Norm resolvent convergence of singularly scaled Schrödinger operators and δ'-potentials

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For a real-valued function V of the Faddeev–Marchenko class, we prove the norm-resolvent convergence, as $\varepsilon \to 0$, of a family S_ε of one-dimensional Schrödinger operators on the line of the form

$$S_\varepsilon := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{\varepsilon^2}V\left(\frac{x}{\varepsilon}\right).$$

Under certain conditions, the functions $\varepsilon^{-2}V(x/\varepsilon)$ converge in the sense of distributions as $\varepsilon \to 0$ to $\delta'(x)$, and then the limit S_0 of S_ε may be considered as a 'physically motivated' interpretation of the one-dimensional Schrödinger operator with potential δ'.

1. Introduction

The aim of this paper is to study convergence as $\varepsilon \to 0$ of the family S_ε of Schrödinger operators on the line given by

$$S_\varepsilon := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{\varepsilon^2}V\left(\frac{x}{\varepsilon}\right) \tag{1.1}$$

for the largest possible class of potentials V, namely, for real-valued V in the Faddeev–Marchenko class $L_1(\mathbb{R}; (1 + |x|)\,\mathrm{d}x)$. The asymptotic behaviour of S_ε and their many-dimensional analogues has been discussed, in both the mathematical and physical literature, in connection with the small-energy scattering [3, 28] and singular perturbations [1, 45, 46], since the 1980s. Recently, the Schrödinger operator family S_ε has been enjoying renewed interest motivated by the problem of the approximation of thin quantum waveguides by quantum graphs [6, 13–15, 19, 22].

In three dimensions, the corresponding family of Hamiltonians

$$H_\varepsilon := -\Delta + \frac{\lambda(\varepsilon)}{\varepsilon^2}V\left(\frac{x}{\varepsilon}\right)$$

was first studied by Albeverio and Høegh-Krohn [1]. Here $\lambda(\varepsilon)$ is a smooth function with $\lambda(0) = 1$ and $\lambda'(0) \neq 0$ and V is short-range and of the Rollnik class [43, ch. X.2]. It was shown that the family H_ε converges as $\varepsilon \to 0$ in the strong resolvent sense to H_0, which is either the free Hamiltonian $-\Delta$ or its perturbation by a delta function (see the pioneering work by Berezin and Faddeev [8]), depending on whether or not there is a zero-energy resonance for $H = -\Delta + V$. We recall that H is said to possess a zero-energy resonance if the equation $H\psi = 0$ has a distributional solution that is bounded but does not belong to $L_2(\mathbb{R}^3)$. Analogous results were established in [1] for V containing finitely or infinitely many summands scaled about different centres.

In [3], Albeverio *et al.* discussed the low-energy scattering in two particle non-relativistic quantum mechanics. They used the results of [1] and the connection between the low-energy behaviour of the scattering amplitude and scattering matrix for the corresponding Hamiltonian $H = -\Delta + V$ in $L_2(\mathbb{R}^3)$ and for the scaled Hamiltonians $(H_\varepsilon = -\Delta + \varepsilon^{-2}V(x/\varepsilon))$ as $\varepsilon \to 0$ to study, in detail, possible resonant and non-resonant cases. A similar problem for Hamiltonians including the Coulomb-type interaction was treated in [4].

Interestingly enough, the low-energy scattering theory for Schrödinger operators in dimensions 1 and 2 is more complicated than in dimension 3. This is connected to, respectively, the square root and logarithmic singularities the Green function of the free Hamiltonian then possesses. The low-energy scattering for the one-dimensional Schrödinger operator S_1 and its connection to the behaviour of S_ε as $\varepsilon \to 0$ was thoroughly investigated by Bollé *et al.*, for both the non-resonant [9] and resonant [10] cases. In dimension 2, the low-energy asymptotics was discussed in [11]. Continuity of the scattering matrix at zero energy for one-dimensional Schrödinger operators with Faddeev–Marchenko potentials in the resonant case was independently established by Guseinov in [25] and by Klaus in [32].

Another reason to study the family S_ε comes from quantum graph theory. One of the fundamental challenges of this theory is to justify the possibility of approximating the dynamics of a quantum particle confined to real-world mesoscopic waveguides of small width ε by its dynamics on the idealized one-dimensional 'manifolds' obtained in the limit as ε vanishes.

For instance, for bent waveguides Ω_ε in dimension 2 that coincide with two straight strips outside a compact 'vertex' region, this question was studied in [6]. Albeverio *et al.* showed that the problem reduces to establishing the norm-resolvent convergence of the operator family S_ε of (1.1) as $\varepsilon \to 0$. Assuming that V has non-zero mean and decays exponentially, i.e. that

$$\int_{\mathbb{R}} V(x)\,\mathrm{d}x \neq 0 \quad \text{and} \quad \mathrm{e}^{a|\cdot|}V \in L_1(\mathbb{R}) \quad \text{for some } a > 0, \tag{1.2}$$

norm-resolvent convergence of S_ε as $\varepsilon \to 0$ was proved and the limiting operator S_0 was identified. Depending on whether or not the potential V is resonant, the operator S_0 is either the direct sum $S_- \oplus S_+$ of two free half-line Schrödinger operators given by $S_\pm := -\mathrm{d}^2/\mathrm{d}x^2$ on $\mathbb{R}_\pm$ and subject to the Dirichlet condition $y(0) = 0$ at the origin, or a singular perturbation $S(\theta)$ of the free Schrödinger operator S, defined by

$$S(\theta)y = -y'', \tag{1.3}$$

on functions y in $W_2^2(\mathbb{R} \setminus \{0\})$ obeying the following non-trivial interface conditions at the origin:

$$y(0+) = \theta y(0-), \qquad \theta y'(0+) = y'(0-). \tag{1.4}$$

The number θ depends on the geometric properties (in particular, on the curvature) of the bent waveguide. The convergence results for bent waveguides were recently re-examined from a different viewpoint in [13].

In [14], the analysis of [6] was further extended to the case of waveguides Ω_ε with non-trivial scaling properties around the vertex region. Namely, Cacciapuoti and Exner demonstrated that in the resonant case the class of limiting Hamiltonians is wider and includes, in particular, perturbations of the free Schrödinger operator S by the delta functions. Results analogous to those of [6] and [14] were also established in [15] in the case of Robin conditions on the boundary of Ω_ε.

The effect of twisting in three-dimensional quantum wave-guides with shrinking non-circular cross-section was investigated in [22]. The problem was reduced to the study of convergence of singularly scaled Schrödinger-type operators $\mathcal{H}_\varepsilon$ in an unbounded tube with potentials containing the singular term $\varepsilon^{-2}\vartheta(x_1/\varepsilon)$, where x_1 is a longitudinal coordinate in the tube. The function ϑ of compact support describes the 'fast' twisting of the cross-section over a small x_1-interval shrinking to a point x_0. It was proved that the family $\mathcal{H}_\varepsilon$ converges as $\varepsilon \to 0$ in the (suitably understood) norm-resolvent sense to the operator $\mathcal{H}_0$, which is a one-dimensional Schrödinger operator subject to the Dirichlet boundary condition at x_0. Previously, the operators $\mathcal{H}_\varepsilon$ appeared in [35] in connection with the problem of large-time behaviour of solutions to the heat equation in a twisted tube, and their strong resolvent convergence was established therein.

The limiting behaviour of the family S_ε is interesting for yet another reason. Namely, in the case where V has a zero mean and its first moment is -1, i.e. where

$$\int_{\mathbb{R}} V(x)\,\mathrm{d}x = 0, \qquad \int_{\mathbb{R}} xV(x)\,\mathrm{d}x = -1,$$

the family of scaled potentials $V_\varepsilon(x) := \varepsilon^{-2}V(\varepsilon^{-1}x)$ converges as $\varepsilon \to 0$ to the derivative δ' of the Dirac delta function. Therefore, the limit S_0 of S_ε can be regarded as a physically motivated realization of the free Hamiltonian S perturbed by the *singular potential* δ'. We notice that there is a related, but completely different, notion of δ'-*interaction*. Namely, following a widely accepted agreement, the Hamiltonian $S_{\beta,\delta'}$ with δ'-interaction, that is given formally by

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \beta\langle\cdot, \delta'\rangle\delta',$$

should be interpreted as the free Schrödinger operator $S_{\beta,\delta'}y = -y''$ acting on the domain

$$\mathrm{dom}\, S_{\beta,\delta'} := \{y \in W_2^2(\mathbb{R}\setminus\{0\}) \mid y'(0-) = y'(0+) =: y'(0),\ y(0+) - y(0-) = \beta y'(0)\} \tag{1.5}$$

(see [2,5]). However, there is no clear physical motivation for this particular choice and, moreover, $S_{\beta,\delta'}$ cannot be used for defining a free Hamiltonian with δ'-*potential* [42]. We note that δ'-interactions and, more generally, singular point interactions for Schrödinger operators in dimension 1 and higher have been widely discussed in

both the mathematical and physical literature; see [7, 12, 20, 27, 34, 36, 41, 44] and the extensive bibliography lists in the monographs [2, 5, 33].

Šeba [45] was seemingly the first to realize the connection between the limiting behaviour of S_ε and δ'-perturbations of the Schrödinger operators. The paper [45] discussed an even wider class of problems, including S_ε as a particular case. However, its results for (1.1) erroneously state that the only possible norm-resolvent limit of S_ε is the direct sum $S_- \oplus S_+$ of the free half-line Schrödinger operators subject to the Dirichlet boundary condition at $x = 0$. Such a result would suggest that in dimension 1 no non-trivial definition of the Schrödinger operator with potential δ' is possible. Recalling that the Schrödinger operators are quantum mechanical Hamiltonians for a particle on the line, one would have to conclude that, in dimension 1, the δ'-potential barrier is completely opaque, i.e. that the particle cannot tunnel through it.

However, such a conclusion is in contradiction with the numerical analysis of exactly solvable models of (1.1), with piecewise constant V of compact support, performed recently by Zolotaryuk *et al.* [17, 48–51]. Namely, Zolotaryuk *et al.* demonstrated that, for resonant V, the limiting value of the transmission coefficient $T_\varepsilon(k)$ of the operator S_ε is different from zero, thus indicating that the limiting operator S_0 cannot be given by $S_- \oplus S_+$. In certain cases, S_0 was identified with the operator $S(\theta)$ of (1.3), (1.4). The operator of the form $S(\theta)$ also appears in [38, 42] as a realization of the pseudo-Hamiltonian $-\mathrm{d}^2/\mathrm{d}x^2 + \alpha\delta'(x)$, by means of the distribution theory over discontinuous test functions. Further evidence that the convergence result of [45] cannot be true was derived in [24], where eigenvalue and eigenfunction asymptotics as $\varepsilon \to 0$ were studied for the full-line Schrödinger operators given by the differential expression

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{\varepsilon^2}V\left(\frac{x}{\varepsilon}\right) + W(x),$$

where V is regular and of compact support and W is unbounded at infinity. Namely, the eigenfunctions were shown to satisfy, in the limit, the Dirichlet condition $y(0) = 0$ in the non-resonant case and interface condition (1.4) in a resonant case, thus once more exhibiting the zero-energy dichotomy.

The above results motivated us to re-examine the convergence of the Schrödinger operator family (1.1) for real-valued V not necessarily satisfying (1.2). In our previous paper [23], we treated the case where V is of compact support, taken in $[-1, 1]$ for definiteness, and, in particular, confirmed the result established in [6, 14, 15] under restriction (1.2). In [23, 24], a real-valued potential V is called *resonant* if the Neumann Sturm–Liouville operator $\mathcal{N}$, given by

$$\mathcal{N}y := -y'' + Vy,$$

on functions in the Sobolev space $W_2^2(-1, 1)$ satisfying the conditions $y'(-1) = 0$, $y'(1) = 0$ has a non-trivial null-space. This definition of a resonant case agrees with the one given in [6]. We proved in [23] that S_ε converge as $\varepsilon \to 0$ in the norm-resolvent sense to the limit operator S_0 that is equal to $S_- \oplus S_+$ in the non-resonance case, and to $S(\theta)$ of (1.3), (1.4) in the resonance case, with $\theta := u(1)/u(-1)$, where u is an eigenfunction of $\mathcal{N}$ corresponding to the eigenvalue 0. The eigenfunction is unique up to a constant factor, so that the number θ is well defined.

The main goal of this paper is to extend the convergence result of [23] to the set of real-valued potentials V of the Faddeev–Marchenko class. In particular, we do not assume that V is of non-zero mean or that V decays exponentially. Nevertheless, we shall prove that the results on convergence of S_ε established in [6, 14, 15, 23] for particular cases remain valid.

Recall [31] that the Schrödinger operator $S_1 = -\mathrm{d}^2/\mathrm{d}x^2 + V$ is said to possess a *zero-energy resonance* (or *half-bound state*) if there exists a solution y to

$$-y'' + Vy = 0$$

that is bounded on the whole line; the corresponding potential V is then called resonant. Such a solution y is then unique up to a scalar factor and has non-zero limits $y(\pm\infty)$ at $\pm\infty$, so that the number

$$\theta := \frac{y(+\infty)}{y(-\infty)} \tag{1.6}$$

is well defined. Our main result reads as follows.

THEOREM 1.1. *Assume that V is real valued and belongs to the Faddeev–Marchenko class. Then operator family (1.1) converges as $\varepsilon \to 0$ in the norm-resolvent sense, and the limit S_0 is equal to the direct sum $S_- \oplus S_+$ of the Dirichlet half-line Schrödinger operators $S_\pm$ in the non-resonant case, and to the operator $S(\theta)$ defined by (1.3), (1.4), with θ of (1.6), in the resonant case.*

We establish theorem 1.1 in two steps. In the first step, we adapt the approach of [23] and establish the theorem with the potential $V_\varepsilon := \varepsilon^{-2}V(\varepsilon^{-1}\cdot)$ truncated onto the contracting intervals $(-x_\varepsilon, x_\varepsilon)$, for a suitably defined x_ε. In the second step, we show that the part of the potential V_ε outside $(-x_\varepsilon, x_\varepsilon)$ introduces a perturbation that is too weak to affect the limit of the resolvents as $\varepsilon \to 0$. In both the resonant and non-resonant cases, the reasoning is based on a careful asymptotic analysis of the Jost solutions for S_ε and S_1 at infinity and in the vicinity of the splitting points $\pm x_\varepsilon$.

It should be noted that the above theorem seemingly cannot be extended beyond the Faddeev–Marchenko class of potentials V. Indeed, take $V(x) = (1 + x^2)^{-1}$. Then the family $V_\varepsilon(x) := (\varepsilon^2 + x^2)^{-1}$ increases as ε decreases to zero. The quadratic forms $\mathfrak{s}_\varepsilon$ generated by the Schrödinger operators S_ε are closed and well defined on $W_2^1(\mathbb{R})$. For $y \in W_2^1(\mathbb{R})$, the limit

$$\lim_{\varepsilon \to 0+} \mathfrak{s}_\varepsilon(y, y) = \int_{\mathbb{R}} |y'(x)|^2 \,\mathrm{d}x + \lim_{\varepsilon \to 0+} \int_{\mathbb{R}} \frac{|y(x)|^2 \,\mathrm{d}x}{\varepsilon^2 + x^2}$$

$$= \int_{\mathbb{R}} |y'(x)|^2 \,\mathrm{d}x + \int_{\mathbb{R}} \frac{|y(x)|^2 \,\mathrm{d}x}{x^2}$$

$$=: \mathfrak{s}_0(y, y)$$

is finite if and only if $y(0) = 0$; the 'only if' part is clear, and the 'if' part follows from the fact that the Hardy operator,

$$g \mapsto \frac{1}{x} \int_0^x g(t) \,\mathrm{d}t,$$

is bounded in $L_2(0,1)$ [26, ch. 9.9]. By [43, theorem S.14], the quadratic form $\mathfrak{s}_0$ is closed on the domain $\{y \in W_2^1(\mathbb{R}) \mid y(0) = 0\}$ and the Schrödinger operators S_ε converge in the strong resolvent sense as $\varepsilon \to 0+$ to the operator S_0 corresponding to the quadratic form $\mathfrak{s}_0$. Since $\operatorname{dom} S_0 \subset \operatorname{dom} \mathfrak{s}_0$ by the first representation theorem [29, theorem VI.2.1], we conclude that S_0 is the direct sum of two half-line Schrödinger operators S_- and S_+, with the Bessel potential $1/x^2$. Thus, there is no resonance effect present here and the limiting operator contains a non-trivial potential, in contrast to the situation of theorem 1.1.

It is worth noting that the operators S_ε given by (1.1) are scale invariant, and therefore it should be expected that the limit operator S_0 possesses the same property. And indeed, in both the non-resonant and resonant cases, the limit operators $S_- \oplus S_+$ and $S(\theta)$ describe the so-called scale-invariant point interactions [21].

We also remark that one can construct infinitely many resonant potentials starting with an arbitrary compactly supported $V \in L_1(\mathbb{R})$ and considering a family $\{\alpha V\}$ with a real coupling constant α. As shown in [23] (see also [17, 47–51] for various step-like potentials V), there is an infinite discrete set Σ of α values such that αV is resonant for every $\alpha \in \Sigma$. The same conclusion holds if V is supported by the whole line but decays rapidly at infinity. We are going to discuss this case elsewhere.

The rest of the paper is structured as follows. In § 2, we recall the definition and some important properties of the Jost solutions of the Schrödinger equation. Also, the low-energy and large-x asymptotics of the Jost solutions and of some special solutions are established therein. In § 3, we study the norm-resolvent convergence of the auxiliary family of Schrödinger operators $\tilde{S}_\varepsilon$ with potentials $\chi_\varepsilon V_\varepsilon$, where χ_ε is the characteristic function of a neighbourhood of the origin squeezing to zero as $\varepsilon \to 0$. Section 4 is devoted to investigation of the resolvent of $\tilde{S}_\varepsilon$, via the subtle asymptotic analysis of the Jost solutions of this operator. Simultaneously, we describe the limit behaviour of scattering coefficients for the operator $\tilde{S}_\varepsilon$. In these three sections, the non-resonant and resonant cases have to be treated separately. Finally, in § 5 we establish proximity of the operator families S_ε and $\tilde{S}_\varepsilon$ in the norm-resolvent sense, which, in combination with theorem 3.2, gives a complete proof of the main result.

Notation

Throughout the paper, C_1, C_2, C_3, C_4 and C_5 shall stand for the constants of proposition 2.1 and lemma 2.5. c_j denote various positive numbers independent of ε, whose values might be different in different proofs and $\|f\|$ stands for the $L_2(\mathbb{R})$-norm of a function f.

2. Asymptotics of Jost solutions

2.1. Large-x behaviour at low energies

Throughout the paper, V denotes a fixed real-valued potential of the Faddeev–Marchenko class, i.e. satisfying

$$\int_{\mathbb{R}} (1 + |t|)|V(t)| \, \mathrm{d}t < \infty.$$

Next, we denote by $f_+(\cdot, k)$ and $f_-(\cdot, k)$, respectively, the *right* and *left Jost solutions* of the Schrödinger equation

$$-y'' + Vy = k^2 y \tag{2.1}$$

for complex k, with $\operatorname{Im} k \geqslant 0$. The Jost solution $f_+(x, k)$ is asymptotic to $\mathrm{e}^{\mathrm{i}kx}$ as $x \to +\infty$, while $f_-(x, k)$ is asymptotic to $\mathrm{e}^{-\mathrm{i}kx}$ as $x \to -\infty$. For non-zero k, the Jost solutions exist whenever the potential V is integrable at $\pm\infty$ [16, ch. I.1.3], or even under somewhat weaker assumptions [30]. However, for V of the Faddeev–Marchenko class, they have some special properties. Namely, set

$$\sigma_-(x) := \int_{-\infty}^x |V(t)|\,\mathrm{d}t, \qquad \tau_-(x) := \int_{-\infty}^x (1 + |t|)|V(t)|\,\mathrm{d}t,$$

$$\sigma_+(x) := \int_x^\infty |V(t)|\,\mathrm{d}t, \qquad \tau_+(x) := \int_x^\infty (1 + |t|)|V(t)|\,\mathrm{d}t.$$

Then, the claims of [18, lemma 1] and [40, lemma 3.1.3] can be stated as follows.

PROPOSITION 2.1. *Assume that the potential V is of the Faddeev–Marchenko class. Then, there are numbers C_j, $j = 1, \ldots, 4$, such that the following holds for every k in the closed upper-half complex plane.*

$$|f_+(x, k) - \mathrm{e}^{\mathrm{i}kx}| \leqslant C_1 |\mathrm{e}^{\mathrm{i}kx}| \tau_+(x), \qquad x \geqslant 0, \tag{2.2}$$

$$|f_+(x, k) - \mathrm{e}^{\mathrm{i}kx}| \leqslant C_2 |\mathrm{e}^{\mathrm{i}kx}| (1 + |x|), \quad x < 0, \tag{2.3}$$

$$|f_+'(x, k) - \mathrm{i}k f_+(x, k)| \leqslant C_3 |\mathrm{e}^{\mathrm{i}kx}| \sigma_+(x), \qquad x \geqslant 0, \tag{2.4}$$

$$|f_+'(x, k) - \mathrm{i}k f_+(x, k)| \leqslant C_4 |\mathrm{e}^{\mathrm{i}kx}|, \qquad x < 0. \tag{2.5}$$

Similar estimates hold for $f_-(\cdot, k)$.

In what follows, we shall consider restrictions of V_ε onto contracting intervals $(x_\varepsilon, x_\varepsilon)$, the choice of x_ε being specifically tailored for the given V. We start with the following observation.

LEMMA 2.2. *For V of the Faddeev–Marchenko class, there exists an even, continuous and positive function $\rho_V : \mathbb{R} \to \mathbb{R}$ such that*

(i) ρ_V *strictly increases for $x > 0$,*

(ii) $|x|^{-1}\rho_V(x) \to +\infty$ *as $|x| \to +\infty$,*

(iii) $\displaystyle\int_{\mathbb{R}} \rho_V(x)|V(x)|\,\mathrm{d}x < \infty$.

Proof. For V of compact support, set $\rho_V(x) := 1 + x^2$. Otherwise, the function

$$\tau(x) := \tau_+(|x|) + \tau_-(-|x|) = \int_{|t| > |x|} (1 + |t|)|V(t)|\,\mathrm{d}t$$

is strictly positive and even. Moreover, it does not increase for $x > 0$ and vanishes at infinity. We set

$$\rho_V(x) := \frac{1 + |x|}{\tau^\alpha(x)}$$

for $\alpha \in (0, 1)$. Then (i) and (ii) are immediate, while (iii) follows from

$$
\begin{aligned}
\int_{\mathbb{R}} \rho_V(x)|V(x)|\,\mathrm{d}x &= \int_0^\infty \frac{(1+x)(|V(x)| + |V(-x)|)}{\tau^\alpha(x)}\,\mathrm{d}x \\
&= -\int_0^\infty \frac{\tau'(x)}{\tau^\alpha(x)}\,\mathrm{d}x = \frac{\tau^{1-\alpha}(0)}{1-\alpha} \\
&< \infty.
\end{aligned}
$$

The proof is complete. □

Fix ρ_V as in lemma 2.2 and for $\varepsilon > 0$ denote by ξ_ε the unique positive solution of $\rho_V(\xi) = 1/\varepsilon$. Such a solution ξ_ε exists for all sufficiently small positive ε. It follows from (i) and (ii) of lemma 2.2 that $\xi_\varepsilon \to +\infty$ and $x_\varepsilon := \varepsilon \xi_\varepsilon \to 0$ as $\varepsilon \to 0$. This choice of x_ε and ξ_ε is fixed for the rest of the paper.

Set $W\{f, g\} := fg' - f'g$ to be the Wronskian of functions f and g and define $D(k)$ as the Wronskian of the Jost solutions $f_+(\cdot, k)$ and $f_-(\cdot, k)$, such that

$$
D(k) := W\{f_+(\cdot, k), f_-(\cdot, k)\} = f_+(x, k)f'_-(x, k) - f'_+(x, k)f_-(x, k).
$$

Fix a non-zero $k \in \overline{\mathbb{C}^+}$. Then, proposition 2.1 implies the following asymptotic behaviour of the Jost solutions.

LEMMA 2.3. *The following hold as $\varepsilon \to 0$:*

$$
\begin{array}{llr}
f_+(\xi_\varepsilon, \varepsilon k) \to 1, & f_-(-\xi_\varepsilon, \varepsilon k) \to 1, & (2.6) \\
\varepsilon f_+(-\xi_\varepsilon, \varepsilon k) \to 0, & \varepsilon f_-(\xi_\varepsilon, \varepsilon k) \to 0, & (2.7) \\
\varepsilon^{-1} f'_+(\xi_\varepsilon, \varepsilon k) \to \mathrm{i}k, & \varepsilon^{-1} f'_-(-\xi_\varepsilon, \varepsilon k) \to -\mathrm{i}k, & (2.8) \\
f'_+(-\xi_\varepsilon, \varepsilon k) \to -D(0), & f'_-(\xi_\varepsilon, \varepsilon k) \to D(0). & (2.9)
\end{array}
$$

Proof. Relations (2.6) follow immediately from (2.2) and an analogous estimate for f_-. Also, (2.3) and a similar inequality for f_- imply that

$$
|f_+(-\xi_\varepsilon, \varepsilon k)| + |f_-(\xi_\varepsilon, \varepsilon k)| \leqslant c(1 + \xi_\varepsilon)
$$

for some $c > 0$ independent of ε. Since $\varepsilon \xi_\varepsilon \to 0$ as $\varepsilon \to 0$, relations (2.7) follow.

Next, (2.4) yields

$$
|\varepsilon^{-1} f'_+(\xi_\varepsilon, \varepsilon k) - \mathrm{i}k| \leqslant C_3 \varepsilon^{-1} \sigma_+(\xi_\varepsilon) + |k||f_+(\xi_\varepsilon, \varepsilon k) - 1|. \tag{2.10}
$$

Now, the choice of ξ_ε and the properties of the function ρ_V justified in lemma 2.2 show that

$$
\begin{aligned}
\varepsilon^{-1} \sigma_+(\xi_\varepsilon) = \varepsilon^{-1} \int_{\xi_\varepsilon}^\infty |V(t)|\,\mathrm{d}t \\
\leqslant \varepsilon^{-1} \int_{\xi_\varepsilon}^\infty \frac{\rho_V(t)|V(t)|}{\rho_V(\xi_\varepsilon)}\,\mathrm{d}t \\
= \int_{\xi_\varepsilon}^\infty \rho_V(t)|V(t)|\,\mathrm{d}t \\
\to 0
\end{aligned}
$$

as $\varepsilon \to 0$. A passage to the limit in (2.10) establishes the first relation of (2.8); the second one is similarly justified.

Finally, combining the equalities

$$D(\varepsilon k) = \begin{vmatrix} \varepsilon f_+(-\xi_\varepsilon, \varepsilon k) & f_-(-\xi_\varepsilon, \varepsilon k) \\ f'_+(-\xi_\varepsilon, \varepsilon k) & \varepsilon^{-1} f'_-(-\xi_\varepsilon, \varepsilon k) \end{vmatrix} = \begin{vmatrix} f_+(\xi_\varepsilon, \varepsilon k) & \varepsilon f_-(\xi_\varepsilon, \varepsilon k) \\ \varepsilon^{-1} f'_+(\xi_\varepsilon, \varepsilon k) & f'_-(\xi_\varepsilon, \varepsilon k) \end{vmatrix}$$

with relations (2.6)–(2.8) proves the above results in (2.9). The proof is complete.
□

2.2. Refinement in the resonant case

In the resonant case, the Jost solutions f_+ and f_- become linearly dependent at $k = 0$, i.e.

$$f_-(\cdot, 0) = \theta f_+(\cdot, 0) \tag{2.11}$$

for some real non-zero θ. The corresponding half-bound state y can be given as $y = c_- f_-(\cdot, 0)$ or $y = c_+ f_+(\cdot, 0)$, with constants $c_\pm$ satisfying the relation $c_+ = \theta c_-$. Since $y(\pm\infty) = c_\pm$ by the definition of the Jost solutions, the number θ, as defined by (2.11), is the same as in (1.6).

Moreover, in this case, more precise information on the asymptotic behaviour of $f_\pm(\cdot, 0)$ at infinity is available. Namely, we shall prove that the solutions $f_\pm(\cdot, \varepsilon k)$ then remain bounded over $(-\xi_\varepsilon, \xi_\varepsilon)$ uniformly in ε, and we shall re-examine the convergence in (2.7) and (2.9). To do this, we need another pair of solutions to (2.1).

LEMMA 2.4. *There exist two solutions, $g_+(\cdot, k)$ and $g_-(\cdot, k)$, of (2.1) satisfying the relations $W\{f_+, g_+\} = 1$ and $W\{f_-, g_-\} = -1$ and such that, for some $x_+ > 0$, $x_- < 0$ and all $k \in \overline{\mathbb{C}^+}$, the following inequalities hold:*

$$|g_+(x, k)| \leqslant 6|\mathrm{e}^{-\mathrm{i}kx}|x, \quad x > x_+, \tag{2.12}$$

$$|g_-(x, k)| \leqslant 6|\mathrm{e}^{\mathrm{i}kx}||x|, \quad x < x_-. \tag{2.13}$$

Proof. It follows from (2.2) that there is an $x_+ > 0$ such that $f_+(x, k) \neq 0$ for every $k \in \overline{\mathbb{C}^+}$ and every $x \geqslant x_+$. Then, the function

$$g_+(x, k) := f_+(x, k) \int_{x_+}^{x} \frac{\mathrm{d}t}{f_+^2(t, k)}$$

is well defined for $x \geqslant x_+$, solves (2.1) and verifies the relation $W\{f_+, g_+\} = 1$. Clearly, g_+ can uniquely be continued to the whole axis as a solution of (2.1).

Without loss of generality, we can (and shall) assume that x_+ is so large that $C_1 \tau_+(x) \leqslant \frac{1}{2}$ for all $k \in \overline{\mathbb{C}^+}$ and all $x > x_+$, C_1 being the constant of (2.2). Then,

$$\tfrac{1}{2}|\mathrm{e}^{\mathrm{i}kx}| \leqslant |f_+(x, k)| \leqslant \tfrac{3}{2}|\mathrm{e}^{\mathrm{i}kx}|$$

for such k and x. Therefore,

$$\frac{|\mathrm{e}^{\mathrm{i}kx} g_+(x, k)|}{x} \leqslant \frac{6|\mathrm{e}^{2\mathrm{i}kx}|}{x - x_+} \int_{x_+}^{x} \frac{\mathrm{d}t}{|\mathrm{e}^{2\mathrm{i}kt}|} \leqslant 6,$$

thus yielding (2.12).

By similar arguments, there exist an $x_- < 0$ and a solution $g_-(\cdot, k)$ of (2.1) that, for all $x < x_-$, is equal to

$$g_-(x, k) := f_-(x, k) \int_x^{x_-} \frac{\mathrm{d}t}{f_-^2(t, k)}$$

and satisfies $W\{f_-, g_-\} = -1$ and (2.13). $\qquad\square$

LEMMA 2.5. *Assume that V is resonant and define θ via (2.11). Then for some $C_5 > 0$ and all ε small enough, we have that*

$$\max_{|x| \leqslant \xi_\varepsilon} |f_\pm(x, \varepsilon k)| \leqslant C_5. \qquad (2.14)$$

Furthermore, the following hold as $\varepsilon \to 0$:

$$f_+(-\xi_\varepsilon, \varepsilon k) \to \theta^{-1}, \qquad f_-(\xi_\varepsilon, \varepsilon k) \to \theta, \qquad (2.15)$$

$$\varepsilon^{-1} f_+'(-\xi_\varepsilon, \varepsilon k) \to \mathrm{i}k\theta, \qquad \varepsilon^{-1} f_-'(\xi_\varepsilon, \varepsilon k) \to -\mathrm{i}k\theta^{-1}. \qquad (2.16)$$

Proof. We shall only prove (2.14) for the solution f_+, since the proof for f_- is completely analogous. Clearly, there are α_ε and β_ε such that

$$f_+(\cdot, \varepsilon k) = \alpha_\varepsilon f_-(\cdot, \varepsilon k) + \beta_\varepsilon g_-(\cdot, \varepsilon k)$$

and it is easy to see that $\beta_\varepsilon = D(\varepsilon k)$. Recall that $D(0) = 0$ in the resonant case. Moreover, [32, (2.29)] implies that D is differentiable at $k = 0$ and that the derivative $\dot{D}(0)$ is

$$\dot{D}(0) = -\mathrm{i}(\theta + \theta^{-1}) \neq 0.$$

It follows that

$$\lim_{\varepsilon \to 0} \frac{D(\varepsilon k)}{\varepsilon} = -\mathrm{i}k(\theta + \theta^{-1}). \qquad (2.17)$$

Thus, $\beta_\varepsilon = O(\varepsilon)$ as $\varepsilon \to 0$ and, in combination with (2.13), this shows that

$$\max_{x \in [-\xi_\varepsilon, x_-]} |f_+(x, \varepsilon k) - \alpha_\varepsilon f_-(x, \varepsilon k)| \leqslant 6|\beta_\varepsilon||\mathrm{e}^{\mathrm{i}x_\varepsilon k}|\xi_\varepsilon = O(x_\varepsilon) \qquad (2.18)$$

as $\varepsilon \to 0$. For each fixed x, $f_\pm(x, k)$ are continuous functions of $k \in \overline{\mathbb{C}^+}$. We conclude from this and the relation (2.18) that $\alpha_\varepsilon \to \theta^{-1}$ as $\varepsilon \to 0$. Therefore, there exists $c > 0$ such that, for all ε small enough,

$$\max_{x \in [-\xi_\varepsilon, x_-]} |f_+(x, \varepsilon k)| \leqslant 2\theta^{-1} \max_{x \in [-\xi_\varepsilon, x_-]} |f_-(x, \varepsilon k)| + 1 \leqslant c.$$

A similar bound over $x_- \leqslant x \leqslant \xi_+$ follows from (2.2) and (2.3), thus establishing (2.14) for f_+.

Now, specifying (2.18) to $x = -\xi_\varepsilon$ and recalling (2.6), we get that

$$\lim_{\varepsilon \to 0} f_+(-\xi_\varepsilon, \varepsilon k) = \theta^{-1}.$$

Observing that

$$f_+(-\xi_\varepsilon, \varepsilon k) \frac{f_-'(-\xi_\varepsilon, \varepsilon k)}{\varepsilon} - \frac{f_+'(-\xi_\varepsilon, \varepsilon k)}{\varepsilon} f_-(-\xi_\varepsilon, \varepsilon k) = \frac{D(\varepsilon k)}{\varepsilon} \to -\mathrm{i}k(\theta + \theta^{-1})$$

as $\varepsilon \to 0$ by (2.17), we deduce from lemma 2.3 and (2.15) that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} f'_+(-\xi_\varepsilon, \varepsilon k) = \mathrm{i} k \theta.$$

The behaviour of the Jost solution f_- on the right-half line can be analysed similarly. □

3. Convergence of the auxiliary operator family $\tilde{S}_\varepsilon$

3.1. A general approach

In what follows, we denote by χ_ε the characteristic function of the interval $[-x_\varepsilon, x_\varepsilon]$ and consider an auxiliary family $\tilde{S}_\varepsilon$ of Schrödinger operators on the line defined via

$$\tilde{S}_\varepsilon := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{\varepsilon^2} V\left(\frac{x}{\varepsilon}\right) \chi_\varepsilon(x).$$

Based on the results of [6, 14, 23, 24], we expect that the family $\tilde{S}_\varepsilon$ converges to a limit S_0 as $\varepsilon \to 0$ in the norm-resolvent sense, and that S_0 is $S_- \oplus S_+$ in the non-resonant case and $S(\theta)$, with θ of (1.6) or (2.11), in the resonant case.

To explain the main idea behind the constructions that follow, we fix a non-real k^2, take an arbitrary $f \in L_2(\mathbb{R})$ and set $y_0 := (S_0 - k^2)^{-1} f$ and $\tilde{y}_\varepsilon := (\tilde{S}_\varepsilon - k^2)^{-1} f$. To show that $\tilde{y}_\varepsilon \to y_0$ as $\varepsilon \to 0$, we construct an auxiliary function y_ε in dom $\tilde{S}_\varepsilon$ such that $q_\varepsilon := y_\varepsilon - y_0 \to 0$ and $r_\varepsilon := (\tilde{S}_\varepsilon - k^2) y_\varepsilon - f \to 0$ in the L_2-norm as $\varepsilon \to 0$. It then follows that

$$\tilde{y}_\varepsilon - y_0 = (y_\varepsilon - y_0) - (y_\varepsilon - \tilde{y}_\varepsilon) = q_\varepsilon - (\tilde{S}_\varepsilon - k^2)^{-1} r_\varepsilon \to 0.$$

A more careful analysis shows that the convergence of $\tilde{y}_\varepsilon$ to y_0 is uniform in f on bounded sets, thus establishing the norm-resolvent convergence of $\tilde{S}_\varepsilon$ to S_0.

We start the construction of the approximation y_ε by observing that the function $\tilde{y}_\varepsilon = (\tilde{S}_\varepsilon - k^2)^{-1} f$ solves

$$-y'' = k^2 y + f$$

for $|x| > x_\varepsilon$ and solves

$$-y'' + \varepsilon^{-2} V(x/\varepsilon) y = k^2 y + f$$

for $|x| < x_\varepsilon$. Setting $\tilde{y}_\varepsilon(x) := w(x/\varepsilon)$ in the 'fast variable' domain $|x| < x_\varepsilon$, we see that w must solve

$$-w''(\xi) + V(\xi) w(\xi) = \varepsilon^2 k^2 w(\xi) + \varepsilon^2 f(\varepsilon \xi)$$

for $\xi \in (-\xi_\varepsilon, \xi_\varepsilon)$; see figure 1. A general solution to this equation has the form $w_\varepsilon + u_\varepsilon$, where

$$w_\varepsilon = a_-(\varepsilon) f_-(\cdot, \varepsilon k) + a_+(\varepsilon) f_+(\cdot, \varepsilon k),$$

with $a_\pm(\varepsilon)$ arbitrary complex numbers, is a solution to the homogeneous problem and

$$u_\varepsilon(\xi) := \frac{\varepsilon^2 f_+(\xi, \varepsilon k)}{D(\varepsilon k)} \int_{-\xi_\varepsilon}^{\xi} f_-(\eta, \varepsilon k) f(\varepsilon \eta) \, \mathrm{d}\eta + \frac{\varepsilon^2 f_-(\xi, \varepsilon k)}{D(\varepsilon k)} \int_{\xi}^{\xi_\varepsilon} f_+(\eta, \varepsilon k) f(\varepsilon \eta) \, \mathrm{d}\eta$$
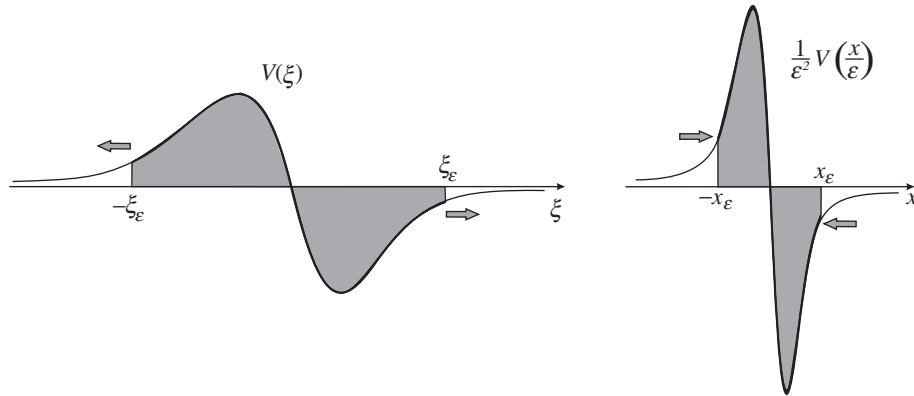
is a particular solution.

Figure 1. Potentials in ξ and x coordinates.

It is therefore natural to take a function

$$z_\varepsilon(x) := y_0(x)[1 - \chi_\varepsilon(x)] + [w_\varepsilon(x/\varepsilon) + u_\varepsilon(x/\varepsilon)]\chi_\varepsilon(x),$$

equal to y_0 for $|x| > x_\varepsilon$ and equal to $w_\varepsilon(x/\varepsilon) + u_\varepsilon(x/\varepsilon)$ for $|x| < x_\varepsilon$, as a leading term of the asymptotic expansion of y_ε as $\varepsilon \to 0$. The function z_ε belongs to W_2^2 outside the points $x = \pm x_\varepsilon$ and has jump discontinuities at these points. Namely, denoting by $[g]_a$ the jump of a function g at a point $x = a$, we see that

$$|[z_\varepsilon]_{\pm x_\varepsilon}| = |y_0(\pm x_\varepsilon) - w_\varepsilon(\pm \xi_\varepsilon) - u_\varepsilon(\pm \xi_\varepsilon)|,$$
$$|[z_\varepsilon']_{\pm x_\varepsilon}| = |y_0'(\pm x_\varepsilon) - \varepsilon^{-1}w_\varepsilon'(\pm \xi_\varepsilon) - \varepsilon^{-1}u_\varepsilon'(\pm \xi_\varepsilon)|. \tag{3.1}$$

Our task is to choose the coefficients $a_+(\varepsilon)$ and $a_-(\varepsilon)$ defining w_ε in such a way that the values of the right-hand sides of (3.1) vanish as $\varepsilon \to 0$. First, we show that the particular solution u_ε does not contribute significantly to these values. Set

$$\Delta_\varepsilon := [-\xi_\varepsilon, \xi_\varepsilon], \qquad \Delta_\varepsilon^- := [-\xi_\varepsilon, 0], \qquad \Delta_\varepsilon^+ := [0, \xi_\varepsilon].$$

LEMMA 3.1. *The following hold as $\varepsilon \to 0$:*

$$|u_\varepsilon(\pm \xi_\varepsilon)| = O(x_\varepsilon^{1/2})\|f\|, \tag{3.2}$$
$$\varepsilon^{-1}|u_\varepsilon'(\pm \xi_\varepsilon)| = O(x_\varepsilon^{1/2})\|f\| \tag{3.3}$$

and, in addition, $\|u_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon\| = O(x_\varepsilon)\|f\|$.

Proof. First, observe that

$$\max_{\Delta_\varepsilon^+} |u_\varepsilon| \leqslant \frac{2\varepsilon^2}{|D(\varepsilon k)|} \max_{\Delta_\varepsilon^+} |f_+(\cdot, \varepsilon k)| \max_{\Delta_\varepsilon} |f_-(\cdot, \varepsilon k)| \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f(\varepsilon \eta)| \, \mathrm{d}\eta,$$

$$\max_{\Delta_\varepsilon^-} |u_\varepsilon| \leqslant \frac{2\varepsilon^2}{|D(\varepsilon k)|} \max_{\Delta_\varepsilon} |f_+(\cdot, \varepsilon k)| \max_{\Delta_\varepsilon^-} |f_-(\cdot, \varepsilon k)| \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f(\varepsilon \eta)| \, \mathrm{d}\eta$$

and that

$$\int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f(\varepsilon \eta)| \, \mathrm{d}\eta \leqslant \sqrt{2}\varepsilon^{-1/2}\xi_\varepsilon^{1/2}\|f\|, \tag{3.4}$$

by the Cauchy–Bunyakowski inequality.

In the non-resonant case, $D(0) \neq 0$, and by virtue of proposition 2.1, we have that

$$\max_{\Delta_\varepsilon} |u_\varepsilon| \leqslant c_1 \varepsilon^{3/2} (1 + \xi_\varepsilon) \xi_\varepsilon^{1/2} \|f\| \leqslant 2c_1 x_\varepsilon^{3/2} \|f\|$$

for all sufficiently small ε and some $c_1 > 0$ independent of ε. In the resonant case, we use (2.17), and the fact that, by lemma 2.5, the Jost solutions $f_\pm(\cdot, \varepsilon k)$ remain bounded over Δ_ε uniformly in ε, to find c_2 independent of ε, such that

$$\max_{\Delta_\varepsilon} |u_\varepsilon| \leqslant c_2 x_\varepsilon^{1/2} \|f\|.$$

Since the last inequality also holds in the non-resonant case for small enough ε, we readily deduce (3.2) and estimate the L_2-norm of $u_\varepsilon(\cdot/\varepsilon) \chi_\varepsilon$ via

$$\int_{-x_\varepsilon}^{x_\varepsilon} |u_\varepsilon(x/\varepsilon)|^2 \, \mathrm{d}x = \varepsilon \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |u_\varepsilon(\xi)|^2 \, \mathrm{d}\xi \leqslant 2c_2^2 \varepsilon x_\varepsilon \xi_\varepsilon \|f\|^2 = O(x_\varepsilon^2) \|f\|^2$$

as $\varepsilon \to 0$.

Next we compute the derivative

$$u_\varepsilon'(\xi_\varepsilon) = \frac{\varepsilon^2 f_+'(\xi_\varepsilon, \varepsilon k)}{D(\varepsilon k)} \int_{-\xi_\varepsilon}^{\xi_\varepsilon} f_-(\eta, \varepsilon k) f(\varepsilon \eta) \, \mathrm{d}\eta$$

and obtain

$$\varepsilon^{-1} |u_\varepsilon'(\xi_\varepsilon)| \leqslant \frac{\varepsilon}{|D(\varepsilon k)|} |f_+'(\xi_\varepsilon, \varepsilon k)| \max_{\Delta_\varepsilon} |f_-(\xi, \varepsilon k)| \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f(\varepsilon \eta)| \, \mathrm{d}\eta$$

$$\leqslant \frac{c_3 |k| \varepsilon x_\varepsilon^{1/2}}{|D(\varepsilon k)|} \max_{\Delta_\varepsilon} |f_-(\xi, \varepsilon k)| \|f\|,$$

due to (2.8) and (3.4). Applying proposition 2.1 in the non-resonant case and lemma 2.5 along with (2.17) in the resonant one, we find that

$$\varepsilon^{-1} |u_\varepsilon'(\xi_\varepsilon^{1/2})| \leqslant c_4 x_\varepsilon \|f\|.$$

Similar considerations yield the estimate for $u_\varepsilon'(-\xi_\varepsilon)$. $\qquad\square$

Next, we need to choose the coefficients $a_\pm(\varepsilon)$ appropriately. This will be done separately in the non-resonant and resonant cases.

3.2. A special solution in the non-resonant case

Observing that $y_0(\pm x_\varepsilon) \to 0$ and that $y_0'(\pm x_\varepsilon) \to y_0'(0\pm)$ as $\varepsilon \to 0$ and recalling that the values $|f_\pm'(-\xi_\varepsilon p, \varepsilon k)|$ converge to the positive number $|D(0)|$ by (2.9), one concludes that a suitable choice of $a_\pm(\varepsilon)$ might be

$$a_+(\varepsilon) := \varepsilon y_0'(-x_\varepsilon)/f_+'(-\xi_\varepsilon, \varepsilon k), \qquad a_-(\varepsilon) := \varepsilon y_0'(x_\varepsilon)/f_-'(\xi_\varepsilon, \varepsilon k).$$

And indeed, the relations of proposition 2.1 and lemma 2.3 then yield the asymptotics

$$
\begin{aligned}
w_\varepsilon(\xi_\varepsilon m) &= \frac{\varepsilon f_-(\pm\xi_\varepsilon, \varepsilon k)}{f'_-(\xi_\varepsilon, \varepsilon k)} y'_0(x_\varepsilon) + \frac{\varepsilon f_+(\pm\xi_\varepsilon, \varepsilon k)}{f'_+(-\xi_\varepsilon, \varepsilon k)} y'_0(-x_\varepsilon) \\
&= O(x_\varepsilon)[y'_0(x_\varepsilon) + y'_0(-x_\varepsilon)], \\
\varepsilon^{-1} w'_\varepsilon(\xi_\varepsilon m) &= \frac{f'_-(\pm\xi_\varepsilon, \varepsilon k)}{f'_-(\xi_\varepsilon, \varepsilon k)} y'_0(x_\varepsilon) + \frac{f'_+(\pm\xi_\varepsilon, \varepsilon k)}{f'_+(-\xi_\varepsilon, \varepsilon k)} y'_0(-x_\varepsilon) \\
&= y'_0(\pm x_\varepsilon) + O(\varepsilon) y'_0(\mp x_\varepsilon).
\end{aligned}
$$

Since the resolvent of S_0 acts continuously from $L_2(\mathbb{R})$ into $W_2^2(\mathbb{R}\setminus\{0\})$, there exists a constant c_1 independent of f such that

$$
\|y_0\|_{W_2^2(\mathbb{R}\setminus\{0\})} \leqslant c_1 \|f\|.
$$

Thus, the Sobolev embedding theorem yields the estimates

$$
\sup_x(|y_0(x)| + |y'_0(x)|) \leqslant c_2 \|y_0\|_{W_2^2(\mathbb{R}\setminus\{0\})} \leqslant c_3 \|f\| \tag{3.5}
$$

as well as the relations

$$
\begin{aligned}
|y_0(\pm x_\varepsilon)| &= \left| \int_0^{\pm x_\varepsilon} y'_0(t)\, \mathrm{d}t \right| \leqslant x_\varepsilon^{1/2} \|y'_0\| \leqslant c_4 x_\varepsilon^{1/2} \|f\|, \\
|y'_0(\pm x_\varepsilon) - y'_0(0\pm)| &= \left| \int_0^{\pm x_\varepsilon} y''_0(t)\, \mathrm{d}t \right| \leqslant x_\varepsilon^{1/2} \|y''_0\| \leqslant c_5 x_\varepsilon^{1/2} \|f\|.
\end{aligned}
$$

Now, returning to (3.1) and combining the above estimates with lemma 3.1, we get that

$$
\begin{aligned}
|[z_\varepsilon]_{\pm x_\varepsilon}| &\leqslant |y_0(\pm x_\varepsilon)| + |w_\varepsilon(\xi_\varepsilon m)| + |u_\varepsilon(\xi_\varepsilon m)| = O(x_\varepsilon^{1/2})\|f\|, \\
|[z'_\varepsilon]_{\pm x_\varepsilon}| &\leqslant |y'_0(\pm x_\varepsilon) - \varepsilon^{-1} w'_\varepsilon(\xi_\varepsilon m)| + \varepsilon^{-1}|u'_\varepsilon(\xi_\varepsilon m)| = O(x_\varepsilon^{1/2})\|f\|.
\end{aligned}
$$

We also need an estimate on the L_2-norm of $w_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon$. Since $a_\pm(\varepsilon) = O(\varepsilon)\|f\|$ and

$$
\int_{-x_\varepsilon}^{x_\varepsilon} |f_\pm(x/\varepsilon, \varepsilon k)|^2\, \mathrm{d}x = \varepsilon \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f_\pm(\xi, \varepsilon k)|^2\, \mathrm{d}\xi \leqslant c_6 \varepsilon (1 + \xi_\varepsilon)^3,
$$

we find that, as $\varepsilon \to 0$,

$$
\|w_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon\| = O(x_\varepsilon^{3/2})\|f\|.
$$

3.3. A special solution in the resonant case

In the resonant case, the limit operator S_0 is expected to be $S(\theta)$, with θ of (2.11). The main difference from the non-resonant case is that the zero energy Jost solutions f_- and f_+ are now linearly dependent and thus follow quite a different asymptotics.

A possible choice for $a_\pm(\varepsilon)$ could be the one resulting in the equalities

$$
w_\varepsilon(-\xi_\varepsilon) = y_0(-x_\varepsilon), \qquad \varepsilon^{-1} w'_\varepsilon(-\xi_\varepsilon) = y'_0(-x_\varepsilon).
$$

However, we find it more convenient to take the limiting values $a_\pm(0)$ instead of such $a_\pm(\varepsilon)$ for all $\varepsilon > 0$, i.e. to set

$$a_-(\varepsilon) \equiv \frac{-1}{ik(\theta + \theta^{-1})}[y_0'(0-)\theta^{-1} - ik\theta y_0(0-)],$$

$$a_+(\varepsilon) \equiv \frac{1}{ik(\theta + \theta^{-1})}[iky_0(0-) + y_0'(0-)].$$

Then, as $\varepsilon \to 0$,

$$
\begin{aligned}
w_\varepsilon(-\xi_\varepsilon) &= y_0(0-)\frac{\theta f_-(-\xi_\varepsilon, \varepsilon k) + f_+(-\xi_\varepsilon, \varepsilon k)}{\theta + \theta^{-1}} \\
&\quad + y_0'(0-)\frac{-\theta^{-1}f_-(-\xi_\varepsilon, \varepsilon k) + f_+(-\xi_\varepsilon, \varepsilon k)}{ik(\theta + \theta^{-1})} \\
&= y_0(0-)(1 + o(1)) + y_0'(0-)o(1) \\
&= y_0(-x_\varepsilon) + o(1)\|f\|
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
\varepsilon^{-1}w_\varepsilon'(-\xi_\varepsilon) &= y_0(0-)\frac{\theta f_-'(-\xi_\varepsilon, \varepsilon k) + f_+'(-\xi_\varepsilon, \varepsilon k)}{\varepsilon(\theta + \theta^{-1})} \\
&\quad + y_0'(0-)\frac{-\theta^{-1}f_-'(-\xi_\varepsilon, \varepsilon k) + f_+'(-\xi_\varepsilon, \varepsilon k)}{i\varepsilon k(\theta + \theta^{-1})} \\
&= y_0(0-)o(1) + y_0'(0-)(1 + o(1)) \\
&= y_0'(-x_\varepsilon) + o(1)\|f\|.
\end{aligned}
$$

Next, recalling the relations $y_0(0-) = \theta^{-1}y_0(0+)$ and $y_0'(0-) = \theta y_0'(0+)$, we see that

$$
\begin{aligned}
w_\varepsilon(\xi_\varepsilon) &= y_0(0+)\frac{f_-(\xi_\varepsilon, \varepsilon k) + \theta^{-1}f_+(\xi_\varepsilon, \varepsilon k)}{\theta + \theta^{-1}} \\
&\quad + y_0'(0+)\frac{-f_-(\xi_\varepsilon, \varepsilon k) + \theta f_+(\xi_\varepsilon, \varepsilon k)}{ik(\theta + \theta^{-1})} \\
&= y_0(0+)(1 + o(1)) + y_0'(0+)o(1) \\
&= y_0(x_\varepsilon) + o(1)\|f\|
\end{aligned}
$$

and

$$
\begin{aligned}
\varepsilon^{-1}w_\varepsilon'(\xi_\varepsilon) &= y_0(0+)\frac{f_-'(\xi_\varepsilon, \varepsilon k) + \theta^{-1}f_+'(\xi_\varepsilon, \varepsilon k)}{\varepsilon(\theta + \theta^{-1})} \\
&\quad + y_0'(0+)\frac{-f_-'(\xi_\varepsilon, \varepsilon k) + \theta f_+'(\xi_\varepsilon, \varepsilon k)}{i\varepsilon k(\theta + \theta^{-1})} \\
&= y_0(0+)o(1) + y_0'(0+)(1 + o(1)) \\
&= y_0'(x_\varepsilon) + o(1)\|f\|.
\end{aligned}
$$

Figure 2. Functions with the prescribed jumps at the origin.

Combining the above relations, we conclude that the jumps of the function z_ε and its derivative at the points $\pm x_\varepsilon$ are small, namely that

$$[z_\varepsilon]_{\pm x_\varepsilon} = o(1)\|f\|,$$
$$[z'_\varepsilon]_{\pm x_\varepsilon} = o(1)\|f\|$$

as $\varepsilon \to 0$.

Finally, using the fact that by lemma 2.5 the Jost solutions $f_\pm(\cdot, \varepsilon k)$ remain uniformly bounded over $(-\xi_\varepsilon, \xi_\varepsilon)$ as $\varepsilon \to 0$, we arrive at the estimate

$$\int_{-x_\varepsilon}^{x_\varepsilon} |f_\pm(x/\varepsilon, \varepsilon k)|^2 \, \mathrm{d}x = \varepsilon \int_{-\xi_\varepsilon}^{\xi_\varepsilon} |f_\pm(\xi, \varepsilon k)|^2 \, \mathrm{d}\xi \leqslant c_7 \varepsilon \xi_\varepsilon = c_7 x_\varepsilon$$

for all small ε. Observing that $a_\pm(\varepsilon) = O(1)\|f\|$ as $\varepsilon \to 0$, we get the estimate

$$\|w_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon\| = O(x_\varepsilon)\|f\|.$$

3.4. Convergence of $\tilde{S}_\varepsilon$

We are now in a position to prove the main result of this section.

THEOREM 3.2. *Denote by S_0 the operator $S_- \oplus S_+$ in the non-resonant case and the operator $S(\theta)$ with $\theta := f_-/f_+$ in the resonant case. Then, the operator family $\tilde{S}_\varepsilon$ converges in the norm-resolvent sense as $\varepsilon \to 0$ to S_0.*

Proof. Fix $k^2 \in \mathbb{C} \setminus \mathbb{R}$, take an arbitrary $f \in L_2(\mathbb{R})$ and set $y_0 := (S_0 - k^2)^{-1}f$. As the first step, we construct a function $q_\varepsilon \in L_2(\mathbb{R})$ with $\|q_\varepsilon\| = o(\|f\|)$ as $\varepsilon \to 0$, such that the function $y_\varepsilon := y_0 + q_\varepsilon$ belongs to $\mathrm{dom}\,\tilde{S}_\varepsilon$ and, as $\varepsilon \to 0$, the relation

$$\|(\tilde{S}_\varepsilon - k^2)y_\varepsilon - f\| = o(\|f\|)$$

holds. We then show that this yields the required norm-resolvent convergence.

To construct such a y_ε, we shall modify z_ε at the points $\pm x_\varepsilon$ to eliminate the jumps. To this end, let us introduce functions φ and ψ, as in figure 2, that are smooth outside the origin, have compact supports contained in $[0, \infty)$ and have the prescribed jumps $[\varphi]_0 = 1$, $[\varphi']_0 = 0$ and $[\psi]_0 = 0$, $[\psi']_0 = 1$. Set

$$\zeta_\varepsilon(x) = [z_\varepsilon]_{-x_\varepsilon}\varphi(-x - x_\varepsilon) - [z'_\varepsilon]_{-x_\varepsilon}\psi(-x - x_\varepsilon) - [z_\varepsilon]_{x_\varepsilon}\varphi(x - x_\varepsilon) - [z'_\varepsilon]_{x_\varepsilon}\psi(x - x_\varepsilon).$$

Then, $\zeta_\varepsilon = 0$ on $(-x_\varepsilon, x_\varepsilon)$ and, in view of the estimates on $[z_\varepsilon]_{\pm x_\varepsilon}$ and $[z'_\varepsilon]_{\pm x_\varepsilon}$ of subsections 3.2 and 3.3,

$$\max_{|x| > x_\varepsilon} |\zeta_\varepsilon^{(k)}(x)| = o(\|f\|) \tag{3.6}$$

as $\varepsilon \to 0$ for $k = 0, 1, 2$.

Clearly, the function $y_\varepsilon := z_\varepsilon + \zeta_\varepsilon$ is continuous on $\mathbb{R}$, along with its derivative, and belongs to $W_2^2(\mathbb{R}) = \mathrm{dom}\, S_\varepsilon$. Observe that $y_\varepsilon = y_0$ for $|x|$ large enough; more exactly, we have that

$$y_\varepsilon = y_0 + q_\varepsilon = (S_0 - k^2)^{-1}f + q_\varepsilon$$

with the function

$$q_\varepsilon := [w_\varepsilon(\cdot/\varepsilon) + u_\varepsilon(\cdot/\varepsilon) - y_0]\chi_\varepsilon + \zeta_\varepsilon$$

of compact support. Taking into account lemma 3.1, the estimates on $\|w_\varepsilon(\cdot/\varepsilon)\chi_\varepsilon\|$ derived above and relations (3.5) and (3.6), we see that the L_2-norm of the function q_ε is $o(\|f\|)$.

Next, we calculate the function $(\tilde{S}_\varepsilon - k^2)y_\varepsilon$ for $|x| < x_\varepsilon$ and $|x| > x_\varepsilon$. If $|x| < x_\varepsilon$, then $\zeta_\varepsilon(x) = 0$, whence $y_\varepsilon(x) = z_\varepsilon(x)$ and

$$(\tilde{S}_\varepsilon - k^2)y_\varepsilon(x) = \varepsilon^{-2}\left[-z_\varepsilon''\left(\frac{x}{\varepsilon}\right) + V\left(\frac{x}{\varepsilon}\right)z_\varepsilon\left(\frac{x}{\varepsilon}\right)\right] - k^2 z_\varepsilon\left(\frac{x}{\varepsilon}\right) = f(x).$$

If $|x| > x_\varepsilon$, then $y_\varepsilon = y_0 + \zeta_\varepsilon$ and, hence,

$$\begin{aligned}
(\tilde{S}_\varepsilon - k^2)y_\varepsilon(x) &= \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - k^2\right)(y_0 + \zeta_\varepsilon)(x) \\
&= f(x) - \zeta_\varepsilon''(x) - k^2\zeta_\varepsilon(x).
\end{aligned}$$

Therefore, $(\tilde{S}_\varepsilon - k^2)y_\varepsilon = f + r_\varepsilon$, with the remainder

$$r_\varepsilon(x) = -\zeta_\varepsilon''(x) - k^2\zeta_\varepsilon(x).$$

Using (3.6), we find that, as $\varepsilon \to 0$,

$$\|r_\varepsilon\| = o(\|f\|).$$

Finally, since $\|(\tilde{S}_\varepsilon - k^2)^{-1}\| \leqslant |\mathrm{Im}\, k^2|^{-1}$, it follows that

$$\begin{aligned}
\|(\tilde{S}_\varepsilon - k^2)^{-1}f - (S_0 - k^2)^{-1}f\| &= \|\tilde{y}_\varepsilon - y_0\| \\
&\leqslant \|q_\varepsilon\| + \|(\tilde{S}_\varepsilon - k^2)^{-1}r_\varepsilon\| \\
&= o(\|f\|)
\end{aligned}$$

as $\varepsilon \to 0$, and the proof is complete. $\qquad\square$

4. Jost solutions and scattering coefficients for the operator $\tilde{S}_\varepsilon$

Our next task is to show that the resolvents of the operators $\tilde{S}_\varepsilon$ and S_ε get closer as ε tends to zero. To do this, we need more information on the resolvent of the operator $\tilde{S}_\varepsilon$.

We denote by $\tilde{f}_\pm(\cdot, \varepsilon, k)$ the Jost solutions of the operator $\tilde{S}_\varepsilon$. Then, the resolvent $\tilde{R}_\varepsilon(k) := (\tilde{S}_\varepsilon - k^2)^{-1}$ is an integral operator with kernel equal to the Green function

$$\tilde{G}_\varepsilon(x, y, k) = \frac{1}{\tilde{D}_\varepsilon(k)}\begin{cases} \tilde{f}_+(x, \varepsilon, k)\tilde{f}_-(y, \varepsilon, k), & x > y, \\ \tilde{f}_-(x, \varepsilon, k)\tilde{f}_+(y, \varepsilon, k), & x < y, \end{cases}$$

where $\tilde{D}_\varepsilon(k)$ is the Wronskian of the Jost solutions $\tilde{f}_+(\cdot, \varepsilon, k)$ and $\tilde{f}_-(\cdot, \varepsilon, k)$, i.e.

$$\tilde{D}_\varepsilon(k) = \tilde{f}_+(\cdot, \varepsilon, k)\tilde{f}'_-(\cdot, \varepsilon, k) - \tilde{f}'_+(\cdot, \varepsilon, k)\tilde{f}_-(\cdot, \varepsilon, k).$$

In the next section, we shall construct the Jost solutions $\tilde{f}_\pm(\cdot, \varepsilon, k)$ of the operator $\tilde{S}_\varepsilon$ for a fixed $k \in \mathbb{C}$ with $\operatorname{Im} k > 0$ and $\operatorname{Re} k > 0$, and then study their behaviour as $\varepsilon \to 0$. The analysis of the right and left Jost solutions is similar; only the case of $\tilde{f}_+$ will be investigated in detail. As a by-product, we establish some estimates on $\tilde{f}_\pm(\cdot, \varepsilon, k)$ that will be used in § 5 to prove the main result.

4.1. Jost solutions: a general construction

Since the potential of $\tilde{S}_\varepsilon$ vanishes for $|x| > x_\varepsilon$, we consider three regions separately: $x > x_\varepsilon$, $|x| < x_\varepsilon$ and $x < -x_\varepsilon$.

Region 1

For $x > x_\varepsilon$, we clearly have that

$$\tilde{f}_+(x, \varepsilon, k) = \mathrm{e}^{\mathrm{i}kx}.$$

Region 2

For $|x| < x_\varepsilon$, the Jost solution satisfies

$$-y'' + \varepsilon^{-2}V(x/\varepsilon)y = k^2 y$$

and thus is equal to

$$\tilde{f}_+(x, \varepsilon, k) = c_\varepsilon^+ f_+(x/\varepsilon, \varepsilon k) + c_\varepsilon^- f_-(x/\varepsilon, \varepsilon k),$$

where $f_\pm$ are the Jost solutions of (2.1), i.e. of the Schrödinger operator S_1. Continuity of $\tilde{f}_+$ and its derivative at the point $x = x_\varepsilon$ results in the system

$$c_\varepsilon^+ f_+(\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f_-(\xi_\varepsilon, \varepsilon k) = \mathrm{e}^{\mathrm{i}kx_\varepsilon},$$
$$c_\varepsilon^+ f'_+(\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f'_-(\xi_\varepsilon, \varepsilon k) = \mathrm{i}\varepsilon k\mathrm{e}^{\mathrm{i}kx_\varepsilon}.$$

Recall that

$$D(k) := f_+(\xi, k)f'_-(\xi, k) - f'_+(\xi, k)f_-(\xi, k)$$

is the Wronskian of the Jost solutions f_+ and f_-, and observe that $D(\varepsilon k) \neq 0$ for every positive ε, since otherwise $f_+(\cdot, \varepsilon k)$ and $f_-(\cdot, \varepsilon k)$ would be linearly dependent and thus $\varepsilon^2 k^2 \in \mathbb{C}^+$ would be an eigenvalue of the self-adjoint operator S_1. Solving the above system for $c_\varepsilon^\pm$, one gets that

$$c_\varepsilon^+ = \frac{\mathrm{e}^{\mathrm{i}kx_\varepsilon}}{D(\varepsilon k)}[f'_-(\xi_\varepsilon, \varepsilon k) - \mathrm{i}\varepsilon k f_-(\xi_\varepsilon, \varepsilon k)],$$

$$c_\varepsilon^- = \frac{\mathrm{e}^{\mathrm{i}kx_\varepsilon}}{D(\varepsilon k)}[\mathrm{i}\varepsilon k f_+(\xi_\varepsilon, \varepsilon k) - f'_+(\xi_\varepsilon, \varepsilon k)].$$

In the non-resonant case, $D(0) \neq 0$, and thus $|D(\varepsilon k)| \geqslant |D(0)|/2$ for all ε small enough by the continuity of D. Then, $c_\varepsilon^- \to 0$ by (2.4), while $c_\varepsilon^+ \to 1$ by lemma 2.3. In the resonant case, we combine lemma 2.3 with (2.17) and get the same convergence $c_\varepsilon^- \to 0$ and $c_\varepsilon^+ \to 1$ as $\varepsilon \to 0$.

Region 3

For $x < -x_\varepsilon$,

$$\tilde{f}_+(x, \varepsilon, k) = a_\varepsilon^+ \mathrm{e}^{\mathrm{i}kx} + b_\varepsilon^+ \mathrm{e}^{-\mathrm{i}kx}$$

for some coefficients a_ε^+ and b_ε^+. Continuity of the Jost solution $\tilde{f}_+$ and its derivative at $x = -x_\varepsilon$ gives the following linear system for a_ε^+ and b_ε^+:

$$a_\varepsilon^+ \mathrm{e}^{-\mathrm{i}kx_\varepsilon} + b_\varepsilon^+ \mathrm{e}^{\mathrm{i}kx_\varepsilon} = \tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k),$$
$$\mathrm{i}k(a_\varepsilon^+ \mathrm{e}^{-\mathrm{i}kx_\varepsilon} - b_\varepsilon^+ \mathrm{e}^{\mathrm{i}kx_\varepsilon}) = \tilde{f}_+'(-x_\varepsilon + 0, \varepsilon, k),$$

solving which, we find that

$$a_\varepsilon^+ = \frac{\mathrm{e}^{\mathrm{i}kx_\varepsilon}}{2\mathrm{i}k}[\mathrm{i}k\tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k) + \tilde{f}_+'(-x_\varepsilon + 0, \varepsilon, k)],$$
$$b_\varepsilon^+ = \frac{\mathrm{e}^{-\mathrm{i}kx_\varepsilon}}{2\mathrm{i}k}[\mathrm{i}k\tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k) - \tilde{f}_+'(-x_\varepsilon + 0, \varepsilon, k)].$$

REMARK 4.1. Calculating the Wronskian $\tilde{D}_\varepsilon(k)$ of the Jost solutions $\tilde{f}_\pm$ at a point x to the left of $-x_\varepsilon$, one immediately sees that $\tilde{D}_\varepsilon(k) = -2\mathrm{i}ka_\varepsilon^+$.

To investigate the behaviour of the coefficients a_ε^+ and b_ε^+, we have to consider the resonant and non-resonant cases separately.

4.2. Scattering coefficients in the non-resonant case

First we note that, as $\varepsilon \to 0$,

$$\tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k) = c_\varepsilon^+ f_+(-\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f_-(-\xi_\varepsilon, \varepsilon k) = O(\xi_\varepsilon)$$

and that, in view of (2.9),

$$\varepsilon \tilde{f}_+'(-x_\varepsilon + 0, \varepsilon, k) = c_\varepsilon^+ f_+'(-\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f_-'(-\xi_\varepsilon, \varepsilon k) \to -D(0).$$

It therefore follows that

$$\varepsilon a_\varepsilon^+ \to -D(0), \qquad \varepsilon b_\varepsilon^+ \to D(0)$$

as $\varepsilon \to 0$. The above analysis remains in force for real k; therefore, one gets the following result.

COROLLARY 4.2. *Assume that the potential V is non-resonant. Then, for every non-zero k with $\mathrm{Im}\, k \geqslant 0$, the reflection and transmission coefficients of the Schrödinger operator $\tilde{S}_\varepsilon$, given by $\tilde{r}_\varepsilon(k) := b_\varepsilon^+/a_\varepsilon^+$ and $\tilde{t}_\varepsilon(k) := 1/a_\varepsilon^+$, respectively, satisfy the asymptotic relations $\tilde{r}_\varepsilon(k) \to -1$ and $\tilde{t}_\varepsilon(k) \to 0$ as $\varepsilon \to 0$.*

This can be compared with the analogous result in [23, 24, 39], proved for the operator family S_ε in the case where the support of V is contained in $[-1, 1]$.

4.3. Scattering coefficients in the resonant case

As $\varepsilon \to 0$, we have that

$$\tilde{f}_+(-x_\varepsilon + 0, \varepsilon, k) = c_\varepsilon^+ f_+(-\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- f_-(-\xi_\varepsilon, \varepsilon k) \to \theta^{-1}$$

by (2.6) and (2.15), while (2.8) and (2.16) yield

$$\tilde{f}'_+(-x_\varepsilon + 0, \varepsilon, k) = c_\varepsilon^+ \varepsilon^{-1} f'_+(-\xi_\varepsilon, \varepsilon k) + c_\varepsilon^- \varepsilon^{-1} f'_-(-\xi_\varepsilon, \varepsilon k) \to \mathrm{i}k\theta.$$

It therefore follows that, as $\varepsilon \to 0$,

$$a_\varepsilon^+ \to \tfrac{1}{2}[\theta^{-1} + \theta], \qquad b_\varepsilon^+ \to \tfrac{1}{2}[\theta^{-1} - \theta].$$

The above analysis remains in force for real k; therefore, one arrives at the following result.

COROLLARY 4.3. *Assume that the potential V is resonant and that θ is given by (1.6) (or, equivalently, by (2.11)). Then, for every non-zero k with $\mathrm{Re}\, k \geqslant 0$, the reflection and transmission coefficients of the Schrödinger operator $\tilde{S}_\varepsilon$, given by $\tilde{r}_\varepsilon(k) := b_\varepsilon^+/a_\varepsilon^+$ and $\tilde{t}_\varepsilon(k) := 1/a_\varepsilon^+$, respectively, satisfy the asymptotic relations*

$$\tilde{r}_\varepsilon(k) \to \frac{1 - \theta^2}{1 + \theta^2}, \quad \tilde{t}_\varepsilon(k) \to \frac{2\theta}{1 + \theta^2} \quad \text{as } \varepsilon \to 0.$$

We note that the above limits are the reflection and transmission coefficients of the Schrödinger operator $S(\theta)$ and that they coincide with the value at $k = 0$ of the reflection and transmission coefficients for the Schrödinger operator S_1; see [32]. This can also be compared with the analogous result in [23, 24, 39], proved for the operator family S_ε in the case where the support of V is contained in $[-1, 1]$.

4.4. Some useful estimates

We conclude § 4 by establishing several estimates that will be used in § 5.

LEMMA 4.4. *There exist constants K_1 and K_2 such that the following hold.*

(i) *For all sufficiently small ε and all x with $|x| > x_\varepsilon$,*

$$|\tilde{D}_\varepsilon^{-1}(k)\tilde{f}_\pm(x, \varepsilon, k)| \leqslant K_1 |\mathrm{e}^{\pm \mathrm{i}kx}|.$$

(ii) *For all sufficiently small ε and all $x \in \mathbb{R}$,*

$$\int_x^\infty \frac{|\tilde{f}_+(t, \varepsilon, k)|^2}{|\tilde{D}_\varepsilon(k)|^2} \, \mathrm{d}t \leqslant K_2 |\mathrm{e}^{2\mathrm{i}kx}|,$$

$$\int_{-\infty}^x \frac{|\tilde{f}_-(t, \varepsilon, k)|^2}{|\tilde{D}_\varepsilon(k)|^2} \, \mathrm{d}t \leqslant K_2 |\mathrm{e}^{-2\mathrm{i}kx}|.$$

Proof. We only give details for the right Jost solution $\tilde{f}_+(\cdot, \varepsilon, k)$, as the case of $\tilde{f}_-(\cdot, \varepsilon, k)$ is completely analogous. Condition (i) follows from the fact that, for $x > x_\varepsilon$, $\tilde{f}(x, \varepsilon, k) = \mathrm{e}^{\mathrm{i}kx}$ and that, for $x < -x_\varepsilon$,

$$\frac{\tilde{f}(x, \varepsilon, k)}{\tilde{D}_\varepsilon(k)} = -\frac{1}{2\mathrm{i}k} \mathrm{e}^{\mathrm{i}kx} - \frac{\tilde{r}_\varepsilon(k)}{2\mathrm{i}k} \mathrm{e}^{-\mathrm{i}kx},$$

where $\tilde{r}_\varepsilon(k)$ has a finite limit as $\varepsilon \to 0$ by corollary 4.2 in the non-resonant case, and by corollary 4.3 in the resonant one.

To prove (ii), we first estimate the integral over $(-x_\varepsilon, x_\varepsilon)$. Denote by P_ε the operator of multiplication by the function χ_ε. Then, P_ε is an orthogonal projector and $\|P_\varepsilon h\| \to 0$ as $\varepsilon \to 0$ for every $h \in L_2(\mathbb{R})$. In view of theorem 3.2, we find that

$$P_\varepsilon(\tilde{S}_\varepsilon - k^2)^{-1}h = P_\varepsilon[(\tilde{S}_\varepsilon - k^2)^{-1} - (S_0 - k^2)^{-1}]h + P_\varepsilon(S_0 - k^2)^{-1}h \to 0$$

in $L_2(\mathbb{R})$ as $\varepsilon \to 0$. Now, take h to be the characteristic function of the interval $[-3, -1]$. Then, for all ε such that $x_\varepsilon < 1$ we calculate that

$$P_\varepsilon(\tilde{S}_\varepsilon - k^2)^{-1}h(x) = \frac{\chi_\varepsilon(x)\tilde{f}_+(x, \varepsilon, k)}{\tilde{D}_\varepsilon(k)} \int_{-3}^{-1} \mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t$$

$$= \frac{2\mathrm{e}^{\mathrm{i}k}\sin k}{k} \frac{\chi_\varepsilon(x)\tilde{f}_+(x, \varepsilon, k)}{\tilde{D}_\varepsilon(k)}.$$

Thus, it follows that

$$\|P_\varepsilon(\tilde{S}_\varepsilon - k^2)^{-1}h\|^2 = \frac{4|\mathrm{e}^{\mathrm{i}k}\sin k|^2}{|k|^2} \int_{-x_\varepsilon}^{x_\varepsilon} \frac{|f_+(x, \varepsilon, k)|^2}{|\tilde{D}_\varepsilon(k)|^2}\,\mathrm{d}x \to 0$$

as $\varepsilon \to 0$.

Returning to (ii), we choose ε_0 so small that $|\mathrm{e}^{2\mathrm{i}kx}| > \frac{1}{2}$ for all $x \leqslant x_{\varepsilon_0}$. If $x > x_{\varepsilon_0}$, the desired inequality holds by (i) for all $\varepsilon < \varepsilon_0$ with $K_2 = K_1^2/(2|\mathrm{Im}\,k|)$. Otherwise, we use (i) and the above limit to get the estimate

$$\int_x^\infty \frac{|\tilde{f}_+(t, \varepsilon, k)|^2}{|\tilde{D}_\varepsilon^2(k)|}\,\mathrm{d}t \leqslant K_1^2 \int_x^\infty |\mathrm{e}^{2\mathrm{i}kt}|\,\mathrm{d}t + \int_{-x_\varepsilon}^{x_\varepsilon} \frac{|f_+(t, \varepsilon, k)|^2}{|\tilde{D}_\varepsilon(k)|^2}\,\mathrm{d}t$$

$$\leqslant K_2|\mathrm{e}^{2\mathrm{i}kx}|,$$

with $K_2 := 1 + K_1^2/(2|\mathrm{Im}\,k|)$ holding for all sufficiently small ε. $\qquad\square$

5. Proximity of the operator families S_ε and $\tilde{S}_\varepsilon$

In this section we shall establish the proximity of the operator families S_ε and $\tilde{S}_\varepsilon$ in the norm-resolvent sense, i.e. we shall prove the following theorem.

THEOREM 5.1. *For every $k^2 \in \mathbb{C} \setminus \mathbb{R}$, it holds that*

$$\|(S_\varepsilon - k^2)^{-1} - (\tilde{S}_\varepsilon - k^2)^{-1}\| \to 0 \tag{5.1}$$

as $\varepsilon \to 0$.

Clearly, combining theorems 3.2 and 5.1 gives a complete proof of the claimed convergence of S_ε to S_0.

Set

$$u_\varepsilon(x) := \varepsilon^{-1}|V(x/\varepsilon)|^{1/2}[1 - \chi_\varepsilon(x)] \quad \text{and} \quad w_\varepsilon(x) := \mathrm{sgn}(V(x/\varepsilon))u_\varepsilon(x)$$

and denote by U_ε and W_ε the operators of multiplications by u_ε and w_ε, respectively. The operators U_ε and W_ε are in general unbounded; note, however, that the L_2-norm of the functions u_ε and w_ε vanishes as $\varepsilon \to 0$. Indeed, in view of lemma 2.2

we deduce that

$$\|u_\varepsilon\|^2 = \|w_\varepsilon\|^2 = \frac{1}{\varepsilon^2} \int_{|t| > x_\varepsilon} \left| V\left(\frac{t}{\varepsilon}\right) \right| \mathrm{d}t$$

$$= \frac{1}{\varepsilon} \int_{|s| > \xi_\varepsilon} |V(s)| \, \mathrm{d}s$$

$$\leqslant \frac{1}{\varepsilon} \int_{|s| > \xi_\varepsilon} \frac{\rho_V(s)}{\rho_V(\xi_\varepsilon)} |V(s)| \, \mathrm{d}s$$

$$= \int_{|s| > \xi_\varepsilon} \rho_V(s) |V(s)| \, \mathrm{d}s$$

$$= o(1) \tag{5.2}$$

as $\varepsilon \to 0$.

The relation $S_\varepsilon = \tilde{S}_\varepsilon + U_\varepsilon W_\varepsilon$ yields the following well-known (formal) representation of the resolvent $R_\varepsilon(k) := (S_\varepsilon - k^2)^{-1}$ of S_ε in terms of the resolvent $\tilde{R}_\varepsilon(k) := (\tilde{S}_\varepsilon - k^2)^{-1}$ of $\tilde{S}_\varepsilon$ and the perturbation:

$$R_\varepsilon(k) - \tilde{R}_\varepsilon(k) = \tilde{R}_\varepsilon(k) W_\varepsilon [I + U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon]^{-1} U_\varepsilon \tilde{R}_\varepsilon(k). \tag{5.3}$$

We shall prove that the norms of the operators $\tilde{R}_\varepsilon(k) W_\varepsilon$, $U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon$ and $U_\varepsilon \tilde{R}_\varepsilon(k)$ vanish as $\varepsilon \to 0$; this will justify both (5.3) and the claim of theorem 5.1.

5.1. The norm of special integral operators

It is immediate to see that the operators $\tilde{R}_\varepsilon(k) W_\varepsilon$, $U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon$ and $U_\varepsilon \tilde{R}_\varepsilon(k)$ are integral ones and act on a function $y \in L_2(\mathbb{R})$ via

$$\frac{\varphi_+(x)}{\tilde{D}_\varepsilon(k)} \int_{-\infty}^x \varphi_-(t) y(t) \, \mathrm{d}t + \frac{\varphi_-(x)}{\tilde{D}_\varepsilon(k)} \int_x^\infty \varphi_+(t) y(t) \, \mathrm{d}t,$$

where φ_+ equals one of the functions $\tilde{f}_+(\cdot, \varepsilon, k)$, $\tilde{f}_+(\cdot, \varepsilon, k) w_\varepsilon$ or $\tilde{f}_+(\cdot, \varepsilon, k) u_\varepsilon$, and similarly for φ_-. To estimate the norm of such operators, we use the following result (see, for example, [37]).

PROPOSITION 5.2. *Assume that ψ_- and ψ_+ are functions belonging to $L_{2,\mathrm{loc}}^2(\mathbb{R})$, and let $T_\pm$ be integral operators defined via*

$$T_+ y(x) = \psi_-(x) \int_x^\infty \psi_+(t) y(t) \, \mathrm{d}t$$

and

$$T_- y(x) = \psi_+(x) \int_{-\infty}^x \psi_-(t) y(t) \, \mathrm{d}t,$$

respectively. Then, $T_\pm$ are bounded in $L_2(\mathbb{R})$ if and only if the quantity

$$K := \sup_{x \in \mathbb{R}} \left(\int_{-\infty}^x |\psi_-(t)|^2 \, \mathrm{d}t \int_x^\infty |\psi_+(t)|^2 \, \mathrm{d}t \right)^{1/2}$$

is finite; in this case $\|T_\pm\| \leqslant 2K$.

5.2. The operator $U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon$

The operator $U_\varepsilon \tilde{R}_\varepsilon W_\varepsilon$ is integral and has the form

$$U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon y(x) = \frac{u_\varepsilon(x)\tilde{f}_+(x,\varepsilon,k)}{\tilde{D}_\varepsilon(k)} \int_{-\infty}^x y(t) w_\varepsilon(t) \tilde{f}_-(t,\varepsilon,k)\,\mathrm{d}t$$
$$+ \frac{u_\varepsilon(x)\tilde{f}_-(x,\varepsilon,k)}{\tilde{D}_\varepsilon(k)} \int_x^\infty y(t) w_\varepsilon(x) \tilde{f}_+(t,\varepsilon,k)\,\mathrm{d}t.$$

By virtue of proposition 5.2 and the relation $|u_\varepsilon| = |w_\varepsilon|$, we conclude that the norm of $U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon$ vanishes as $\varepsilon \to 0$, provided this is true for the quantity $\sup_{x \in \mathbb{R}} K_\varepsilon(x)$, with

$$K_\varepsilon(x) := \frac{1}{|\tilde{D}_\varepsilon(k)|^2} \int_{-\infty}^x |u_\varepsilon(t)\tilde{f}_-(t,\varepsilon,k)|^2\,\mathrm{d}t \int_x^\infty |u_\varepsilon(t)\tilde{f}_+(t,\varepsilon,k)|^2\,\mathrm{d}t.$$

For $x \geqslant x_\varepsilon$, we have that

$$\int_x^\infty |u_\varepsilon(t)\tilde{f}_+(t,\varepsilon,k)|^2\,\mathrm{d}t = \int_x^\infty |u_\varepsilon(t)\mathrm{e}^{\mathrm{i}kt}|^2\,\mathrm{d}t \leqslant |\mathrm{e}^{2\mathrm{i}kx}| \|u_\varepsilon\|^2.$$

Similarly, by lemma 4.4(i),

$$\frac{1}{|\tilde{D}_\varepsilon(k)|^2} \int_{-\infty}^x |u_\varepsilon(t)\tilde{f}_-(t,\varepsilon,k)|^2\,\mathrm{d}t \leqslant K_1^2 |\mathrm{e}^{-2\mathrm{i}kx}| \|u_\varepsilon\|^2. \tag{5.4}$$

Therefore,

$$\sup_{x \geqslant x_\varepsilon} K_\varepsilon(x) \leqslant K_1^2 \|u_\varepsilon\|^4 \to 0$$

as $\varepsilon \to 0$, due to (5.2). For $x \leqslant -x_\varepsilon$, one can use the relation $\tilde{f}_-(x,\varepsilon,k) = \mathrm{e}^{-\mathrm{i}kx}$ and the estimate of lemma 4.4(i) to conclude that

$$\sup_{x \leqslant -x_\varepsilon} K_\varepsilon(x) \leqslant K_1^2 \|u_\varepsilon\|^4 \to 0$$

as $\varepsilon \to 0$. Finally,

$$\sup_{|x| < x_\varepsilon} K_\varepsilon(x) \leqslant \frac{|\mathrm{e}^{4\mathrm{i}kx_\varepsilon}| \|u_\varepsilon\|^4}{|\tilde{D}_\varepsilon(k)|^2} \to 0$$

as $\varepsilon \to 0$, thus proving the following statement.

LEMMA 5.3. *The norm of the operator $U_\varepsilon \tilde{R}_\varepsilon(k) W_\varepsilon$ vanishes as $\varepsilon \to 0$.*

5.3. The operators $\tilde{R}_\varepsilon(k) W_\varepsilon$ and $U_\varepsilon \tilde{R}_\varepsilon(k)$

We need to show that both $\sup_{x \in \mathbb{R}} K_\varepsilon^{(1)}(x)$ and $\sup_{x \in \mathbb{R}} K_\varepsilon^{(2)}(x)$ vanish as $\varepsilon \to 0$, with

$$K_\varepsilon^{(1)}(x) := \frac{1}{\tilde{D}_\varepsilon^2(k)} \int_{-\infty}^x |u_\varepsilon(t)\tilde{f}_-(t,\varepsilon,k)|^2\,\mathrm{d}t \int_x^\infty |\tilde{f}_+(t,\varepsilon,k)|^2\,\mathrm{d}t$$

and

$$K_\varepsilon^{(2)}(x) := \frac{1}{\tilde{D}_\varepsilon^2(k)} \int_{-\infty}^x |\tilde{f}_-(t,\varepsilon,k)|^2\,\mathrm{d}t \int_x^\infty |u_\varepsilon(t)\tilde{f}_+(t,\varepsilon,k)|^2\,\mathrm{d}t.$$

Since both quantities can be estimated in a similar way, only the first will be treated in detail.

For $x \geqslant x_\varepsilon$, we have that

$$\int_x^\infty |\tilde{f}_+(t, \varepsilon, k)|^2 \, \mathrm{d}t = \frac{1}{2|\operatorname{Im} k|} |\mathrm{e}^{2\mathrm{i}kx}|,$$

which together with (5.4) results in

$$\sup_{x \geqslant x_\varepsilon} K_\varepsilon^{(1)}(x) \leqslant \frac{K_1^2}{2|\operatorname{Im} k|} \|u_\varepsilon\|^2.$$

For $x < x_\varepsilon$, we use lemma 4.4(ii) and the inequality

$$\int_{-\infty}^x |u_\varepsilon(t)\tilde{f}_-(t, \varepsilon, k)|^2 \, \mathrm{d}t \leqslant |\mathrm{e}^{-2\mathrm{i}kx}| \|u_\varepsilon\|^2$$

to conclude that

$$\sup_{x < x_\varepsilon} K_\varepsilon^{(1)}(x) \leqslant K_2 \|u_\varepsilon\|^2.$$

Combining the above estimates with (5.2), we arrive at the following conclusion.

LEMMA 5.4. *Under the standing assumptions,*

$$\lim_{\varepsilon \to 0} (\|U_\varepsilon \tilde{R}_\varepsilon(k)\| + \|\tilde{R}_\varepsilon(k) W_\varepsilon(k)\|) = 0.$$

Proof of theorem 5.1. It suffices to note that lemmata 5.3 and 5.4 justify (5.3) and, in turn, establish the convergence in (5.1). $\qquad\square$

The main result of the paper is now easy to justify.

Proof of theorem 1.1. Clearly, the statements of theorems 3.2 and 5.1 immediately yield the norm-resolvent convergence to the limiting operator S_0, as $\varepsilon \to 0$, of the family of Schrödinger operators S_ε given by (1.1). $\qquad\square$

Acknowledgements

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