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Asymptotic behaviour of solutions of Fisher–KPP equation with free boundaries in time-periodic environment†

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We study a free boundary problem of the form: $u_t = u_{xx} + f(t, u)$ $(g(t) < x < h(t))$ with free boundary conditions $h'(t) = -u_x(t, h(t)) - \alpha(t)$ and $g'(t) = -u_x(t, g(t)) + \beta(t)$, where $\beta(t)$ and $\alpha(t)$ are positive T-periodic functions, $f(t, u)$ is a Fisher–KPP type of nonlinearity and T-periodic in t. This problem can be used to describe the spreading of a biological or chemical species in time-periodic environment, where free boundaries represent the spreading fronts of the species. We study the asymptotic behaviour of bounded solutions. There are two T-periodic functions $\alpha_0(t)$ and $\alpha^*(t; \beta)$ with $0 < \alpha_0 < \alpha^*$ which play key roles in the dynamics. More precisely, (i) in case $0 < \beta < \alpha_0$ and $0 < \alpha < \alpha^*$, we obtain a trichotomy result: (i-1) spreading, that is, $h(t) - g(t) \to +\infty$ and $u(t, \cdot + ct) \to 1$ with $c \in (-\bar{l}, \bar{r})$, where $\bar{l} := \frac{1}{T}\int_0^T l(s)ds$, $\bar{r} := \frac{1}{T}\int_0^T r(s)ds$, the T-periodic functions $-l(t)$ and $r(t)$ are the asymptotic spreading speeds of $g(t)$ and $h(t)$, respectively (furthermore, $r(t) > 0 > -l(t)$ when $0 < \beta < \alpha < \alpha_0$; $r(t) = 0 > -l(t)$ when $0 < \beta < \alpha = \alpha_0$; $0 > \bar{r} > -\bar{l}$ when $0 < \beta < \alpha_0 < \alpha < \alpha^*$); (i-2) vanishing, that is, $\lim_{t\to\mathcal{T}} h(t) = \lim_{t\to\mathcal{T}} g(t)$ and $\lim_{t\to\mathcal{T}} \max_{g(t)\le x\le h(t)} u(t, x) = 0$, where $\mathcal{T}$ is some positive constant; (i-3) transition, that is, $g(t) \to -\infty$, $h(t) \to -\infty$, $0 < \lim_{t\to\infty}[h(t) - g(t)] < +\infty$ and $u(t, \cdot) \to V(t, \cdot)$, where V is a T-periodic solution with compact support. (ii) in case $\beta \ge \alpha_0$ or $\alpha \ge \alpha^*$, vanishing happens for any solution.

Key words: Fisher–KPP equation, free boundary problem, periodic travelling wave, asymptotic behaviour.

2010 Mathematics Subject Classification: 35K20, 35K55, 35B40, 35R35.

1 Introduction

In this paper, we consider the following problem:

$$\begin{cases} u_t = u_{xx} + f(t, u), & g(t) < x < h(t),\ t > 0, \\ u\big(t, g(t)\big) = u\big(t, h(t)\big) = 0, & t > 0, \\ g'(t) = -u_x\big(t, g(t)\big) + \beta(t), & t > 0, \\ h'(t) = -u_x\big(t, h(t)\big) - \alpha(t), & t > 0, \\ -g(0) = h(0) = h_0,\quad u(0, x) = u_0(x), & -h_0 \le x \le h_0, \end{cases} \tag{1.1}$$

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where $x = g(t)$ and $x = h(t)$ are moving boundaries to be determined together with $u(t, x)$, and $\alpha(t)$ and $\beta(t)$ are given positive T-periodic functions with $0 < \beta(t) < \alpha(t)$ for $t \in [0, T]$, where $T > 0$ is a given constant. Our basic assumptions on $f(t, u)$ are the following:

$$\begin{cases} f(t,u) \in C^{\nu/2,1+\nu/2}\big([0,T] \times \mathbb{R}\big) \text{ for some } \nu \in (0,1), T\text{-periodic in } t;\ f(t,0) \equiv 0; \\ a(t) := f_u(t,0) > 0 \text{ is } T\text{-periodic}, a(t) \in C^{\nu/2}\big([0,T]\big); \\ f(t,u)/u \text{ is strictly decreasing in } u > 0;\ \text{and for any } t \in [0,T], f(t,u) < 0 \text{ for } u > 1. \end{cases} \quad (1.2)$$

This condition on f implies that $f(t, u) > 0$ for small $u > 0$ and $f(t, u) < 0$ for $u > 1$. Hence $f(t, u)$ is a Fisher–Kolmogorov, Petrovsky and Piskunov (KPP) type of nonlinearity. A typical example is $f(t, u) = u(a(t) - b(t)u)$ for some positive T-periodic functions $a(t)$ and $b(t)$.

The initial function u_0 belongs to $\mathscr{X}(h_0)$ for some $h_0 > 0$, where

$$\mathscr{X}(h_0) := \Big\{\phi \in C^2\big([-h_0, h_0]\big) : \phi(-h_0) = \phi(h_0) = 0,\ \phi(x) \geq (\not\equiv) 0 \text{ in } (-h_0, h_0)\Big\}. \quad (1.3)$$

The problem (1.1) may be used to model the spreading of a new or invasive species in time-periodic environment. The density of the species is represented by $u(t, x)$, and the free boundaries $x = h(t)$ and $x = g(t)$ represent the expanding fronts of the species.

In homogenous environment, that is, the special case of the problem (1.1) independent of time t is studied by many authors. Such as, in the case $\alpha = \beta = 0$, in Refs. [12, 13], the authors proved that, when $h_0 \geq \pi/\big(2\sqrt{f'(0)}\big)$, spreading happens (i.e., $u(t, \cdot) \to 1$ and $-g(t)$, $h(t) \to +\infty$); when $h_0 < \pi/\big(2\sqrt{f'(0)}\big)$, they obtained a spreading–vanishing dichotomy result, namely the species either spreads or it vanishes (i.e., $u(t, \cdot) \to 0$, $g(t) \to g_\infty$ and $h(t) \to h_\infty$ for some $-g_\infty, h_\infty \in (0, +\infty)$). This result shows that free boundary problem has advantages in explaining the spreading of species compared with the Cauchy problems (the Cauchy problems have hair-trigger effect: spreading always happens no matter how small the positive initial data are, cf. [1, 2]). Recently, [13, 14] also considered the bistable and combustion types of f in homogenous environment and gave the descriptions on the asymptotic behaviour of solutions. In addition, in Ref. [22], the authors considered the corresponding problem of (1.1) with a fixed boundary $g(t) \equiv 0$. In Ref. [19], the authors studied the problem with advection term in homogenous environment and obtained a complete description in the long-time behaviour of the solutions. Moreover, in Refs. [10, 15, 17], the authors also studied the corresponding problem of (1.1) in higher-dimensional spaces with $\alpha = \beta = 0$. In Refs. [4, 6], the authors considered the problem of (1.1) (which is also independent of t) with $\alpha = \beta =$ const.; they studied the asymptotic behaviour of solutions when $\alpha > 0$ is not large (i.e., $\alpha < \alpha_0 := \sqrt{2\int_0^1 f(s)\,ds}$) and obtained a trichotomy result: besides spreading and vanishing ($u \to 0$ and $\lim_{t\to T_0}\big[h(t) - g(t)\big] = 0$ for some $0 < T_0 < +\infty$, the third possibility may happen, namely the transition case (both $\lim_{t\to\infty} g(t)$ and $\lim_{t\to\infty} h(t)$ are *finite* numbers, and the solution $u(t, x)$ tends to a *stationary solution* of equation $(1.1)_1$, where $(1.1)_1$ means the first equation in system (1.1)). Later, in Ref. [5], the authors studied the problem (1.1) when α and β are different constants (i.e., $0 < \beta < \alpha$). They obtained complicated but interesting results: there are two parameters α_0 and α^* which play key roles in the dynamics. They obtained a trichotomy result:

(i) Spreading, that is, $h(t) - g(t) \to +\infty$ and $u(t, \cdot + ct) \to 1$ for some constant $c \in (c_l, c_r)$, where the constants c_l and c_r are the asymptotic spreading speed of $g(t)$ and $h(t)$, respectively. Moreover, using phase plane analysis, they have $c_r > 0 > c_l$ when $0 < \alpha < \alpha_0$; $c_r = 0 > c_l$ when $\alpha = \alpha_0$; $c_l < c_r < 0$ when $\alpha_0 < \alpha < \alpha^*$.

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(ii) Vanishing, that is, $u \to 0$ and $h(t) - g(t) \to 0$ as $t \to \mathcal{T}$ for some $\mathcal{T} \in (0, +\infty)$.

(iii) In the transition case that u converges to a compactly supported travelling wave V^*, that is, $\lim_{t\to\infty} h(t) \to -\infty$, $\lim_{t\to\infty} g(t) \to -\infty$, $0 < \lim_{t\to\infty} \big[h(t) - g(t)\big] < +\infty$ and $u(t, x) \to V^*(x - c^* t)$ with $c^* < 0$, where V^* is a travelling wave with compact support and it satisfies the free boundary conditions.

Recently, in inhomogenous environment, some authors studied the problem in time-dependent environments with the case $\alpha = \beta = 0$. Such as, in Ref. [11], the authors considered the time-periodic problem and obtained a spreading–vanishing result; in Ref. [27], the authors studied the problem with a time-periodic advection term; they discussed the influences of the advection term on the asymptotic behaviour of the solutions and obtained almost all the results as in temporally homogeneous case, while in Ref. [23], the authors considered the almost time-periodic problem. Other problems in time-periodic environment can be seen in Refs. [8, 9], so on.

Besides the asymptotic behaviour of solutions, many authors also considered the asymptotic spreading speed of free boundaries; in Refs. [12, 13], they also obtained the estimates:

$$k^* := \lim_{t\to\infty} \frac{h(t)}{t} = \lim_{t\to\infty} \frac{g(t)}{t} > 0.$$

Later, Ref. [16] improved this result to sharper estimates. Moreover, Refs. [5, 6, 19, 21, 27] also give the estimates of asymptotic spreading speed of free boundaries.

In this paper, we will examine the problem (1.1) with $0 < \beta(t) < \alpha(t)$ in the time-periodic case, which more closely reflects the periodic variation of the natural environment, such as daily or seasonal changes. We use the parameters $\alpha(t)$ and $\beta(t)$ to denote the decay rates at the boundaries since there is a force resistant to spreading at the front for some species. Intuitively, the presence of $\alpha > 0$ makes the solution more difficult to spread than the case $\alpha = 0$. Indeed, $h'(t) > 0$ only if $u_x(t, h(t)) < -\alpha$. Such free boundary conditions are widely used in many biological models. For example, in Refs. [7, 18, 25, 26, 29], the authors considered protocell models which mimic the biological process of growth and dissolution of an organism with external nutrient supply. The nutrient is converted to building material of liquid phase as it is metabolised. The liquid building material then diffuses within the protocell and it is polymerised into solid phase (more precisely, plastic phase) which builds the protocell. Besides polymerisation, the liquid building material also undergoes disintegration due to some factors such as aging, which causes the cell to shrink. On the other hand, the flux of building material at the boundary causes the cell to grow. The total result of these two effects is

$$V_n = -\frac{\partial C}{\partial n} - \gamma,$$

where n is the exterior normal, V_n is the velocity of the boundary points in the direction n, C is the concentration of building materials of the cell and $\gamma > 0$ is the disintegration rate at the boundary. In addition, the free boundary conditions used in our problem can also be deduced by some competition systems of three species (cf. [5, 28]). Moreover, Bao et al. [3] used a similar free boundary condition to describe the spreading of mosquito driven by climate warming.

Our main purpose in this paper is to study the asymptotic behaviour of solutions of the problem (1.1) in time-periodic environment with $0 < \beta(t) < \alpha(t)$, where $\alpha(t)$ and $\beta(t)$ are T-periodic functions. Throughout this paper, we assume that $T > 0$ is a given constant and using the following notation:

$$\mathcal{K} := \Big\{p(t) \in C^{\nu/2}\big([0, T]\big) : p(0) = p(T) \text{ for some } \nu \in (0, 1)\Big\};$$

$$\mathcal{K}^+ := \Big\{p(t) \in \mathcal{K} : p(t) > 0 \text{ for all } t > 0\Big\}; \text{ for each } p \in \mathcal{K}, \text{ denote } \overline{p} := \frac{1}{T}\int_0^T p(s)\, ds.$$

In this paper, we assume that $\beta(t), \alpha(t) \in \mathcal{K}^+$.

To sketch the influences of $\alpha(t)$ and $\beta(t)$ on the asymptotic behaviour of solutions of the problem (1.1), we need three special solutions (see details in Section 3). (1) The unique positive T-periodic solution $P(t)$ of the ordinary differential equation $u_t = f(t, u)$. (2) Periodic rightward travelling semi-wave $Q_R(t, R(t) - x)$, which is defined by the solution of the problem

$$\begin{cases} U_t = U_{zz} - r(t)U_z + f(t, U), & t \in [0, T],\ z > 0, \\ U(t, 0) = 0,\ U(t, \infty) = P(t), & t \in [0, T], \\ U(0, z) = U(T, z),\ U_z(t, z) > 0, & t \in [0, T],\ z \geq 0, \\ r(t) = U_z(t, 0) - \alpha(t), & t \in [0, T]. \end{cases} \tag{1.4}$$

In Section 3, we will show that for any given $\alpha(t) \in \mathcal{K}^+$, the problem (1.4) has a unique solution pair $(r, U) = (r, Q_R)$ with $r = r(t; \alpha) \in \mathcal{K}$, then with $R(t) := \int_0^t r(s; \alpha)ds$, the function $u(t, x) = Q_R(t, R(t) - x)$ satisfies $(1.1)_1$, $u(t, R(t)) = 0$ and $R'(t) = -u_x(t, R(t)) - \alpha(t)$. As in Refs. [13, 27], we also call $u(t, x) = Q_R(t, R(t) - x)$ a periodic rightward travelling semi-wave since it is defined only for $x \leq R(t)$ and $Q(t, z)$ is periodic in t. Moreover, for some special $\alpha(t)$, there is a unique solution $(r, U) = (r, Q_R)$ of (1.4) with $r = 0$, denote this $\alpha(t)$ by $\alpha_0(t)$, and $Q_R(t, z)$ by $Q_0(t, z)$, so $\alpha_0(t) = Q_{0z}(t, 0)$. (3) Periodic leftward travelling semi-wave $Q_L(t, L(t) + x)$, where $L(t) := \int_0^t l(s; \beta)ds$ for some function $l = l(s; \beta) \in \mathcal{K}$, and $(l, U) = (l, Q_L)$ is the unique solution of

$$\begin{cases} U_t = U_{zz} - l(t)U_z + f(t, U), & t \in [0, T],\ z > 0, \\ U(t, 0) = 0,\ U(t, \infty) = P(t), & t \in [0, T], \\ U(0, z) = U(T, z),\ U_z(t, z) > 0, & t \in [0, T],\ z \geq 0, \\ l(t) = U_z(t, 0) - \beta(t), & t \in [0, T]. \end{cases} \tag{1.5}$$

$u(t, x) = Q_L(t, L(t) + x)$ also satisfies the equation $u_t = u_{xx} + f(t, u)$ with $u(t, -L(t)) = 0$ and $-L'(t) = -u_x(t, -L(t)) + \beta(t)$.

The asymptotic behaviour of solutions of (1.1) is different when β and α are small, or medium-sized, or large (more on this below). However, β and α have both the size ($\overline{\beta}$ and $\overline{\alpha}$) and the shape ($\widetilde{\beta} := \beta - \overline{\beta}$ and $\widetilde{\alpha} := \beta - \overline{\alpha}$). Then the partition of β and α is more complicated than the homogeneous (cf. [5]). According to [27] and our analysis, we consider only the shape, namely, $\beta = b + \theta$, $\alpha = a + \theta$ for $a, b \in (0, +\infty)$ and $\theta \in \mathcal{K}$ with $\overline{\theta} = 0$.

We are now ready to explain the asymptotic behaviour of solutions intuitively. We need to consider a solution of (1.1) with the right front (the part of the solution near $h(t)$) and the left front (the part of the solution near $g(t)$). As we will see in Theorem 2.4, when spreading happens, the right front has a shape like the periodic rightward travelling semi-wave $Q_R(t, R(t) - x)$ and moves at a speed $\approx \overline{r(t; \alpha)}$; the left front has a shape like the periodic leftward travelling semi-wave $Q_L(t, L(t) + x)$ and moves at a speed $\approx -\overline{l(t; \beta)}$. Thus we need to consider the relationship between $r(t; \alpha)$ and $-l(t; \beta)$, which can determine the asymptotic locations of two free boundaries. By Section 3, we can derive that, for any fixed β (with $\beta(t) < \alpha_0(t)$) there exists a unique $\alpha^*(t;\beta)$ such that when $\alpha(t) = \alpha^*(t;\beta)$,

$$\overline{r(t;\alpha)} = -\overline{l(t;\beta)}; \tag{1.6}$$

when $\alpha(t) < \alpha^*(t)$, $\overline{r(t;\alpha)} > -\overline{l(t;\beta)}$; when $\alpha(t) > \alpha^*(t)$, $\overline{r(t;\alpha)} < -\overline{l(t;\beta)}$.

The precise relationship between $r(t;\alpha)$ and $l(t;\beta)$ can be seen in Proposition 3.8 below. According to their relations, we have the following situations. *Case 1.* When $0 < \beta(t) < \alpha(t) < \alpha_0(t)$, by Proposition 3.8, we have $-l(t;\beta) < 0 < r(t;\alpha)$, so the free boundary $h(t)$ may move rightwards to $+\infty$ and $g(t)$ moves leftwards to $-\infty$ as $t \to \infty$, respectively. The solution u with large initial data can grow up and converges to $P(t)$. *Case 2.* Assume that $0 < \beta(t) < \alpha_0(t) < \alpha(t) < \alpha^*(t)$. If the initial data are sufficiently large, the solution $u(t,x)$ can also grow up between the left front (with speed $\approx -\bar{l}$) and the right front (its speed $\approx \bar{r}$). In this case, $-\overline{l(t;\beta)} < \overline{r(t;\alpha)} < 0$ (cf. Proposition 3.8), so both $g(t)$ and $h(t)$ tend to $-\infty$, but the free boundary $g(t)$ travels leftwards faster than $h(t)$, so their distance becomes larger and lager and tends to $+\infty$ as time increases. *Case 3.* When $0 < \beta(t) < \alpha_0(t) < \alpha^*(t) < \alpha(t)$, we have $\overline{r(t;\alpha)} < \overline{l(t;\beta)} < 0$ (cf. Proposition 3.8); this implies that both the left front and the right front move leftwards, but the right front moves faster than the left front, which forces the solution to vanish since there is not enough space for the solution to grow up. *Case 4.* When $\alpha_0(t) < \beta(t) < \alpha(t)$, it follows from Proposition 3.8 that $\overline{r(t;\alpha)} < 0 < \overline{l(t;\beta)}$, so the two free boundaries move relative to one another and they move to the same point within a finite time. Hence, vanishing happens for any solution.

This paper is organised as follows. In Section 2 we present the main results. In Section 3 we give some basic results including the comparison principles, several types of travelling waves, and the relationships of the speeds of two travelling semi-waves. These are fundamental for this research. In Section 4 we give some sufficient conditions for vanishing and spreading. In Section 5 we prove the main theorems.

2 Main results

In this section, we give the main results. Using a similar argument as in Refs. [6, 12, 13], one can show that (1.1) has a unique solution (u,g,h) defined on the maximal interval $[0,\mathcal{T})$ (for some $\mathcal{T} \in (0,+\infty]$) with $(u,g,h) \in C^{1+\gamma/2,2+\gamma}(\overline{D}_{\mathcal{T}}) \times C^{1+\gamma/2}([0,\mathcal{T}]) \times C^{1+\gamma/2}([0,\mathcal{T}])$ for any $\gamma \in (0,1)$, where $\mathcal{T} := \{s \mid u \in C^{1+\gamma/2,2+\gamma}(\overline{D}_s)\}$ with $D_s := \{(t,x) \in \mathbb{R}^2 : x \in [g(t),h(t)],\ t \in (0,s]\}$, $D_{\mathcal{T}} := \{(t,x) \in \mathbb{R}^2 :\ x \in [g(t),h(t)], t \in (0,\mathcal{T}]\}$. Moreover, as in the proof of [5] and [6, Lemma 2.9], one can show that

$$h_{\mathcal{T}} := \lim_{t\to\mathcal{T}} h(t) \in [-\infty,+\infty] \quad \text{and} \quad g_{\mathcal{T}} := \lim_{t\to\mathcal{T}} g(t) \in [-\infty,+\infty]$$

exist. As we will see below, $\mathcal{T} < +\infty$ if and only if vanishing happens, that is, $h(t) - g(t) \to 0$ and $u \to 0$. Even, such a phenomena may happen within a period, that is, $\mathcal{T} < T$ (e.g. in the case $\alpha, \beta \gg 1$). On the other hand, when $\mathcal{T} = +\infty$, we also write $h_{\mathcal{T}}$ and $g_{\mathcal{T}}$ as h_∞ and g_∞, respectively. Moreover, using the similar arguments as in [13, Lemma 2.8] with obvious modifications, we have

$$g(t) + h(t) < 2h_0 \text{ for all } t \in [0,\mathcal{T}). \tag{2.1}$$

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In order to study the asymptotic behaviour of the problem (1.1) for large $\alpha > 0$, we need the following condition:

$$\varrho(t) := P(t)f_u\big(t, P(t)\big) - f\big(t, P(t)\big) < 0 \quad \text{for } t \in [0, T], \tag{2.2}$$

where $P(t)$ is the T-periodic solution of $u_t = f(t, u)$. A typical example of such f is $f(t, u) = u(a(t) - b(t)u)$ with T-periodic functions $a(t) > 0$ and $b(t) > 0$. When $f = f(u)$, the condition (2.2) reduces to $f'(1) < 0$.

The following main theorems (Theorem 2.1, Theorem 2.3) give a rather complete description for the asymptotic behaviour of the solution (u, g, h) of the problem (1.1).

We first consider the case where β and α are not large: $0 < \beta(t) < \alpha_0(t)$ and $0 < \alpha(t) < \alpha^*(t; \beta)$ for $t \in [0, T]$, where $\alpha^*(t; \beta)$ is the unique root of (1.6). We also write $\alpha^*(t; \beta)$ as $\alpha^*(t)$ for simplicity. As given in the introduction, $\alpha_0(t) := Q_{0z}(t, 0)$. In order to study the influence of the initial data on the asymptotic behaviour of solutions, we consider the initial data $u_0 = \sigma\phi$ for any fixed $\phi \in \mathscr{X}(h_0)$, $\sigma \in (0, +\infty)$ are constant. We analyse the asymptotic behaviour of solutions by changing σ.

Theorem 2.1 *Assume $\beta(t), \alpha(t) \in \mathcal{K}^+$ and $0 < \beta(t) < \alpha_0(t)$, $0 < \alpha(t) < \alpha^*(t)$ for $t \in [0, T]$, and $u_0 = \sigma\phi$ for some $\phi \in \mathscr{X}(h_0)$. Let (u, g, h) be a solution of the problem (1.1) defined on some maximal time interval $[0, \mathcal{T})$. Then there exist $\sigma_* = \sigma_*(h_0, \phi)$ and $\sigma^* = \sigma^*(h_0, \phi)$ with $0 < \sigma_* \leq \sigma^* \leq \infty$ such that*

(i) **spreading** *happens when $\sigma > \sigma^*$, that is, $\mathcal{T} = +\infty$, $g_\infty = -\infty$ and*

$$\lim_{t\to\infty} u\big(t, \cdot + K(t)\big) = P(t) \quad \textit{locally uniformly in } \mathbb{R}, \tag{2.3}$$

where $K(t) := \int_0^t k(s)ds$ for any $k(t) \in \mathcal{K}$ with $\overline{k(t)} \in (-\overline{l(t; \beta)}, \overline{r(t; \alpha)})$, $r(t; \alpha)$ and $l(t; \beta)$ are the speeds of travelling semi-waves in (1.4) and (1.5), respectively, with $-\overline{l(t; \beta)} < \overline{r(t; \alpha)}$, moreover,

(i-a) *when $0 < \alpha(t) < \alpha_0(t)$ for $t \in [0, T]$, $h_\infty = +\infty$;*

(i-b) *when $\alpha_0(t) < \alpha(t) < \alpha^*(t)$ for $t \in [0, T]$, $h_\infty = -\infty$;*

(i-c) *when $\alpha(t) = \alpha_0(t)$ for $t \in [0, T]$, $-\infty < h_\infty < +\infty$, and more precisely,*

$$\lim_{t\to\infty} u(t, \cdot) = Q_0(t, h_\infty - \cdot) \quad \textit{locally uniformly in } (-\infty, h_\infty], \tag{2.4}$$

where $Q_0(t, z)$ is the unique solution of (1.4) with $r(t; \alpha) \equiv 0$,

(ii) **vanishing** *happens when $\sigma < \sigma_*$, that is, $\mathcal{T} < +\infty$,*

$$\lim_{t\to\mathcal{T}} g(t) = \lim_{t\to\mathcal{T}} h(t) \in (-\infty, +\infty) \quad \textit{and} \quad \lim_{t\to\mathcal{T}} \max_{g(t)\leq x\leq h(t)} u(t, x) = 0,$$

(iii) *in the* **transition** *case when $\sigma \in [\sigma_*, \sigma^*]$, that is, $\mathcal{T} = +\infty$,*

$$g_\infty = h_\infty = -\infty, \ \lim_{t\to\infty} \big[h(t) - g(t)\big] = \ell \ \textit{for some } \ell > 0,$$

and $u(t, x) \to V(t, x - ct - x_1)$ as $t \to \infty$ for some $x_1 \in \mathbb{R}$, where V is the positive periodic solution of

$$\begin{cases} V_t = V_{xx} - cV_x + f(t, V), & 0 < x < \ell, \ t > 0, \\ V(t, 0) = 0, \ V(t, \ell) = 0, \ t > 0, \end{cases} \tag{2.5}$$

with $c \in (-\bar{l}, 0)$.

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Remark 2.2 *(1) Obviously, $k(t)$ in (2.3) can be replaced by any constant $c \in (-\bar{l}, \bar{r})$.*

(2) In the spreading case, when $0 < \alpha(t) < \alpha_0(t)$, from Proposition 3.8 we have $-\bar{l} < 0 < \bar{r}$, so we can choose $k(t) \equiv 0 \in (-\bar{l}, \bar{r})$ in the above theorem, then (2.3) reduces to

$$\lim_{t\to\infty} u(t,\cdot) = P(t) \quad \text{locally uniformly in } \mathbb{R}.$$

(3) In the transition case, we show that neither vanishing nor spreading happens. In homogeneous case (i.e. $f(t,u) = f(u)$, $\alpha(t) = C_1$ and $\beta(t) = C_2$, where C_1 and C_2 are constants with $0 < C_2 < C_1$), Cai and Gu [5] proved that $\sigma_* = \sigma^*$ and the transition solution converges to a compactly supported travelling wave $V^*(x - c^*t - x_1)$, where V^* is a unique solution of $(1.1)_1$ with positive compactly support and satisfies both the left and right free boundary conditions, and it travels leftwards at a speed $c^* < 0$. We guess that in the transition case in our theorem, we also have $\sigma_* = \sigma^*$ and the solution converges to such a compactly supported travelling wave $\mathcal{V}$, which has a time-periodic profile and satisfies the two free boundary conditions. Similarly with the existence of tadpole-like travelling wave in Ref. [27], the main difficulty is the existence of $\mathcal{V}$ itself. This problem also remains open now.

We now show that only vanishing happens when $\beta(t)$ is large, or $\alpha(t)$ is large.

Theorem 2.3 *Assume that $\beta(t) < \alpha_0(t)$ and $\alpha(t) \geq \alpha^*(t)$, or $\beta(t) \geq \alpha_0(t)$ for all $t \in [0,T]$. Let (u, g, h) be a solution of (1.1) with initial data $u_0 \in \mathscr{X}(h_0)$. Then vanishing happens.*

For large α or/and β, the environment at the boundary is so bad that the free boundaries cannot expand outwards and their separation becomes smaller and smaller, so vanishing happens. Actually, we can construct two upper solutions on two sides of the solution moving relatively to one another. Thus the two free boundaries are forced by these two upper solutions to go to one point at a finite time and then the solution tends to 0.

Finally we show that when spreading happens as in Theorem 2.1, the spreading speed and the asymptotic profiles of the fronts are characterised by (1.4) and (1.5).

Theorem 2.4 *Assume spreading happens for a solution $u(t,x)$ of the problem (1.1) as in Theorem 2.1. Let (r, Q_R) and (l, Q_L) be the solutions of (1.4) and (1.5), respectively. Then there exist H_1, $G_1 \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \left[h(t) - R(t) - H_1\right] = 0, \qquad \lim_{t\to\infty} h'(t) = r(t;\alpha),$$

$$\lim_{t\to\infty} \left[g(t) + L(t) - G_1\right] = 0, \qquad \lim_{t\to\infty} g'(t) = -l(t;\beta),$$

and

$$\lim_{t\to\infty} \sup_{x\in[ct,h(t)]} \left|u(t,x) - Q_R\big(t, R(t) + H_1 - x\big)\right| = 0, \tag{2.6}$$

$$\lim_{t\to\infty} \sup_{x\in[g(t),ct]} \left|u(t,x) - Q_L\big(t, x + L(t) - G_1\big)\right| = 0 \tag{2.7}$$

for any c with $-\overline{l(t;\beta)} < c < \overline{r(t;\alpha)}$.

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Remark 2.5 *Finally, we remark that all the conclusions in Theorems 2.1, 2.3 and 2.4 are also true if the condition* $0 < \beta(t) < \alpha(t)$ *is relaxed to* $0 \le (\not\equiv)\beta(t) \le (\not\equiv)\alpha(t)$, $\beta(t) < \alpha_0(t)$ *is relaxed to* $\beta(t) \le (\not\equiv)\alpha_0(t)$ *and* $\alpha(t) < \alpha^*(t)$ *is relaxed to* $\alpha(t) \le\not\equiv \alpha^*(t)$. *For convenience, we do not use the relaxed conditions in this paper.*

Remark 2.6 *When* $\alpha =$ *const,* $\beta =$*const, that is, the problem (1.1) is homogenous, one can use the phase plane analysis to consider the existence and some essential properties of travelling semi-waves and other travelling waves. But, when* α *and* β *depend on t, that is, the problem (1.1) is inhomogeneous, and the phase plane analysis is not useful. So the methods we used in this paper are completely different from those in Ref. [5].*

Moreover, when $\alpha(t) = \beta(t)$ *we have the property* $-2h_0 < g(t) + h(t) < 2h_0$, *so the long-time behaviour of* $h(t)$ *and* $g(t)$ *is similar. Consequently, in the spreading case in Theorem 2.1, we only have (i-a), but (i-b) and (i-c) do not happen. Moreover, the transition case is also a different one, and the corresponding analysis is also very different from this paper but it is not difficult. So we will not consider the case* $\alpha(t) = \beta(t)$ *in this paper.*

3 Some basic results

In this section we first give the comparison principle and the definitions of upper solutions and lower solutions, then we present the existence of two types of travelling semi-waves and travelling waves with compact supports.

3.1 The comparison principle

We first give the following comparison theorems which can be proved similarly to [6, Lemma 2.3–2.4] and [12, Lemma 5.7].

Lemma 3.1 *Let* $\beta(t), \alpha(t) \in \mathcal{K}^+$. *Suppose that* $\mathcal{T} \in (0,\infty)$, $\bar{g}, \bar{h} \in C^1([0,\mathcal{T}])$, $\bar{u} \in C(\overline{D}_{\mathcal{T}}) \cap C^{1,2}(D_{\mathcal{T}})$ *with* $D_{\mathcal{T}} = \{(t,x) \in \mathbb{R}^2 : 0 < t \le \mathcal{T},\ \bar{g}(t) < x < \bar{h}(t)\}$, *and*

$$\begin{cases} \bar{u}_t \ge \bar{u}_{xx} + f(t,\bar{u}), & 0 < t \le \mathcal{T},\ \bar{g}(t) < x < \bar{h}(t), \\ \bar{u} = 0, \quad \bar{g}'(t) \le -\bar{u}_x + \beta(t), & 0 < t \le \mathcal{T},\ x = \bar{g}(t), \\ \bar{u} = 0, \quad \bar{h}'(t) \ge -\bar{u}_x - \alpha(t), & 0 < t \le \mathcal{T},\ x = \bar{h}(t). \end{cases}$$

If $[-h_0, h_0] \subseteq [\bar{g}(0), \bar{h}(0)]$, $u_0(x) \le \bar{u}(0,x)$ *in* $[-h_0, h_0]$, *and if* (u, g, h) *is a solution of (1.1) with initial data* $u_0(x)$, *then*

$$g(t) \ge \bar{g}(t),\ h(t) \le \bar{h}(t),\ u(t,x) \le \bar{u}(t,x) \quad \text{for } t \in (0,\mathcal{T}] \text{ and } x \in (g(t), h(t)).$$

Lemma 3.2 *Let* $\beta(t), \alpha(t) \in \mathcal{K}^+$. *Suppose that* $\mathcal{T} \in (0,\infty)$, $\bar{g}, \bar{h} \in C^1([0,\mathcal{T}])$, $\bar{u} \in C(\overline{D}_{\mathcal{T}}) \cap C^{1,2}(D_{\mathcal{T}})$ *with* $D_{\mathcal{T}} = \{(t,x) \in \mathbb{R}^2 : 0 < t \le \mathcal{T},\ \bar{g}(t) < x < \bar{h}(t)\}$, *and*

$$\begin{cases} \bar{u}_t \ge \bar{u}_{xx} + f(t,\bar{u}), & 0 < t \le \mathcal{T},\ \bar{g}(t) < x < \bar{h}(t), \\ \bar{u} \ge u, & 0 < t \le \mathcal{T},\ x = \bar{g}(t), \\ \bar{u} = 0, \quad \bar{h}'(t) \ge -\bar{u}_x - \alpha(t), & 0 < t \le \mathcal{T}, x = \bar{h}(t), \end{cases}$$

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with $h(t) > \overline{g}(t) \geq g(t)$ *in* $[0, \mathcal{T}]$, $h_0 \leq \overline{h}(0)$, $u_0(x) \leq \overline{u}(0, x)$ *in* $[\overline{g}(0), h_0]$, *where* (u, g, h) *is a solution of* (1.1). *Then*

$$h(t) \leq \overline{h}(t),\ u(t, x) \leq \overline{u}(t, x) \ \textit{for}\ t \in (0, \mathcal{T}] \ \textit{and}\ \ \overline{g}(t) < x < h(t).$$

Remark 3.3 *The function* $\overline{u}$ *or the triple* $(\overline{u}, \overline{g}, \overline{h})$ *in Lemmas 3.1 and 3.2 is usually called an upper solution of* (1.1) *and one can define a lower solution by reversing all the inequalities. There is a symmetric version of Lemma 3.2, where the conditions on the left and right boundaries are interchanged. We also have corresponding comparison results for lower solutions in each case.*

3.2 Positive solutions on bounded and unbounded intervals

In this subsection we present servals kinds of positive solutions of $(1.1)_1$. First, for any given $k(t) \in \mathcal{K}$ and $\ell > 0$, we consider the following T-periodic solutions of the following problem:

$$\begin{cases} v_t = v_{zz} + k(t)v_z + f(t, v), & t \in [0, T],\ z \in (0, \ell), \\ v(t, 0) = v(t, \ell) = 0, & t \in [0, T], \\ v(0, z) = v(T, z), & z \in (0, \ell). \end{cases} \tag{3.1}$$

Recall that we write $a(t) := f_u(t, 0)$ throughout this paper, and we denote $\overline{a} := \frac{1}{T}\int_0^T a(s)ds$.

Lemma 3.4 ([20, 24]) *Assume that* $k(t) \in \mathcal{K}$ *satisfies* $|\overline{k}| < \overline{c} := 2\sqrt{\overline{a}}$. *Then there exits a real number* $\ell^* := \ell(k, a) > 0$ *such that, when* $\ell > \ell^*$ *the problem* (3.1) *has a unique solution* $v = U_0(t, z; k, \ell)$ *which satisfies* $0 < U_0(t, z; k, \ell) < P(t)$ *in* $[0, T] \times (0, \ell)$ *and* $(U_0)_z(t, z; k, \ell) > 0$ *for* $t \in [0, T]$. *Moreover,* $U_0(t, z; k, \ell)$ *is strictly increasing in* ℓ *and* $U_0(t, z + \ell/2; k, \ell) \to P(t)$ *as* $\ell \to \infty$ *in* $L^\infty_{loc}(\mathbb{R})$ *topology; when* $\ell \leq \ell^*$, *the problem* (3.1) *has only zero solution.*

Assume that $k(t)$ *satisfies* $|\overline{k}| \geq \overline{c}$. *Then for any* $\ell > 0$, *the problem* (3.1) *has only zero solution.*

We next consider the following problem on the half-line:

$$\begin{cases} v_t = v_{zz} + k(t)v_z + f(t, v), & t \in [0, T],\ z > 0, \\ v(t, 0) = 0, & t \in [0, T], \\ v(0, z) = v(T, z), & z \geq 0. \end{cases} \tag{3.2}$$

Lemma 3.5 ([27]) *For any* $k(t) \in \mathcal{K}$, *the problem* (3.2) *has a bounded and nonnegative solution* $U(t, z; k)$. *Moreover,*

(i) *if* $\overline{k} > -\overline{c}$, *then*

$$U_z(t, z; k) > 0 \ \textit{in}\ [0, T] \times [0, +\infty), \quad U(t, z; k) - P(t) \to 0 \ \textit{as}\ z \to \infty; \tag{3.3}$$

$U(t, z; k)$ is the unique solution of (3.2) *satisfying* (3.3); $U_z(t, 0; k)$ *has a positive lower bound* δ *(independent of* t*), and it is strictly increasing in* k: $U_z(t, 0; k_1) < U_z(t, 0; k_2)$ *for any* $k_1(t), k_2(t) \in \mathcal{K}$ *satisfying* $k_1 \leq, \not\equiv k_2$ *and* $\overline{k}_1, \overline{k}_2 > -\overline{c}$; $U_z(t, 0; k)$ *is continuous in* k *in the sense that, for any* $\{k_1, k_2, ...\} \subset \mathcal{K}$ *satisfying* $\overline{k}_i > -\overline{c}\,(i = 1, 2, ...)$, $U_z(t, 0; k_n) \to U_z(t, 0; k)$ *in* $C^{\nu/2}([0, T])$ *if* $k_n \to k$ *in* $C^{\nu/2}([0, T])$.

(ii) *when* $\overline{k} \leq -\overline{c}$, (3.2) *has only trivial solution* 0.

(iii) *$U_0(t,z;k,\ell)$ converges as $\ell\to\infty$ to $U(t,z;k)$ in the topology of $L^\infty_{loc}([0,T]\times[0,\infty))$, where U_0 is the solution of (3.1). Moreover, $||(U_0)_z(t,z;k,\ell)-U_z(t,z;k)||_{C([0,T]\times[0,1])}\to 0$ as $\ell\to\infty$.*

3.3 Periodic travelling semi-waves

Based on the results in the previous subsection, we now construct periodic travelling semi-waves of $(1.1)_1$.

(I) Periodic rightward travelling semi-waves $Q_R(t,R(t;\alpha)-x)$. In this part we construct a periodic rightward travelling semi-wave which is periodic in time and is used to characterise solutions near the right boundaries.

We will use the above lemmas and the Schauder fixed point theorem to prove the existence of periodic rightward travelling semi-waves. We need define the following operator:

$$A[k](t):=U_z(t,0;k)-\alpha(t),$$

where $U(t,z;k)$ is the solution of the problem (3.2). From Lemma 3.5 we see that $A[k](t)$ is strictly increasing in k when $\overline{k}>-\overline{c}$.

Lemma 3.6 *Assume $\alpha(t)\in\mathcal{K}^+$. Then there exists a unique $r(t;\alpha)\in\mathcal{K}$ with $\overline{r(t;\alpha)}<\overline{c}$ such that $u(t,x)=Q_R(t,R(t;\alpha)-x)$ (with $R(t;\alpha):=\int_0^t r(s;\alpha)\,ds$) solves equation $(1.1)_1$ for $t\in\mathbb{R}$, $x<R(t;\alpha)$ and $r(t;\alpha)=-u_x(t,R(t;\alpha))-\alpha(t)=A[-r(t;\alpha)]$.*

Moreover, $r(t;\alpha)$ is strictly decreasing in $\alpha(t)>0$ in the sense that $r(t;\alpha_1)>r(t;\alpha_2)$ if $\alpha_1(t),\alpha_2(t)\in\mathcal{K}^+$ satisfy $\alpha_1\le\not\equiv\alpha_2$. In addition, $r(t;\alpha)$ is continuous in α.

Proof We give the proof of the lemma in three steps:

(i) Firstly, we prove the existence of the solution (r,Q_R) of the problem (1.4). By Lemma 3.5, for any $r(t)\in\mathcal{K}$, the problem (3.2) has a maximal (bounded and nonnegative) solution $U(t,z;-r)$, and $A[-r](t)=U_z(t,0;-r)-\alpha(t)$ is non-increasing in r.

When $r=r_*:=-\alpha(t)<0$ (note that $-r_*=\alpha(t)>0>-\overline{c}$) we have

$$A[-r_*]=U_z\big(t,0;\alpha(t)\big)-\alpha(t)>-\alpha(t)=r_* \tag{3.4}$$

and $A[-r_*]<U_z(t,0;\alpha(t))+\overline{c}$ (note that $-\alpha(t)<0<\overline{c}$). When $r=r^*:=U_z(t,0;\alpha(t))+\overline{c}$ we have $-\overline{r^*}=-\overline{U_z(t,0;\alpha(t))}-\overline{c}<-\overline{c}$. It follows from Lemma 3.5 (ii) that $A[-r^*]=U_z(t,0;-r^*)-\alpha(t)=0-\alpha(t)\in[r_*,r^*]$. Set

$$\Sigma:=\big\{r\in\mathcal{K}:r_*\le r\le r^*\big\}.$$

By the monotonicity of $A[-r]$ on r and the proof of [11, Theorem 2.4] one can show that the mapping $A[-\cdot]$ maps Σ continuously into a precompact set in Σ. Using the Schauder fixed point theorem we see that there exists $r(t;\alpha)\in\Sigma$ such that $r(t;\alpha)=A[-r(t;\alpha)]$. Clearly, equation (3.4) implies that $r(t;\alpha)\ge\not\equiv-\alpha(t)$, combining this and the definition of A we have $U_z(t,0;-r)=A[-r](t)+\alpha(t)\ge\not\equiv 0$. This implies by Lemma 3.5 that $-\overline{r}>-\overline{c}$ and $U_z(t,0;-r)>0$ for all $t\in[0,T]$. This yields $\overline{r}<\overline{c}$. We denote $U(t,z;-r)$ by $Q_R(t,z)$. Then a direct calculation shows that the function $(r,Q_R(t,R(t)-x))$ solves the problem (1.4) with $R(t)=R(t;\alpha):=\int_0^t r(s;\alpha)\,ds$. Moreover, let $u(t,x)=Q_R(t,R(t)-x)$, then $r(t;\alpha)=A[-r(t;\alpha)]=(Q_R)_z(t,0)-\alpha(t)=-u_x(t,R(t))-\alpha(t)$.

(ii) We now prove that $r(t;\alpha)$ is decreasing in α. Assume that $\alpha_1(t) \leq\not\equiv \alpha_2(t)$. Denote $r_i = r(t;\alpha_i)$ $(i=1,2)$ for convenience. Then $r_i = A[-r_i]$. We show that $r_1 \leq r_2$ is impossible. Otherwise, $-r_1 \geq\not\equiv -r_2$ and so $r_1 = A[-r_1] > A[-r_2] = r_2$ by Lemma 3.5 and the definition of A, contradicting our assumption $r_1 \leq r_2$. Therefore, $r_1 \not\leq r_2$ and so

$$r_2(t_0) < r_1(t_0) \text{ for some } t_0 \in [0,T]. \tag{3.5}$$

We now prove $r_2(t) \leq r_1(t)$ for all $t \in [0,T]$. Suppose by way of contradiction that $r_2(t_1) > r_1(t_1)$ for some $t_1 \in [0,T)$, then

$$t_2 := \sup\left\{\tau \in \left(t_1, t_1+T\right) : r_2(t) > r_1(t) \text{ for } t \in [t_1,\tau)\right\}$$

is well defined by (3.5). Hence, we have

$$r_2(t) > r_1(t) \text{ for } t \in \left[t_1,t_2\right], \quad r_2\left(t_2\right) = r_1(t_2). \tag{3.6}$$

Set

$$X := \int_{t_1}^{t_2}\left[r_2(s) - r_1(s)\right]ds, \quad R_1(t) := \int_{t_1}^{t} r_1(s)ds + X, \quad R_2(t) := \int_{t_1}^{t} r_2(s)ds.$$

Then $R_1(t) > R_2(t)$ for $t \in [t_1,t_2]$ and $R_1(t_2) = R_2(t_2)$ (denote it by x_2). For $i=1,2$, the problem (3.2) with $k_i(t) = -r_i$ (note that $-\overline{r_i} > -\overline{c}$) has a maximal bounded solution $U(t,z;-r_i) > 0$ $(i=1,2)$ for $z>0$. To get a contradiction, let us consider the functions $u_i(t,x) := U(t,R_i(t)-x;-r_i)$. It is easy to check that u_i satisfies

$$\begin{cases} u_{it} = u_{ixx} + f(t,u_i), & x < R_i(t),\ t \in [t_1,t_2], \\ u_i\left(t,R_i(t)\right) = 0,\ u_i(t,-\infty) = P(t), & t \in [t_1,t_2]. \end{cases} \tag{3.7}$$

Moreover, by the definition of r_i,

$$r_1(t) = -u_{1x}\left(t,R_1(t)\right) - \alpha_1(t) = A[-r_1](t), \quad r_2(t) = -u_{2x}\left(t,R_2(t)\right) - \alpha_2(t) = A[-r_2](t). \tag{3.8}$$

Set $\eta(t,x) := u_2(t,x) - u_1(t,x)$ for $(t,x) \in \Omega := \{(t,x) : x < R_2(t),\ t \in [t_1,t_2]\}$. By [27, Lemma 3.6, Step2] and [11, Theorem 2.5], we have

$$\eta(t,x) < 0 \text{ in } \Omega \setminus \left\{(t_2,x_2)\right\} \text{ and } \eta_x(t_2,x_2) > 0,$$

and so $u_{2x}(t_2,x_2) > u_{1x}(t_2,x_2)$, combining with $\alpha_1(t) \leq, \not\equiv \alpha_2(t)$ and (3.8), we have $r_2(t_2) < r_1(t_2)$, contradicts (3.6). Therefore, $r_1(t) \geq r_2(t)$ for all $t \in [0,T]$.

(iii) The uniqueness of $r(t;\alpha)$. One can derive this conclusion directly from Step 2. We now prove the continuity of $r(t;a+\theta)$ in a when $\alpha(t) = a+\theta(t)$. Assume $a_n > 0$ decreases and converges to $\widetilde{a}$ as $n \to \infty$. It then follows from the monotonicity of $r(t;\alpha)$ with respect to α that $r_n := r(t;a_n+\theta) < \widetilde{r} := r(t;\widetilde{a}+\theta)$. By this, the definition of r and Lemma 3.5 we have

$$0 < r\left(t;\widetilde{a}+\theta\right) - r\left(t;a_n+\theta\right) = U_z\left(t,0;-\widetilde{r}\right) - \widetilde{a} - U_z\left(t,0;-r_n\right) + a_n < a_n - \widetilde{a} \to 0\ (n\to\infty).$$

□

(II) Periodic leftward travelling semi-waves $Q_L\left(t,x+L(t;\beta)\right)$.

Lemma 3.7 *Assume $\beta(t) \in \mathcal{K}^+$. Then there exists a unique $l(t;\beta) \in \mathcal{K}$ with $\overline{l(t;\beta)} < \overline{c}$ such that $u(t,x) := Q_L(t,x+L(t;\beta))$ with $L(t;\beta) := \int_0^t l(s;\beta)ds$ solves $(1.1)_1$ for $t \in \mathbb{R}$ and $-l(t;\beta) =$*

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$-u_x(t, -L(t;\beta)) + \beta(t)$. *Moreover,* $-l(t;\beta)$ *is strictly increasing in* β *in the sense that* $-l(t;\beta_1) < -l(t;\beta_2)$ *if* $\beta_1 \leq, \not\equiv \beta_2$. *In addition,* $l(t;\beta)$ *is continuous with respect to* β.

Proof Consider the problem (3.2) and define an operator $B[k](t) := U_z\big(t, 0; k(t)\big) - \beta(t)$, where $U\big(t, z; k(t)\big)$ is the unique solution of the problem (3.2). Note that $B[-l]$ maps the set $S := \{l \in \mathcal{K} : -\beta(t) \leq l(t) \leq \overline{c} + U_z\big(t, 0; \beta(t)\big)$ for all $t \in [0, T]\}$ continuously into a precompact set in S. The rest proof is similar with that in Lemma 3.6. □

Proposition 3.8 *Assume* $0 < \beta(t) < \alpha(t)$. *Then*

- (i) *Let* $Q_0(t, z)$ *be the unique positive solution of the problem* (3.2) *with* $k = 0$. *Denote* $\alpha_0(t) := Q_{0z}(t, 0)$, *then* $\alpha_0(t)$ *is the unique zero point of* $r(t;\alpha)$ *and* $-l(t;\beta)$.
- (ii) *when* $\beta(t) \geq, \not\equiv \alpha_0(t)$, *we have* $r(t;\alpha) < 0 < -l(t;\beta)$. *Assume* $\beta < \alpha$ *and* $\alpha = \alpha_0$, *then* $-l(t;\beta) < 0 = r(t;\alpha)$.
- (iii) *For any fixed* $\theta \in \mathcal{K}$ *with* $\overline{\theta} = 0$, *when* $0 < \beta(t) := b + \theta < \alpha_0(t)$ $(b \in (0, \infty))$, *there exists a unique* T*-periodic function* $\alpha^* = \alpha^*(t;\beta) > \alpha_0(t)$ *such that*

$$\begin{cases} -l(t;\beta) < 0 < r(t;\alpha), & \alpha \in (\beta, \alpha_0), \\ \overline{-l(t;\beta)} < \overline{r(t;\alpha)} < 0, & \alpha \in (\alpha_0, \alpha^*), \\ \overline{-l(t;\beta)} = \overline{r(t;\alpha)} < 0, & \alpha = \alpha^*, \\ \overline{r(t;\alpha)} < \overline{-l(t;\beta)} < 0, & \alpha > \alpha^*. \end{cases} \tag{3.9}$$

Proof Based on Lemmas 3.6 and 3.7, we give the proof of this proposition.

- (i) By Lemma 3.5, the problem (3.2) has a unique solution $Q_0(t, z)$ with $k = 0$, and $Q_0(t, z) > 0$ for all $t \in [0, T]$ and $z > 0$. Denote $\alpha_0(t) := Q_{0z}(t, 0)$. It follows from the definition of $r(t;\alpha)$, we have $r(t;\alpha_0) = 0$. By Lemma 3.6, one can then get that α_0 is the unique zero point of $r(t;\alpha)$. Using similar arguments, α_0 is also the unique zero point of $l(t;\beta)$.
- (ii) By the monotonicity of $r(t;\alpha)$ and $-l(t;\beta)$ in α and β, respectively, we can derive from $\beta(t) \geq, \not\equiv \alpha_0(t)$ and $\beta < \alpha$ that $r(t;\alpha) < r(t;\alpha_0) = 0$ and $-l(t;\beta) > -l(t;\alpha_0) = 0$. Moreover, by (i), $\beta < \alpha$ and $\alpha = \alpha_0$, we have $-l(t;\beta) < -l(t;\alpha_0) = 0 = r(t;\alpha)$.
- (iii) When β and α have the same shape, that is, $\beta(t) = b + \theta(t)$, $\alpha(t) = a + \theta(t)$(cf. Section 1), from $\beta(t) := b + \theta(t) < \alpha_0(t)$ we can choose a_1 satisfying $a_1 > b$ and $\alpha_1(t) := a_1 + \theta(t) < \alpha_0(t)$ for all $t \in [0, T]$, then we have $-l(t;\beta) < -l(t;\alpha_0) = 0 = r(t;\alpha_0) < r(t;\alpha_1)$, so

  $$\overline{r(t;\alpha_1)} > \overline{-l(t;\beta)}. \tag{3.10}$$

  On the other hand,

  $$U_z(t, 0; \overline{c}) = -\overline{c} + \alpha(t) \text{ when } \alpha(t) = U_z(t, 0; \overline{c}) + \overline{c}, \tag{3.11}$$

  where $U(t, z; \overline{c})$ is the unique positive solution of the problem (3.2) with $k = \overline{c}$. Combining (3.11) with the definition of $r(t;\alpha)$ and Lemma 3.7, we have

  $$r\big(t; U_z(t, 0; \overline{c}) + \overline{c}\big) = -\overline{c} < -\overline{l(t;\beta)}. \tag{3.12}$$

  We now choose $a_2 > b$ large such that $\alpha_2(t) := a_2 + \theta(t) > U_z(t, 0; \overline{c}) + \overline{c}$. It follows from the monotonicity of $r(t;\alpha)$ in α; we have

  $$r(t, \alpha_2) < r\big(t; U_z(t, 0; \overline{c}) + \overline{c}\big) = -\overline{c} < -\overline{l(t;\beta)}.$$

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By this, one can obtain

$$\overline{r(t;\alpha_2)} < -\bar{c} < \overline{-l(t;\beta)}. \tag{3.13}$$

By (3.10), (3.13), the monotonicity and continuity of $r(t;\alpha)$ in α (see Lemma 3.6), there exists a unique $a^*(b,\theta) \in (a_1, a_2)$ such that when $\alpha^*(t;\beta) := a^*(b,\theta) + \theta(t)$, we have $\overline{r(t;\alpha^*)} = \overline{-l(t;\beta)}$.

Moreover, since $r(t;\alpha)$ is strictly decreasing in α, we have

$$\overline{-l(t;\beta)} < \overline{r(t;\alpha)} \text{ when } \alpha < \alpha^*; \quad \overline{r(t;\alpha)} < \overline{-l(t;\beta)} \text{ when } \alpha > \alpha^*$$

and

$$r(t;\alpha) < 0 = r(t;\alpha_0) \text{ when } \alpha > \alpha_0; \quad r(t;\alpha) > 0 = r(t;\alpha_0) \text{ when } \alpha < \alpha_0. \qquad \square$$

(III) Travelling waves with compact supports. In this subsection we present some periodic travelling waves with compact supports, which will be used as lower solutions of (1.1) to prove spreading happens.

Proposition 3.9 *Assume* $0 < \beta < \alpha < \alpha^*$.

(i) For any small $\varepsilon > 0$*, when* ℓ *is sufficiently large, the problem* (3.1) *has a unique solution* $U_0(t,z;-r^\varepsilon;\ell)$*, where* $r^\varepsilon := r(t;\alpha) - \varepsilon$*. Then the function* $V_R^\varepsilon(t,x) := U_0(t, R^\varepsilon(t) - x; -r^\varepsilon, \ell)$ *(with* $R^\varepsilon(t) := \int_0^t r^\varepsilon(s)ds$*) solves* $(1.1)_1$ *in* $\mathbb{R} \times (R^\varepsilon(t) - \ell, R^\varepsilon(t))$*, and* $r^\varepsilon < -(V_R^\varepsilon)_x\big(t, R^\varepsilon(t)\big) - \alpha(t)$*.*

(ii) For any small $\varepsilon > 0$*, when* $\ell > 0$ *is sufficiently large, the function* $u = V_L^\varepsilon(t,x) := U_0(t, x + L^\varepsilon(t); -l^\varepsilon, \ell)$ *(where* $l^\varepsilon := l(t;\beta) - \varepsilon$ *and* $L^\varepsilon(t) := \int_0^t l^\varepsilon(s;\beta)ds$*) solves* $(1.1)_1$ *in* $\mathbb{R} \times (-L^\varepsilon(t), \ell - L^\varepsilon(t))$*, and* $-l^\varepsilon(t) > -(V_L^\varepsilon)_x(t, -L^\varepsilon(t)) + \beta(t)$*.*

Proof We only give the proof of (i) since (ii) can be proved similarly. By Lemmas 3.6 and 3.7 and the assumption $0 < \alpha < \alpha^*$ we have $-\bar{c} < -\bar{l} < \bar{r} < \bar{c}$, so $|\bar{r}| < \bar{c}$. Consequently, there exists $\varepsilon_0 > 0$ small such that for any $\varepsilon \in (0, \varepsilon_0)$, $r^\varepsilon := r - \varepsilon$ satisfies $|\overline{r^\varepsilon}| < \bar{c}$. Then it follows from Lemma 3.4 that the problem (3.1) has a unique positive solution $U_0(t,z;-r^\varepsilon,\ell)$ for each $\ell > 0$. A direct calculation shows that $V_R^\varepsilon(t,x) := U_0(t, R^\varepsilon(t) - x; -r^\varepsilon, \ell)$ solves $(1.1)_1$ in $\mathbb{R} \times (R^\varepsilon(t) - \ell, R^\varepsilon(t))$, where $R^\varepsilon(t) := \int_0^t r^\varepsilon(s)\,ds$. Moreover, by the definition and monotonicity of A, we have

$$r^\varepsilon := r - \varepsilon < r = A[-r] < A[-r^\varepsilon] = U_z\big(t, 0; -r^\varepsilon\big) - \alpha(t).$$

This, together with Lemma 3.5 (iii), implies that $r^\varepsilon < -(V_R^\varepsilon)_x\big(t, R^\varepsilon(t)\big) - \alpha(t)$ when ℓ is sufficiently large. $\square$

Remark 3.10 *By the definition of lower solutions (see Remark 3.3)* V_R^ε *will be a lower solution of the problem* (1.1) *if* $g(t)$ *satisfies* $g(t) \le R^\varepsilon(t) - \ell$ *for all* $t \in \mathbb{R}$*. Similarly, when* $h(t) \ge \ell - L^\varepsilon(t)$ *for all* $t \in \mathbb{R}$*,* V_L^ε *is also a lower solution of the problem* (1.1)*.*

4 Vanishing and spreading phenomena

In this section, we give some necessary and sufficient conditions for vanishing and spreading phenomena.

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4.1 Vanishing phenomenon

We firstly give the following equivalent conditions for vanishing.

Lemma 4.1 *Let* (u, g, h) *be the solution of* (1.1) *in some maximal interval* $[0, \mathcal{T})$ *with* $\mathcal{T} \in (0, +\infty]$. *Then the following arguments are equivalent:*

(i) $\lim_{t\to\mathcal{T}} \max_{g(t)<x<h(t)} u(t,x) = 0$;
(ii) $\mathcal{T} < +\infty$;
(iii) $\lim_{t\to\mathcal{T}} h(t) = \lim_{t\to\mathcal{T}} g(t)$.

Proof Firstly, the conclusion of (i) implies (ii) and (iii). We will construct an upper solution to block the spreading of two free boundaries; then one can show that vanishing happens.

Under the assumption (1.2), we can construct a Fisher–KPP type of nonlinearity $f_1(u) \geq f(t,u)$ for $t \in [0,T]$, $u \geq 0$. $f_1'(0) = a_1 := \max_{t\in[0,T]} a(t)$ and $f_1(0) = f(b) = 0$ for some $b > 1$. Take $\gamma > 0$ small and $\gamma < \min_{t\in[0,T]} \{\beta(t), \alpha(t)\}$. Then the following problem

$$\begin{cases} v'' + f_1(v) = 0, \quad -\ell < x < 0, \\ v(0) = v(-\ell) = 0,\ v(x) > 0,\ x \in (-\ell, 0),\ t > 0, \\ -v'(0) = v'(-\ell) = \gamma \end{cases} \tag{4.1}$$

has a unique solution $(\ell, v) = (\ell_\gamma, v_\gamma)$ for each small $\gamma < \sqrt{2\int_0^b f_1(s)ds}$ (see section 3.4 in [5]). When (i) holds, there exists a large time T_0 and X such that

$$u(t,x) \leq m := v_\gamma(-X),\ x \in \big[g(t), h(t)\big],\ t > T_0.$$

Choose a large $b_1 > 0$, then the function $v_\gamma(x - b_1)$ is an upper solution of the problem (1.1) on $[-X + b_1, b_1]$, and by Lemma 3.2, it blocks the extension of $h(t)$. Therefore, $h(t) < b_1$ for all $t > T_0$. Similarly, there is a $b_2 \in \mathbb{R}$ such that $g(t) > b_2$ for all $t > T_0$.

Set $L := 2(1 + b_1 + |b_2|)$ and

$$\eta_0(x) := \frac{2\varepsilon}{L^2}(L^2 - x^2),\ x \in [-L, L],$$

where $K_1 := \max_{0\leq u\leq b} |f_1'(u)|$, and $\varepsilon > 0$ is small such that

$$8\Big(\alpha_1 + \sqrt{\alpha_1^2 + 4K_1}\Big)\varepsilon \leq \alpha_1, \quad 32\varepsilon \leq \alpha_1,$$

where $\alpha_1 := \min_{t\in[0,T]} \alpha(t)$.

Consider the following problem:

$$\begin{cases} \eta_t = \eta_{xx} + \bar{f}(\eta), & \bar{g}(t) < x < \bar{h}(t),\ t > 0, \\ \eta\big(t, \bar{g}(t)\big) = \eta\big(t, \bar{h}(t)\big) = 0, & t > 0, \\ \bar{g}'(t) = -\eta_x\big(t, \bar{g}(t)\big) + \beta_1, & t > 0, \\ \bar{h}'(t) = -\eta_x\big(t, \bar{h}(t)\big) - \alpha_1, & t > 0, \\ -\bar{g}(0) = \bar{h}(0) = L,\ \ \eta(0,x) = \eta_0(x), & -L \leq x \leq L, \end{cases} \tag{4.2}$$

where $\beta_1 := \min_{t\in[0,T]} \beta(t)$, and

$$\bar{f}(\eta) := 2K_1\eta\Big(1 - \frac{\eta}{2\varepsilon}\Big) \quad (\geq f_1(\eta) \text{ for } 0 \leq \eta \leq \varepsilon\ ).$$

Clearly, $\eta(t,x) \leq 2\varepsilon$ for $t \geq 0$. By the definitions of $\bar{f}$ and η_0, we see that

$$U^\varepsilon(t,x) := 2\varepsilon\Big[2M\big(\bar{h}(t)-x\big) - M^2\big(\bar{h}(t)-x\big)^2\Big]$$

is an upper solution of (4.2) over $\overline{Q} := \{(t,x) : t > 0, \max\{\bar{g}(t), \bar{h}(t) - M^{-1}\} \leq x \leq \bar{h}(t)\}$ with $M := \max\{\alpha_1 + \sqrt{\alpha_1^2 + 4K_1}, 4\}$. Hence,

$$-\eta_x\big(t, \bar{h}(t)\big) \leq -U_x^\varepsilon\big(t, \bar{h}(t)\big) \leq 4M\varepsilon \leq \frac{\alpha_1}{2}.$$

Therefore, $\bar{h}'(t) \leq -\frac{\alpha_1}{2}$. Similarly, $\bar{g}'(t) \geq \frac{\beta_1}{2}$. Thus $\bar{h}(t) - \bar{g}(t) \to 0$ as $t \to T_1 \leq \frac{4L}{\alpha_1+\beta_1}$. On the other hand, by $\lim_{t\to\mathcal{T}} \max_{g(t)\leq x\leq h(t)} u(t,x) = 0$, there exists a large time $T^* > 0$ such that $u(t,x) \leq \varepsilon$ for all $t > T^*$, then $\eta(t,x)$ is an upper solution of (1.1) for $t > T^*$. Hence, $h(t+T^*) - g(t+T^*) \leq \bar{h}(t) - \bar{g}(t) \to 0$ as $t \to T_1$, which implies $\mathcal{T} < +\infty$ and (iii) holds.

Secondly, (ii) implies (iii). Assume on the contrary that $\inf_{0<t<\mathcal{T}} \big[h(t) - g(t)\big] > 0$, then by standard L^p estimates, the Sobolev embedding theorem and the Hölder estimates for parabolic equations, we can extend the solution to some larger interval $(0, \overline{T})$ with $\overline{T} > \mathcal{T}$ as long as $\inf_{0<t<\mathcal{T}} \big[h(t) - g(t)\big] > 0$. This contradicts that $(0, \mathcal{T})$ is the maximal existence interval of the solution $u(t,x)$.

Thirdly, (iii) implies (i). By (1.2), one can find a constant $C > 0$ such that $u(t,x) \leq C$ for $t \in [0, \mathcal{T})$ and $x \in [g(t), h(t)]$. Construct a function

$$U(t,x) = C\Big[2M\big(h(t)-x\big) - M^2\big(h(t)-x\big)^2\Big]$$

over the region $Q := \{(t,x) : t > 0, \max\{h(t) - M^{-1}, g(t)\} < x < h(t)\}$. When $M > 0$ is large, $U(t,x)$ is an upper solution of (1.1) in $\overline{Q}$. By $\lim_{t\to\mathcal{T}} \big[h(t) - g(t)\big] = 0$, there is $t_0 \in (0, \mathcal{T})$ such that $g(t) > h(t) - M^{-1}$. Hence, $u(t,x) \leq U(t,x)$ for $g(t) \leq x \leq h(t)$ and $t_0 < t < \mathcal{T}$. Combining with $U(t,x) \to 0$ $(x \to h(t))$, we have

$$\lim_{t\to\mathcal{T}} \max_{g(t)\leq x\leq h(t)} u(t,x) \leq \lim_{t\to\mathcal{T}} \max_{g(t)\leq x\leq h(t)} U(t,x) = 0.$$

□

We now give a sufficient condition for vanishing as follows.

Lemma 4.2 *Let $h_0 > 0$ and $u_0 \in \mathscr{X}(h_0)$, then u vanishes if $\|u_0\|_{L^\infty([-h_0,h_0])}$ is sufficiently small.*

Proof Since f is Fisher–KPP type, there exists $K > 0$ such that $f(t,u) \leqslant Ku$ $(u \geqslant 0)$. Choose $C > 0$ such that

$$2\Big(\alpha_2 + \sqrt{\alpha_2^2 + 2K}\Big)C \leqslant \alpha_1, \quad 3C \leqslant 1, \tag{4.3}$$

where α_1 is defined in Lemma 4.1, $\alpha_2 := \max_{t\in[0,T]} \alpha(t)$.

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For this C, we take $\varepsilon > 0$ sufficiently small such that

$$\varepsilon < \min\left\{\frac{2h_0\alpha_1}{\pi C},\ Ce^{-\frac{4h_0K}{\alpha_1+\beta_1}}\right\}, \quad 16\varepsilon^2\left(1+\frac{\pi}{2h_0}\right) \leqslant 3\alpha_1. \tag{4.4}$$

Now we consider the problem

$$\begin{cases} \eta_t = \eta_{xx} + K\eta, & -h_0 < x < h_0,\ t > 0, \\ \eta(t, \pm h_0) = 0, & t > 0, \\ \eta(0, x) = \tilde{\phi}(x), & -h_0 \leqslant x \leqslant h_0, \end{cases} \tag{4.5}$$

where $\tilde{\phi}(x) := \varepsilon^2 \cos\frac{\pi x}{2h_0}$. Clearly $\|\tilde{\phi}\|_{C^1([-h_0,h_0])} \leqslant 3\alpha_1/16$ by the choice of ε. The solution of (4.5) is

$$\eta(t,x) = \varepsilon^2 e^{\left(K-\frac{\pi^2}{4h_0^2}\right)t}\cos\frac{\pi x}{2h_0}.$$

Set

$$T_1 := \frac{1}{K}\log\frac{C}{\varepsilon} > \frac{4h_0}{\alpha_1+\beta_1}.$$

Here we have used the choice of ε in the last inequality. Denote the solution of (1.1) with initial data $u_0(x) = \tilde{\phi}(x)$ by $u(t,x)$. Since

$$-\eta_x(t,h_0) \leqslant \varepsilon^2 e^{KT_1}\frac{\pi}{2h_0} = \frac{\pi C\varepsilon}{2h_0} < \alpha_1 \le \alpha(t) \quad \text{for } 0 \leqslant t \leqslant T_1,$$

by the choice of ε, η is an upper solution of (1.1) and

$$u(t,x) \leqslant \eta(t,x) \leqslant \eta(t,0) \leqslant \varepsilon^2 e^{Kt} \leqslant C \quad \text{for } 0 \leqslant t \leqslant T_1.$$

Now we construct function

$$U(t,x) := C\left[2M\big(h(t)-x\big) - M^2\big(h(t)-x\big)^2\right]$$

over $Q := \{(t,x) : 0 \leqslant t < T_1, \max\{h(t)-M^{-1}, g(t)\} \leqslant x \leqslant h(t)\}$, where

$$M := \max\left\{\frac{\alpha_2+\sqrt{\alpha_2^2+2K}}{2},\ \frac{4\|\tilde{\phi}(x)\|_{C^1}}{3C}\right\}.$$

A direct calculation (cf. [6, Lemma 2.5]) shows that, $U(t,x)$ is an upper solution of (1.1) over Q. So $u(t,x) \leqslant U(t,x)$ in Q and

$$-u_x\big(t,h(t)\big) \leqslant -U_x\big(t,h(t)\big) = 2MC \leqslant \frac{\alpha_1}{2}$$

by the choice of C. Therefore,

$$h'(t) = -u_x\big(t,h(t)\big) - \alpha(t) \leqslant -\frac{\alpha_1}{2},$$

$g'(t) \geqslant \frac{\beta_1}{2}$ (where $\beta_1 := \min_{t\in[0,T]}\beta(t)$) is proved similarly, so

$$h(t) - g(t) \leqslant 2h_0 - \frac{\alpha_1+\beta_1}{2}t \to 0 \quad \text{as } t \to \frac{4h_0}{\alpha_1+\beta_1} < T_1.$$

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Therefore, vanishing happens for (u,g,h) by Lemma 4.1. Finally any solution of (1.1) with sufficiently small initial data (such as, less than ϕ) also vanishes in finite time. □

4.2 Spreading phenomenon

To give some necessary and sufficient conditions for spreading, we first prove the following lemma for $0<\alpha(t)<\alpha^*(t)$ for all $t\in[0,T]$.

Lemma 4.3 *Assume $0<\beta<\alpha_0$ and $0<\alpha<\alpha^*$. Let (u,g,h) be a solution of (1.1). Suppose that there exist x_i, $\varepsilon>0$, $\ell_i>0$ and an integer $m\geq 0$ such that*

$$u(mT,x)\geq V_R^\varepsilon(0,x_1-x)\text{ and }u(mT,x)\geq V_L^\varepsilon(0,x-x_2)\text{ for }x\in\big[x_2,x_2+\ell_2\big]\cup\big[x_1-\ell_1,x_1\big]. \tag{4.6}$$

Here, $\varepsilon>0$ is sufficiently small and x_1 (with $x_1>x_2$) is large such that $x_1\gg x_2+\ell_2$. V_L^ε and V_R^ε are the compactly supported travelling waves given in Proposition 3.9. Then

$$\lim_{t\to\infty}u\big(t,x+K(t)\big)=P(t)\quad\text{locally uniformly in }\mathbb{R}, \tag{4.7}$$

where $K(t):=\int_0^t k(s)ds$ for any $k(t)\in\mathcal{K}$ satisfying $\overline{k}\in(-\overline{l},\overline{r})$, and

$$g_\infty=-\infty,\qquad \lim_{t\to\infty}\big[h(t)-g(t)\big]=+\infty. \tag{4.8}$$

Proof We first compare u with V_R^ε and V_L^ε. By Proposition 3.9, $R^\varepsilon(t):=\int_0^t r^\varepsilon(s)ds$ (with $r^\varepsilon:=r-\varepsilon$) satisfies

$$(R^\varepsilon)'(t)<-(V_R^\varepsilon)_x\big(t,R^\varepsilon(t)\big)-\alpha(t), \tag{4.9}$$

and $L^\varepsilon(t):=\int_0^t l^\varepsilon(s)ds$ (with $l^\varepsilon:=l-\varepsilon$) satisfies

$$(-L^\varepsilon)'(t)>-(V_L^\varepsilon)_x\big(t,-L^\varepsilon(t)\big)+\beta(t). \tag{4.10}$$

On the other hand, $0<\alpha<\alpha^*$ implies that $\overline{r}>-\overline{l}$ (cf. Proposition 3.8), and so $\overline{r^\varepsilon}=\overline{r}-\varepsilon>-\overline{l}+\varepsilon=-\overline{l^\varepsilon}$ for each small $\varepsilon>0$. Combining this with the periodicity of $r^\varepsilon(t)$ and $l^\varepsilon(t)$, we can choose x_1 large such that

$$-L^\varepsilon(t)+x_2<R^\varepsilon(t)+x_1-\ell_1\quad\text{for all }t\geq 0 \tag{4.11}$$

and

$$-L^\varepsilon(t)+x_2+\ell_2<R^\varepsilon(t)+x_1\quad\text{for all }t\geq 0. \tag{4.12}$$

We use (4.11) and (4.12) to ensure that the left point $R^\varepsilon(t)+x_1-\ell_1$ of $V_R^\varepsilon(t,R^\varepsilon(t)+x_1-x)$ is always on the right side of the left point $-L^\varepsilon(t)+x_2$ of $V_L^\varepsilon(t,x+L^\varepsilon(t)-x_2)$ for all $t\geq 0$. Hence, (4.6) and (4.9)–(4.12) imply that $V_R^\varepsilon(t,R^\varepsilon(t)+x_1-x)$ and $V_L^\varepsilon(t,x+L^\varepsilon(t)-x_2)$ are lower solutions of the problem (1.1). Therefore, by the comparison principle Lemma 3.2, we have

$$u(t+mT,x)>V_R^\varepsilon\big(t,R^\varepsilon(t)+x_1-x\big)\quad\text{for}\quad x\in\big[R^\varepsilon(t)+x_1-\ell_1,R^\varepsilon(t)+x_1\big],\ t>0 \tag{4.13}$$

and

$$u(t+mT,x)>V_L^\varepsilon\big(t,x+L^\varepsilon(t)-x_2\big)\quad\text{for}\quad x\in\big[-L^\varepsilon(t)+x_2,-L^\varepsilon(t)+x_2+\ell_2\big],\ t>0, \tag{4.14}$$

with

$$h(t+mT) > R^{\varepsilon}(t) + x_1, \quad g(t+mT) < -L^{\varepsilon}(t) + x_2 \quad \text{for all } t \geq 0. \tag{4.15}$$

By Proposition 3.8, our assumption $\beta < \alpha_0$ implies that $-l^{\varepsilon}(t) < 0$ for small $\varepsilon > 0$, and $0 < \alpha < \alpha^*$ implies that $\overline{r^{\varepsilon}} > -\overline{l^{\varepsilon}}$ for small $\varepsilon > 0$. Combining these and (4.15) one can derive that $g_\infty = -\infty$ and $\lim_{t\to\infty} \big[h(t) - g(t)\big] = +\infty$.

Moreover, for $k^{\varepsilon}(t) \in \mathcal{K}$ satisfying $\overline{k^{\varepsilon}} \in (-\overline{l^{\varepsilon}}, \overline{r^{\varepsilon}})$, we have, as $t \to \infty$,

$$H(t) := h(t+mT) - K^{\varepsilon}(t) > R^{\varepsilon}(t) + x_1 - K^{\varepsilon}(t) \to +\infty,$$

$$G(t) := g(t+mT) - K^{\varepsilon}(t) < -L^{\varepsilon}(t) + x_2 - K^{\varepsilon}(t) \to -\infty,$$

where $K^{\varepsilon}(t) := \int_0^t k^{\varepsilon}(s)ds$. Define

$$w(t,x) := u\big(t+mT, x+K^{\varepsilon}(t)\big) \quad \text{for} \quad G(t) \leq x \leq H(t), \ t > 0.$$

Then w satisfies

$$w_t = w_{xx} + k^{\varepsilon}(t)w_x + f(t,w), \ t > 0, \ G(t) < x < H(t), \ t > 0.$$

By Lemmas 3.6 and 3.7 we have $|\overline{k^{\varepsilon}}| < \bar{c}$. We next prove that (4.7) holds.

First, using a similar argument to that used in proving [11, Lemma 3.4] one can show that, for positive integer n,

$$\liminf_{n\to\infty} w(t+nT, x) \geq P(t) \ \text{ locally uniformly for } (t,x) \in [0,T] \times \mathbb{R}. \tag{4.16}$$

On the other hand, consider the problem

$$\begin{cases} v_t = v_{xx} + k^{\varepsilon}(t)v_x + f(t,v), & x \in \mathbb{R}, \ t > 0, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{4.17}$$

where $v_0(x) = w(0,x)$ for $x \in [G(0), H(0)]$, and $v_0(x) = 0$ for $x > H(0)$ and $x < G(0)$. By the comparison principle we have

$$w(t,x) \leq v(t,x) \text{ for all } (t,x) \in [0,+\infty) \times [G(t), H(t)]. \tag{4.18}$$

Since $|\overline{k^{\varepsilon}}| < \bar{c}$, it follows from [24, Theorem 1.6] that

$$\lim_{t\to\infty} [v(t,x) - P(t)] = 0 \text{ locally uniformly for } x \in \mathbb{R}.$$

Combining this and (4.18), one can derive that

$$\limsup_{n\to\infty} w(t+nT, x) \leq P(t) \text{ locally uniformly for } t \in [0,T], \ x \in \mathbb{R}.$$

By this and (4.16) we have

$$\lim_{n\to\infty} w(t+nT, x) = P(t) \text{ locally uniformly for } t \in [0,T], \ x \in \mathbb{R}.$$

Hence, using the periodicity of $P(t)$ one can obtain

$$\lim_{t\to\infty} w(t,x) = P(t) \text{ locally uniformly in } \mathbb{R}. \tag{4.19}$$

Therefore, by the arbitrary of ε, the definition of w and (4.19) one can derive that (4.7) holds. □

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Remark 4.4 *When $0 < \alpha < \alpha^*$, it follows from Proposition 3.8 that $-\bar{l} < \bar{r}$, and so (4.7) also holds for $K(t) := ct$ for any constant $c \in (-\bar{l}, \bar{r})$.*

Now, we are ready to give a necessary and sufficient condition for spreading.

Lemma 4.5 *Assume $0 < \beta < \alpha_0$ and $0 < \alpha < \alpha^*$. Let (u, g, h) be a solution of (1.1). Then spreading happens if and only if there exist x_i, $\varepsilon > 0$, ℓ_i and an integer $m \geq 0$ such that (4.6) holds, where $\varepsilon > 0$ is sufficiently small and x_1 (with $x_1 > x_2$) is large such that $x_1 \gg x_2 + \ell_2$.*

Proof The inequality (4.6) follows from the definition of spreading (cf. Theorem 2.1) immediately. We only need to prove that (4.6) is a sufficient condition for spreading.

(1) The case $0 < \alpha < \alpha_0$. It follows from Proposition 3.8 that $r > 0$. This and (4.15) (one can choose $\varepsilon > 0$ small such that $r^\varepsilon > 0$) imply that $h_\infty = +\infty$.

(2) The case $\alpha_0 < \alpha < \alpha^*$ and $0 < \beta < \alpha_0$. By Lemma 4.3, we only need to prove that $h_\infty = -\infty$. For convenience, we normalise the problem (1.1) by setting

$$v(t, x) := \frac{u(t, x)}{P(t)}.$$

Then the problem (1.1) is converted to

$$\begin{cases} v_t = v_{xx} + F(t, v), & g(t) < x < h(t),\ t > 0, \\ v(t, x) = 0,\ g'(t) = -P(t)v_x(t, x) + \beta(t), & x = g(t),\ t > 0, \\ v(t, x) = 0,\ h'(t) = -P(t)v_x(t, x) - \alpha(t), & x = h(t),\ t > 0, \\ v_0(x) := v(0, x) = u_0(x)/P(0), & -h_0 \leq x \leq h_0, \end{cases} \tag{4.20}$$

where $F(t, v) := \frac{1}{P(t)}[f(t, P(t)v) - f(t, P(t))v]$ satisfies

$$\begin{cases} F(t, v) \in C^{\nu/2, 1+\nu/2}\big([0, T] \times \mathbb{R}\big) \text{ for some } \nu \in (0, 1),\ T\text{-periodic in } t, \\ F(t, 0) = F(t, 1) \equiv 0,\ \ F(t, v)/v \text{ is decreasing in } v > 0, \\ \text{for any } t \in [0, T],\ \ F(t, v) > 0 \text{ for } 0 < v < 1,\ \ F(t, v) < 0 \text{ for } v > 1, \\ a_1(t) := F_v(t, 0) = a(t) - f\big(t, P(t)\big)/P(t),\ \ \varrho_1(t) := F_v(t, 1) = \varrho(t)/P(t). \end{cases}$$

Clearly, $\overline{a_1} = \bar{a}$, the condition (2.2) implies that $\varrho_1(t) < -\kappa$ for some $\kappa > 0$. So, for some small $\varepsilon > 0$, there holds

$$F_v(t, v) \leq -\kappa \quad \text{for } t \in [0, T],\ v \in [1 - \varepsilon, 1 + \varepsilon]. \tag{4.21}$$

Set $A := 2\max\{1, ||v_0||_{L^\infty([-h_0, h_0])}\}$. For some $\delta \in (0, \kappa)$, define

$$\bar{g}(t) := g(t),\ \bar{h}(t) := R(t) - Me^{-\delta t} + H,\ \bar{v}(t, x) := \big(1 + Ae^{-\delta t}\big)\widetilde{Q}_R\big(t, \bar{h}(t) - x\big),$$

for $\bar{g}(t) < x < \bar{h}(t)$, $t > 0$, where $\widetilde{Q}_R := Q_R/P(t)$, Q_R is the periodic rightward travelling semi-wave (the solution of (1.4)) with speed $R'(t)$. Then a direct calculation as in the proof of [16, Lemma 3.2] shows that $(\bar{v}, \bar{g}, \bar{h})$ is an upper solution of (4.20), provided that H, M are large. Therefore, we have

$$v(t, x) < \bar{v}(t, x) \ \text{ for } x \in [g(t), h(t)],\ t > 0$$

and

$$h(t) < \overline{h}(t) \quad \text{for large } t > 0. \tag{4.22}$$

On the other hand, by Lemma 4.3 we see that vanishing cannot happen when (4.6) holds. Hence (4.22) and $r(t) < 0$ ($\alpha > \alpha_0$ implies $r < 0$, one can see the details in Proposition 3.8) imply that $h_\infty = -\infty$.

(3) The case $\alpha = \alpha_0$. By Lemma 4.3, it suffices to prove that $-\infty < h_\infty < +\infty$ and (2.4) holds. Consider the problem (4.20) again, by (4.7) and Remark 4.4, for any constant $c \in (-\bar{l}, \bar{r})$, we have

$$\lim_{t\to\infty} v(t, x + ct) = 1 \quad \text{locally uniformly in } \mathbb{R}.$$

Combining this and [13, Lemma 6.5], there exists $\delta \in (0, \kappa)$, $M > 0$ and $T_0 > 0$ such that for $c \in (-\bar{l}, \bar{r})$,

$$v(t, x + ct) \geq 1 - Me^{-\delta t},\ x \in [-\ell, \ell] \subset \big(g(t) - ct, h(t) - ct\big),\ t \geq T_0 \tag{4.23}$$

for any fixed $\ell > 0$, and

$$v(t, x + ct) \leq 1 + Me^{-\delta t},\ x \in \big[g(t) - ct, h(t) - ct\big],\ t \geq T_0.$$

Define

$$\underline{g}(t) := ct, \quad \underline{h}(t) := \sigma e^{-\delta t} + cT_0,$$

$$\underline{v}(t, x) = \big(1 - Me^{-\delta t}\big)\widetilde{Q}_0\big(t, \underline{h}\,(t) - x\big),$$

where $\sigma > 0$ is some constant to be determined later, and $\widetilde{Q}_0 := \frac{Q_0}{P(t)}$, Q_0 is the solution of (1.4) with $r = 0$ (cf. Proposition 3.8). Using the similar arguments as in Ref. [16, Lemma 3.3], one can derive that

$$\underline{v}_t - \underline{v}_{xx} - F(t, \underline{v}) \leq 0 \quad \text{for } x \in \big(\underline{g}(t), \underline{h}(t)\big),\ t > T_0, \tag{4.24}$$

provided that T_0 and $\sigma > 0$ (independent of T_0) are sufficiently large.

Next we choose $\ell > \sigma$ in (4.23), then for sufficiently large $T_0 > 0$, we have

$$v(t, x + ct) \geq 1 - Me^{-\delta t} \quad \text{for } x \in \big[0, \sigma e^{-\delta t}\big] \subset [-\ell, \ell],\ t \geq T_0.$$

In particular, when $t = T_0$,

$$v(T_0, x) \geq 1 - Me^{-\delta T_0} \quad \text{for } x \in \big[\underline{g}(T_0), \underline{h}(T_0)\big].$$

Moreover, by the definition of $\underline{g}(t)$, (4.15) and (4.23) we have, for $t > T_0$,

$$g(t) < \underline{g}(t),\ \underline{v}\big(t, \underline{g}(t)\big) = \big(1 - Me^{-\delta t}\big)\widetilde{Q}_0\big(t, \underline{h}(t) - \underline{g}(t)\big) < 1 - Me^{-\delta t} \leq v\big(t, \underline{g}(t)\big). \tag{4.25}$$

It is obvious that $\underline{v}(t, \underline{h}(t)) = 0$, and a direct calculation shows that

$$\underline{h}'(t) \leq -P(t)\underline{v}_x\big(t, \underline{h}(t)\big) - \alpha_0(t) \ \text{ for } t > T_0 \tag{4.26}$$

provided $\sigma > M \max\limits_{t\in[0,T]} \alpha_0(t)/\delta$.

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Consequently, $(\underline{v}, \underline{g}, \underline{h})$ is a lower solution of (4.20). Therefore, $h(t) \geq \underline{h}(t)$ for $t > T_0$; this implies that

$$h_\infty > -\infty \quad \text{and} \quad \liminf_{t\to\infty} v(t,x) \geq \widetilde{Q}_0\big(t, cT_0 - x\big). \tag{4.27}$$

Next we show that $h_\infty < +\infty$. We need to construct an upper solution. Define, for some $\sigma_1 > 0$, $\delta_1 \in (0, \kappa)$ and $H_0 > 0$,

$$\overline{g}(t) := g(t),\ \overline{h}(t) := H_0 + \sigma_1 M\big(1 - e^{-\delta_1 t}\big),$$

and

$$\overline{v}(t,x) := \big(1 + Me^{-\delta_1 t}\big)\widetilde{Q}_0\big(t, \overline{h}(t) - x\big) \quad \text{for } \overline{g}(t) < x < \overline{h}(t),\ t > 0.$$

One can calculate directly as in [16, Lemma 3.2] to prove that, when σ_1 and H_0 are large, $(\overline{v}, \overline{g}, \overline{h})$ is an upper solution of (4.20) for large t. Therefore, $h(t) \leq \overline{h}(t)$ for large t, which implies that $h_\infty < +\infty$. To prove (2.4), we need to construct the following more precise upper solution and lower solution of the problem (4.20). For large $t > 0$ and some $\delta_2 \in (0, \kappa)$, define

$$\overline{g}_1(t) := g(t),\ \overline{h}_1(t) := h_\infty + \varepsilon + \sigma_2 M\varepsilon\big(1 - e^{-\delta_2 t}\big)$$

and

$$\overline{v}_1(t,x) := \big(1 + M\varepsilon e^{-\delta_2 t}\big)\widetilde{Q}_0\big(t, \overline{h}_1(t) - x\big) \text{ for } \overline{g}_1(t) < x < \overline{h}_1(t).$$

Using the similar arguments as above, $(\overline{v}_1, \overline{g}_1, \overline{h}_1)$ is an upper solution of the problem (4.20) when $\sigma_2 > 0$ is large. Hence

$$\limsup_{t\to\infty} v(t,x) \leq \widetilde{Q}_0\big(t, h_\infty + \varepsilon + \sigma_2 M\varepsilon - x\big). \tag{4.28}$$

On the other hand, $(\underline{g}_1, \underline{h}_1, \underline{v}_1)$ is a lower solution of the problem (1.1) when $\sigma_3 > 0$ is large, where

$$\underline{g}_1(t) := ct,\ \underline{h}_1(t) := h_\infty - \varepsilon - \sigma_3 M\varepsilon\big(1 - e^{-\delta t}\big),\ \underline{v}_1(t,x) := \big(1 - M\varepsilon e^{-\delta t}\big)\widetilde{Q}_0\big(t, \underline{h}_1(t) - x\big).$$

Then

$$\liminf_{t\to\infty} v(t,x) \geq \widetilde{Q}_0\big(t, h_\infty - \varepsilon - \sigma_3 M\varepsilon - x\big). \tag{4.29}$$

By the arbitrary of ε, (4.28) and (4.29), we have

$$\lim_{t\to\infty} v(t,x) = \widetilde{Q}_0(t, h_\infty - x) \text{ locally uniformly in } (-\infty, h_\infty].$$

By this equation, the definitions of $v(t,x)$ and $\widetilde{Q}_0$, one can derive that (2.4) holds. □

5 Proof of main theorems

In this section, we will prove the main theorems. For any given $h_0 > 0$ and $\phi \in \mathscr{X}(h_0)$, denote the solution (u, g, h) of (1.1) also by $(u(t,x;\sigma\phi), g(t;\sigma\phi), h(t;\sigma\phi))$ to emphasise the dependence on the initial data $u_0 = \sigma\phi$. Define

$$\sigma_* = \sigma_*(h_0, \phi) := \sup\big\{\sigma \geq 0:\ \text{vanishing happens for } (u,g,h)\big\}$$

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and

$$\sigma^* = \sigma^*(h_0, \phi) := \inf\{\sigma > 0 : \text{ spreading happens for } (u, g, h)\}.$$

It follows from the comparison principle that $\sigma_* \le \sigma^* \le +\infty$. Moreover, Lemma 4.2 implies that the solution $u(t,x;\sigma\phi)$ vanishes provided $\sigma > 0$ small. Therefore, $\sigma_* \in (0, +\infty]$. We next prove Theorem 2.1.

5.1 Proof of Theorem 2.1

If $\sigma_* = \infty$, then there is nothing left to prove. Thus we assume that $\sigma_* > 0$ is a finite number. We divide the proof into three steps:

Step 1. Vanishing happens for $\sigma \in (0, \sigma_*)$. This can be proved directly by the comparison principle and the definition of σ_*.

Step 2. Transition happens when $\sigma \in [\sigma_*, \sigma^*]$. We first prove that vanishing and spreading cannot happen for any $\sigma \in [\sigma_*, \sigma^*]$. Suppose on the contrary that vanishing happens for some $\sigma \in [\sigma_*, \sigma^*]$, then there exists t_0 such that $||u(t_0, x)||_{L^\infty([g(t_0),h(t_0)])}$ is sufficiently small. By the continuous dependence of the solution on the initial data value, there is $\varepsilon > 0$ sufficiently small such that the solution $(u_\varepsilon, g_\varepsilon, h_\varepsilon)$ of (1.1) with initial data $u_0 = (\sigma + \varepsilon)\,\phi$ satisfies

$$||u_\varepsilon(t_0, x)||_{L^\infty([g(t_0),h(t_0)])} \text{ is sufficiently small.}$$

Then it follows from Lemma 4.2 that vanishing happens for $u(t,x;(\sigma+\varepsilon)\phi)$; this contradicts the definition of σ_*.

On the other hand, spreading cannot happen for any $\sigma \in [\sigma_*, \sigma^*]$. Otherwise there are some x_i, $\varepsilon > 0$, ℓ_i and an integer $m \ge 0$ such that (4.6) holds, where $\varepsilon > 0$ is sufficiently small and x_1 is large. By the continuous dependence of the solution on the initial data again, we can find a small $\varepsilon > 0$ such that the solution $(u^\varepsilon, g^\varepsilon, h^\varepsilon)$ of (1.1) with initial data $u_0 = (\sigma - \varepsilon)\phi$ satisfies (4.6). Hence, by Lemma 4.5, one can show that spreading happens for $(u^\varepsilon, g^\varepsilon, h^\varepsilon)$; this contradicts the definition of σ^*.

Therefore, vanishing and spreading cannot happen when $\sigma \in [\sigma_*, \sigma^*]$. We next prove that $u(t,x;\sigma\phi)$ is in the transition case. It remains to prove $h_\infty = -\infty$, $g_\infty = -\infty$.

We first prove that $g_\infty = -\infty$. Suppose on the contrary that $-\infty < g_\infty < +\infty$, thus (2.1) implies that $-\infty < h_\infty < +\infty$. Then it follows from [20, Theorem 28.1] or the proof in [11, Lemma 3.3] that $u(t+nT, x)$ converges to a function $v(t,x)$ in $C^{1,2}([0,T] \times (g_\infty, h_\infty))$, where $v(t,x)$ is the unique positive solution of the problem

$$\begin{cases} v_t = v_{xx} + f(t, v), & g_\infty < x < h_\infty, \quad t > 0, \\ v\big(t, g_\infty\big) = v\big(t, h_\infty\big) = 0, & \qquad\qquad\qquad t > 0. \end{cases} \tag{5.1}$$

Moreover, by making a change of the variable x to reduce $[g(t), h(t)]$ to the fixed finite interval $[-h_0, h_0]$ and applying the L^p estimates (as well as Sobolev embeddings) on the reduced equation with Dirichlet boundary conditions, we have

$$\|u(t+nT, x) - v(t,x)\|_{C^{1,2}([g(t+nT),h(t+nT)])} \to 0 \quad \text{as } n \to \infty. \tag{5.2}$$

From this, we have $h'(t+nT) = -u_x(t+nT, h(t+nT)) - \alpha(t+nT) \to -v_x(t, h_\infty) - \alpha(t)$ and $g'(t+nT) = -u_x(t+nT, g(t+nT)) + \beta(t+nT) \to -v_x(t, g_\infty) + \beta(t)$ as $n \to \infty$. Since $h(t)$ and

$g(t)$ are Hölder continuous, combining this and $g_\infty, h_\infty \in (-\infty, +\infty)$, we have $h'(t) \to 0$ and $g'(t) \to 0$ (cf. the proof of [6, Theorem 1.3]). Therefore, we have $-v_x(t, h_\infty) = \alpha(t)$, $v_x(t, g_\infty) = \beta(t)$. However, this is impossible since $(5.1)_1$ is symmetric while $\beta \neq \alpha$. Hence this contradiction implies that $g_\infty = -\infty$.

We now prove $h_\infty = -\infty$. For any $c \in [-\bar{l}, \bar{r}]$, set $w(t,x) := u(t, x+ct)$ and $G(t) := g(t) - ct$, $H(t) := h(t) - ct$, then w satisfies

$$\begin{cases} w_t = w_{xx} + cw_x + f(t,w), \quad G(t) < x < H(t), & t > 0, \\ w\big(t, G(t)\big) = 0,\ G'(t) = -w_x\big(t, G(t)\big) + \beta(t) - c, & t > 0, \\ w\big(t, H(t)\big) = 0, H'(t) = -w_x\big(t, H(t)\big) - \alpha(t) - c, & t > 0, \\ -G(0) = H(0) = h_0,\ w(0,x) = u(0,x),\ -h_0 \le x \le h_0. \end{cases} \tag{5.3}$$

Then $\lim_{t\to\infty} H(t) \in [-\infty, +\infty]$ and $\lim_{t\to\infty} G(t) \in [-\infty, +\infty]$ exist (cf. [6, Lemma 2.9]). On the other hand, the proof of Lemma 4.5 implies that $h(t) < R(t) + H$ for some constant H. Combining with the definition of $R(t)$ and the periodicity of r, we can get $h(t) < R(t) + H = \bar{r}t + \Theta_1(t)$, where $\Theta_1(t) := \int_0^t (r(s) - \bar{r})ds + H$ (it is bounded). Similarly, there is a constant G such that $g(t) > -L(t) + G = -\bar{l}t + \Theta_2(t)$, where $\Theta_2(t) := \int_0^t (-l(s) + \bar{l})ds + G$ (which is also bounded). Therefore, there are $c_1, c_2 \in [-\bar{l}, \bar{r}]$ and constants $H_\infty, G_\infty \in (-\infty, +\infty)$ such that $\lim_{t\to\infty}[h(t) - c_1 t] = H_\infty$, $\lim_{t\to\infty}[g(t) - c_2 t] = G_\infty$. Clearly, $c_2 \le c_1$. We claim that $c_1 = c_2$, for otherwise, choose $c := (c_1 + c_2)/2$ in the problem (5.3), then $H(t) \to +\infty$ and $G(t) \to -\infty$ as $t \to \infty$. Using the arguments as in the proof of Lemma 4.3 we have spreading happens; this contradicts that spreading cannot happen for any $\sigma \in [\sigma_*, \sigma^*]$. This proves $c_1 = c_2$. So $g_\infty = -\infty$ implies $h_\infty = -\infty$. Moreover, from $c_1 \in (-\bar{l}, \bar{r})$ and Lemmas 3.6 and 3.7 we have $|c_1| < \bar{c}$. Consider the problem (5.3) again (with $c := c_1$), we have $w(t + nT, x) \to V(t,x)$ as $n \to \infty$, where V is the positive periodic solution of

$$\begin{cases} V_t = V_{xx} + cV_x + f(t,V), \quad G_\infty < x < H_\infty,\ t > 0, \\ V(t, G_\infty) = 0,\ V(t, H_\infty) = 0,\ t > 0. \end{cases} \tag{5.4}$$

Since vanishing cannot happen, Lemma 3.4 implies that $H_\infty - G_\infty > \ell^*(c)$.

Step 3. Spreading happens for $\sigma \in (\sigma^*, \infty)$. By the definition of σ^* and the comparison principle we derive that spreading happens when $\sigma > \sigma^*$. □

5.2 Proof of Theorem 2.3

We prove this theorem by constructing two upper solutions. Consider (4.20) again. Define, for some $M_1, X_1, \sigma_1 > 0$ and $\delta_1 \in (0, \kappa)$,

$$\bar{g}_1(t) := g(t),\ \bar{h}_1(t) := R(t) - \sigma_1 M_1 e^{-\delta_1 t} + X_1,\ \bar{v}_1(t,x) := \big(1 + M_1 e^{-\delta_1 t}\big)\widetilde{Q}_R\big(t, \bar{h}_1(t) - x\big),$$

where $\widetilde{Q}_R := Q_R/P(t)$ and for some $M_2, X_2 \ll X_1, \sigma_2 > 0$, $\delta_2 \in (0, \kappa)$,

$$\bar{h}_2(t) := h(t),\ \bar{g}_2(t) := -L(t) + \sigma_2 M_2 e^{-\delta_2 t} + X_2,\ \bar{v}_2(t,x) := \big(1 + M_2 e^{-\delta_2 t}\big)\widetilde{Q}_L\big(t, x - \bar{g}_2(t)\big),$$

where $\widetilde{Q}_L := Q_L/P(t)$. One can calculate directly as in [16, Lemma 3.2] to show that, when σ_1, σ_2 are large, $(\bar{v}_1, \bar{g}_1, \bar{h}_1)$ and $(\bar{v}_2, \bar{g}_2, \bar{h}_2)$ are upper solutions of (4.20) for large $t > 0$. Therefore,

$$h(t) \le \bar{h}_1(t) \quad \text{and} \quad g(t) \ge \bar{g}_2(t) \tag{5.5}$$

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for large t. In case that $\beta < \alpha_0$ and $\alpha > \alpha^*$, or $\beta \geq \alpha_0$, Proposition 3.8 implies that $-\bar{l} > \bar{r}$. Hence, (5.5) implies that $h(t) - g(t) \to 0$ as $t \to T^*$ for some $T^* < +\infty$, that is, vanishing happens by Lemma 4.1.

Now we consider the case $\alpha = \alpha^*$. In this case, $\bar{r} = -\bar{l}$. It follows from (5.5) that $0 < h_\infty - g_\infty < +\infty$ or $h(t) - g(t) \to 0$ as $t \to T_0$ for some $T_0 < +\infty$. By Lemma 4.1, the later one implies vanishing. We next prove that the former case is impossible. Otherwise, define

$$H_1(t) := R(t) - h(t), \quad G_1(t) := R(t) - g(t), \ t > 0$$

and

$$w_1(t,x) := u\big(t, R(t) - x\big) \quad \text{for } H_1(t) < x < G_1(t), \ t > 0.$$

Then $w_1(t,x)$ satisfies

$$\begin{cases} v_t = v_{xx} - r(t)v_x + f(t,v), & H_1(t) < x < G_1(t), \ \ t > 0, \\ v\big(t, G_1(t)\big) = 0, \ G_1'(t) = -v_x\big(t, G_1(t)\big) - \beta(t) + r(t), & t > 0, \\ v\big(t, H_1(t)\big) = 0, H_1'(t) = -v_x\big(t, H_1(t)\big) + \alpha(t) + r(t), & t > 0, \\ -H_1(0) = G_1(0) = h_0, \ v(0,x) = u(0,-x), \ -h_0 \leq x \leq h_0. \end{cases} \tag{5.6}$$

Compared with $Q_R(t,z)$ and $Q_L(t,z)$, one can prove that $H_1(t)$, as well as $G_1(t)$, does not move across any fixed point for infinitely many times (cf. [6, Lemma 2.9] and [5]), that is, $(H_1)_\infty := \lim_{t\to\infty} H_1(t) \in [-\infty, +\infty]$ and $(G_1)_\infty := \lim_{t\to\infty} G_1(t) \in [-\infty, +\infty]$ exist. Moreover, it deduced by (5.5) that $\bar{g}_2(t) \leq g(t) < h(t) \leq \bar{h}_1(t)$ for large t. Therefore, $(H_1)_\infty, (G_1)_\infty \in (-\infty, +\infty)$. Using the similar arguments as in Step 2 in the proof of Theorem 2.1, one can show that $w_1(t + nT, x)$ converges as $n \to \infty$ to V uniformly in $[(H_1)_\infty, (G_1)_\infty]$, where V satisfies:

$$\begin{cases} V_t = V_{xx} - r(t)V_x + f(t,V), \quad x \in \big((H_1)_\infty, (G_1)_\infty\big), \\ V\big(t, (G_1)_\infty\big) = V\big(t, (H_1)_\infty\big) = 0, \ t > 0. \\ r(t) = w_x\big(t, (H_1)_\infty\big) - \alpha(t), \ r(t) = w_x\big(t, (G_1)_\infty\big) + \beta(t). \end{cases} \tag{5.7}$$

This is a contradiction, since (5.7) has no positive solution by Lemma 3.6 and the uniqueness of the solution of (1.4). □

5.3 Proof of Theorem 2.4

We give the outline of the the proof. In this subsection, we also consider the normalisation version of (1.1), that is, the problem (4.20).

Step 1. Boundedness of $h(t) - R(t)$ and $g(t) + L(t)$. By Theorem 2.1, using the similar arguments as in [21, Proposition A] and [27], for any fixed $c \in (-\bar{l}, \bar{r})$, there exist $\delta \in (0, \kappa)$, a large integer $m > 0$ and $M > 0$ such that for $t > mT$, $ct \subset [g(t), h(t)]$ and

$$v(t, ct) \geq 1 - Me^{-\delta t}, \quad v(t,x) \leq 1 + Me^{-\delta t} \text{ for } x \in \big[g(t), h(t)\big]. \tag{5.8}$$

Define, for some $N_1, L_1(mT) > 0$,

$$\bar{g}(t) := g(t), \ \ \bar{h}(t) := \int_T^t r(s)ds + h(T) - \sigma N_1 e^{-\delta t} + L_1(mT), \ \ \bar{v}(t,x) := \big(1 + N_1 e^{-\delta t}\big)\widetilde{Q}_R\big(t, \bar{h}(t) - x\big),$$

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and for some $N_2, L_2(mT) > 0$,

$$\underline{g}(t) := ct, \quad \underline{h}(t) := \int_T^t r(s)ds + \sigma N_2 e^{-\delta t} + L_2(mT), \quad \underline{v}(t,x) := \big(1 - N_2 e^{-\delta t}\big)\widetilde{Q}_R\big(t, \underline{h}(t) - x\big),$$

where $\widetilde{Q}_R := Q_R/P(t)$. One can calculate directly as in [16] to prove that, $(\bar{v}, \bar{g}, \bar{h})$ is an upper solution and $(\underline{v}, \underline{g}, \underline{h})$ is a lower solution of (1.1) provided that $\sigma > 0$ is sufficiently large. Therefore, $\underline{h}(t) \le h(t) \le \bar{h}(t)$ for large t. This implies the boundedness of $h(t) - R(t)$. The boundedness of $g(t) + L(t)$ is proved similarly.

Step 2. Convergence of $h(t) - R(t)$, $g(t) + L(t)$, $h'(t)$, $g'(t)$. Define, for $t > 0$,

$$H(t) := R(t) - h(t),\ G(t) := R(t) - g(t),\ w(t,z) := u\big(t, R(t) - z\big),\ z \in \big[H(t), G(t)\big].$$

Then w satisfies (5.6). $H := \lim_{t\to\infty} H(t)$ exists and it is finite by the boundedness of $H(t)$. Moreover, by the limit of $H(t)$ and the uniform Hölder estimate for $H'(t)$: $\|H'(t)\|_{C^{\nu/2}([1,\infty))} \le C$ for C independent of t (cf. the proof of Theorem 1.6 in [6]), it is easy to show that $\lim_{t\to\infty} H'(t) = 0$, that is, $\lim_{t\to\infty} h'(t) = r(t)$. Similarly, $\lim_{t\to\infty} g'(t) = -l(t)$.

Step 3. Equations (2.6) *and* (2.7) *hold.* Define, for any small $\varepsilon > 0$, and some $H_1, N, B, K > 0$, $m_0 > 0$, $\delta \in (0, \kappa)$,

$$\begin{aligned}
&\bar{g}_1(t) = g(t),\\
&\bar{h}_1(t) = R(t) + H_1 + N\varepsilon + N\varepsilon B\big(1 - e^{-\delta(t - m_0 T)}\big),\\
&\bar{v}_1(t,x) = \big(1 + K\varepsilon e^{-\delta(t-m_0T)}\big)\widetilde{Q}_R\big(t, \bar{h}_1(t) - x\big).
\end{aligned}$$

One can calculate directly as in [16, section 3.3] to show that, when $N > 1$, m_0, K and B are sufficiently large, $(\bar{v}_1, \bar{g}_1, \bar{h}_1)$ is an upper solution of (4.20). Then by the monotonicity of $\widetilde{Q}_R$, we obtain

$$v(t,x) \le \widetilde{Q}_R\big(t, R(t) + H_1 + N\varepsilon(1+B) - x\big) + \varepsilon K e^{-\delta(t-m_0T)}. \tag{5.9}$$

Similarly, define a lower solution as follows:

$$\begin{aligned}
&\underline{g}_1(t) = ct,\\
&\underline{h}_1(t) = R(t) + H_1 - N\varepsilon - N\varepsilon B\big(1 - e^{-\delta(t - m_0 T)}\big),\\
&\underline{v}_1(t,x) = \big(1 - K\varepsilon e^{-\delta(t-m_0T)}\big)\widetilde{Q}_R\big(t, \underline{h}_1(t) - x\big).
\end{aligned}$$

We may use the comparison principle to obtain $\underline{v}(t,x) \le v(t,x)$ for $x \in [\underline{g}(t), \underline{h}(t)]$ and $t > 0$. More precisely,

$$\widetilde{Q}_R\big(t, R(t) + H_1 - N\varepsilon(1+B) - x\big) - \varepsilon K e^{-\delta(t-m_0T)} \le v(t,x). \tag{5.10}$$

By (5.9), (5.10), the mean value theorem and the monotonicity of $\widetilde{Q}_R$, we have

$$\limsup_{t\to\infty} \sup_{x\in[ct,h(t)]} |v(t,x) - \widetilde{Q}_R\big(t, R(t) - x + H_1\big)| \le \widetilde{C}\varepsilon,$$

where $\widetilde{C}$ is dependent of $||(\widetilde{Q}_R)_z(t,z)||_\infty$ but independent of ε. Let $\varepsilon \to 0$ we deduce

$$\limsup_{t\to\infty} \sup_{x\in[ct,h(t)]} |v(t,x) - \widetilde{Q}_R\big(t, R(t) - x + H_1\big)| = 0.$$

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One can similarly show that, for some $G_1 \in \mathbb{R}$,

$$\limsup_{t\to\infty} \sup_{x\in[g(t),ct]} |v(t,x) - \widetilde{Q}_L\big(t, x + L(t) - G_1\big)| = 0,$$

where $\widetilde{Q}_L := Q_L/P(t)$. Combining with the definitions of v, $\widetilde{Q}_R$ and $\widetilde{Q}_L$, we prove (2.6) and (2.7). □

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Conflicts of interest

None.

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