form f of $\text{SL}_2(\mathbb{Z})$ of weight $2n + 2$ with $r_f^+(x, y) = \sum c_a x^{2a} y^{2n-2a}$.

Proof. The formulas (27.3) and the definition of $\check{\epsilon}_{2n+2}$ imply that

$$\check{\epsilon}_{2m+2}(A) = -A^{2m+2} \cdot T = A^{2m+1} \cdot R_1 \quad \text{and} \quad \check{\epsilon}_{2m+2}(T) \in D^2\mathbb{L}(A, T).$$

Consequently

$$\begin{aligned} \check{\epsilon}_{2m+2}(R_0) &= \check{\epsilon}_{2m+2}(A - [T, A]/2 + \frac{1}{12}T \cdot [T, A] + \dots) \\ &\equiv A^{2m+1} \cdot R_1 \pmod{D^2\mathfrak{p}(E')} \\ &\equiv R_0^{2m+1} \cdot R_1 \pmod{D^2\mathfrak{p}(E')} \end{aligned}$$

Since $\mathfrak{p}(U) \rightarrow \mathfrak{p}(E')$ is strict with respect to D^\bullet (Proposition 28.2), Proposition 29.4 implies that

$$\phi_{\mathbb{P}}(\sigma_{2m+1})(X_0) \equiv X_0^{2m+1} \cdot X_1 \pmod{D^2\mathfrak{p}(E')}.$$

Since $\phi_E(\sigma_{2m+1})(T) \in D^2\mathbb{L}(A, T)^\wedge$, we have

$$\begin{aligned} \sum_{a+b=n} c_a[\phi_{\mathbb{P}}(\sigma_{2a+1}), \phi_{\mathbb{P}}(\sigma_{2b+1})] &\equiv 0 \pmod{D^3 \text{Der } \mathfrak{p}(U)} \\ \iff \sum_{a+b=n} c_a[\phi_{\mathbb{P}}(\sigma_{2a+1}), \phi_{\mathbb{P}}(\sigma_{2b+1})](X_0) &\in D^3\mathfrak{p}(U) \\ \iff \sum_{a+b=n} c_a[\phi_E(\sigma_{2a+1}), \phi_E(\sigma_{2b+1})](R_0) &\in D^3\mathfrak{p}(E') \\ \iff \sum_{a+b=n} c_a[\phi_E(\sigma_{2a+1}), \phi_E(\sigma_{2b+1})](A) &\in D^3\mathfrak{p}(E') \\ \iff \sum_{a+b=n} c_a[\phi_E(\sigma_{2a+1}), \phi_E(\sigma_{2b+1})] &\equiv 0 \pmod{D^3 \text{Der}^0 \mathbb{L}(A, T)}. \end{aligned}$$

The result now follows from the elliptic case, Theorem 29.5. □

Acknowledgments. The authors gratefully acknowledge the support of MSRI in 2008 where some of the initial work was done. The first author also gratefully acknowledges the generous support of the Friends of the Institute for Advanced Study and Duke University which allowed him to spend his sabbatical year 2014–2015 at the IAS where part of this paper was written. The first author would like to thank F. Brown for numerous stimulating discussion during which the material in Parts 3 and 4 evolved. The authors would also thank T. Terasoma for his correspondence related to the computation of the cup product in Deligne cohomology of Kuga–Sato varieties. They would also like to thank F. Brown and the referee for their detailed and constructive comments on the manuscript, and A. Pollack for his contributions to this project during the early stages of this work.

Appendix A. Relative weight filtrations

Universal mixed elliptic motives are mixed Tate motives with additional structure. This extra structure includes a second weight filtration W_\bullet and a nilpotent endomorphism N of V that preserves W_\bullet . Every universal mixed elliptic motive V has two weight filtrations: its weight filtration M_\bullet as an object of MTM and the second weight filtration W_\bullet . One of the axioms of a universal mixed elliptic motive is that M_\bullet be the relative weight filtration of the nilpotent endomorphism N of the filtered vector space (V, W_\bullet) . In this section we review Deligne’s definition [14] of the relative weight filtration of a nilpotent endomorphism of a filtered vector space. More information about relative weight filtrations can be found in [56]. A concise exposition is given in [24, § 7].

A.1. The weight filtration of a nilpotent endomorphism

There is a natural weight filtration of a vector space associated to a nilpotent endomorphism N of it.

Proposition A.1. *If N is a nilpotent endomorphism of a finite dimensional vector space V over a field of characteristic zero, then there is a unique filtration*

$$0 = W(N)_{-m-1} \subseteq W(N)_{-m} \subseteq W(N)_{-m+1} \subseteq \cdots \subseteq W(N)_{m-1} \subseteq W(N)_m = V$$

of V such that

- (i) for all $n \in \mathbb{Z}$, $NW(N)_n \subseteq W(N)_{n-2}$;
- (ii) for each $k \in \mathbb{Z}$, $N^k : \text{Gr}_k^{W(N)} V \rightarrow \text{Gr}_{-k}^{W(N)} V$ is an isomorphism.

The filtration $W(N)_\bullet$ of V is called the *weight filtration* of V .

Note that $W(N)_\bullet$ is centered at 0. When V is a motive of weight m , it is natural to reindex the weight filtration of a nilpotent endomorphism N of V so that it is centered at m . The shifted filtration

$$M_k V := W(N)_{k-m}$$

is centered at m . The reindexed filtration M_\bullet satisfies $NM_k \subseteq M_{k-2}$ and

$$N^k : \text{Gr}_{m+k}^M V \xrightarrow{\cong} \text{Gr}_{m-k}^M V$$

is an isomorphism for all $k \in \mathbb{Z}$. We will call the shifted weight filtration M_\bullet the monodromy weight filtration of $N : V \rightarrow V$.

Example A.2. Let $H = \mathbb{C}\mathbf{w} \oplus \mathbb{C}\mathbf{a}$, regarded as a vector space of weight 1. Let N be the nilpotent endomorphism $\mathbf{a} \frac{\partial}{\partial \mathbf{w}}$ of H . It induces a nilpotent endomorphism of $V = S^n H$, the space of homogeneous polynomials in \mathbf{a} and \mathbf{w} of degree n , which we regard as a vector space of weight n . The shifted monodromy weight filtration M_\bullet of V is obtained by giving \mathbf{a} weight 0 and \mathbf{w} weight 2. The monomial $\mathbf{a}^{n-j} \mathbf{w}^j$ has weight $2j$. Then

$$M_k V = \text{span of the monomials } \mathbf{a}^{n-j} \mathbf{w}^j \text{ of weight } \leq k.$$

A.2. The weight filtration of a nilpotent endomorphism of a filtered vector space

Now suppose that N is a nilpotent endomorphism of a *filtered* finite dimensional vector space V over a field of characteristic zero. That is, V has a filtration

$$0 \subseteq \dots \subseteq W_{m-1} V \subseteq W_m V \subseteq W_{m+1} V \subseteq \dots \subseteq V$$

which is stable under N .

Since N preserves the weight filtration, it induces a nilpotent endomorphism

$$N_m := \text{Gr}_m^W N : \text{Gr}_m^W V \rightarrow \text{Gr}_m^W V.$$

of the m th weight graded quotient of V . Proposition A.1 implies that each graded quotient has a weight filtration $W(N_m)$. The reindexed filtration $W(N_m)[m]_\bullet$ is centered at m . Denote it by $M_\bullet^{(m)}$.

Definition A.3. A filtration M_\bullet of V is called a *relative weight filtration* of $N : (V, W_\bullet) \rightarrow (V, W_\bullet)$ if

- (i) for each $k \in \mathbb{Z}$, $N M_k \subseteq M_{k-2}$;
- (ii) the filtration induced by M_\bullet on $\text{Gr}_m^W V$ is the reindexed weight filtration $M_\bullet^{(m)}$.

Relative weight filtrations, if they exist, are unique. (Cf. [56]).

Example A.4. If $N : (V, W_\bullet) \rightarrow (V, W_\bullet)$ satisfies $N(W_m V) \subseteq W_{m-2} V$ for all $m \in \mathbb{Z}$, then each $N_m = 0$ and the relative weight filtration M_\bullet of N exists and equals the original weight filtration W_\bullet .

Even though the weight filtration of a nilpotent endomorphism of a finite dimensional vector space always exists, the relative weight filtration of a nilpotent endomorphism of a *filtered* vector space (V, W_\bullet) does not. Necessary and sufficient conditions for the existence of a relative weight filtration are given in [56]. They imply that the generic nilpotent endomorphism of (V, W_\bullet) does not have a relative weight filtration.

Example A.5. Let E be a compact Riemann surface of genus 1 and P, Q two distinct points of E . Let $V = H_1(E, \{P, Q\}; \mathbb{Q})$. Then one has the exact sequence

$$0 \rightarrow H_1(E; \mathbb{Q}) \rightarrow V \rightarrow \tilde{H}_0(\{P, Q\}) \rightarrow 0.$$

Define a filtration W_\bullet on V by $W_{-2}V = 0, W_{-1}V = H_1(E)$, and $W_0V = V$. Choose any path γ from P to Q . It determines a class $[\gamma]$ in V . Let u be any non-trivial element of $H_1(E; \mathbb{Q})$. Define a nilpotent automorphism N of (V, W_\bullet) by insisting that N be trivial on $\text{Gr}_\bullet^W V$ and that $N[\gamma] = u$. Since N is trivial on $\text{Gr}_\bullet^W V$, the relative weight filtration M_\bullet , should it exist, would equal W_\bullet . But W_\bullet is not a relative weight filtration because NW_0V is not contained in $W_{-2}V$.

Appendix B. Splitting the weight filtrations M_\bullet and W_\bullet .

In this section, we show that both weight filtrations of an object of MEM_* can be simultaneously split and that such splittings are compatible with tensor products and duals, and are preserved by morphisms in MEM_* . Moreover, such a splitting of M_\bullet can be chosen to agree with any given natural splitting of the weight filtration in MTM . The existence of such splittings implies that $\text{Gr}_\bullet^M, \text{Gr}_\bullet^W$ and $\text{Gr}_\bullet^W \text{Gr}_\bullet^M$ are exact functors on MEM_* .

Fix a fiber functor $\omega : \text{MTM} \rightarrow \text{Vec}_F$, where F is a field of characteristic zero. This induces the fiber functor

$$\text{MEM}_* \xrightarrow{\tilde{v}_\omega} \text{MTM} \xrightarrow{\omega} \text{Vec}_F$$

which we also denote by ω . We will regard objects of MEM_* as F -vector spaces with an action of $\pi_1(\text{MEM}_*, \omega)$. Similarly for mixed Tate motives.

Suppose that we have chosen a natural splitting

$$V \cong \bigoplus_m \text{Gr}_\bullet^M V$$

of the weight filtration M_\bullet of each object V of MTM . That is, we have chosen a splitting of the canonical homomorphism $\pi_1(\text{MTM}, \omega) \rightarrow \mathbb{G}_m$. The weight filtration M_\bullet of a mixed Tate motive V then splits under the \mathbb{G}_m -action.

Proposition B.1. *For each $* \in \{1, \bar{1}, 2\}$, the F -vector space V underlying an object of MEM_* has a bigraded splitting*

$$V = \bigoplus V_{m,n}$$

in the category of \mathbb{Q} -vector spaces with the property that

$$M_m V = \bigoplus_{\substack{r \leq m \\ n \in \mathbb{Z}}} V_{r,n} \quad \text{and} \quad W_n V = \bigoplus_{\substack{r \leq n \\ m \in \mathbb{Z}}} V_{m,r}.$$

This bigrading is natural in the sense that it is preserved by morphisms $\phi : V \rightarrow V'$ of MEM_* :

$$\phi(V_{m,n}) \subseteq V'_{m,n}$$

and also by the functors $\text{MEM}_1 \rightarrow \text{MEM}_2 \rightarrow \text{MEM}_1$. It is compatible with tensor products and duals. Finally, this splitting of (V, M_\bullet) can be chosen so that it agrees with the splitting of M_\bullet when (V, M_\bullet) is viewed as an object of MTM via the action

$$\pi_1(\text{MTM}, \omega) \xrightarrow{s_{\check{V}}} \pi_1(\text{MEM}_*, \omega) \longrightarrow \text{Aut } V.$$

This result will follow from the following lemma. Set

$$\text{diag}(t_1, t_2) := \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}.$$

Define a homomorphism $c : \mathbb{G}_m \rightarrow \text{GL}_2$ by $c(t) = \text{diag}(1, t)$. This is a section of $\det : \text{GL}_2 \rightarrow \mathbb{G}_m$.

Lemma B.2. *Suppose that F is a field of characteristic 0 and that*

$$\begin{array}{ccccccc} 1 & \longrightarrow & U & \longrightarrow & G & \xrightarrow{\rho_G} & \text{GL}_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & \curvearrowright \sigma & \downarrow \det & & \\ 1 & \longrightarrow & K & \longrightarrow & A & \longrightarrow & \mathbb{G}_m & \longrightarrow & 1 \end{array}$$

is a commutative diagram of affine F -groups with exact rows and where K and U are pronipotent and σ is a section. If $s_A : \mathbb{G}_m \rightarrow A$ is a section of $A \rightarrow \mathbb{G}_m$ such that $\rho_G \circ \sigma \circ s_A = c$, then there exists a section $s_G : \text{GL}_2 \rightarrow G$ that extends $\sigma \circ s_A$ in the sense that the diagram

$$\begin{array}{ccc} & & \mathbb{G}_m \\ & \swarrow^{s_G \circ c} & \downarrow c \\ G & \xleftarrow{s_G} & \text{GL}_2 \\ \uparrow \sigma & & \downarrow \det \\ A & \xleftarrow{s_A} & \mathbb{G}_m \\ & \searrow^{\text{id}} & \end{array}$$

commutes. Moreover, any two such sections s_G are conjugate by an element of U that centralizes σ .

Proof. Denote $\rho_G^{-1} \text{im } c$ by G_c . This is an extension of \mathbb{G}_m by U . Choose a splitting $s : \text{GL}_2 \rightarrow G$ of ρ_G . Then s induces a splitting $s' : \mathbb{G}_m \rightarrow G_c$. Since $\rho_G \circ \sigma \circ s_A = c$, the image of σ lies in G_c . Levi's theorem implies that there is an element u of U such that $\sigma = us'u^{-1}$. Now define $s_G = usu^{-1}$. Levi's theorem applied to G implies that any two such sections s_B are conjugate by an element of U that centralizes with the image of σ . \square

Proof of Proposition B.1. Recall that we have fixed an ordered basis \mathbf{b}, \mathbf{a} of \check{H} . This gives an identification of GL_2 with $\text{GL}(\check{H})$. Denote the diagonal torus by T :

$$\text{diag}(t_1, t_2) : \mathbf{b} \mapsto t_1 \mathbf{b} \quad \text{and} \quad \text{diag}(t_1, t_2) : \mathbf{a} \mapsto t_2 \mathbf{a}.$$

Now take $A = \pi_1(\text{MTM}, \omega)$, $G = \pi_1(\text{MEM}_*, \omega)$, s_A to be any splitting (such as the \mathbb{Q} de Rham splitting), and $\sigma = s_{\check{v}}$. The summand $V_{2m,n}$ of an object of MEM_* is the subspace of $V_{\mathbb{Q}}$ on which $(t_1, t_2) \in T$ acts by $t_1^{m-n} t_2^{-m}$. \square

Such bigraded splittings of M_{\bullet} and W_{\bullet} correspond to homomorphisms $\mathbb{G}_m \times \mathbb{G}_m \rightarrow \pi_1(\text{MEM}_*)$ whose composition with the natural surjection $\pi_1(\text{MEM}_*) \rightarrow \text{GL}(H)$ is the inclusion of the maximal torus associated to the basis \mathbf{a}, \mathbf{b} of H . Since $V_{m,n} \cong \text{Gr}_m^M \text{Gr}_n^W V$, we have:

Corollary B.3. *Each mixed elliptic motive V is naturally isomorphic to its associated bigraded*

$$V \cong \bigoplus_{m,n} \text{Gr}_m^M \text{Gr}_n^W V$$

in the category of \mathbb{Q} -vector spaces. Consequently, the functors Gr_{\bullet}^M , Gr_{\bullet}^W and $\text{Gr}_{\bullet}^M \text{Gr}_{\bullet}^W$ are exact functors from MEM_* to the category of (bi)graded \mathbb{Q} -vector spaces and are compatible with \otimes and Hom .

The Lie algebra $\mathfrak{g}_*^{\text{MEM}}$ of $\pi_1(\text{MEM}_*)$ is a pro-object of MEM_* . Its weight filtration W_{\bullet} satisfies

$$\mathfrak{g}_*^{\text{MEM}} = W_0 \mathfrak{g}_*^{\text{MEM}}, \quad W_{-1} \mathfrak{g}_*^{\text{MEM}} \text{ is prounipotent, and } \text{Gr}_0^W \mathfrak{g}_*^{\text{MEM}} \cong \mathfrak{gl}(H).$$

It follows that if V is an object of MEM_* , then each $\text{Gr}_{\bullet}^W V$ is a $\mathfrak{gl}(H)$ -module. The spaces can be split according to their $\mathfrak{sl}(H)$ -weight. To make this precise, we will view $\mathfrak{sl}(H)$ as a subalgebra of $\text{End}(H)$. We fix a Cartan by specifying that \mathbf{a} has $\mathfrak{sl}(H)$ -weight 1 and \mathbf{b} has $\mathfrak{sl}(H)$ -weight -1 .

Lemma B.4. *If V is an object of MEM_* , then, under the identification of V with $\bigoplus \text{Gr}_{\bullet}^M \text{Gr}_{\bullet}^W V$, the $\mathfrak{sl}(H)$ -weight of $V_{m,n}$ is $m - n$.*

In other words, the three notions of weight in $\text{Gr}_{\bullet}^M \text{Gr}_{\bullet}^W V$ are related by

$$M\text{-weight} = \mathfrak{sl}(H)\text{-weight} + W\text{-weight}.$$

Appendix C. Index of principal notation

E	an elliptic curve, often the fiber of the Tate curve over $\check{\mathfrak{t}}$	
E'	$E - \{0\}$	
e_o	the unique cusp $q = 0$ of $\overline{\mathcal{M}}_{1,1}$	p. 675
$\check{\mathfrak{t}}$	the integral tangent vector $\partial/\partial q$ of $\overline{\mathcal{M}}_{1,1}$ at the cusp e_o	p. 676
\vec{w}_o	the integral tangent vector $\partial/\partial w$ of 1 in \mathbb{P}^1	p. 676
\vec{v}_o	the generic notation for the integral tangent vector of $\overline{\mathcal{M}}_{1,*}$	p. 676
\mathbb{H}	the local system $R^1 f_* \mathbb{Q}$ associated to $f : \mathcal{E} \rightarrow \mathcal{M}_{1,1}^{\text{an}}$	p. 680
\mathcal{H}	the canonical extension of $\mathbb{H} \otimes \mathcal{O}_{\mathcal{M}_{1,1}^{\text{an}}}$ to $\overline{\mathcal{M}}_{1,1}^{\text{an}}$	p. 681
∇_0	the canonical connection on $S^m \mathcal{H}$	p. 681

H	the fiber of \mathcal{H} over \vec{v}_o , an object of MTM	p. 683
MTM	the category of mixed Tate motives over \mathbb{Z}	p. 674
MEM_*	the category of universal mixed elliptic motives of type $*$	p. 687
MEM_*^{SS}	the category of semi-simple universal mixed elliptic motives	p. 691
MHS ($\mathcal{M}_{1,*}$, \mathbb{H})	the category of admissible VMHS over $\mathcal{M}_{1,*}$ generated by \mathbb{H}	p. 693
\mathcal{K}	the prounipotent radical of $\pi_1(\mathbf{MTM})$	p. 675
\mathfrak{k}	the Lie algebra of \mathcal{K}	p. 675
$\pi_1^{\text{geom}}(\mathbf{MEM}_*)$	the kernel of $\pi_1(\mathbf{MEM}_*) \rightarrow \pi_1(\mathbf{MTM})$	p. 690
$\mathcal{G}_*^{\text{rel}}$	the relative completion of $\pi_1(\mathcal{M}_{1,*}^{\text{an}}, \vec{v}_o)$	p. 700
$\mathcal{U}_*^{\text{rel}}$	its prounipotent radical	p. 700
$\mathcal{U}_*^{\text{MEM}}$	prounipotent radical of $\pi_1(\mathbf{MEM}_*)$	p. 703
$\mathcal{U}_*^{\text{geom}}$	the prounipotent radical of $\pi_1^{\text{geom}}(\mathbf{MEM}_*)$	p. 703
$\mathfrak{u}_*^{\text{geom}}$	the Lie algebra of $\mathcal{U}_*^{\text{geom}}$	p. 703
$\widehat{\mathcal{G}}_*$	$\pi_1(\mathbf{MHS}) \ltimes \mathcal{G}_*^{\text{rel}}$	p. 704
$\mathcal{G}_*^{\text{cris}, \ell}$	the ℓ -adic crystalline completion of $\pi_1(\mathcal{M}_{1,*}/\mathbb{Z}[1/\ell], \vec{v}_o)$	p. 709
$\mathcal{G}_*^{\text{MEM}}$	notation for $\pi_1(\mathbf{MEM}_*)$	p. 729
ψ_{2n}	differential form associated to the Eisenstein series G_{2n}	p. 695
V_f	real two-dimensional Hodge structure associated to eigenform f	p. 696
M_f	the simple \mathbb{Q} -Hodge structure associated to the eigenform f	p. 696
\mathcal{F}_∞	involution induced by complex conjugation on $\mathcal{M}_{1,*}^{\text{an}}$ and \mathbb{H}	p. 706
$\overline{\mathcal{F}}_\infty$	\mathcal{F}_∞ composed with complex conjugation on $\mathbb{H}_{\mathbb{C}}$	p. 706
V^\pm	the eigenspaces of the involution $\overline{\mathcal{F}}_\infty : V \rightarrow V$	p. 706
r_f	modular symbol of the cusp form f	p. 739
r_f^\pm	the even and odd degree parts of r_f	p. 739
a, b	\mathbb{Q} -Betti basis of H	p. 677
a, w	\mathbb{Q} -de Rham basis of H	p. 681
\check{H}	$H(1)$	p. 726
A, T	\mathbb{Q} -de Rham basis of \check{H}	p. 726
$\mathbb{L}(V)$	free Lie algebra generated by a vector space V	p. 674
\mathfrak{e}_0	the weight lowering nilpotent in $\mathfrak{sl}(H)$	p. 727
\mathfrak{e}_{2n}	the generator of $\mathfrak{u}_1^{\text{MEM}}$ dual to G_{2n} when $n > 0$	p. 728
\mathfrak{z}_{2m-1}	a generator of \mathfrak{k} and also certain of its lifts to $\mathfrak{u}_*^{\text{MEM}}$	p. 728
\mathfrak{f}_*	the free Lie algebra generated by $H_1(\mathfrak{u}_*^{\text{MEM}})$	p. 729
$\mathfrak{f}_*^{\text{geom}}$	the free Lie algebra generated by $H_1(\mathfrak{u}_*^{\text{geom}})$	p. 729
$L^r \mathfrak{g}$	the r th terms of the lower central series of the Lie algebra \mathfrak{g}	p. 737
$\text{Der}^0 \mathbb{L}(A, T)$	the derivations of $\mathbb{L}(A, T)$ that annihilate $[T, A]$	p. 733
$\epsilon_{2n}(\mathbf{v}_1, \mathbf{v}_2)$	a derivation of $\mathbb{L}(H)$ depending on $\mathbf{v}_1, \mathbf{v}_2 \in H$	p. 734
ϵ_{2n}	$\epsilon_{2n}(T, A)$	p. 734
$\check{\epsilon}_{2n}$	$\epsilon_{2n}(A, T)$	p. 753

$\mathfrak{p}(E')$	the Lie algebra of $\pi_1^{\text{un}}(E', \vec{\mathbf{w}}_o)$	p. 748
$\mathfrak{p}(U)$	the Lie algebra of $\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \vec{\mathbf{w}}_o)$	p. 748
$\text{Der}^N \mathfrak{p}(E')$	derivations that commute with N and kill cuspidal loop	p. 751
$\text{Der}^E \mathfrak{p}(U)$	the derivations of $\mathfrak{p}(U)$ that extend to $\mathfrak{p}(E')$, etc.	p. 751

References

1. D. ARAPURA, An abelian category of motivic sheaves, *Adv. Math.* **233** (2013), 135–195.
2. J. AYOUB, *A Guide to (Étale) Motivic Sheaves*, Proceedings of the International Congress of Mathematicians, Seoul 2014, Volume II, pp. 1101–1124 (Kyung Moon Sa, Seoul).
3. S. BAUMARD AND L. SCHNEPS, On the derivation representation of the fundamental Lie algebra of mixed elliptic motives, *Ann. Math. Qué.* **41** (2017), 4362.
4. A. BEILINSON, *Higher Regulators and Values of L-Functions*, Current Problems in Mathematics, Volume 24, pp. 181–238 (Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984).
5. A. BEILINSON, Higher regulators of modular curves, in *Applications of Algebraic K-Theory to Algebraic Geometry and Number theory, Parts I, II (Boulder, CO, 1983)*, Contemporary Mathematics, Volume 55, pp. 1–34 (American Mathematical Society, Providence, RI, 1986).
6. A. BEILINSON, Notes on absolute Hodge cohomology, in *Applications of Algebraic K-Theory to Algebraic Geometry and Number Theory, Parts I, II (Boulder, CO, 1983)*, Contemporary Mathematics, Volume 55, pp. 35–68 (American Mathematical Society, Providence, RI, 1986).
7. A. BEILINSON AND A. LEVIN, The elliptic polylogarithm, in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math., Volume 55, Part 2, pp. 123–190 (American Mathematical Society, Providence, RI, 1994).
8. A. BOREL, Cohomologie de SL_n et valeurs de fonctions zeta aux points entiers, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* **4** (1977), 613–636.
9. F. BROWN, Multiple zeta values and periods of moduli spaces $\overline{\mathcal{M}}_{0,n}$, *Ann. Sci. Éc. Norm. Supér. (4)* **42** (2009), 371–489.
10. F. BROWN, Mixed Tate motives over \mathbb{Z} , *Ann. of Math. (2)* **175** (2012), 949–976.
11. F. BROWN, Multiple modular values for $\text{SL}_2(\mathbb{Z})$, Preprint, 2014, [arXiv:1407.5167](https://arxiv.org/abs/1407.5167).
12. F. BROWN, Zeta elements in depth 3 and the fundamental Lie algebra of the infinitesimal Tate curve, *Forum Math. Sigma* **5** (2017), e1, 56 pp.
13. D. CALAQUE, B. ENRIQUEZ AND P. ETINGOF, Universal KZB equations I: the elliptic case, in *Algebra, Arithmetic, and Geometry: in Honor of Yu. I. Manin, Volume I*, Progr. Math., Volume 269, pp. 165–266 (Birkhäuser, Boston, 2009).
14. P. DELIGNE, La conjecture de Weil, II, *Publ. Math. Inst. Hautes Études Sci.* **52** (1980), 137–252.
15. P. DELIGNE, Le groupe fondamental de la droite projective moins trois points, in *Galois Groups Over \mathbb{Q} (Berkeley, CA, 1987)*, Math. Sci. Res. Inst. Publ., Volume 16, pp. 79–297 (Springer, 1989).
16. P. DELIGNE, Le groupe fondamental unipotent motivique de $\mathbb{G}_m - \mu_N$, pour $N = 2, 3, 4, 6$ ou 8, *Publ. Math. Inst. Hautes Études Sci.* **112** (2010), 101–141.

17. P. DELIGNE AND A. GONCHAROV, Groupes fondamentaux motiviques de Tate mixte, *Ann. Sci. Éc. Norm. Supér. (4)* **38** (2005), 1–56.
18. V. DRINFELD, On quasitriangular quasi-Hopf algebras and on a group that is closely connected with $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, *Algebra i Analiz* **2** (1990), 149–181; translation in *Leningrad Math. J.* **2** (1991), 829–860.
19. B. ENRIQUEZ, Elliptic associators, *Selecta Math. (N.S.)* **20** (2014), 491–584.
20. J.-M. FONTAINE AND B. PERRIN-RIOU, Autour des conjectures de Bloch et Kato: cohomologie galoisienne et valeurs de fonctions L , in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math., Volume 55, Part 1, pp. 599–706 (American Mathematical Society, 1994).
21. A. GONCHAROV, Mixed elliptic motives, in *Galois Representations in Arithmetic Algebraic Geometry (Durham, 1996)*, London Mathematical Society Lecture Note Series, Volume 254, pp. 147–221 (Cambridge University Press, 1998).
22. A. GONCHAROV, The dihedral Lie algebras and Galois symmetries of $\pi_1^{(\ell)}(\mathbb{P}^1 - (\{0, \infty\} \cup \mu_N))$, *Duke Math. J.* **110** (2001), 397–487.
23. R. HAIN, Hodge–de Rham theory of relative Malcev completion, *Ann. Sci. Éc. Norm. Supér.* **31** (1998), 47–92.
24. R. HAIN, Relative weight filtrations on completions of mapping class groups, in *Groups of Diffeomorphisms*, Adv. Stud. Pure Mathematics, Volume 52, pp. 309–368 (Math. Soc., Japan, Tokyo, 2008).
25. R. HAIN, Letter to P. Deligne, December, 2009.
26. R. HAIN, Lectures on moduli spaces of elliptic curves, in *Transformation Groups and Moduli Spaces of Curves*, Adv. Lect. Math. (ALM), Volume 16, pp. 95–166 (International Press, 2011).
27. R. HAIN, *Notes on the Universal Elliptic KZB Equation*, Pure and Applied Mathematics Quarterly, Volume 12, no. 2, (International Press, Somerville, MA, 2016).
28. R. HAIN, The Hodge–de Rham theory of modular groups, in *Recent Advances in Hodge Theory Period Domains, Algebraic Cycles, and Arithmetic* (ed. M. KERR AND G. PEARLSTEIN), LMS Lecture Notes Series, Volume 427, pp. 422–514 (Cambridge University Press, Cambridge, 2016).
29. R. HAIN, Deligne–Beilinson cohomology of affine groups, in *Hodge Theory and L^2 Methods* (ed. L. JI AND S. ZUCKER), pp. 377–418 (International Press, Somerville, MA).
30. R. HAIN, Unipotent path torsors of Ihara curves, in preparation.
31. R. HAIN AND M. MATSUMOTO, Weighted completion of Galois groups and Galois actions on the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$, *Compositio Math.* **139** (2003), 119–167.
32. R. HAIN AND M. MATSUMOTO, Tannakian fundamental groups associated to Galois groups, in *Galois Groups and Fundamental Groups*, Math. Sci. Res. Inst. Publ., Volume 41, pp. 183–216 (Cambridge University Press, 2003).
33. R. HAIN AND M. MATSUMOTO, Relative pro- l completions of mapping class groups, *J. Algebra* **321** (2009), 3335–3374.
34. R. HAIN AND S. ZUCKER, Unipotent variations of mixed Hodge structure, *Invent. Math.* **88** (1987), 83–124.
35. M. HANAMURA, Mixed motives and algebraic cycles, I, *Math. Res. Lett.* **2** (1995), 811–821.
36. Y. IHARA, Some arithmetic aspects of Galois actions in the pro- p fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$, in *Arithmetic Fundamental Groups and Noncommutative Algebra (Berkeley, CA, 1999)*, Proceedings of Symposia in Pure Mathematics, Volume 70, pp. 247–273 (American Mathematical Society, 2002).

37. J. C. JANTZEN, *Representations of Algebraic Groups*, Pure and Applied Mathematics, Volume 131 (Academic Press, 1987).
38. N. KATZ AND B. MAZUR, *Arithmetic Moduli of Elliptic Curves*, Annals of Mathematics Studies, Volume 108 (Princeton University Press, 1985).
39. F. KNUDSEN, The projectivity of the moduli space of stable curves, III. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic 0, *Math. Scand.* **52** (1983), 200–212.
40. S. LANG, *Introduction to Modular Forms, with appendixes by D. Zagier and Walter Feit*, Grundlehren der Mathematischen Wissenschaften, Volume 222 (Springer, 1995). Corrected reprint of the 1976 original.
41. M. LEVINE, Tate motives and the vanishing conjectures for algebraic K -theory, in *Algebraic K-Theory and Algebraic Topology (Lake Louise, AB, 1991)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Volume 407, pp. 167–188 (Kluwer, 1993).
42. M. LEVINE, *Mixed Motives*, Mathematical Surveys and Monographs, Volume 57 (American Mathematical Society, 1998).
43. A. LEVIN AND G. RACINET, Towards multiple elliptic polylogarithms, Preprint, 2007, [arXiv:math/0703237](https://arxiv.org/abs/math/0703237).
44. M. LUO, The elliptic KZB connection and algebraic de Rham theory for unipotent fundamental groups of elliptic curves, Preprint, 2017, [arXiv:1710.07691](https://arxiv.org/abs/1710.07691).
45. H. NAKAMURA, Tangential base points and Eisenstein power series, in *Aspects of Galois Theory (Gainesville, FL, 1996)*, London Mathematical Society Lecture Note Series, Volume 256, pp. 202–217 (Cambridge University Press, 1999).
46. P. MAY, Matric Massey products, *J. Algebra* **12** (1969), 533–568.
47. B. NOOHI, Fundamental groups of algebraic stacks, *J. Inst. Math. Jussieu* **3** (2004), 69–103.
48. M. OLSSON, Towards non-abelian p -adic Hodge theory in the good reduction case, *Mem. Amer. Math. Soc.* **210**(990) (2011), vi+157 pp.
49. A. POLLACK, Relations between derivations arising from modular forms, undergraduate thesis, Duke University, 2009. Available at: <http://dukespace.lib.duke.edu/dspace/handle/10161/1281>.
50. W. SCHMID, Variation of Hodge structure: the singularities of the period mapping, *Invent. Math.* **22** (1973), 211–319.
51. L. SCHNEPS, On the Poisson bracket on the free Lie algebra in two generators, *J. Lie Theory* **16** (2006), 19–37.
52. A. SCHOLL, Motives for modular forms, *Invent. Math.* **100** (1990), 419–430.
53. J. SILVERMAN, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, Volume 106 (Springer, 1986).
54. J. SILVERMAN, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics, Volume 151 (Springer, New York, 1994).
55. C. SOULÉ, On higher p -adic regulators, in *Algebraic K-Theory, Evanston 1980 (Proc. Conf., Northwestern University, Evanston, IL, 1980)*, Lecture Notes in Mathematics, Volume 854, pp. 372–401 (Springer, 1981).
56. J. STEENBRINK AND S. ZUCKER, Variation of mixed Hodge structure, I, *Invent. Math.* **80** (1985), 489–542.
57. N. TAKAO, Braid monodromies on proper curves and pro- ℓ Galois representations, *J. Inst. Math. Jussieu* **11** (2012), 161–181.

58. T. TERASOMA, Relative Deligne cohomologies and higher regulators for Kuga–Sato fiber spaces, Preprint, January 2011.
59. H. TSUNOGAI, On some derivations of Lie algebras related to Galois representations, *Publ. Res. Inst. Math. Sci.* **31** (1995), 113–134.
60. V. VOEVODSKY, A. SUSLIN AND E. FRIEDLANDER, *Cycles, Transfers, and Motivic Homology Theories*, Annals of Mathematics Studies, Volume 143 (Princeton University Press, 2000).
61. W. WASOW, *Asymptotic Expansions for Ordinary Differential Equations*, Pure and Applied Mathematics, Volume XIV (Interscience Publishers, 1965).
62. S. ZUCKER, Hodge theory with degenerating coefficients, L_2 cohomology in the Poincaré metric, *Ann. of Math. (2)* **109** (1979), 415–476.