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A SPHERICAL VERSION OF THE KOWALSKI–SŁODKOWSKI THEOREM AND ITS APPLICATIONS

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Dedicated to Professor Osamu Hatori on the occasion of his retirement from Niigata University

Abstract

Li *et al.* ['Weak 2-local isometries on uniform algebras and Lipschitz algebras', *Publ. Mat.* **63** (2019), 241–264] generalized the Kowalski–Słodkowski theorem by establishing the following spherical variant: let A be a unital complex Banach algebra and let $\Delta : A \to \mathbb{C}$ be a mapping satisfying the following properties:

(a) Δ is 1-homogeneous (that is, $\Delta(\lambda x) = \lambda\Delta(x)$ for all $x \in A$, $\lambda \in \mathbb{C}$);
(b) $\Delta(x) - \Delta(y) \in \mathbb{T}\sigma(x - y), \quad x, y \in A$.

Then Δ is linear and there exists $\lambda_0 \in \mathbb{T}$ such that $\lambda_0\Delta$ is multiplicative. In this note we prove that if (a) is relaxed to $\Delta(0) = 0$, then Δ is complex-linear or conjugate-linear and $\overline{\Delta(\mathbf{1})}\Delta$ is multiplicative. We extend the Kowalski–Słodkowski theorem as a conclusion. As a corollary, we prove that every 2-local map in the set of all surjective isometries (without assuming linearity) on a certain function space is in fact a surjective isometry. This gives an affirmative answer to a problem on 2-local isometries posed by Molnár ['On 2-local *-automorphisms and 2-local isometries of *B(H)*', *J. Math. Anal. Appl.* **479**(1) (2019), 569–580] and also in a private communication between Molnár and O. Hatori, 2018.

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1. Introduction

One of the basic problems in operator theory is to find sufficient conditions to deduce linearity and multiplicativity of maps between Banach algebras. As a generalization of the Gleason–Kahane–Żelazko theorem [6, 14, 35], Kowalski and Słodkowski [16] proved the linearity and the multiplicativity of a functional Δ on a Banach algebra A under the spectral condition; $\Delta(a) - \Delta(b) \in \sigma(a - b)$ for $a, b \in A$. Recently, Li *et al*. proved interesting spherical variants of the Gleason–Kahane–Żelazko theorem and the Kowalski–Słodkowski theorem [17]. They proved that a 1-homogeneous functional on a unital Banach algebra that satisfies a mild spectral condition is linear. Applying it, they studied 2-local and weak 2-local complex-linear isometries.

Motivated by the Kowalski–Słodkowski theorem, the concept of a 2-local map was introduced by Šemrl [34], who proved the first results on 2-local automorphisms and derivations on algebras of operators. Molnár [22] began to study 2-local complex-linear isometries. Given a Banach space $\mathfrak{M}_j$ for $j = 1, 2$, an isometry from $\mathfrak{M}_1$ into $\mathfrak{M}_2$ is a distance-preserving map. The set of all surjective complex-linear isometries from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ is denoted by $\mathrm{Iso}_{\mathbb{C}}(\mathfrak{M}_1, \mathfrak{M}_2)$. The set of all maps from $\mathfrak{M}_1$ into $\mathfrak{M}_2$ is denoted by $M(\mathfrak{M}_1, \mathfrak{M}_2)$. We say that a map $T \in M(\mathfrak{M}_1, \mathfrak{M}_2)$ is a 2-local complex-linear isometry if for every $x, y \in \mathfrak{M}_1$ there is a $T_{x,y} \in \mathrm{Iso}_{\mathbb{C}}(\mathfrak{M}_1, \mathfrak{M}_2)$ such that $T(x) = T_{x,y}(x)$ and $T(y) = T_{x,y}(y)$. Molnár [22] proved that each 2-local complex-linear isometry on certain C^*-algebras is a surjective complex-linear isometry. Initiated by his result, there are a lot of studies on 2-local complex-linear isometries on operator algebras and function spaces assuring that each 2-local complex-linear isometry is in fact a surjective complex-linear isometry [1, 3, 7, 9, 12, 13, 17, 22, 24].

Molnár raised a problem on 2-local isometries [25, 26]. The set of all (non-necessarily linear) surjective isometries from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ is denoted by $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$. We say that $T \in M(\mathfrak{M}_1, \mathfrak{M}_2)$ is a 2-local isometry or T is 2-local in $\mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ if for every $x, y \in \mathfrak{M}_1$ there is a $T_{x,y} \in \mathrm{Iso}(\mathfrak{M}_1, \mathfrak{M}_2)$ such that

$$T(x) = T_{x,y}(x) \text{ and } T(y) = T_{x,y}(y).$$

The problem asks whether a 2-local isometry is in fact a surjective isometry or not. One may expect that the problems on 2-local *complex-linear* isometries and 2-local isometries are not so different. But the problem on 2-local isometries is very different from the one on 2-local complex-linear isometries. To clarify the situation, we exhibit an example showing that the assumption of linearity makes a quite big difference in the conclusion for 2-local maps. Let $\mathrm{A}(\mathbb{C}, \mathbb{C}) = \{T : \mathbb{C} \to \mathbb{C}; Tx = ax + b \ (\exists a, b \in \mathbb{C})\}$. Since any map $T : \mathbb{C} \to \mathbb{C}$ is 2-local in $\mathrm{A}(\mathbb{C}, \mathbb{C})$, T need not be in $\mathrm{A}(\mathbb{C}, \mathbb{C})$ in general. However, let

$$\mathrm{A}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) = \{T; T \in \mathrm{A}(\mathbb{C}, \mathbb{C}), \mathrm{T \ is \ } \mathbb{C}\text{-linear}\} = \{T : \mathbb{C} \to \mathbb{C}; Tx = ax \ (\exists a \in \mathbb{C})\}.$$

Then we get that every 2-local map in $\mathrm{A}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$ is an element of $\mathrm{A}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})$. We can easily prove that a 2-local isometry is necessarily an isometry. What we need to prove is that a 2-local isometry is surjective. One may think that it is not a difficult problem, but it is. Molnár [26] worked quite hard to prove that for each separable complex Hilbert space H, every 2-local isometry on $B(H)$ is in fact a surjective isometry on $B(H)$. The author believes that this is the first result on the problem of 2-local isometries. Molnár asked whether a 2-local map in $\mathrm{Iso}(C([0, 1]), C([0, 1]))$ is an element in $\mathrm{Iso}(C([0, 1]), C([0, 1]))$ or not [25]. Inspired by his problem, Hatori and the author proved that a 2-local map in $\mathrm{Iso}(B, B)$ is an element of $\mathrm{Iso}(B, B)$, where B is the Banach space of all continuously differentiable functions or the Banach space of Lipschitz functions on the closed unit interval equipped with a certain norm [10].

The aim of this paper is to establish a generalization of the spherical variant of the Kowalski–Słodkowski theorem exhibited in [17]. Applying it, we prove that 2-local isometries on several function spaces are surjective isometries. In particular, we give an affirmative answer to the problem posed by Molnár (Corollary 4.3). We remark that Mori [29] also got an affirmative answer to the problem by a different approach applying the theory of operator algebras.

In this paper, we denote the unit circle on the complex plane by $\mathbb{T} = \{z \in \mathbb{C}; |z| = 1\}$. For simplicity of notation, we denote $[f]^1 = f$ and $[f]^{-1} = \overline{f}$, the complex-conjugate of f for any complex-valued function f. For any unital Banach algebra, $\mathbf{1}$ stands for its unit element. The identity map is denoted by Id.

2. Generalization of the Kowalski–Słodkowski theorem

Li *et al.* [17] proved the spherical variant of the Kowalski–Słodkowski theorem; a 1-homogeneous functional that satisfies a certain spectral condition is complex-linear. The concrete result reads as follows.

THEOREM 2.1 (Li *et al.* [17]). *Let A be a unital complex Banach algebra and let $\Delta : A \to \mathbb{C}$ be a mapping satisfying the following properties:*

(a) *Δ is 1-homogeneous, that is, $\Delta(\lambda x) = \lambda\Delta(x)$ for all $x \in A$, $\lambda \in \mathbb{C}$;*
(b) $\Delta(x) - \Delta(y) \in \mathbb{T}\sigma(x - y), \quad x, y \in A.$

Then Δ is linear and there exists $\lambda_0 \in \mathbb{T}$ such that $\lambda_0\Delta$ is multiplicative.

In this note, we consider the case that the hypothesis (a) is relaxed to $\Delta(0) = 0$. This hypothesis is closer to the one of the original Kowalski–Słodkowski theorem; however; the conclusion also admits conjugete-linear maps.

THEOREM 2.2. *Let A be a unital complex Banach algebra. Suppose that a map $\Delta : A \to \mathbb{C}$ satisfies the conditions:*

(a) $\Delta(0) = 0$;
(b) $\Delta(x) - \Delta(y) \in \mathbb{T}\sigma(x - y), \quad x, y \in A.$

Then Δ is complex-linear or conjugate-linear and $\overline{\Delta(\mathbf{1})}\Delta$ is multiplicative.

Fix $a \in A$; we define a map $f : \mathbb{C} \to \mathbb{C}$ by $f(\lambda) = \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a)$. For any $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\Delta(a + \lambda_1 \cdot \mathbf{1}) - \Delta(a + \lambda_2 \cdot \mathbf{1}) \in \mathbb{T}\sigma((\lambda_1 - \lambda_2) \cdot \mathbf{1}) = (\lambda_1 - \lambda_2)\mathbb{T}\sigma(\mathbf{1}) = (\lambda_1 - \lambda_2)\mathbb{T},$$

by the assumption (b). Thus,

$$\begin{aligned} |f(\lambda_1) - f(\lambda_2)| &= |\Delta(a + \lambda_1 \cdot \mathbf{1}) - \Delta(a) - (\Delta(a + \lambda_2 \cdot \mathbf{1}) - \Delta(a))| \\ &= |\Delta(a + \lambda_1 \cdot \mathbf{1}) - \Delta(a + \lambda_2 \cdot \mathbf{1})| \\ &= |\lambda_1 - \lambda_2|. \end{aligned}$$

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This implies that the map f is an isometry on $\mathbb{C}$. The form of an isometry on $\mathbb{C}$ is well known. Without assuming surjectivity on the isometry, there exist $\alpha, \beta \in \mathbb{C}$ with $|\alpha| = 1$ such that $f(\lambda) = \beta + \lambda\alpha$ ($\lambda \in \mathbb{C}$) or $f(\lambda) = \beta + \bar{\lambda}\alpha$ ($\lambda \in \mathbb{C}$). Since

$$f(0) = \Delta(a + 0 \cdot \mathbf{1}) - \Delta(a) = \Delta(a) - \Delta(a) = 0,$$

$$f(\lambda) = \lambda\alpha, \quad \lambda \in \mathbb{C},$$

or

$$f(\lambda) = \bar{\lambda}\alpha, \quad \lambda \in \mathbb{C}.$$

In addition, we have $\alpha = f(1) = \Delta(a + \mathbf{1}) - \Delta(a)$ and we infer that

$$\Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \lambda(\Delta(a + \mathbf{1}) - \Delta(a)), \quad \lambda \in \mathbb{C},$$

or

$$\Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \bar{\lambda}(\Delta(a + \mathbf{1}) - \Delta(a)), \quad \lambda \in \mathbb{C}.$$

Let

$$A_1 = \{a \in A; \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \lambda(\Delta(a + \mathbf{1}) - \Delta(a)),\ \lambda \in \mathbb{C}\}$$

and

$$A_{-1} = \{a \in A; \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \bar{\lambda}(\Delta(a + \mathbf{1}) - \Delta(a)),\ \lambda \in \mathbb{C}\}.$$

For any $a \in A$, the map $\lambda \mapsto \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a)$ is an isometry on $\mathbb{C}$, so we have $A = A_1 \cup A_{-1}$.

Lemma 2.3. *We have $A = A_1$ or $A = A_{-1}$.*

Proof. We have proved that $A = A_1 \cup A_{-1}$. We prove that A_1 and A_{-1} are closed subsets of A. Let $\{a_n\}$ be a sequence in A_1 converging to a point $a_0 \in A$. By assumption (b), we have $\Delta(a_n) - \Delta(a_0) \in \mathbb{T}\sigma(a_n - a_0)$. Hence, $|\Delta(a_n) - \Delta(a_0)| \le r(a_n - a_0)$ for the spectral radius $r(\cdot)$. Since $r(\cdot) \le \|\cdot\|$ for the original norm $\|\cdot\|$ on A, we get that $\Delta(a_n) - \Delta(a_0) \to 0$ as $n \to \infty$. In the same way, we have $\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_n + \lambda \cdot \mathbf{1})$ converges to 0 for $\lambda \in \mathbb{C}$. Thus, for any $\lambda \in \mathbb{C}$,

$$\begin{aligned}
&|\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_0) - \lambda(\Delta(a_0 + \mathbf{1}) - \Delta(a_0))| \\
&\quad = |\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_0) - (\Delta(a_n + \lambda \cdot \mathbf{1}) - \Delta(a_n)) \\
&\qquad + \lambda(\Delta(a_n + \mathbf{1}) - \Delta(a_n)) - \lambda(\Delta(a_0 + \mathbf{1}) - \Delta(a_0))| \\
&\quad \le |\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_n + \lambda \cdot \mathbf{1})| + |\Delta(a_0) - \Delta(a_n)| \\
&\qquad + |\lambda||\Delta(a_n + \mathbf{1}) - \Delta(a_0 + \mathbf{1})| + |\lambda||\Delta(a_n) - \Delta(a_0)| \\
&\quad \to 0
\end{aligned}$$

as $n \to \infty$. This implies that $\Delta(a_0 + \lambda \cdot \mathbf{1}) - \Delta(a_0) = \lambda(\Delta(a_0 + \mathbf{1}) - \Delta(a_0))$ for any $\lambda \in \mathbb{C}$. Since $a_0 \in A_1$, we have that A_1 is closed. We can prove that A_{-1} is also closed in the same way. In addition, suppose that $a \in A_1 \cap A_{-1}$. Then, for any $\lambda \in \mathbb{C}$,

$$\lambda(\Delta(a + \mathbf{1}) - \Delta(a)) = \Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \bar{\lambda}(\Delta(a + \mathbf{1}) - \Delta(a)).$$

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This shows that $\Delta(a+\mathbf{1})-\Delta(a)=0$. On the other hand,

$$\Delta(a+\mathbf{1})-\Delta(a)\in\mathbb{T}\sigma(\mathbf{1})=\mathbb{T}.$$

We arrive at a contradiction. Therefore, $A_1\cap A_{-1}=\emptyset$. Since A is connected, we conclude that $A_1=A$ or $A_{-1}=A$. □

Proof of Theorem 2.2. Lemma 2.3 shows that one of $A=A_1$ and $A=A_{-1}$ occurs. We consider first the case in which $A=A_1$.

(i) Let us assume that A is separable. By the definition of A_1, for any $a\in A_1$,

$$\Delta(a+\lambda\cdot\mathbf{1})-\Delta(a)=\lambda(\Delta(a+\mathbf{1})-\Delta(a)),\quad \lambda\in\mathbb{C}. \tag{2-1}$$

By assumption (b),

$$|\Delta(a)-\Delta(b)|\le\|a-b\|,\quad a,b\in A,$$

which implies that Δ is a Lipschitz map. Kowalski and Słodkowski [16, Theorem 2.3] (see also [17, Theorem 3.4]) showed that Δ has real differentials except for some zero set. We say that Δ has a real differential at a point of $a\in A$ if for every $x\in A$ the derivative $\Delta'_x(a)=\lim_{\mathbb{R}\ni r\to 0}(\Delta(a+rx)-\Delta(a))/r$ exists and the map $(D\Delta)_a:A\to\mathbb{C}$, defined by $(D\Delta)_a(x)=\Delta'_x(a)$, is real-linear and continuous (cf. [16–18]). Since

$$\frac{\Delta(a+rx)-\Delta(a)}{r}\in\frac{\mathbb{T}\sigma(rx)}{r}=\frac{r\mathbb{T}\sigma(x)}{r}=\mathbb{T}\sigma(x),\quad r\in\mathbb{R}\setminus\{0\},$$
$$(D\Delta)_a(x)=\lim_{\mathbb{R}\ni r\to 0}\frac{\Delta(a+rx)-\Delta(a)}{r}\in\mathbb{T}\sigma(x).$$

As $(D\Delta)_a$ is real-linear, [17, Lemma 3.3] implies that $(D\Delta)_a$ is complex-linear or conjugate-linear. Since $a\in A=A_1$, Δ satisfies (2-1) and thus

$$\begin{aligned}(D\Delta)_a(\mathbf{1})=\lim_{r\to 0}\frac{\Delta(a+r\mathbf{1})-\Delta(a)}{r}&=\lim_{r\to 0}\frac{r(\Delta(a+\mathbf{1})-\Delta(a))}{r}\\&=\Delta(a+\mathbf{1})-\Delta(a)\in\mathbb{T}\sigma(\mathbf{1})=\mathbb{T}\end{aligned}$$

and

$$\begin{aligned}(D\Delta)_a(i\mathbf{1})=\lim_{r\to 0}\frac{\Delta(a+ri\mathbf{1})-\Delta(a)}{r}&=\lim_{r\to 0}\frac{ri(\Delta(a+\mathbf{1})-\Delta(a))}{r}\\&=i(\Delta(a+\mathbf{1})-\Delta(a)).\end{aligned}$$

It follows that $(D\Delta)_a(i\mathbf{1})=i(D\Delta)_a(\mathbf{1})$ and $(D\Delta)_a(\mathbf{1})\neq 0$. We conclude that $(D\Delta)_a$ is complex-linear. We have proved that if Δ has a real differential at a point $a\in A=A_1$, then $(D\Delta)_a$ is complex-linear. We conclude that Δ is holomorphic on A by applying [16, Lemma 2.4]. For $a,b\in A$, we define a map $f_{a,b}:\mathbb{C}\to\mathbb{C}$ by

$$f_{a,b}(\lambda)=\Delta(\lambda a+b)-\Delta(b).$$

Since Δ is holomorphic on A, $f_{a,b}$ is entire. Moreover, for any $\lambda\in\mathbb{C}\setminus\{0\}$,

$$\frac{f_{a,b}(\lambda)}{\lambda}=\frac{\Delta(\lambda a+b)-\Delta(b)}{\lambda}\in\frac{\mathbb{T}\sigma(\lambda a)}{\lambda}=\frac{\lambda\mathbb{T}\sigma(a)}{\lambda}=\mathbb{T}\sigma(a)$$

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and

$$\left|\frac{f_{a,b}(\lambda)}{\lambda}\right| \le \|a\|.$$

By Liouville's theorem, there exists $M \in \mathbb{C}$ such that $f_{a,b}(\lambda) = \lambda M$ for all $\lambda \in \mathbb{C}$. As $M = f_{a,b}(1) = \Delta(a + b) - \Delta(b)$,

$$\Delta(\lambda a + b) - \Delta(b) = \lambda(\Delta(a + b) - \Delta(b)), \quad \lambda \in \mathbb{C},$$

and

$$\Delta(\lambda a + b) = \lambda(\Delta(a + b) - \Delta(b)) + \Delta(b), \quad \lambda \in \mathbb{C}. \tag{2-2}$$

Taking $b = 0$ in (2-2),

$$\Delta(\lambda a) = \lambda\Delta(a), \quad \lambda \in \mathbb{C}, \tag{2-3}$$

by the hypothesis (a). We have shown that Δ is 1-homogeneous. We can therefore apply Theorem 2.1 (see also [17, Proposition 3.2]) to conclude that Δ is complex-linear.

(ii) We consider the case in which A is not separable. If we fix $a \in A$ and consider the subalgebra generated by a and $\mathbf{1}$, it follows from the above that $\Delta(\lambda a) = \lambda\Delta(a)$, that is, Δ is 1-homogeneous, and we finish by Theorem 2.1.

In addition, since $\Delta(a) = \Delta(a) - \Delta(0) \in \mathbb{T}\sigma(a)$, we apply [17, Proposition 2.2] to conclude that $\overline{\Delta(\mathbf{1})}\Delta$ is multiplicative.

Secondly, we assume that $A = A_{-1}$. We define the map $\overline{\Delta} : A \to \mathbb{C}$ by

$$\overline{\Delta}(a) = \overline{\Delta(a)}, \quad a \in A.$$

In the case in which $A = A_{-1}$, Δ satisfies, for any $a \in A$,

$$\Delta(a + \lambda \cdot \mathbf{1}) - \Delta(a) = \overline{\lambda}(\Delta(a + \mathbf{1}) - \Delta(a)), \quad \lambda \in \mathbb{C}.$$

Thus,

$$\overline{\Delta}(a + \lambda \cdot \mathbf{1}) - \overline{\Delta}(a) = \lambda(\overline{\Delta}(a + \mathbf{1}) - \overline{\Delta}(a)), \quad \lambda \in \mathbb{C}.$$

Moreover, it is clear that $\overline{\Delta}(0) = \overline{\Delta(0)} = 0$. Therefore, the map $\overline{\Delta} : A \to \mathbb{C}$ satisfies the conditions for Δ in the case of $A = A_1$. This in turn implies that $\overline{\Delta}$ is complex-linear and $\overline{\overline{\Delta}(\mathbf{1})}\overline{\Delta}$ is multiplicative. Thus, we conclude that Δ is conjugate-linear and $\overline{\Delta(\mathbf{1})}\Delta$ is multiplicative. □

3. 2-local maps in GWC

In this section B_j is a unital semisimple commutative Banach algebra with maximal ideal space M_j for $j = 1, 2$. The Gelfand transform $\hat{\cdot} : B_j \to \widehat{B_j} \subset C(M_j)$ is a continuous isomorphism. Identifying B_j with $\widehat{B_j}$, we consider that B_j is a subalgebra of $C(M_j)$. We say that $f \in B_j$ is unimodular if $|f| = 1$ on M_j. Since M_j is a maximal ideal space and a unimodular element f of B_j has no zeros on M_j, $\overline{f} = 1/f \in B_j$.

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An interesting generalization of the concept of 2-local maps is weak 2-locality. There are some papers dealing with weak 2-local maps, not only with 2-local maps (see, for example, [5, 17, 30, 31]). We define next *pointwise* 2-local maps.

DEFINITION 3.1. Let $\mathcal{S} \subset M(B_1, B_2)$. We say that $T \in M(B_1, B_2)$ is pointwise 2-local in $\mathcal{S}$ if for every trio $f, g \in B_1$ and $x \in M_2$ there exists $T_{f,g,x} \in \mathcal{S}$ such that

$$(T(f))(x) = (T_{f,g,x}(f))(x) \text{ and } (T(g))(x) = (T_{f,g,x}(g))(x).$$

Note that if a map T is 2-local, then T is weak 2-local. If T is weak 2-local, then T is pointwise 2-local. We say that $T \in M(B_1, B_2)$ is a pointwise 2-local isometry if T is pointwise 2-local in $\mathrm{Iso}(B_1, B_2)$. Our interest is whether a pointwise 2-local isometry in $\mathrm{Iso}(B_1, B_2)$ is in fact a surjective isometry from B_1 onto B_2 or not. Simple examples show that a pointwise 2-local isometry need not be a surjection or an isometry. We show three of them.

- A map on $C[0, 1]$. We denote the algebra of all complex-valued continuous functions on $[0, 1]$ by $C[0, 1]$. The supremum norm $\|\cdot\|_\infty$ makes it a Banach algebra. Let $\pi : [0, 1] \to [0, 1]$ be a continuous function such that $\pi(0) = 0$, $\pi(1) = 1$ and $0 < \pi(x) < 1$ for $x \in (0, 1)$. Put $T(f) = f \circ \pi$, $f \in C[0, 1]$. It is easy to see that T is pointwise 2-local in $\mathrm{Iso}(C[0, 1], C[0, 1])$ while it is not surjective when π is not a homeomorphism.
- A map on $C^1[0, 1]$. We denote the algebra of all continuously differentiable functions defined on the closed unit interval $[0, 1]$ by $C^1[0, 1]$. With the norm $\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty$ for $f \in C^1[0, 1]$, $C^1[0, 1]$ is a unital semisimple commutative Banach algebra with maximal ideal space $[0, 1]$. Let $T : C^1[0, 1] \to C^1[0, 1]$ stand for $T(f) = \exp(i\cdot)f$, $f \in C^1[0, 1]$. By [33, Theorem 4.1], every surjective complex-linear isometry on $C^1[0, 1]$ is of the form $f(x) \mapsto e^{i\theta} f(x)$ or $f(x) \mapsto e^{i\theta} f(1 - x)$ and, conversely, $\theta \in [-\pi, \pi]$, and therefore T is pointwise 2-local in $\mathrm{Iso}(C^1[0, 1], C^1[0, 1])$. However, T is not an isometry since $\|\mathbf{1}\|_\Sigma = 1$ and $\|T(\mathbf{1})\|_\Sigma = 2$.
- A map on the disk algebra $A(\bar{\mathbb{D}})$. The disk algebra $A(\bar{\mathbb{D}})$ on the closed unit disk $\bar{\mathbb{D}}$ is the algebra of all continuous functions on $\bar{\mathbb{D}}$ that are analytic on the open unit disk $\mathbb{D}$. The disk algebra on $\bar{\mathbb{D}}$ is a uniform algebra on $\bar{\mathbb{D}}$. It is well known that the maximal ideal space of $A(\bar{\mathbb{D}})$ is $\bar{\mathbb{D}}$. Let $\pi_0(z) = z^2$, $z \in \bar{\mathbb{D}}$. Then the map $T : A(\bar{\mathbb{D}}) \to A(\bar{\mathbb{D}})$ is defined by $T(f) = f \circ \pi_0$, $f \in A(\bar{\mathbb{D}})$. Trivially, T is not surjective and hence $T \notin \mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$. On the other hand, T is pointwise 2-local in $\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$. The reason is as follows. Let $f, g \in A(\bar{\mathbb{D}})$ and $x \in \bar{\mathbb{D}}$ be arbitrary. If $|x| = 1$, then put $\varphi_x(z) = xz$. If $|x| < 1$, then it is well known that there is a Möbius transformation φ_x such that $\varphi_x(x) = x^2$ since both of x and x^2 are in $\mathbb{D}$. Put $T_{f,g,x}(h) = h \circ \varphi_x$, $h \in A(\bar{\mathbb{D}})$. We infer by a calculation that $(T(f))(x) = (T_{f,g,x}(f))(x)$ and $(T(g))(x) = (T_{f,g,x}(g))(x)$. Thus, T is pointwise 2-local in $\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$.

It is interesting to point out that a pointwise 2-local isometry is in fact a surjective isometry for some Banach algebras (see Subsections 4.4 and 4.5). A simple example is a pointwise 2-local isometry on the annulus algebra.

- Let $0 < r < 1$ and $\Omega = \{z : r \le |z| \le 1\}$ be an annulus. Let $A(\Omega)$ be the algebra of all complex-valued continuous functions that are analytic on the interior of Ω. It is well known that $A(\Omega)$ is a uniform algebra on Ω whose maximal ideal space is homeomorphic to Ω. A pointwise 2-local map in $\mathrm{Iso}(A(\Omega), A(\Omega))$ is a surjective isometry (cf. Corollary 4.14).

Recall that for an $\epsilon \in \{\pm 1\}$ and $f \in B_j$, $[f]^\epsilon = f$ if $\epsilon = 1$ and $[f]^\epsilon = \bar{f}$ if $\epsilon = -1$. Let

$$\begin{aligned} \mathrm{GWC} = \{T \in M(B_1, B_2);\ & \text{there exist a } \beta \in B_2, \\ & \text{an } \alpha \in B_2 \text{ with } |\alpha| = 1 \text{ on } M_2, \\ & \text{a continuous map } \pi : M_2 \to M_1 \\ & \text{and a continuous map } \epsilon : M_2 \to \{\pm 1\} \\ & \text{such that } T(f) = \beta + \alpha[f \circ \pi]^\epsilon \text{ for every } f \in B_1\}. \end{aligned}$$

Applying Theorem 2.2, we show that a pointwise 2-local map in GWC is also in GWC.

THEOREM 3.2. *Suppose that $T \in M(B_1, B_2)$ is pointwise 2-local in* GWC. *Then there exist a continuous map $\pi : M_2 \to M_1$ and a continuous map $\epsilon : M_2 \to \{\pm 1\}$ such that*

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^\epsilon, \quad f \in B_1, \tag{3-1}$$

where $T(\mathbf{1}) - T(0)$ is a unimodular element in B_2. In particular, a pointwise 2-local map in GWC *is an element in* GWC.

PROOF. Put $T_0 = T - T(0)$. We infer that $T_0(0) = 0$. Since T is pointwise 2-local in GWC, it is obvious that T_0 is also pointwise 2-local in GWC. Let $x \in M_2$. There exist $\beta_{0,\mathbf{1},x}, \alpha_{0,\mathbf{1},x} \in B_2$ with $|\alpha_{0,\mathbf{1},x}| = 1$ on M_2, a continuous map $\pi_{0,\mathbf{1},x} : M_2 \to M_1$ and a continuous map $\epsilon_{0,\mathbf{1},x} : M_2 \to \{\pm 1\}$ such that

$$T_0(\mathbf{1})(x) = \beta_{0,\mathbf{1},x}(x) + \alpha_{0,\mathbf{1},x}(x)[\mathbf{1} \circ \pi_{0,\mathbf{1},x}]^{\epsilon_{0,\mathbf{1},x}(x)}(x) = \beta_{0,\mathbf{1},x}(x) + \alpha_{0,\mathbf{1},x}(x)$$

and

$$0 = T_0(0)(x) = \beta_{0,\mathbf{1},x}(x) + \alpha_{0,\mathbf{1},x}(x)[0 \circ \pi_{0,\mathbf{1},x}]^{\epsilon_{0,\mathbf{1},x}(x)}(x) = \beta_{0,\mathbf{1},x}(x).$$

It follows that $T_0(\mathbf{1})(x) = \alpha_{0,\mathbf{1},x}(x)$. As $x \in M_2$ is arbitrary,

$$|T_0(\mathbf{1})(x)| = 1, \quad x \in M_2. \tag{3-2}$$

Hence, $T_0(\mathbf{1})$ has no zeros on M_2, so $\overline{T_0(\mathbf{1})} = T_0(\mathbf{1})^{-1} \in B_2$. We define $T_1 \in M(B_1, B_2)$ by

$$T_1 = \overline{T_0(\mathbf{1})}T_0. \tag{3-3}$$

We see that

$$T_1(0) = \overline{T_0(\mathbf{1})}T_0(0) = 0, \quad T_1(\mathbf{1}) = \overline{T_0(\mathbf{1})}T_0(\mathbf{1}) = 1, \tag{3-4}$$

by (3-2). To proceed with the proof of Theorem 3.2, we need some claims.

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Claim 1. There exist a map $\pi : M_2 \to M_1$ and a map $\epsilon : M_2 \to \{\pm 1\}$ such that

$$T_1(f) = [f \circ \pi]^{\epsilon}, \quad f \in B_1.$$

PROOF. Let $f, g \in \mathrm{GWC}$ and $x \in M_2$. Since T_0 is pointwise 2-local in GWC, there exist $\beta_{f,g,x}, \alpha_{f,g,x} \in B_2$ with $|\alpha_{f,g,x}| = 1$ on M_2, a continuous map $\pi_{f,g,x} : M_2 \to M_1$ and a continuous map $\epsilon_{f,g,x} : M_2 \to \{\pm 1\}$ such that

$$T_0(f)(x) = \beta_{f,g,x}(x) + \alpha_{f,g,x}(x)[f \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x)$$

and

$$T_0(g)(x) = \beta_{f,g,x}(x) + \alpha_{f,g,x}(x)[g \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x).$$

As $T_1 = \overline{T_0(\mathbf{1})}T_0$,

$$T_1(f)(x) = \overline{T_0(\mathbf{1})}(x)\beta_{f,g,x}(x) + \overline{T_0(\mathbf{1})}(x)\alpha_{f,g,x}(x)[f \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x)$$

and

$$T_1(g)(x) = \overline{T_0(\mathbf{1})}(x)\beta_{f,g,x}(x) + \overline{T_0(\mathbf{1})}(x)\alpha_{f,g,x}(x)[g \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x).$$

By (3-2), $\overline{T_0(\mathbf{1})}\alpha_{f,g,x}$ is a unimodular function and thus T_1 is pointwise 2-local in GWC by the definition of GWC. Fix $x \in M_2$. We define $\Delta_x : B_1 \to \mathbb{C}$ by

$$\Delta_x(f) = (T_1(f))(x), \quad f \in B_1.$$

As T_1 is pointwise 2-local in GWC, for any $f, g \in B_1$, there exists $T_{f,g,x} \in \mathrm{GWC}$ such that

$$\begin{aligned}\Delta_x(f) &= (T_1(f))(x)\\ &= T_{f,g,x}(f)(x) = \beta_{f,g,x}(x) + \alpha_{f,g,x}(x)[f \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x)\end{aligned}$$

and

$$\begin{aligned}\Delta_x(g) &= (T_1(g))(x)\\ &= T_{f,g,x}(g)(x) = \beta_{f,g,x}(x) + \alpha_{f,g,x}(x)[g \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x).\end{aligned}$$

We infer that

$$\Delta_x(f) - \Delta_x(g) = \alpha_{f,g,x}(x)[(f-g) \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x).$$

If $x \in \epsilon_{f,g,x}^{-1}(1)$,

$$[(f-g) \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x) = (f-g)(\pi_{f,g,x}(x)) \in \sigma(f-g).$$

If $x \in \epsilon_{f,g,x}^{-1}(-1)$,

$$[(f-g) \circ \pi_{f,g,x}]^{\epsilon_{f,g,x}(x)}(x) = \overline{(f-g)(\pi_{f,g,x}(x))} \in \mathbb{T}\sigma(f-g).$$

Therefore,

$$\Delta_x(f) - \Delta_x(g) \in \mathbb{T}\sigma(f-g), \quad f, g \in B_1.$$

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By (3-4), we have $\Delta_x(0) = T_1(0)(x) = 0$. Applying Theorem 2.2, we obtain that Δ_x is complex-linear or conjugate-linear and $\overline{\Delta_x(\mathbf{1})}\Delta_x$ is multiplicative. As $\overline{\Delta_x(\mathbf{1})} = \overline{T_1(\mathbf{1})(x)} = 1$ by (3-4), we conclude that Δ_x is multiplicative. In addition, $\Delta_x(\mathbf{1}) = 1$ implies that $\Delta_x \neq 0$. Therefore, for any $x \in M_2$, one of the following (i) and (ii) occurs:

(i) Δ_x is a nonzero multiplicative complex-linear functional;
(ii) Δ_x is a nonzero multiplicative conjugate-linear functional.

In the case (i), by Gelfand theory, there exists $\pi(x) \in M_1$ such that

$$\Delta_x(f) = f(\pi(x)), \quad f \in B_1.$$

In the case (ii), $\overline{\Delta_x}$ is a nonzero multiplicative complex-linear functional. Thus, there exists $\pi(x) \in M_1$ such that

$$\overline{\Delta_x}(f) = f(\pi(x)), \quad f \in B_1,$$

and hence

$$\Delta_x(f) = \overline{f(\pi(x))}, \quad f \in B_1.$$

Recalling that $\Delta_x(f) = (T_1(f))(x)$,

$$T_1(f)(x) = \begin{cases} f \circ \pi(x) & (\Delta_x \text{ is complex-linear}), \\ \overline{f \circ \pi(x)} & (\Delta_x \text{ is conjugate-linear}). \end{cases}$$

We define a map $\epsilon : M_2 \to \{\pm 1\}$ by

$$\epsilon(x) = \begin{cases} 1 & (\Delta_x \text{ is complex-linear}), \\ -1 & (\Delta_x \text{ is conjugate-linear}). \end{cases} \tag{3-5}$$

Then

$$T_1(f)(x) = [f \circ \pi]^{\epsilon(x)}(x), \quad f \in B_1,\ x \in M_2.$$

□

Let

$$K_1 = \{x \in M_2; \Delta_x \text{ is complex-linear}\}$$

and

$$K_{-1} = \{x \in M_2; \Delta_x \text{ is conjugate-linear}\}.$$

Rewriting (3-5),

$$\epsilon(x) = \begin{cases} 1 & (x \in K_1), \\ -1 & (x \in K_{-1}). \end{cases}$$

Claim 2. We have $K_1 = \{x \in M_2; \Delta_x(i) = i\}$ and $K_{-1} = \{x \in M_2; \Delta_x(i) = -i\}$. In addition, $M_2 = K_1 \cup K_{-1}$, $K_1 \cap K_{-1} = \emptyset$ and K_1 and K_{-1} are closed subsets of M_2.

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PROOF. Since, for any $x \in M_2$, Δ_x is complex-linear or conjugate-linear, it is clear that $M_2 = K_1 \cup K_{-1}$. By the definition of K_1 and $\Delta_x(\mathbf{1}) = 1$, if $x \in K_1$, then we have $x \in \{y \in M_2; \Delta_y(i) = i\}$. Suppose that $x \in \{y \in M_2; \Delta_y(i) = i\}$. Then $\Delta_x(i) = i\Delta_x(\mathbf{1})$. This implies that $x \in K_1$. We conclude that $K_1 = \{x \in M_2; \Delta_x(i) = i\}$. We can also prove that $K_{-1} = \{x \in M_2; \Delta_x(i) = -i\}$ with a similar argument. Therefore, it is easy to see that $K_1 \cap K_{-1} = \emptyset$. Let $\{\boldsymbol{x}_\alpha\} \subset K_1$ be a net with $x_\alpha \to x_0 \in M_2$. We get

$$i = \Delta_{x_\alpha}(i) = (T_1(i))(x_\alpha) \to (T_1(i))(x_0) = \Delta_{x_0}(i).$$

This implies that $\Delta_{x_0}(i) = i$ and $x_0 \in K_1$. We have that K_1 is closed in M_2. We also get that K_{-1} is closed in the same way. □

Claim 2 shows that $\epsilon : M_2 \to \{\pm 1\}$ is continuous.

Claim 3. The mapping $\pi : M_2 \to M_1$ is continuous.

PROOF. Let $\{x_\alpha\} \subset M_2$ be a net with $x_\alpha \to x_0 \in M_2$. By Claim 2, K_1 and K_{-1} are closed and $K_1 \cap K_{-1} = \emptyset$. Thus, there is no loss of generality to assume that:

(i) $\{x_\alpha\} \subset K_1$ and $x_0 \in K_1$;
(ii) $\{x_\alpha\} \subset K_{-1}$ and $x_0 \in K_{-1}$.

First, we consider the case (i). Then

$$T_1(f)(x_\alpha) \to T_1(f)(x_0), \quad f \in B_1,$$

and hence

$$(f \circ \pi)(x_\alpha) \to (f \circ \pi)(x_0), \quad f \in B_1.$$

This implies that $\pi(x_\alpha) \to \pi(x_0)$ with the Gelfand topology. In case (ii) a similar argument gives that π is continuous and therefore we finish the proof. □

CONTINUATION OF PROOF OF THEOREM 3.2. By (3-3), we get $T_0 = T_0(\mathbf{1})T_1$. By applying $T_0 = T - T(0)$ and Claim 1,

$$\begin{aligned} T(f) &= T_0(f) + T(0) \\ &= T_0(\mathbf{1})T_1(f) + T(0) \\ &= T_0(\mathbf{1})[f \circ \pi]^\epsilon + T(0), \quad f \in B_1. \end{aligned}$$

Putting $f = \mathbf{1}$, we have $T_0(\mathbf{1}) = T(\mathbf{1}) - T(0)$ and

$$T(f) = (T(\mathbf{1}) - T(0))[f \circ \pi]^\epsilon + T(0).$$

In addition, by (3-2), we have $|T_0(\mathbf{1})| = 1$. We obtain that $T_0(\mathbf{1}) = T(\mathbf{1}) - T(0)$ is a unimodular element in B_2. □

REMARK. Even though a map $T \in M(B_1, B_2)$ is a 2-local map in GWC, it is not always the case that $\pi : M_2 \to M_1$ is a homeomorphism. In fact, the map T_0 in [9, Theorem 2.3] is a 2-local automorphism and hence 2-local in $\mathrm{Iso}_{\mathbb{C}}(C(\bar{\mathcal{K}}), C(\bar{\mathcal{K}}))$.

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On the other hand, the corresponding continuous map is not injective and hence it is not a homeomorphism.

COROLLARY 3.3. *Suppose that $T \in M(B_1, B_2)$ is pointwise 2-local in* GWC *and T is injective. Then $\pi(M_2)$ is a uniqueness set for B_1, that is, if $g \in B_1$ and $g = 0$ on $\pi(M_2)$, then $g = 0$.*

PROOF. Suppose that $g \in B_1$ and $g = 0$ on $\pi(M_2)$. Substituting g in (3-1),

$$T(g) = T(0) + (T(\mathbf{1}) - T(0))[g \circ \pi]^{\epsilon} = T(0) + (T(\mathbf{1}) - T(0))[0]^{\epsilon} = T(0).$$

Since T is injective, we have that $g = 0$. Hence, $\pi(M_2)$ is a uniqueness set for B_1. □

Let

$$\begin{aligned}\mathrm{WC}_{\mathbb{C}} = \{T \in M(B_1, B_2);\ &\text{there exist}\\ &\text{an } \alpha \in B_2 \text{ with } |\alpha| = 1 \text{ on } M_2\\ &\text{and a continuous map } \pi : M_2 \to M_1\\ &\text{such that } T(f) = \alpha f \circ \pi \text{ for every } f \in B_1\}.\end{aligned}$$

Then $\mathrm{WC}_{\mathbb{C}}$ is a set of weighted composition operators. We see that a pointwise 2-local weighted composition operator is a weighted composition operator.

COROLLARY 3.4. *Suppose that $T \in M(B_1, B_2)$ is pointwise 2-local in $\mathrm{WC}_{\mathbb{C}}$. Then $T \in \mathrm{WC}_{\mathbb{C}}$.*

PROOF. Let $T \in M(B_1, B_2)$ be pointwise 2-local in $\mathrm{WC}_{\mathbb{C}}$. Since $\mathrm{WC}_{\mathbb{C}} \subset \mathrm{GWC}$, we see by Theorem 3.2 that there exist a continuous map $\pi : M_2 \to M_1$ and a continuous map $\epsilon : M_2 \to \{\pm 1\}$ such that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in B_1, \tag{3-6}$$

where $T(\mathbf{1}) - T(0)$ is a unimodular element in B_2. Since any map in $\mathrm{WC}_{\mathbb{C}}$ is complex-linear, we infer by a simple calculation that $T(0) = 0$ and T is homogeneous with respect to a complex scalar. We see by (3-6) that

$$T(f) = T(\mathbf{1}) f \circ \pi, \quad f \in B_1,$$

where $T(\mathbf{1})$ is a unimodular function. Thus, $T \in \mathrm{WC}_{\mathbb{C}}$. □

4. Applications

In this section we study 2-local isometries on several function spaces by applying Theorem 3.2.

4.1. Uniform algebras. Let X be a compact Hausdorff space. The algebra of all complex-valued continuous functions on X is denoted by $C(X)$, which is a Banach algebra with respect to the supremum norm $\|\cdot\|_\infty$ on X. We say that A is a uniform algebra on X if A is a uniformly closed subalgebra of $C(X)$ that contains constant

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functions and separates the points of X. As the Gelfand transformation on a uniform algebra is an isometric isomorphism, a uniform algebra is isometrically isomorphic to its Gelfand transform. We may suppose that X is a subset of the maximal ideal space M_A and A is a uniform algebra on M_A. The Banach algebra $C(X)$ is a uniform algebra on X whose maximal ideal space is X. The next result is obtained in [8, Theorem 2.1 and Corollary 3.4]. Note that we denote the maximal ideal space of a uniform algebra A_j by M_j for $j = 1, 2$.

THEOREM 4.1. *Let A_j be a uniform algebra on a compact Hausdorff space X_j for $j = 1, 2$. Suppose that $U : A_1 \to A_2$ is a surjective isometry from A_1 onto A_2. Then there exist a homeomorphism $\pi : M_2 \to M_1$, an $\alpha \in A_2$ with $|\alpha| = 1$ on M_2 and a continuous map $\epsilon : M_2 \to \{\pm 1\}$ such that*

$$U(f) = U(0) + \alpha[f \circ \pi]^{\epsilon}, \quad f \in A_1. \tag{4-1}$$

If $A_j = C(X_j)$, the map U defined by (4-1) is a surjective isometry from $C(X_1)$ onto $C(X_2)$.

By Theorem 4.1,

$$\mathrm{Iso}(A_1, A_2) \subset \mathrm{GWC}$$

for uniform algebras A_1 and A_2. As a direct consequence of Theorem 3.2, we get Corollary 4.2, which is a generalization of Theorem 3.10 of [17].

COROLLARY 4.2. *Let A_j be a uniform algebra on a compact Hausdorff space X_j for $j = 1, 2$. Suppose that $T \in M(A_1, A_2)$ is pointwise 2-local in* Iso(A_1, A_2). *Then there exist a continuous map $\pi : M_2 \to M_1$ and a continuous map $\epsilon : M_2 \to \{\pm 1\}$ such that*

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in A_1,$$

where $T(\mathbf{1}) - T(0)$ is a unimodular function.

We also obtain the following corollary.

COROLLARY 4.3. *Let X_j be a first countable compact Hausdorff space for $j = 1, 2$. Suppose that $T \in M(C(X_1), C(X_2))$ is 2-local in* Iso$(C(X_1), C(X_2))$. *Then we have $T \in$* Iso$(C(X_1), C(X_2))$.

PROOF. Let T be 2-local in Iso$(C(X_1), C(X_2))$. By Corollary 4.2, there exist a continuous map $\pi : X_2 \to X_1$ and a continuous map $\epsilon : X_2 \to \{\pm 1\}$ such that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in C(X_1). \tag{4-2}$$

We prove that π is an injection. Suppose that $y_1, y_2 \in X_2$ are such that $\pi(y_1) = \pi(y_2) = x \in X_1$. Since X_1 is first countable, there exists $g \in C(X_1)$ such that $g^{-1}(0) = \{x\}$. (For example, in [23, Page 117, line 19–21], we see the existence of such a function g.) Since, for $T_0 = T - T(0)$, $T_1 = \overline{T_0(\mathbf{1})}T_0$ is 2-local in Iso$(C(X_1), C(X_2))$,

$$\begin{aligned} 0 = T_1(0) &= T_{0,g}(0) \\ &= \beta_{0,g} + \alpha_{0,g}[0 \circ \pi_{0,g}]^{\epsilon_{0,g}} = \beta_{0,g} \end{aligned}$$

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and

$$\begin{aligned}T_1(g) &= T_{0,g}(g)\\ &= \beta_{0,g} + \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}}.\end{aligned}$$

Hence,

$$T_1(g) = \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}}$$

and thus

$$(T_1(g))^{-1}(0) = (g \circ \pi_{0,g})^{-1}(0) = \pi_{0,g}^{-1}(x).$$

Since $\pi_{0,g}$ is homeomorphism, the set $\pi_{0,g}^{-1}(x)$ is a singleton. Moreover, applying (4-2),

$$T_1(g) = [g \circ \pi]^{\epsilon}.$$

Thus,

$$(T_1(g))^{-1}(0) = (g \circ \pi)^{-1}(0) = \pi^{-1}(x) \ni \{y_1, y_2\}.$$

As we have already proved that the set $(T_1(g))^{-1}(0) = \pi_{0,g}^{-1}(x)$ is a singleton, we infer that $y_1 = y_2$. Thus, π is injective. Since T is a 2-local isometry, T is an isometry by definition of 2-local isometries. Hence, T is injective. By Corollary 3.3, $\pi(X_2)$ is a uniqueness set for $C(X_1)$, which must coincide with X_1 itself. Since a one-to-one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism, we infer that π is a homeomorphism. It follows that $T \in \mathrm{Iso}(C(X_1), C(X_2))$ □

Corollary 4.3 gives an affirmative answer to the problem mentioned by Molnár. Mori proved the same statement in [29, Theorem 4.6] by a different argument.

Next we consider the disk algebra. Let $\bar{\mathbb{D}}$ be the closed unit disk.

COROLLARY 4.4. *Suppose that U is a surjective isometry from the disk algebra $A(\bar{\mathbb{D}})$ onto itself. Then there exist a Möbius transformation φ on $\bar{\mathbb{D}}$ and a unimodular constant α such that*

$$U(f) = U(0) + \alpha f \circ \varphi, \quad f \in A(\bar{\mathbb{D}}),$$

or

$$U(f) = U(0) + \alpha \overline{f \circ \bar{\varphi}}, \quad f \in A(\bar{\mathbb{D}}).$$

Conversely, if one of the above equations holds, then U is a surjective isometry from the disk algebra onto itself.

PROOF. Applying Theorem 4.1, we have a homeomorphism $\pi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$, a unimodular function $\alpha \in A(\bar{\mathbb{D}})$ and a continuous map $\epsilon : \bar{\mathbb{D}} \to \{\pm 1\}$ such that

$$U(f) = U(0) + \alpha[f \circ \pi]^{\epsilon}, \quad f \in A(\bar{\mathbb{D}}). \tag{4-3}$$

Due to the maximum modulus principle for analytic functions, α is a unimodular constant. Since $\bar{\mathbb{D}}$ is connected, $\epsilon = 1$ on $\bar{\mathbb{D}}$ or $\epsilon = -1$ on $\bar{\mathbb{D}}$. Letting $f = \mathrm{Id}$, the identity

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function, in (4-3),

$$\bar{\alpha}(U(\mathrm{Id}) - U(0)) = \pi \quad \text{if } \epsilon = 1, \tag{4-4}$$

$$\bar{\alpha}(U(\mathrm{Id}) - U(0)) = \bar{\pi} \quad \text{if } \epsilon = -1. \tag{4-5}$$

Suppose that $\epsilon = 1$. Then π is analytic on $\mathbb{D}$ by (4-4). As π is a homeomorphism, we conclude that π is a Möbius transformation. In the same way, $\bar{\pi}$ is a Möbius transformation if $\epsilon = -1$. Letting $\varphi = \pi$ if $\epsilon = 1$, and $\varphi = \bar{\pi}$ if $\epsilon = -1$, φ is a Möbius transformation. It follows that

$$U(f) = U(0) + \alpha f \circ \varphi, \quad f \in A(\bar{\mathbb{D}})$$

if $\epsilon = 1$ and

$$U(f) = U(0) + \alpha \overline{f \circ \bar{\varphi}}, \quad f \in A(\bar{\mathbb{D}})$$

if $\epsilon = -1$.

The converse statement is trivial. □

By Corollary 4.4,

$$\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}})) \subset \mathrm{GWC}$$

for the disk algebra $A(\bar{\mathbb{D}})$.

COROLLARY 4.5. *Suppose that $T \in M(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$ is 2-local in* $\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$. *Then* $T \in \mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$.

PROOF. Corollary 4.2 asserts that there exist a continuous map $\pi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ and a continuous map $\epsilon : \bar{\mathbb{D}} \to \{\pm 1\}$ such that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^{\epsilon}, \quad f \in A(\bar{\mathbb{D}}), \tag{4-6}$$

where $T(\mathbf{1}) - T(0)$ is a unimodular function. By the same way as in the proof of Corollary 4.4, we see that $T(\mathbf{1}) - T(0)$ is a unimodular constant. We also see that $\epsilon = 1$ on $\bar{\mathbb{D}}$ or $\epsilon = -1$ on $\bar{\mathbb{D}}$ because $\bar{\mathbb{D}}$ is connected and ϵ is continuous. Letting $f = \mathrm{Id}$ in (4-6), we have that π is analytic on $\mathbb{D}$ if $\epsilon = 1$, and $\bar{\pi}$ is analytic on $\mathbb{D}$ if $\epsilon = -1$. Put $\varphi = \pi$ if $\epsilon = 1$, $\varphi = \bar{\pi}$ if $\epsilon = -1$ and $T_1 = \overline{T(\mathbf{1}) - T(0)}(T - T(0))$. Then

$$T_1(f) = f \circ \varphi, \quad f \in A(\bar{\mathbb{D}})$$

if $\epsilon = 1$ and

$$T_1(f) = \overline{f \circ \bar{\varphi}}, \quad f \in A(\bar{\mathbb{D}})$$

if $\epsilon = -1$. Since T_1 is 2-local in $\mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$, we apply Corollary 4.4 and thus there exist a Möbius transform φ_0, $u \in A(\bar{\mathbb{D}})$ and a unimodular constant α such that

$$\varphi = T_1(\mathrm{Id}) = u + \alpha\varphi_0 \text{ and } 0 = T_1(0) = u.$$

It follows that $\varphi = \alpha\varphi_0$. As $|\alpha| = 1$, we infer that φ is a Möbius transformation on $\bar{\mathbb{D}}$. We infer by Corollary 4.4 that $T \in \mathrm{Iso}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$. □

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4.2. Lipschitz algebras. Let (X_j, d) be a compact metric space for $j = 1, 2$. Let

$$\mathrm{Lip}(X_j) = \left\{ f \in C(X_j) : L(f) = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} < \infty \right\}.$$

We say that $L(f)$ is the Lipschitz constant for f. With the norm $\|f\|_\Sigma = \|f\|_\infty + L(f)$ for $f \in \mathrm{Lip}(X_j)$, the algebra $\mathrm{Lip}(X_j)$ is a unital semisimple commutative Banach algebra. In addition, the maximal ideal space of $\mathrm{Lip}(X_j)$ can be identified with X_j.

COROLLARY 4.6. *Let $\|\cdot\|_j$ be any norm on $\mathrm{Lip}(X_j)$. We do not assume that $\|\cdot\|_j$ is complete. Suppose that*

$$\begin{aligned} &\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2)) \\ &\quad = \{T \in M(\mathrm{Lip}(X_1), \mathrm{Lip}(X_2)); \\ &\qquad \textit{there exist } \beta \in \mathrm{Lip}(X_2),\ \alpha \in \mathbb{T}, \\ &\qquad \textit{a surjective isometry } \pi : X_2 \to X_1 \textit{ and } \epsilon = \pm 1 \\ &\qquad \textit{such that } T(f) = \beta + \alpha[f \circ \pi]^\epsilon \textit{ for every } f \in \mathrm{Lip}(X_1)\}. \end{aligned} \tag{4-7}$$

Let $T \in M((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$ be 2-local in $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$. Then $T \in \mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$.

PROOF. Suppose that T is 2-local in $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$. The equality (4-7) implies that $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2)) \subset \mathrm{GWC}$. Applying Theorem 3.2, there exist a continuous map $\pi : X_2 \to X_1$ and a continuous map $\epsilon : X_2 \to \{\pm 1\}$ such that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^\epsilon, \quad f \in \mathrm{Lip}(X_1). \tag{4-8}$$

Recall that $T_1 = \overline{T_0(\mathbf{1})} T_0$ for $T_0 = T - T(0)$. Since T_0 is 2-local,

$$T_0(\mathbf{1}) = \beta_{0,\mathbf{1}} + \alpha_{0,\mathbf{1}}[\mathbf{1} \circ \pi_{0,\mathbf{1}}]^{\epsilon_{0,\mathbf{1}}}$$

and

$$0 = T_0(0) = \beta_{0,\mathbf{1}} + \alpha_{0,\mathbf{1}}[0 \circ \pi_{0,\mathbf{1}}]^{\epsilon_{0,\mathbf{1}}} = \beta_{0,\mathbf{1}}.$$

It follows that $T(\mathbf{1}) - T(0) = T_0(\mathbf{1})$ is a unimodular constant. Thus, $T_1 = \overline{T_0(\mathbf{1})} T_0$ is 2-local in $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$. We get

$$\begin{aligned} 0 = T_1(0) &= T_{0,i}(0) \\ &= \beta_{0,i} + \alpha_{0,i}[0 \circ \pi_{0,i}]^{\epsilon_{0,i}} = \beta_{0,i} \end{aligned}$$

and

$$\begin{aligned} T_1(i) &= T_{0,i}(i) \\ &= \beta_{0,i} + \alpha_{0,i}[i \circ \pi_{0,i}]^{\epsilon_{0,i}}. \end{aligned}$$

Therefore,

$$T_1(i) = \alpha_{0,i}[i \circ \pi_{0,i}]^{\epsilon_{0,i}}.$$

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Since $\alpha_{0,i}$ is a unimodular constant and $\epsilon_{0,i} = \pm 1$, we obtain that $T_1(i)$ is a constant. Moreover, applying (4-8),

$$T_1(i) = [i \circ \pi]^{\epsilon}.$$

Thus, we conclude that $\epsilon = 1$ or $\epsilon = -1$. As T is a 2-local isometry, T is an isometry and hence T is injective. Corollary 3.3 asserts that $\pi(X_2)$ is a uniqueness set for $\mathrm{Lip}(X_1)$. Thus, we have $\pi(X_2) = X_1$. This implies that π is surjective. Finally, we shall prove that π is an isometry. Let $x_0 \in X_2$. We define a Lipschitz function g on X_1 by

$$g(x) = d(x, \pi(x_0)), \quad x \in X_1.$$

As T_1 is 2-local in $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_1), (\mathrm{Lip}(X_2), \|\cdot\|_2))$, there exists $\alpha_{0,g} \in \mathbb{T}$ and $\pi_{0,g} : X_2 \to X_1$ is a surjective isometry such that

$$\begin{aligned} 0 = T_1(0) &= T_{0,g}(0) \\ &= \beta_{0,g} + \alpha_{0,g}[0 \circ \pi_{0,g}]^{\epsilon_{0,g}} = \beta_{0,g} \end{aligned}$$

and

$$\begin{aligned} T_1(g) &= T_{0,g}(g) \\ &= \beta_{0,g} + \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}} = \beta_{0,g} + \alpha_{0,g} g \circ \pi_{0,g} \end{aligned}$$

because g is a real-valued function. It follows that

$$(T_1(g))(z) = \alpha_{0,g} g(\pi_{0,g}(z)), \quad z \in X_2.$$

By (4-8), for any $z \in X_2$,

$$\begin{aligned} d(\pi(z), \pi(x_0)) &= [g(\pi(z))]^{\epsilon} \\ &= (T_1(g))(z) = \alpha_{0,g} g(\pi_{0,g}(z)) = \alpha_{0,g} d(\pi_{0,g}(z), \pi(x_0)). \end{aligned} \tag{4-9}$$

We may suppose that X_1 is not a singleton. (Otherwise, X_2 is a singleton since $\pi_{0,g}$ is a surjective isometry. Then π is automatically a surjective isometry.) Hence, there exists $z_0 \in X_2$ such that $d(\pi_{0,g}(z_0), \pi(x_0)) \neq 0$. By (4-9) with $z = z_0$,

$$\alpha_{0,g} = \frac{d(\pi(z_0), \pi(x_0))}{d(\pi_{0,g}(z_0), \pi(x_0))} \geq 0$$

and we obtain $\alpha_{0,g} = 1$. Hence, by (4-9),

$$d(\pi(z), \pi(x_0)) = d(\pi_{0,g}(z), \pi(x_0)), \quad z \in X_2. \tag{4-10}$$

Putting $z = x_0$ in (4-10),

$$0 = d(\pi(x_0), \pi(x_0)) = d(\pi_{0,g}(x_0), \pi(x_0)).$$

It follows $\pi_{0,g}(x_0) = \pi(x_0)$. By (4-10),

$$d(\pi(z), \pi(x_0)) = d(\pi_{0,g}(z), \pi(x_0)) = d(\pi_{0,g}(z), \pi_{0,g}(x_0)) = d(z, x_0)$$

since $\pi_{0,g}$ is an isometry. As z and x_0 are arbitrary, we conclude that π is an isometry. This completes the proof. □

For an arbitrary compact metric space X_j for $j = 1, 2$, Hatori and the author [10, Theorem 6] showed that $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_\Sigma), (\mathrm{Lip}(X_2), \|\cdot\|_\Sigma))$ fulfills the condition of Corollary 4.6. Thus, we have the following corollary.

COROLLARY 4.7. *Let* $T \in M(\mathrm{Lip}(X_1), \mathrm{Lip}(X_2))$ *be 2-local in* $\mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_\Sigma), (\mathrm{Lip}(X_2), \|\cdot\|_\Sigma))$. *Then* $T \in \mathrm{Iso}((\mathrm{Lip}(X_1), \|\cdot\|_\Sigma), (\mathrm{Lip}(X_2), \|\cdot\|_\Sigma))$.

Corollary 4.7 generalizes Theorem 8 in [10], where the case $X_1 = X_2 = [0, 1]$ is proved.

4.3. The algebra of continuously differentiable functions. We denote the algebra of all continuously differentiable functions by $C^1([0, 1])$. It is a unital semisimple commutative Banach algebra with the norm $\|\cdot\|_\Sigma$ defined by

$$\|f\|_\Sigma = \|f\|_\infty + \|f'\|_\infty, \quad f \in C^1([0, 1]).$$

The maximal ideal space of $C^1([0, 1])$ is homeomorphic to $[0, 1]$. We have the following corollary.

COROLLARY 4.8. *Let* $\|\cdot\|_j$ *be any norm on* $C^1([0, 1])$ *for* $j = 1, 2$. *We do not assume that* $\|\cdot\|_j$ *is complete. Suppose that*

$$\begin{aligned}
&\mathrm{Iso}((C^1([0, 1]), \|\cdot\|_1), (C^1([0, 1]), \|\cdot\|_2)) \\
&\quad = \{T \in M(C^1([0, 1]), C^1([0, 1])); \\
&\qquad \textit{there exist } \beta \in C^1([0, 1]),\ \alpha \in \mathbb{T}, \\
&\qquad \pi = \mathrm{Id} \textit{ or } \pi = 1 - \mathrm{Id} \textit{ and } \epsilon = \pm 1 \\
&\qquad \textit{such that } T(f) = \beta + \alpha[f \circ \pi]^\epsilon \textit{ for every } f \in C^1([0, 1])\}.
\end{aligned} \tag{4-11}$$

Suppose that $T \in M(C^1([0, 1]), C^1([0, 1]))$ *is 2-local in* $\mathrm{Iso}((C^1([0, 1]), \|\cdot\|_1), (C^1([0, 1]), \|\cdot\|_2))$. *Then* $T \in \mathrm{Iso}((C^1([0, 1]), \|\cdot\|_1), (C^1([0, 1]), \|\cdot\|_2))$.

PROOF. Let T be 2-local in $\mathrm{Iso}((C^1([0, 1]), \|\cdot\|_1), (C^1([0, 1]), \|\cdot\|_2))$. By (4-11), $\mathrm{Iso}((C^1([0, 1]), \|\cdot\|_1), (C^1([0, 1]), \|\cdot\|_2)) \subset \mathrm{GWC}$. Theorem 3.2 asserts that there exist a continuous map $\pi : [0, 1] \to [0, 1]$ and a continuous map $\epsilon : [0, 1] \to \{\pm 1\}$ such that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0))[f \circ \pi]^\epsilon, \quad f \in C^1([0, 1]). \tag{4-12}$$

Since $\epsilon : [0, 1] \to \{\pm 1\}$ is continuous and $[0, 1]$ is connected, we conclude that $\epsilon = \pm 1$. As T is a 2-local isometry, we get that T is an isometry. This implies that T is injective. Corollary 3.3 asserts that $\pi([0, 1])$ is a uniqueness set for $C^1([0, 1])$, which is $[0, 1]$. Thus, we have that π is surjective. To complete the proof, we prove that π is an isometry. Let $x_0 \in [0, 1]$. We define the function $g(x) = x - \pi(x_0) \in C^1([0, 1])$. Define $T_1 = \overline{T_0(\mathbf{1})} T_0$ for $T_0 = T - T(0)$. It is easy to see that T_0 is 2-local in $\mathrm{Iso}((C^1([0, 1]), \|\cdot\|_1), (C^1([0, 1]), \|\cdot\|_2))$:

$$T_0(\mathbf{1}) = \beta_{0,\mathbf{1}} + \alpha_{0,\mathbf{1}}[\mathbf{1} \circ \pi_{0,\mathbf{1}}]^{\epsilon_{0,\mathbf{1}}}$$

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and

$$0 = T_0(0) = \beta_{0,\mathbf{1}} + \alpha_{0,\mathbf{1}}[0 \circ \pi_{0,\mathbf{1}}]^{\epsilon_{0,\mathbf{1}}} = \beta_{0,\mathbf{1}}.$$

It follows that $T(\mathbf{1}) - T(0) = T_0(\mathbf{1})$ is a unimodular constant. We have that $T_1 = \overline{T_0(\mathbf{1})}T_0$ is 2-local in $\mathrm{Iso}((C^1([0,1]), \|\cdot\|_1), (C^1([0,1]), \|\cdot\|_2))$. Hence,

$$\begin{aligned} 0 = T_1(0) &= T_{0,g}(0) \\ &= \beta_{0,g} + \alpha_{0,g}[0 \circ \pi_{0,g}]^{\epsilon_{0,g}} = \beta_{0,g} \end{aligned}$$

and

$$\begin{aligned} T_1(g) &= T_{0,g}(g) \\ &= \beta_{0,g} + \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}}. \end{aligned}$$

It follows that

$$(T_1(g))(z) = \alpha_{0,g}[g \circ \pi_{0,g}]^{\epsilon_{0,g}}(z) = \alpha_{0,g}[g(\pi_{0,g}(z))]^{\epsilon_{0,g}}, \quad z \in [0,1].$$

Thus, by (4-12),

$$\begin{aligned} [\pi(z) - \pi(x_0)]^{\epsilon} = [g(\pi(z))]^{\epsilon} &= (T_1(g))(z) \\ &= \alpha_{0,g}[g(\pi_{0,g}(z))]^{\epsilon_{0,g}} = \alpha_{0,g}[\pi_{0,g}(z) - \pi(x_0)]^{\epsilon_{0,g}} \end{aligned}$$

for any $z \in [0,1]$, where $\alpha_{0,g} \in \mathbb{T}$ and $\pi_{0,g} = \mathrm{Id}$ or $\pi_{0,g} = 1 - \mathrm{Id}$. Putting $z = x_0$,

$$0 = [\pi(x_0) - \pi(x_0)]^{\epsilon} = \alpha_{0,g}[\pi_{0,g}(x_0) - \pi(x_0)]^{\epsilon_{0,g}}.$$

It follows that $\pi_{0,g}(x_0) = \pi(x_0)$. Thus,

$$[\pi(z) - \pi(x_0)]^{\epsilon} = \alpha_{0,g}[\pi_{0,g}(z) - \pi(x_0)]^{\epsilon_{0,g}} = \alpha_{0,g}[\pi_{0,g}(z) - \pi_{0,g}(x_0)]^{\epsilon_{0,g}}$$

and

$$|\pi(z) - \pi(x_0)| = |\pi_{0,g}(z) - \pi_{0,g}(x_0)| = |z - x_0|.$$

As z and x_0 are arbitrary, we conclude that π is an isometry. This completes the proof. □

In [15, 20], the authors gave a complete characterization of all surjective isometries on $C^1([0,1])$ with respect to various norms. There are many norms with which the groups of surjective isometries on $C^1([0,1])$ fulfill the condition of Corollary 4.8. We present one of them.

COROLLARY 4.9. *Suppose that* $T \in M((C^1([0,1]), \|\cdot\|_\Sigma), (C^1([0,1]), \|\cdot\|_\Sigma))$ *and* T *is 2-local in* $\mathrm{Iso}((C^1([0,1]), \|\cdot\|_\Sigma), (C^1([0,1]), \|\cdot\|_\Sigma))$. *We conclude that* $T \in \mathrm{Iso}((C^1([0,1]), \|\cdot\|_\Sigma), (C^1([0,1]), \|\cdot\|_\Sigma))$.

Corollary 4.9 has been also obtained in [10, Theorem 9] with a different argument.

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4.4. The algebra $S^\infty(\mathbb{D})$. As we stated in the beginning of Section 3, for some Banach algebras B_j, a pointwise 2-local map in $\mathrm{Iso}(B_1, B_2)$ is not always a surjective isometry. But, in this subsection and the next, we show examples of Banach algebras B_j in which every pointwise 2-local map in $\mathrm{Iso}(B_1, B_2)$ is always a surjective isometry.

Let

$$S^\infty(\mathbb{D}) = \{ f \in H(\mathbb{D});\ f' \in H^\infty(\mathbb{D})\},$$

where $H(\mathbb{D})$ is the linear space of all analytic functions on $\mathbb{D}$ and $H^\infty(\mathbb{D})$ is the algebra of all bounded analytic functions on $\mathbb{D}$. The algebra $S^\infty(\mathbb{D})$ equipped with the norm $\|f\|_\Sigma = \sup_{z\in\mathbb{D}} |f(z)| + \sup_{w\in\mathbb{D}} |f'(w)|$ for $f \in S^\infty(\mathbb{D})$ is a unital semisimple commutative Banach algebra. As is described in [19], $S^\infty(\mathbb{D})$ coincides with the space of all Lipschitz functions in the linear space of all analytic functions on $\mathbb{D}$ and each $f \in S^\infty(\mathbb{D})$ is continuously extended to the closed unit disk $\bar{\mathbb{D}}$. Hence, we may suppose that $S^\infty(\mathbb{D})$ is a unital subalgebra of the disk algebra on $\bar{\mathbb{D}}$. Trivially, all analytic polynomials are in $S^\infty(\mathbb{D})$.

THEOREM 4.10. *The maximal ideal space M_∞ of $S^\infty(\mathbb{D})$ is homeomorphic to the closed unit disk $\bar{\mathbb{D}}$.*

PROOF. To prove that $\bar{\mathbb{D}} = M_\infty$, firstly we show that if $f_1, \dots, f_n$ are arbitrary functions in $S^\infty(\mathbb{D})$ such that

$$\sum_{j=1}^n |f_j| > 0 \text{ on } \bar{\mathbb{D}},$$

then there exist $g_1, \dots, g_n \in S^\infty(\mathbb{D})$ such that

$$\sum_{j=1}^n f_j g_j = 1.$$

It is well known that the maximal ideal space of the disk algebra $A(\bar{\mathbb{D}})$ is $\bar{\mathbb{D}}$. As $f_1, \dots, f_n \in S^\infty(\mathbb{D}) \subset A(\bar{\mathbb{D}})$, there exist $h_1, \dots, h_n \in A(\bar{\mathbb{D}})$ such that

$$\sum_{j=1}^n f_j h_j = 1.$$

As functions in $A(\bar{\mathbb{D}})$ are uniformly approximated by analytic polynomials, there exists a sequence of polynomials $\{p_m^{(j)}\}_{m=1}^\infty$ such that $\|p_m^{(j)} - h_j\|_\infty \to 0$ as $m \to \infty$ for every $j = 1, \dots, n$. Hence, for sufficiently large m_0,

$$\Big\| 1 - \sum_{j=1}^n f_j p_{m_0}^{(j)} \Big\| < 1/2.$$

In particular, $\sum_{j=1}^n f_j p_{m_0}^{(j)}$ has no zeros on $\bar{\mathbb{D}}$. Then $1/\sum_{j=1}^n f_j p_{m_0}^{(j)} \in S^\infty(\mathbb{D})$. Put $g_j = p_{m_0}^{(j)} / \sum_{k=1}^n f_k p_{m_0}^{(k)}$ for $j = 1, \dots, n$. Then $g_j \in S^\infty(\mathbb{D})$ and $\sum_{j=1}^n f_j g_j = 1$ by a simple calculation.

For each $p \in \bar{\mathbb{D}}$, the point evaluation δ_p on $S^\infty(\mathbb{D})$ which takes the value at p is a nontrivial complex homomorphism. Hence, we have that $\bar{\mathbb{D}} \subset M_\infty$. Suppose that there exists $\delta \in M_\infty$ such that $\delta \neq \delta_p$ for any $p \in \bar{\mathbb{D}}$. It follows that for any $p \in \bar{\mathbb{D}}$, there exists $f_p \in S^\infty(\mathbb{D})$ such that

$$\delta(f_p) = 0, \quad \delta_p(f_p) \neq 0. \tag{4-13}$$

For every $p \in \bar{\mathbb{D}}$, we define an open subset of $\bar{\mathbb{D}}$ by $V_p = \{x \in \bar{\mathbb{D}};\ f_p(x) \neq 0\}$ and we have $p \in V_p$. Since $\bar{\mathbb{D}}$ is compact, let $p_1, \ldots, p_n$ be the corresponding elements in $\bar{\mathbb{D}}$; thus, we have $\sum_{j=1}^n |f_{p_j}| > 0$. By the above arguments, there exist $g_1, \ldots, g_n \in S^\infty(\mathbb{D})$ such that $\sum_{j=1}^n f_{p_j} g_j = 1$. By (4-13),

$$1 = \delta(\mathbf{1}) = \delta\Big(\sum_{j=1}^n f_{p_j} g_j\Big) = \sum_{j=1}^n \delta(f_{p_j})\delta(g_j) = 0,$$

which is a contradiction. It follows that $\bar{\mathbb{D}} = M_\infty$. □

Miura [19, Theorem 1] determined the form of all surjective isometries on $S^\infty(\mathbb{D})$.

THEOREM 4.11 (Miura [19]). *Suppose that $U : S^\infty(\mathbb{D}) \to S^\infty(\mathbb{D})$ is a surjective isometry with respect to the norm $\|\cdot\|_\Sigma$. Then there exist unimodular constants $\alpha, \lambda \in \mathbb{C}$ such that*

$$U(f) = U(0) + \alpha f(\lambda\cdot), \quad f \in S^\infty(\mathbb{D}),$$

or

$$U(f) = U(0) + \alpha \overline{f(\overline{\lambda\cdot})}, \quad f \in S^\infty(\mathbb{D}).$$

Conversely, each mapping of the above form is a surjective isometry from $S^\infty(\mathbb{D})$ onto $S^\infty(\mathbb{D})$.

COROLLARY 4.12. *Suppose that $T \in M(S^\infty(\mathbb{D}), S^\infty(\mathbb{D}))$ is pointwise 2-local in $\mathrm{Iso}(S^\infty(\mathbb{D}), S^\infty(\mathbb{D}))$. Then $T \in \mathrm{Iso}(S^\infty(\mathbb{D}), S^\infty(\mathbb{D}))$.*

PROOF. Suppose that $T \in M(S^\infty(\mathbb{D}), S^\infty(\mathbb{D}))$ is pointwise 2-local in $\mathrm{Iso}(S^\infty(\mathbb{D}), S^\infty(\mathbb{D}))$. By Theorem 4.11, $\mathrm{Iso}(S^\infty(\mathbb{D}), S^\infty(\mathbb{D})) \subset \mathrm{GWC}$. Then Theorem 3.2 asserts that there exist a continuous map $\pi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ and a continuous map $\epsilon : \bar{\mathbb{D}} \to \{\pm 1\}$ such that

$$T(f) = T(0) + \alpha[f \circ \pi]^\epsilon, \quad f \in S^\infty(\mathbb{D}),$$

where $\alpha = T(\mathbf{1}) - T(0)$ is a unimodular constant since $T(\mathbf{1}) - T(0)$ is a unimodular function and it is analytic on $\mathbb{D}$. Furthermore, $\epsilon = 1$ on $\bar{\mathbb{D}}$ or $\epsilon = -1$ on $\bar{\mathbb{D}}$. Put $T_1 = \bar{\alpha}(T - T(0))$. Then

$$T_1 = f \circ \pi, \quad f \in S^\infty(\mathbb{D})$$

if $\epsilon = 1$ and

$$T_1(f) = \overline{f \circ \pi}, \quad f \in S^\infty(\mathbb{D})$$

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if $\epsilon = -1$. Letting $f = \mathrm{Id}$, the identity function, we see that $\pi \in S^\infty(\mathbb{D})$ if $\epsilon = 1$ and $\bar{\pi} \in S^\infty(\mathbb{D})$ if $\epsilon = -1$. Put $\varphi = \pi$ if $\epsilon = 1$ and $\varphi = \bar{\pi}$ if $\epsilon = -1$. Then we have that $\varphi \in S^\infty(\mathbb{D})$ and

$$T_1(f) = f \circ \varphi, \quad f \in S^\infty(\mathbb{D})$$

if $\epsilon = 1$ and

$$T_1(f) = \overline{f \circ \bar{\varphi}}, \quad f \in S^\infty(\mathbb{D})$$

if $\epsilon = -1$. In particular,

$$T_1(\mathrm{Id}) = \varphi \tag{4-14}$$

either for $\epsilon = 1$ or for $\epsilon = -1$. Since T_1 is pointwise 2-local in $\mathrm{Iso}(S^\infty(\mathbb{D}), S^\infty(\mathbb{D}))$ by the definition of T_1, for every $x \in \bar{\mathbb{D}}$ there exist $u_x \in S^\infty(\mathbb{D})$ and unimodular constants α_x, λ_x such that

$$(T_1(\mathrm{Id}))(x) = u_x(x) + \alpha_x \,\mathrm{Id}(\lambda_x x)$$

and

$$0 = (T_1(0))(x) = u_x(x),$$

or

$$(T_1(\mathrm{Id}))(x) = u_x(x) + \alpha_x \overline{\mathrm{Id}(\overline{\lambda_x x})} = u_x(x) + \alpha_x \,\mathrm{Id}(\lambda_x x)$$

and

$$0 = (T_1(0))(x) = u_x(x).$$

In any case,

$$(T_1(\mathrm{Id}))(x) = \alpha_x \lambda_x x. \tag{4-15}$$

Combining (4-14) and (4-15),

$$\varphi(x) = \alpha_x \lambda_x x$$

for every $x \in \bar{\mathbb{D}}$. Then we have $\varphi(0) = 0$ and $|\varphi(x)| = |x|$ for every $x \in \bar{\mathbb{D}}$. Since $\varphi : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$ is analytic in $\mathbb{D}$, the Schwartz lemma asserts that there is a unimodular constant λ_0 such that

$$\varphi(x) = \lambda_0 x, \quad x \in \bar{\mathbb{D}}.$$

It follows that

$$T(f) = T(0) + (T(\mathbf{1}) - T(0)) f(\lambda_0 \cdot), \quad f \in S^\infty(\mathbb{D})$$

or

$$T(f) = T(0) + (T(\mathbf{1}) - T(0)) \overline{f(\overline{\lambda_0 \cdot})}, \quad f \in S^\infty(\mathbb{D}).$$

By Theorem 4.11, we conclude that $T \in \mathrm{Iso}(S^\infty(\mathbb{D}), S^\infty(\mathbb{D}))$. □

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4.5. The algebra $A(\dot{})$. Let $0 < r < 1$ and $\Omega = \{z : r \le |z| \le 1\}$ be an annulus. Let $A(\Omega)$ be the algebra of all complex-valued continuous functions that are analytic on the interior of Ω. It is well known that $A(\Omega)$ is a uniform algebra on Ω whose maximal ideal space is homeomorphic to Ω.

COROLLARY 4.13. *Suppose that U is a surjective isometry from the annulus algebra $A(\Omega)$ onto itself. Then there exist unimodular constants $\alpha, \lambda \in \mathbb{T}$ such that*

$$U(f) = U(0) + \alpha f(\lambda\cdot), \quad f \in A(\Omega),$$

or

$$U(f) = U(0) + \alpha\overline{f(\overline{\lambda\cdot})}, \quad f \in A(\Omega).$$

Conversely, each mapping of the form is a surjective isometry from $A(\Omega)$ onto $A(\Omega)$.

PROOF. Let π be a homeomorphism on Ω that is analytic on the interior of Ω. Then there exists a unimodular constant $\lambda \in \mathbb{T}$ such that $\pi(z) = \lambda z$ for any $z \in \Omega$. The desired statement follows from a similar argument to that of Corollary 4.4, in which we get a characterization of all surjective isometries on the disk algebra. □

COROLLARY 4.14. *Suppose that $T \in M(A(\Omega), A(\Omega))$ is pointwise 2-local in* $\mathrm{Iso}(A(\Omega), A(\Omega))$. *Then $T \in \mathrm{Iso}(A(\Omega), A(\Omega))$.*

PROOF. By Corollary 4.13, we get $\mathrm{Iso}(A(\Omega), A(\Omega)) \subset \mathrm{GWC}$. A homeomorphism on Ω which is analytic on the interior is just a rotation and thus we finish by a similar argument to Corollary 4.12. □

5. Iso-reflexivity

Many references in the literature study isometries from the point of view of how they are determined by their local actions [2, 4, 11, 21, 27, 28, 32]. By Theorem 3.2, we have that several 2-local maps are linear and hence they are local maps. In this section we prove that a local isometry in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ is 2-local in $\mathrm{Iso}(B_1, B_2)$. Applying corollaries of the above section, we see the reflexivity of $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ for several Banach spaces of continuous functions.

DEFINITION 5.1. Put

$$M_{\mathbb{C}}(B_1, B_2) = \{T \in M(B_1, B_2);\ T \text{ is complex-linear}\},$$

$$\mathrm{Iso}_{\mathbb{C}}(B_1, B_2) = \{T \in \mathrm{Iso}(B_1, B_2);\ T \text{ is complex-linear}\}.$$

Recall that $T \in M_{\mathbb{C}}(B_1, B_2)$ is local in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ if, for every $f \in B_1$, there exists $T_f \in \mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ such that

$$T(f) = T_f(f).$$

We say that $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ is iso-reflexive if every local map in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ is an element in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$.

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PROPOSITION 5.2. *Suppose that* $T \in M_{\mathbb{C}}(B_1, B_2)$ *is local in* $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$*. Then* T *is 2-local in* $\mathrm{Iso}(B_1, B_2)$*.*

PROOF. Let $f, g \in B_1$ be arbitrary. Then there exists $T_{f,g} \in \mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ such that

$$T(f - g) = T_{f,g}(f - g).$$

As T and $T_{f,g}$ are complex-linear,

$$T(f) - T(g) = T_{f,g}(f) - T_{f,g}(g). \tag{5-1}$$

Put

$$h_{f,g} = T(f) - T_{f,g}(f).$$

By (5-1),

$$T(f) = h_{f,g} + T_{f,g}(f),$$

$$T(g) = h_{f,g} + T_{f,g}(g).$$

It is easy to see that $h_{f,g} + T_{f,g}(\cdot) \in \mathrm{Iso}(B_1, B_2)$. It follows that T is 2-local in $\mathrm{Iso}(B_1, B_2)$. □

COROLLARY 5.3. *Suppose that every 2-local map in* $\mathrm{Iso}(B_1, B_2)$ *is an element in* $\mathrm{Iso}(B_1, B_2)$*. Then* $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$ *is iso-reflexive.*

PROOF. Suppose that $T \in M_{\mathbb{C}}(B_1, B_2)$ is local in $\mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$. Then, by Proposition 5.2, T is 2-local in $\mathrm{Iso}(B_1, B_2)$. By assumption, we have $T \in \mathrm{Iso}(B_1, B_2)$. Since T is complex-linear, we infer that $T \in \mathrm{Iso}_{\mathbb{C}}(B_1, B_2)$. □

Applying Corollaries 4.3, 4.5, 4.7, 4.9, 4.12 and 4.14, we obtain $\mathrm{Iso}_{\mathbb{C}}(C(X_1), C(X_2))$ for first countable compact Hausdorff spaces X_1 and X_2 and $\mathrm{Iso}_{\mathbb{C}}(A(\bar{\mathbb{D}}), A(\bar{\mathbb{D}}))$, $\mathrm{Iso}_{\mathbb{C}}(\mathrm{Lip}(X_1), \mathrm{Lip}(X_2))$, $\mathrm{Iso}_{\mathbb{C}}(C^1[0,1], C^1[0,1])$, $\mathrm{Iso}_{\mathbb{C}}(S^\infty(\mathbb{D}), S^\infty(\mathbb{D}))$ and $\mathrm{Iso}_{\mathbb{C}}(A(\Omega), A(\Omega))$ are iso-reflexive.

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