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# Asymptotic behaviour for a non-local parabolic problem

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In this paper, we consider the asymptotic behaviour for the non-local parabolic problem

$$u_t = \Delta u + \frac{\lambda f(u)}{(\int_\Omega f(u)dx)^p}, \quad x \in \Omega, \ t > 0,$$

with a homogeneous Dirichlet boundary condition, where $\lambda > 0$, $p > 0$ and $f$ is non-increasing. It is found that (a) for $0 < p \leqslant 1$, $u(x,t)$ is globally bounded and the unique stationary solution is globally asymptotically stable for any $\lambda > 0$; (b) for $1 < p < 2$, $u(x,t)$ is globally bounded for any $\lambda > 0$; (c) for $p = 2$, if $0 < \lambda < 2|\partial\Omega|^2$, then $u(x,t)$ is globally bounded; if $\lambda = 2|\partial\Omega|^2$, there is no stationary solution and $u(x,t)$ is a global solution and $u(x,t) \to \infty$ as $t \to \infty$ for all $x \in \Omega$; if $\lambda > 2|\partial\Omega|^2$, there is no stationary solution and $u(x,t)$ blows up in finite time for all $x \in \Omega$; (d) for $p > 2$, there exists a $\lambda^* > 0$ such that for $\lambda > \lambda^*$, or for $0 < \lambda \leqslant \lambda^*$ and $u_0(x)$ sufficiently large, $u(x,t)$ blows up in finite time. Moreover, some formal asymptotic estimates for the behaviour of $u(x,t)$ as it blows up are obtained for $p \geqslant 2$.

## 1 Introduction

In this paper we study the asymptotic behaviour for the non-local parabolic problem

$$\begin{aligned} u_t &= \Delta u + \frac{\lambda f(u)}{(\int_\Omega f(u)dx)^p}, && x \in \Omega, \ t > 0, \\ u(x,t) &= 0, && x \in \partial\Omega, \ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain, $\lambda > 0$ and $p > 0$.

Problem (1.1) arises, for example, in the analytical study of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates [2, 12], in modelling the phenomena of Ohmic heating [1, 10, 11], in the investigation of the fully turbulent behaviour of a real flow, using invariant measures for Euler equation [3], and in the theory of gravitational equilibrium of polytropic stars [9]. If $p = 2$ and $n = 1, 2$, problem (1.1) models Ohmic heating (see [5, 10, 14]), where $u(x,t) = u(x,t;\lambda)$ stands for the dimensionless temperature of a conductor when an electric current flows through it, and $f(s)$ represents, depending on the problem, either the electrical conductivity or the electrical resistance of the conductor, satisfying the condition

$$f(s) > 0,\ f'(s) < 0,\ s \geqslant 0,\ \int_0^\infty f(s)\,ds < \infty. \tag{1.2}$$

Condition (1.2) permits us to use comparison methods (see [10, 11, 13, 14]). Also, for simplicity, we assume $u_0(x)$ is continuous with $u_0(x) = 0, x \in \partial\Omega$ and $u_0(x) \geqslant 0, x \in \Omega$. Without loss of generality, we may assume that $\int_0^\infty f(s)ds = 1$.

For problem (1.1) with $p = 2$, Lacey [10, 11] and Tzanetis [14] proved the occurrence of blow-up for the one-dimensional problem and for the two-dimensional radially symmetric problem, respectively. First they estimated the supremum $\lambda^*$ of the spectrum of the related steady-state equations and then they proved the blow-up, for $\lambda > \lambda^*$, by constructing some blowing-up lower solutions. In [1], Bebernes and Lacey considered problem (1.1) and its associated steady-state equations with $p > 0$, $n \geqslant 1$ and $f(s)$ positive and locally Lipschitz continuous. Existence/non-existence results were proven when $f(s) = e^s$ or $e^{-s}$ and $\Omega$ is a ball or star-shaped domain. Using some ideas of [1], Kavallaris and Nadzieja [8] generalized the blow-up results for $\lambda > \lambda^*$ and $n \geqslant 2$ if $u_0$ is sufficiently large and $f(s)$ satisfies

$$\int_0^\infty (sf(s) - s^2 f'(s))ds < \infty. \tag{1.3}$$

Kavallaris *et al.* [7] showed that the solution $u^*(x,t) = u(x,t;\lambda^*)$ is global in time and diverges in the sense that $\|u^*(\cdot,t)\|_\infty \to \infty$ as $t \to \infty$ when $n = 1$, $\Omega = (-1,1)$ and $f(s)$ satisfies (1.2) or $n = 2$, $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and $f(s) = e^{-s}$. Moreover, it was proved that this divergence is global, i.e. $u^*(x,t) \to \infty$ as $t \to \infty$ for all $x \in \Omega$.

The main purpose of this paper is to generalize and improve the results for dimensions $n \geqslant 2$ and $p > 0$ obtained in [7, 8, 10, 11, 14]. Throughout this paper, we have assumed that the domain $\Omega$ satisfies the following condition:

(H) for any point $y_0 \in \partial\Omega$, there exist two balls $B_1$ and $B_2$ such that $B_1 \subset \Omega \subset B_2$ and $\partial B_1 \cap \partial\Omega \cap \partial B_2 = \{y_0\}$.

Our main results read as follows:

- If $0 < p \leqslant 1$, then $u(x,t)$ is globally bounded and there exists a unique stationary solution which is globally asymptotically stable for any $\lambda > 0$.
- If $1 < p < 2$, then $u(x,t)$ is globally bounded for any $\lambda > 0$.
- Assume $p = 2$, and let $\lambda^* = 2|\partial\Omega|^2$. If $0 < \lambda < \lambda^*$, $u(x,t)$ is globally bounded. If $\lambda = \lambda^*$, there is no stationary solution and $u^*(x,t)$ is a global-in-time solution and $u^*(x,t) \to \infty$ as $t \to \infty$ for all $x \in \Omega$. If $\lambda > \lambda^*$, there is no stationary solution and $u(x,t)$ blows up globally in finite time $T$ without requiring (1.3) and $u_0$ sufficiently large.
- If $p > 2$, then there exists a critical value $\lambda^*$ such that for $\lambda > \lambda^*$ or for any $0 < \lambda \leqslant \lambda^*$ and $u_0(x)$ sufficiently large, $u(x,t)$ blows up globally in finite time $T$.
- We also obtain some formal asymptotic estimates for the local behaviour of $u(x,t)$ as it blows up for $p \geqslant 2$.

This paper is organized as follows. In Section 2 we consider the steady-state problem corresponding to (1.1). In Section 3, we investigate the asymptotic behaviour of some critical solutions of (1.1) for $p = 2$. Section 4 is devoted to some formal asymptotic estimates for the local behaviour of $u(x,t)$ as it blows up in finite time for $p \geqslant 2$.

## 2 Steady-state problem

The steady states of the problem (1.1) play an important role in the description of the asymptotic behaviour of the solutions of (1.1), and hence we first consider the stationary problem of (1.1). The stationary problem corresponding to (1.1) is

$$\Delta w + \frac{\lambda f(w)}{(\int_\Omega f(w)dx)^p} = 0, \quad x \in \Omega; \quad w = 0, \quad x \in \partial\Omega. \tag{2.1}$$

In order to study the non-local problem (2.1), let us first consider the following local problem:

$$\Delta w + \mu f(w) = 0, \quad x \in \Omega; \quad w = 0, \quad x \in \partial\Omega, \tag{2.2}$$

where $\mu \geqslant 0$ and $f(s)$ satisfies (1.2). It is well known that the basic theory of monotone schemes can be carried out for the problem (2.2). Therefore, there exists a solution in $H_0^1(\Omega)$. Moreover, the straightforward argument, based on the coercivity of $-\Delta$ with Dirichlet boundary condition, implies that (2.2) has a unique positive solution $w_\mu^\Omega$ in $H_0^1(\Omega)$. The above arguments are classical and known in the literature [6]. In order to establish a relationship between the local problem (2.2) and the non-local problem (2.1), we define a real function $\lambda(\mu)$ by

$$\lambda(\mu) = \mu \left( \int_\Omega f\left(w_\mu^\Omega\right) dx \right)^p \tag{2.3}$$

for any $\mu \geqslant 0$. This function is well defined due to the positive character of $w_\mu^\Omega$. From the analyticity of the solutions $w_\mu^\Omega$ on $\mu$, we deduce that the function $\lambda(\mu)$ is analytic on $\mu$. It is easy to see the relation between the solutions of problem (2.2) and problem (2.1).

**Theorem 2.1** *If $w$ is a solution of problem (2.1) for $\lambda = \lambda_0$, then $w$ is a solution of problem (2.2) for $\mu = \lambda_0 / (\int_\Omega f(w)dx)^p$. Conversely, if $w$ is a solution of problem (2.2) for $\mu = \mu_0$, then $w$ is a solution of problem (2.1) for $\lambda = \lambda(\mu_0)$.*

Theorem 2.1 allows us to study problem (2.1) by analysing the behaviour of the function $\lambda(\mu)$. Now we give some qualitative properties of the profile of the bifurcation diagram of the local problem (2.2).

**Lemma 2.2** *Let $w_\mu^\Omega$ be the solution of (2.2), then*

(1) $\partial w_\mu^\Omega / \partial\mu > 0$ *for $x \in \Omega$.*

(2) $\lim_{\mu\to\infty} w_\mu^\Omega(x)/\Phi_1^\Omega(x) \to \infty$, *uniformly in $\Omega$, where $\Phi_1^\Omega(x)$ is the first normalized eigenfunction of $-\Delta$ in $H_0^1(\Omega)$.*

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The proof follows the same lines as in [4] so we omit it.

Now we prove that the solution of (2.1) is unique for any $0 < p \leqslant 1$.

**Theorem 2.3** *For any $0 < p \leqslant 1$, there exists a unique solution of the problem (2.1) for any $\lambda \geqslant 0$.*

**Proof** Let us prove that $\lambda(\mu)$ is strictly increasing. Integrating (2.2) over $\Omega$, we have

$$\int_{\partial\Omega} \frac{\partial w}{\partial \nu} ds + \lambda^{\frac{1}{p}} \mu^{\frac{p-1}{p}} = 0,$$

where $\partial/\partial\nu$ is the outward normal derivative, which implies

$$\lambda(\mu) = \mu^{1-p} \left( - \int_{\partial\Omega} \frac{\partial w}{\partial \nu} ds \right)^p . \tag{2.4}$$

By $0 < p \leqslant 1$, $w_\mu = \partial w_\mu^\Omega / \partial \mu = 0$ on $\partial\Omega$ and Lemma 2.2, we get

$$\lambda'(\mu) > 0 \quad \text{for} \quad \mu > 0 \quad \text{and} \quad \lim_{\mu \to \infty} \lambda(\mu) = \infty.$$

The proof is completed. □

The following results give us a way to construct a sub-solution of $w_\mu^\Omega$ in order to estimate from the above function $\lambda(\mu)$.

**Lemma 2.4** *Let $\Omega' \subset \Omega$. Then $w_\mu^{\Omega'} \leqslant w_\mu^\Omega$ on $\Omega'$ for any $\mu > 0$.*

We omit the proof.

We need a lemma concerning the solution to the problem in a ball

$$\Delta w + \mu f(w) = 0, \quad x \in B; \quad w = 0, \quad x \in \partial B. \tag{2.5}$$

**Lemma 2.5** (*see* [14, *Lemma 5.1*]). *Let $f(s)$ satisfy (1.2), $\int_0^\infty f(s) ds = 1$ and $w_\mu^B$ be a solution of (2.5); then we have*

$$-\frac{1}{\sqrt{\mu}} \frac{dw_\mu^B}{dr}\Big|_{\partial B} < \sqrt{2}, \quad -\lim_{\mu \to \infty} \frac{1}{\sqrt{\mu}} \frac{dw_\mu^B}{dr}\Big|_{\partial B} = \sqrt{2}, \tag{2.6}$$

*where $B = \{x \in R^n : | x - x_0 | < R\}$, $r = |x - x_0|$.*

**Theorem 2.6** *Let $f(s)$ satisfy (1.2), $\int_0^\infty f(s) ds = 1$ and $\Omega$ be a bounded domain satisfying (H). Then the following assertions hold:*

(1) *For $1 < p < 2$, there exists at least one solution of the problem (2.1) for any value $\lambda > 0$.*

(2) *For $p = 2$, let $\lambda^* = 2|\partial\Omega|^2$; then there exists at least one solution of the problem (2.1) for $0 < \lambda < \lambda^*$ and no solution for $\lambda \geqslant \lambda^*$. Moreover, $\lambda(\mu) < 2|\partial\Omega|^2$ for $\mu > 0$ and $\lim_{\mu \to \infty} \lambda(\mu) = 2|\partial\Omega|^2$.*

(3) *For $p > 2$, there exists a critical value $\lambda^* > 0$ such that there exist at least two solutions of the problem (2.1) for $0 < \lambda < \lambda^*$, at least one solution for $\lambda = \lambda^*$ and no solution for $\lambda > \lambda^*$. Moreover,* $\lim_{\mu\to\infty} \lambda(\mu) = 0$.

**Proof** Let $y_0 \in \partial\Omega$. Without loss of generality we assume that $y_0 = 0$. By (H), there exist two balls $\Omega_1$, $\Omega_2$ ($\Omega_1 \subset \Omega \subset \Omega_2$) which are tangent to $\Omega$ at $y_0$, where $\Omega_i = \{x \in R^n : |x - y_i| < R_i, y_i = (L_i, 0')\}$. Lemma 2.4 implies that $w_\mu^\Omega \geqslant w_\mu^{\Omega_1}$ on $\Omega_1$ and $w_\mu^{\Omega_2} \geqslant w_\mu^\Omega$ on $\Omega$. Applying Lemma 2.5, we conclude that

$$\sqrt{2} > -\frac{1}{\sqrt{\mu}}\frac{dw_\mu^{\Omega_2}(0)}{dx_1} \geqslant -\frac{1}{\sqrt{\mu}}\frac{dw_\mu^{\Omega}(0)}{dx_1} \geqslant -\frac{1}{\sqrt{\mu}}\frac{dw_\mu^{\Omega_1}(0)}{dx_1}, \quad \mu > 0,$$

and

$$\sqrt{2} = -\lim_{\mu\to\infty}\frac{1}{\sqrt{\mu}}\frac{dw_\mu^{\Omega_2}(0)}{dx_1} \geqslant -\lim_{\mu\to\infty}\frac{1}{\sqrt{\mu}}\frac{dw_\mu^{\Omega}(0)}{dx_1} \geqslant -\lim_{\mu\to\infty}\frac{1}{\sqrt{\mu}}\frac{dw_\mu^{\Omega_1}(0)}{dx_1} = \sqrt{2},$$

which imply

$$-\frac{1}{\sqrt{\mu}}\frac{dw_\mu^{\Omega}(0)}{dx_1} < \sqrt{2} \quad \text{and} \quad -\lim_{\mu\to\infty}\frac{1}{\sqrt{\mu}}\frac{dw_\mu^{\Omega}(0)}{dx_1} = \sqrt{2}.$$

Since $y_0$ is arbitrary, it follows that

$$-\frac{1}{\sqrt{\mu}}\int_{\partial\Omega}\frac{\partial w_\mu^\Omega}{\partial\nu}ds < \sqrt{2}|\partial\Omega| \quad \text{for} \quad \mu > 0 \quad \text{and} \quad -\lim_{\mu\to\infty}\frac{1}{\sqrt{\mu}}\int_{\partial\Omega}\frac{\partial w_\mu^\Omega}{\partial\nu}ds = \sqrt{2}|\partial\Omega|.$$

By (2.4), we obtain the following:

(i) If $0 < p < 2$, then $\lim_{\mu\to\infty} \lambda(\mu) = \infty$.

(ii) If $p = 2$, then $\lambda(\mu) < 2|\partial\Omega|^2$ for $\mu > 0$ and $\lim_{\mu\to\infty} \lambda(\mu) = 2|\partial\Omega|^2$.

(iii) If $p > 2$, then $\lim_{\mu\to\infty} \lambda(\mu) = 0$.

The proof is completed. □

Using Theorems 2.3 and 2.6, similarly as in [11], we can prove the following global existence results.

**Theorem 2.7**

(i) *If $0 < p \leqslant 1$, then $u$ is globally bounded and the unique steady state is globally asymptotically stable for any $\lambda > 0$.*

(ii) *If $1 < p < 2$ and $\int_0^\infty f(s)ds = 1$, then $u(x,t)$ is globally bounded for any $\lambda > 0$.*

(iii) *If $p = 2$, $\int_0^\infty f(s)ds = 1$ and $0 < \lambda < 2|\partial\Omega|^2$, then $u$ is globally bounded for any initial data.*

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## 3 Asymptotic behaviour of solutions of problem (1.1) for $p = 2$

In this section, we study the asymptotic behaviour of solutions of the following non-local parabolic problem:

$$
\begin{aligned}
u_t = \Delta u + \frac{2|\partial\Omega|^2 f(u)}{(\int_\Omega f(u)dx)^2}, &\quad x \in \Omega,\ t > 0,\\
u(x,t) = 0, &\quad x \in \partial\Omega,\ t > 0,\\
u(x,0) = u_0(x), &\quad x \in \Omega,
\end{aligned}
$$

where $f$ satisfies (1.2) and $\int_0^\infty f(s)ds = 1$. By Theorem 2.6, it follows that $\lambda(\mu) < 2|\partial\Omega|^2$ for all $\mu > 0$; then we can find an increasing lower solution $v = w(x;\mu(t))$ with $\mu(t) \to \infty$ as $t \to T \leqslant \infty$. Thus $u(x,t)$ is unbounded. Moreover, $u(x,t)$ is globally unbounded. Indeed, if $T = \infty$, from Lemma 2.2, $u(x,t)$ is globally unbounded; if $T < \infty$, $u(x,t)$ blows up globally (see the proof of Theorem 4.1 for details).

Now we prove that $||u(\cdot,t)||_\infty \to \infty$ as $t \to \infty$, i.e. $T = \infty$. It is sufficient to construct an upper solution $V(x,t)$ to problem (3.1) which is global in time and unbounded. Without loss of generality, we assume that the hyperplane $\{x : x_1 = 1\}$ is tangent to $\Omega$ at $(1, 0')$, and $\Omega$ lies in the half-space $\{x : x_1 < 1\}$. Let $d(x) = \text{dist}(x, \partial\Omega)$. Set

$$
\begin{aligned}
V(x,t) = w(y(x,t);\mu(t)), &\quad 0 \leqslant d(x) \leqslant \varepsilon(t), \quad x \in \Omega,\ t > 0,\\
V(x,t) = M(t) = \max_{0\leqslant d(x)\leqslant\varepsilon(t)} w(y(x,t);\mu(t)), &\quad d(x) \geqslant \varepsilon(t), \quad x \in \Omega,\ t > 0,
\end{aligned}
\tag{3.1}
$$

where $0 \leqslant y(x,t) = d(x)/\varepsilon(t) \leqslant 1$, $\varepsilon(t) > 0$ is a function to be chosen later and $w(y;\mu(t))$ satisfies

$$
w_{yy} + \mu(t)f(w) = 0, \quad 0 < y < 1,\ t > 0; \quad w(0;\mu(t)) = w'(1;\mu(t)) = 0, \tag{3.2}
$$

or equivalently

$$
w_{rr} + \frac{\mu(t)}{\varepsilon^2(t)} f(w) = 0, \quad r = d(x), \quad 0 \leqslant r \leqslant \varepsilon(t),\ t > 0; \quad w(0) = \frac{dw}{dr}|_{r=\varepsilon(t)} = 0, \tag{3.3}
$$

which implies

$$
\begin{aligned}
\Delta w - \frac{\Delta d}{\varepsilon}\frac{dw}{dy} + \frac{\mu}{\varepsilon^2} f(w) = 0, &\quad 0 \leqslant d(x) \leqslant \varepsilon(t),\ t > 0,\\
w(y(x,t);\mu(t)) = 0,\ x \in \partial\Omega,\ t > 0,\ &\frac{dw}{dr}|_{r=\varepsilon(t)} = 0,
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
\frac{d^2 w(y((x_1,0'),t);\mu(t))}{dx_1^2} + \frac{\mu}{\varepsilon^2} f(w(y((x_1,0'),t);\mu(t))) = 0, &\quad \delta(t) < x_1 < 1,\ t > 0,\\
w(y((1,0'),t);\mu(t)) = 0, \quad \frac{dw(y((\delta(t),0'),t);\mu(t))}{dx_1} = 0, &
\end{aligned}
\tag{3.5}
$$

where $\varepsilon(t) = 1 - \delta(t)$.

From the definition of $w$, it is obvious that $w$, $w_r$ are continuous at $r = \varepsilon(t)$. We can choose $\mu(0)$ (or equivalently $M(0)$) sufficiently large so that $V(x,0) \geqslant u_0(x)$ (such a choice is possible since $w \to \infty$ as $\mu \to \infty$ and provided that $u_0(x), u_0'(x)$ are bounded).

For any $\varepsilon > 0$, set $\Omega_\varepsilon = \{x \in \Omega : 0 < d(x) < \varepsilon(t)\}$. To prove that $V(x,t)$ is an upper solution, we need some preliminary results.

Problems (3.3) and (3.5) imply that

$$w_r(0) = \frac{\sqrt{2\mu}}{\varepsilon}\sqrt{\int_0^M f(s)ds} \tag{3.6}$$

and

$$\int_{\delta(t)}^1 f(w(y((x_1,0'),t);\mu(t)))dx_1 = -\frac{\varepsilon^2}{\mu}\frac{dw(y((1,0'),t);\mu(t))}{dx_1}. \tag{3.7}$$

From (3.3), we get

$$\frac{w_r}{\sqrt{F(w)-F(M)}} = \frac{\sqrt{2\mu}}{\varepsilon}, \tag{3.8}$$

where $F(s) = \int_s^\infty f(\sigma)d\sigma > 0$. Relation (3.8) gives

$$\sqrt{\mu(M)} = \frac{\sqrt{2}}{2}\int_0^M \frac{ds}{\sqrt{F(s)-F(M)}}. \tag{3.9}$$

For $s \leqslant M$, we have $F(s) - F(M) = f(\theta)(M-s),\ \theta \in [s,M]$ and due to $f'(s) < 0$ for $s \geqslant 0$, we get

$$(M-s)f(M) \leqslant F(s) - F(M) \leqslant (M-s)f(s). \tag{3.10}$$

Then

$$\sqrt{\mu(M)} \leqslant \frac{\sqrt{2}}{2}\int_0^M (M-s)^{-\frac{1}{2}} f^{-\frac{1}{2}}(M)ds \leqslant \sqrt{\frac{2M}{f(M)}},$$

and hence

$$\mu(M)f(M) \leqslant 2M \quad \text{for} \quad M > 0. \tag{3.11}$$

However,

$$Mf(M) \leqslant 2\int_{M/2}^M f(s)ds \leqslant 2\int_{M/2}^\infty f(s)ds \quad \text{and} \quad \int_{M/2}^\infty f(s)ds \to 0 \quad \text{as} \quad M \to \infty,$$

so $Mf(M) \to 0$ as $M \to \infty$ and due to (3.11) we finally get

$$\sqrt{\mu(M)}f(M) \to 0 \quad \text{as} \quad M \to \infty. \tag{3.12}$$

Next we claim that $\lim_{\mu\to\infty} \sqrt{2\mu}/M = \infty$. Indeed, by (1.2) and (3.9), we obtain

$$\frac{\sqrt{2\mu}}{M} \geqslant \frac{\int_0^M (M-s)^{-\frac{1}{2}} f^{-\frac{1}{2}}(s)ds}{M} = \int_0^1 \frac{s^{\frac{1}{2}}(1-s)^{-\frac{1}{2}}}{(Msf(Ms))^{\frac{1}{2}}}ds.$$

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Taking into account $sf(s) \to 0$ as $s \to \infty$, we deduce that $\lim_{\mu\to\infty} \sqrt{2\mu}/M = \infty$, i.e.

$$\lim_{M\to\infty} M/\sqrt{2\mu} = 0. \tag{3.13}$$

As indicated in [1], $d(x)$ is smooth and, more precisely, $|\Delta d| \leqslant K$, for some $K$, in a neighbourhood of the boundary if $\partial\Omega$ is smooth. In particular, such a neighbourhood $\Omega_\varepsilon$ consists of all $x \in \Omega$ such that $d(x, \partial\Omega) \leqslant \varepsilon(t)$, where $\varepsilon(t)$ is chosen small enough.

Integrating (3.4) over $\Omega_\varepsilon$ we obtain

$$\begin{aligned}
\int_{\Omega_\varepsilon} f(w)dx &= -\frac{\varepsilon^2}{\mu}\int_{\partial\Omega}\frac{\partial w}{\partial v}ds + \frac{\varepsilon}{\mu}\int_{\Omega_\varepsilon}\Delta d\frac{dw}{dy}dx \\
&= \frac{\varepsilon^2|\partial\Omega|}{\mu}w_r(0) + \frac{\varepsilon}{\mu}\int_{\Omega_\varepsilon}\Delta d\frac{dw}{dy}dx \\
&= \varepsilon|\partial\Omega|\sqrt{\frac{2}{\mu}}\sqrt{\int_0^M f(s)ds} + \frac{\varepsilon}{\mu}\int_{\Omega_\varepsilon}\Delta d\frac{dw}{dy}dx \\
&\geqslant \varepsilon|\partial\Omega|\sqrt{\frac{2}{\mu}}\sqrt{\int_0^M f(s)ds} - \frac{\varepsilon K}{\mu}\int_{\Omega_\varepsilon}\frac{dw}{dy}dx \quad \left(\text{using } \frac{dw}{dy} \geqslant 0\right) \\
&\geqslant \varepsilon|\partial\Omega|\sqrt{\frac{2}{\mu}}\sqrt{\int_0^M f(s)ds} + \frac{\varepsilon^2|\partial\Omega|K}{\mu}\int_{\delta(t)}^1\frac{dw((x_1,0');\mu(t))}{dx_1}dx_1 \\
&= \varepsilon|\partial\Omega|\sqrt{\frac{2}{\mu}}\sqrt{\int_0^M f(s)ds} - \varepsilon^2|\partial\Omega|K\frac{M}{\mu},
\end{aligned}$$

which implies

$$\begin{aligned}
\int_\Omega f(V)dx &= \int_{\Omega\setminus\Omega_\varepsilon} f(M)dx + \int_{\Omega_\varepsilon} f(w)dx \\
&\geqslant |\Omega\setminus\Omega_\varepsilon|f(M) + \varepsilon|\partial\Omega|\sqrt{\frac{2}{\mu}}\sqrt{\int_0^M f(s)ds} - \varepsilon^2|\partial\Omega|K\frac{M}{\mu}.
\end{aligned} \tag{3.14}$$

Our construction of the upper solution $V$ depends strongly on the behaviour of the function

$$g(s) = \frac{f(s)\sqrt{\mu(s)}}{F(s)} > 0.$$

Since (3.12) holds and $F(M) \to 0$ as $M \to \infty$, we distinguish two cases for the behaviour of $g(M)$. More precisely the following holds:

**Theorem 3.1** *Let $f(s)$ satisfy (1.2), $\int_0^\infty f(s)ds = 1$, $\Omega$ satisfy (H) and*

$$\liminf_{s\to\infty} g(s) = C > 0.$$

*If*

$$\lim_{s\to\infty} \mu(s)f(s) = C_0 > 0 \quad (e.g.\ f(s) = e^{-s})$$

*or*

$$\liminf_{s\to\infty} \mu(s)f(s)/s = C_1 > 0 \quad (C_1 \leqslant 2,\ e.g.\ f(s) = b(1+s)^{-1-b},\ b > 0).$$

*Then the function $V(x,t)$ is an upper solution to problem (3.1) and exists for all $t > 0$.*

In order to prove Theorem 3.1, we first derive a number of preliminary facts on $d(x)$.

**Lemma 3.2** *Assume $x_0 = (x_{10}, x_{20}, \ldots, x_{n0})$, $\Omega_i = \{x \in \mathbb{R}^n : |x - x_0| < R_i\}(i = 1,2)$ and $R_1 > R_2$. Let $d(x) = dist(x, \partial\Omega_1), x \in \Omega_1 \setminus \Omega_2$. Then $\Delta d(x) = (1-n)/(|x - x_0|)$.*

**Lemma 3.3** *$\Omega$ is a bounded domain satisfying (H). Then there exists $\varepsilon > 0$ such that $\Delta d \leqslant 0$ for $x \in \Omega_\varepsilon$.*

**Proof** Here we only consider the case of $n = 2$. As for $n = 1$ or $n \geqslant 3$, the proof is very similar. Divide $\partial\Omega$ into $m$ parts and take $m$ large enough such that the largest arc is sufficiently small. Let $A_1, A_2, \ldots, A_m$ be the division points. For any arc $\widehat{A_iA_{i+1}}$ $(1 \leqslant i \leqslant m-1)$, choose $C \in \widehat{A_iA_{i+1}}$ such that $|A_iC| = |\widehat{CA_{i+1}}|$. By the definition of $\Omega$, there exists a circle $\Omega_1 = \{x \in R^2 : |x - x_0| < R_1\}$ such that $\Omega_1$ $(\Omega \subset \Omega_1)$ is tangent to $\Omega$ at the point $C$. Take $A_i', A_{i+1}' \in \partial\Omega_1$ such that the segments $A_i'x_0, A_{i+1}'x_0$ intersect $\partial\Omega$ at $A_i, A_{i+1}$, respectively. Since $\widehat{A_iA_{i+1}}$ is sufficiently small, we have $\widehat{A_iA_{i+1}} \sim \widehat{A_i'A'}_{i+1}$. From Lemma 3.2, there exists a constant $\varepsilon_{\widehat{A_iA_{i+1}}} > 0$ such that

$$\Delta d(x) \leqslant \frac{-1}{2|x - x_0|} < 0, \quad x \in \{x \in \Omega : d(x, \widehat{A_iA_{i+1}}) < \varepsilon_{\widehat{A_iA_{i+1}}}\}.$$

Set $\varepsilon = \min\{\varepsilon_{\widehat{A_iA_{i+1}}}, \varepsilon_{\widehat{A_1A_m}}, i = 1, 2, \ldots, m-1\}$. Then

$$\Delta d(x) \leqslant 0, \quad x \in \Omega_\varepsilon = \{x \in \Omega : d(x, \partial\Omega) < \varepsilon\}.$$

The proof is completed. □

Now we give the proof of Theorem 3.1.

**Proof** *Case 1*: We assume $f(s)$ to be such that $\liminf_{s\to\infty} g(s) > C > 0$ and $\lim_{s\to\infty} \mu(s)f(s) = C_0 > 0$. Then taking into account the relation (3.14), for $d(x) \geqslant \varepsilon(t)$, we

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get

$$
\begin{aligned}
\mathscr{F}(V) &\equiv V_t - \Delta V - \frac{2|\partial\Omega|^2 f(V)}{(\int_\Omega f(V)dx)^2} \\
&\geqslant \dot{M}(t) - \frac{2|\partial\Omega|^2 f(M)}{\left(|\Omega\setminus\Omega_\varepsilon| f(M) + \varepsilon|\partial\Omega|\sqrt{\frac{2}{\mu}}\sqrt{\int_0^M f(s)ds} - \varepsilon^2|\partial\Omega| K\frac{M}{\mu}\right)^2} \\
&\geqslant \dot{M}(t) - \frac{\mu(M)f(M)}{\varepsilon^2\left(\frac{|\Omega\setminus\Omega_\varepsilon|\sqrt{\mu}f(M)}{\sqrt{2}\varepsilon|\partial\Omega|} + \int_0^M f(s)ds - \frac{K\varepsilon M}{\sqrt{2\mu}}\right)^2} \\
&\geqslant \dot{M}(t) - \frac{\mu(M)f(M)}{\varepsilon^2\left(\frac{|\Omega|\sqrt{\mu}f(M)}{2\sqrt{2}\varepsilon|\partial\Omega|} + \int_0^M f(s)ds - \frac{K\varepsilon M}{\sqrt{2\mu}}\right)^2} \qquad \text{for} \quad \varepsilon(t) \ll 1.
\end{aligned}
$$

Choosing $K_1 = (C_0|\Omega|)/(8K|\partial\Omega|)$ and $\varepsilon(t) = (K_1/M)^{1/2}$, we have $0 < \varepsilon(M) \ll 1$ for $M \gg 1$. Moreover, from (3.12), (3.13) and $\lim_{M\to\infty} \mu(M)f(M) = C_0$, we obtain

$$
\begin{aligned}
&\frac{|\Omega|\sqrt{\mu}f(M)}{2\sqrt{2}\varepsilon|\partial\Omega|} + \int_0^M f(s)ds - \frac{K\varepsilon M}{\sqrt{2\mu}} \\
&\qquad \geqslant \frac{|\Omega|\sqrt{\mu}f(M)}{2\sqrt{2}\varepsilon|\partial\Omega|} + \int_0^M f(s)ds - \frac{\sqrt{2}K_1K\sqrt{\mu}f(M)}{C_0\varepsilon} \\
&\qquad = \frac{|\Omega|\sqrt{\mu}f(M)}{4\sqrt{2}\varepsilon|\partial\Omega|} + \int_0^M f(s)ds \quad \text{for} \quad M \gg 1.
\end{aligned}
$$

Since

$$
\frac{|\Omega|\sqrt{\mu}f(M)}{4\sqrt{2}\varepsilon|\partial\Omega|F(M)} = \frac{|\Omega|\sqrt{\mu}f(M)}{4\sqrt{2}\varepsilon|\partial\Omega|\left(1-\int_0^M f(s)ds\right)} \geqslant \frac{|\Omega|C}{4\sqrt{2}\varepsilon|\partial\Omega|} > 1 \quad \text{for} \quad M \gg 1,
$$

it implies that

$$
\frac{|\Omega|\sqrt{\mu}f(M)}{4\sqrt{2}\varepsilon|\partial\Omega|} + \int_0^M f(s)ds > 1 \quad \text{for} \quad M \gg 1. \tag{3.15}
$$

Taking $M(t)$ to satisfy

$$
\dot{M}(t) = \frac{\mu(M)f(M)}{\varepsilon^2(M)}, \qquad t > 0, \tag{3.16}
$$

we obtain

$$
\mathscr{F}(V) > \dot{M}(t) - \frac{\mu(M)f(M)}{\varepsilon^2(M)} = 0 \quad \text{for } d(x) \geqslant \varepsilon(t)\ (x \in \Omega) \ \text{ and } \ M \gg 1.
$$

By integrating (3.16), we have

$$
\int_{M(0)}^{M(t)} \frac{\varepsilon^2(s)}{\mu(s)f(s)} ds = t,
$$

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and taking into account $\lim_{s\to\infty}\mu(s)f(s)=C_0$, we obtain

$$\frac{K_1}{1+C_0}\int_{M(0)}^{M(t)}\frac{1}{s}ds<t \quad \text{for} \quad M(0)\gg 1.$$

The last inequality implies that if $M(t)\to\infty$ then $t\to\infty$.

Also for $0<d(x)\leqslant\varepsilon(t)$ $(x\in\Omega)$, we have

$$\begin{aligned}\mathscr{F}(V)&\equiv w_\mu(y(x,t);\mu(t))\dot{\mu}(t)+\frac{dw(y(x,t);\mu(t))}{dy}\dot{y}(t)-\Delta w-\frac{2|\partial\Omega|^2 f(w)}{(\int_\Omega f(V)dx)^2}\\ &=w_\mu(y(x,t);\mu(t))\dot{\mu}(t)-\frac{dw(y(x,t);\mu(t))}{dy}\frac{d(x)}{\varepsilon^2}\dot{\varepsilon}(t)\\ &\quad-\frac{\Delta d}{\varepsilon}\frac{dw}{dy}+\frac{\mu}{\varepsilon^2}f(w)-\frac{2|\partial\Omega|^2 f(w)}{(\int_\Omega f(V)dx)^2}.\end{aligned}$$

Since $w_\mu>0$, $\dot{\mu}(t)>0$, $\dot{\varepsilon}(t)<0$, $dw/dy\geqslant 0$ and $\Delta d(x)\leqslant 0$ for $M\gg 1$, we have

$$\begin{aligned}\mathscr{F}(V)&\geqslant\frac{\mu f(w)}{\varepsilon^2}-\frac{2|\partial\Omega|^2 f(w)}{(\int_\Omega f(V)dx)^2}\\ &\geqslant\frac{\mu f(w)}{\varepsilon^2}\left(1-\frac{1}{\left(\frac{|\Omega|\sqrt{\mu}f(M)}{4\sqrt{2}\varepsilon|\partial\Omega|}+\int_0^M f(s)ds\right)^2}\right)>0 \quad \text{for} \quad M\gg 1.\end{aligned}$$

*Case 2*: Now let $f$ be such that $\liminf_{s\to\infty}\mu(s)f(s)/s=C_1>0\,(C_1\leqslant 2)$ and $\liminf_{s\to\infty}g(s)>C>0$. For this case it is enough to consider $\varepsilon(t)$ to be constant such that $\Delta d\leqslant 0$ for $x\in\Omega_\varepsilon$. Moreover, we choose $\varepsilon$ to satisfy

$$\frac{|\Omega\setminus\Omega_\varepsilon|}{\sqrt{2}\varepsilon|\partial\Omega|}-\frac{\sqrt{2}K\varepsilon}{C_1}>\frac{1}{C}.$$

For $d(x)\geqslant\varepsilon$ $(x\in\Omega)$, we have

$$\begin{aligned}\mathscr{F}(V)&\geqslant\dot{M}(t)-\frac{2|\partial\Omega|^2 f(M)}{\left(|\Omega\setminus\Omega_\varepsilon|f(M)+\varepsilon|\partial\Omega|\sqrt{\frac{2}{\mu}}\sqrt{\int_0^M f(s)ds}-\varepsilon^2|\partial\Omega|K\frac{M}{\mu}\right)^2}\\ &\geqslant\dot{M}(t)-\frac{\mu(M)f(M)}{\varepsilon^2\left(\frac{|\Omega\setminus\Omega_\varepsilon|\sqrt{\mu}f(M)}{\sqrt{2}\varepsilon|\partial\Omega|}+\int_0^M f(s)ds-\frac{K\varepsilon M}{\sqrt{2\mu}}\right)^2}\\ &\geqslant\dot{M}(t)-\frac{\mu(M)f(M)}{\varepsilon^2\left(\frac{|\Omega\setminus\Omega_\varepsilon|\sqrt{\mu}f(M)}{\sqrt{2}\varepsilon|\partial\Omega|}+\int_0^M f(s)ds-\frac{\sqrt{2}K\varepsilon f(M)\sqrt{\mu}}{C_1}\right)^2}\\ &\geqslant\dot{M}(t)-\frac{f(M)\mu(M)}{\varepsilon^2\left(\frac{f(M)\sqrt{\mu(M)}}{C}+\int_0^M f(s)ds\right)^2} \quad \text{for} \quad M\gg 1.\end{aligned}$$

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Since

$$\frac{f(M)\sqrt{\mu(M)}}{CF(M)} = \frac{f(M)\sqrt{\mu(M)}}{C\left(1 - \int_0^M f(s)ds\right)} > \frac{1}{C}C = 1,$$

it implies that

$$\frac{f(M)\sqrt{\mu(M)}}{C} + \int_0^M f(s)ds > 1.$$

Hence $\mathscr{F}(V) > 0$ for $d(x) \geqslant \varepsilon$ and $M \gg 1$, provided that $M(t)$ satisfies

$$\dot{M}(t) = \frac{\mu(M)f(M)}{\varepsilon^2}, \quad t > 0. \tag{3.17}$$

By integrating (3.17), we have

$$\int_{M(0)}^{M(t)} \frac{\varepsilon^2}{\mu(s)f(s)} ds = t,$$

and taking into account (3.11) we obtain

$$\frac{\varepsilon^2}{2} \int_{M(0)}^{M(t)} \frac{1}{s} ds \leqslant t,$$

which implies that if $M(t) \to \infty$ then $t \to \infty$.

For $0 \leqslant d(x) \leqslant \varepsilon$, we have

$$\begin{aligned}\mathscr{F}(V) &\equiv w_\mu\big(y(x,t);\mu(t)\big)\dot{\mu}(t) - \frac{\Delta d}{\varepsilon}\frac{dw}{dy} + \frac{\mu}{\varepsilon^2}f(w) - \frac{2|\partial\Omega|^2 f(w)}{(\int_\Omega f(V)dx)^2} \\ &\geqslant \frac{\mu(M)f(w)}{\varepsilon^2} - \frac{f(w)\mu(M)}{\varepsilon^2\left(\frac{f(M)\sqrt{\mu(M)}}{C} + \int_0^M f(s)ds\right)^2} > 0 \quad \text{for} \quad M \gg 1.\end{aligned}$$

Therefore, we finally get that, in each case, $V(x,t)$ is an upper solution to problem (3.1) for all $t > 0$. The proof is completed. □

Thus we formulate this main result of this section in the following theorem.

**Theorem 3.4** *If $f(s)$ satisfies the hypotheses of Theorem 3.1, and $\Omega$ satisfies (H), then $u(x,t)$ is a global-in-time solution to problem (3.1) and $u(x,t) \to \infty$ as $t \to \infty$, i.e. $u(x,t)$ diverges globally in $\Omega$.*

## 4 Asymptotic behaviour of the blow-up solutions

In this section, we deal with the blow-up solutions of problem (1.1). We do calculations similar to those for the one-dimensional case (see [11], and also [14]).

**Theorem 4.1** *Let $f(s)$ satisfy (1.2), $\int_0^\infty f(s)ds = 1$, $p = 2$ and $\Omega$ satisfy (H). If $\lambda > \lambda^* = 2|\partial\Omega|^2$, the solution of the problem (1.1) blows up globally in finite time $T$.*

**Proof** By Theorem 2.6, in the case of $\lambda > \lambda^* = 2|\partial\Omega|^2$ and $\int_0^\infty f(s)ds = 1$, there is no steady solution to (2.2). Since $\lambda(\mu) < \lambda$ for any $\mu > 0$, we can find an increasing lower solution $v(x,t) = w(x;\mu(t))$ with $\mu$ and $v \to \infty$ for all $x \in \Omega$ as $t \to T \leqslant \infty$. Thus $u(x,t)$ is globally unbounded. We shall show that $T < \infty$. Therefore, we look for a lower solution $V(x,t)$ which blows up at a finite time ($V(x,t)$ satisfy (3.1)–(3.5)). From (3.5) and (3.7), we have

$$\begin{aligned}\int_\Omega f(V)dx &= \int_{\Omega\setminus\Omega_\varepsilon} f(M)dx + \int_{\Omega_\varepsilon} f(w)dx \\ &\leqslant |\Omega|f(M) + |\partial\Omega| \int_{\delta(t)}^1 f(w(y((x_1,0'),t);\mu(t)))dx_1 \\ &\leqslant |\Omega|f(M) + |\partial\Omega|\varepsilon\sqrt{\frac{2}{\mu}} = \sqrt{2}|\partial\Omega|f(M)\left(\frac{|\Omega|}{\sqrt{2}|\partial\Omega|}+\alpha\right),\end{aligned}$$

on choosing $\alpha = \varepsilon/(\sqrt{\mu}f(M))$, where $\alpha$ is a suitable chosen constant; in particular choose $\alpha > |\Omega|/(\sqrt{\lambda} - \sqrt{2}|\partial\Omega|)$ for $\lambda > \lambda^* = 2|\partial\Omega|^2$. Such an $\alpha$ gives

$$3\Lambda = \frac{\lambda}{(|\Omega| + \sqrt{2}\alpha|\partial\Omega|)^2} - \frac{1}{\alpha^2} > 0.$$

From (3.12), we also note that with such a fixed $\alpha$, $\varepsilon \to 0$ as $M \to \infty$. Integrating (3.8) on $(0,r)$, we get

$$\int_0^w \frac{ds}{\sqrt{F(s)-F(M)}} = \frac{\sqrt{2\mu}r}{\varepsilon} = \frac{\sqrt{2}r}{\alpha f(M)}. \tag{4.1}$$

For $x \in \Omega \setminus \Omega_\varepsilon$,

$$\mathscr{F}(V) = \dot{M} - \frac{\lambda f(M)}{(\int_\Omega f(V)dx)^2} \leqslant \dot{M} - \frac{\lambda}{2|\partial\Omega|^2 f(M)\left(\frac{|\Omega|}{\sqrt{2}|\partial\Omega|}+\alpha\right)^2} \leqslant \dot{M} - \frac{\Lambda}{f(M)} \leqslant 0,$$

on choosing $\dot{M} \leqslant \Lambda/f(M)$.

For $x \in \Omega_\varepsilon$, we first differentiate (4.1) with respect to $t$ and get

$$\begin{aligned} w_t &= -\frac{f'(M)}{f(M)}\dot{M}(t)\sqrt{F(w)-F(M)}\int_0^w \frac{ds}{\sqrt{F(s)-F(M)}} \\ &\quad + \frac{1}{2}f(M)\dot{M}(t)\sqrt{F(w)-F(M)}\int_0^w \frac{ds}{(F(s)-F(M))^{3/2}} \\ &:= A + B.\end{aligned}$$

For $A$, from (3.10) we have

$$\begin{aligned} A &= -\frac{f'(M)}{f(M)}\dot{M}(t)\sqrt{F(w)-F(M)}\int_0^w \frac{ds}{\sqrt{F(s)-F(M)}} \\ &\leqslant -\frac{2f'(M)}{f^{3/2}(M)}M\dot{M}(t)f^{1/2}(w) \leqslant \frac{\Lambda f(w)}{f^2(M)},\end{aligned}$$

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provided that

$$\dot{M}(t) \leqslant -\frac{\Lambda}{2Mf'(M)}$$

and taking into account that $f'(s) \leqslant 0$ so that $f(w)/f(M) \geqslant 1$ for $w \leqslant M$. For $B$ we have

$$B = \frac{1}{2}f(M)\dot{M}(t)\sqrt{F(w)-F(M)}\int_0^w \frac{ds}{(F(s)-F(M))^{3/2}}$$
$$\leqslant \frac{f^{1/2}(w)}{f^{1/2}(M)}\dot{M}(t) \leqslant \frac{\Lambda f(w)}{f^2(M)},$$

provided that

$$\dot{M}(t) \leqslant \frac{\Lambda}{f(M)}.$$

Also, using (3.4) and (3.8), we have the estimate

$$-\Delta w = -w_r\Delta d + \frac{\mu}{\varepsilon^2}f(w) \leqslant Kw_r + \frac{\mu}{\varepsilon^2}f(w) \quad (\text{using } |\Delta d| \leqslant K)$$
$$= \frac{K\sqrt{2\mu}}{\varepsilon}\sqrt{F(w)-F(M)} + \frac{f(w)}{\alpha^2 f^2(M)}$$
$$\leqslant \frac{\sqrt{2}K}{\alpha}\frac{(Mf(M))^{1/2}f(w)}{f^2(M)} + \frac{f(w)}{\alpha^2 f^2(M)}$$
$$\leqslant \frac{\Lambda f(w)}{f^2(M)} + \frac{f(w)}{\alpha^2 f^2(M)}, \quad \text{for } M \gg 1,$$

since $Mf(M) \to 0$ as $M \to \infty$. Thus for $x \in \Omega_\varepsilon$ if

$$0 \leqslant \dot{M}(t) = \min\left\{\frac{\Lambda}{f(M)},\ -\frac{\Lambda}{2Mf'(M)}\right\}, \tag{4.2}$$

and using the previous estimate, we obtain

$$\mathscr{F}(V) = w_t - \Delta w - \frac{\lambda f(w)}{(\int_\Omega f(V)dx)^2}$$
$$= A + B - w_r\Delta d + \frac{\mu}{\varepsilon^2}f(w) - \frac{\lambda f(w)}{(\int_\Omega f(V)dx)^2}$$
$$\leqslant \frac{3\Lambda f(w)}{f^2(M)} + \frac{f(w)}{\alpha^2 f^2(M)} - \frac{\lambda f(w)}{2|\partial\Omega|^2 f^2(M)\left(\frac{|\Omega|}{\sqrt{2}|\partial\Omega|}+\alpha\right)^2} = 0.$$

Also $V(x,t) = u(x,t) = 0$ on the boundary $\partial\Omega$ and taking $V(x,0) \leqslant u_0(x)$, the function $V(x,t)$ is a lower solution to the problem (1.1). Hence $u(x,t) \geqslant V(x,t)$ for $M$ large enough (after some time at which $u(x,t)$ is sufficiently large if $T = \infty$).

Now we show that $u(x,t)$ blows up in finite time. Indeed, from (4.2) we have

$$\Lambda\frac{dt}{dM} = \max\{f(M),\ -2Mf'(M)\} \leqslant f(M) - 2Mf'(M) \quad (f'(s) \leqslant 0)$$

or

$$At \leqslant \int^{M} (f(s) - 2sf'(s))ds < \infty,$$

since $Mf(M) \to 0$ as $M \to \infty$ and $\int_0^\infty f(s)ds = 1$. Hence $V(x,t)$ blows up at $t^* < \infty$, and $u(x,t)$ must blow up at $T \leqslant t^* < \infty$.

The global blow-up is due to

$$\int_\Omega f(u)dx \to 0 \quad \text{as} \quad t \to T.$$

Indeed,

$$\dot{M} \leqslant \frac{\lambda f(M)}{(\int_\Omega f(u)dx)^2} = h(t),$$

giving

$$M(t) - M(0) \leqslant \int_0^t h(s)ds \to \infty \quad \text{as} \quad t \to T.$$

This implies $\int_\Omega f(u)dx \to 0$ as $t \to T$ since $f(s)$ is bounded. Thus, for $\lambda > \lambda^* = 2|\partial\Omega|^2$, $u(x,t)$ blows up globally. The proof is completed. □

Now we examine the case $p > 2$ and we have:

**Theorem 4.2** *Let $f(s)$ satisfy (1.2), $\int_0^\infty f(s)ds = 1$, $p > 2$ and $\Omega$ satisfy $(H)$. Then there exists a critical value $\lambda^*$ such that for $\lambda > \lambda^*$ or for any $0 < \lambda \leqslant \lambda^*$ but with initial data sufficiently large, the solution of the problem (1.1) blows up globally in finite time $T$.*

**Proof** Using Theorem 2.6, we know that for $\lambda > \lambda^*$ or for any $0 < \lambda \leqslant \lambda^*$ but with initial data $u_0$ more than the greater steady state $u(x,t)$ is globally unbounded (see [11]). In order to prove $u(x,t)$ blows up in finite time $T < \infty$, we also look for a lower solution $V(x,t)$ to satisfy (3.1)–(3.5). Then

$$\begin{aligned}\int_\Omega f(V)dx &= \int_{\Omega\setminus\Omega_\varepsilon} f(M)dx + \int_{\Omega_\varepsilon} f(w)dx \\ &\leqslant |\Omega|f(M) + |\partial\Omega| \int_{\delta(t)}^1 f(w(y((x_1,0'),t);\mu(t)))dx_1 \\ &\leqslant |\Omega|f(M) + |\partial\Omega|\varepsilon\sqrt{\frac{2}{\mu}} = \sqrt{2}|\partial\Omega|f(M)\left(\frac{|\Omega|}{\sqrt{2}|\partial\Omega|} + 1\right),\end{aligned}$$

on choosing $\varepsilon = \sqrt{\mu}f(M)$. From (3.12), we also note that $\varepsilon \to 0$ as $M \to \infty$.

For $x \in \Omega \setminus \Omega_\varepsilon$,

$$\begin{aligned}\mathscr{F}(V) = \dot{M} - \frac{\lambda f(M)}{(\int_\Omega f(V)dx)^p} &\leqslant \dot{M} - \frac{\lambda}{(\sqrt{2}|\partial\Omega|)^p f^{p-1}(M)\left(\frac{|\Omega|}{\sqrt{2}|\partial\Omega|} + 1\right)^p} \\ &\leqslant \dot{M} - \frac{1}{f(M)} \leqslant 0 \quad \text{for } M \gg 1,\end{aligned}$$

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on choosing $\dot{M} \leqslant 1/f(M)$ and taking into account $p > 2$ and $f(M) \to 0$ as $M \to \infty$.

For $x \in \Omega_\varepsilon$, similar to the proof of Theorem 4.1, we have $w_t = A + B$. For $A$, from (3.10) we have

$$A = -\frac{f'(M)}{f(M)}\dot{M}(t)\sqrt{F(w) - F(M)}\int_0^w \frac{ds}{\sqrt{F(s) - F(M)}}$$
$$\leqslant -\frac{2f'(M)}{f^{3/2}(M)}M\dot{M}(t)f^{1/2}(w) \leqslant \frac{f(w)}{f^2(M)},$$

provided that

$$\dot{M}(t) \leqslant -\frac{1}{2Mf'(M)}.$$

For $B$ we have

$$B = \frac{1}{2}f(M)\dot{M}(t)\sqrt{F(w) - F(M)}\int_0^w \frac{ds}{(F(s) - F(M))^{3/2}}$$
$$\leqslant \frac{f^{\frac{1}{2}}(w)}{f^{\frac{1}{2}}(M)}\dot{M}(t) \leqslant \frac{f(w)}{f^2(M)},$$

provided that

$$\dot{M}(t) \leqslant \frac{1}{f(M)}.$$

Also, using (3.4) and (3.8), we have the estimate

$$-\Delta w = -w_r\Delta d + \frac{\mu}{\varepsilon^2}f(w) \leqslant Kw_r + \frac{\mu}{\varepsilon^2}f(w) \quad (\text{using } |\Delta d| \leqslant K)$$
$$= \frac{K\sqrt{2\mu}}{\varepsilon}\sqrt{F(w) - F(M)} + \frac{f(w)}{f^2(M)}$$
$$\leqslant \sqrt{2}K\frac{(Mf(M))^{1/2}f(w)}{f^2(M)} + \frac{f(w)}{f^2(M)}$$
$$\leqslant \frac{2f(w)}{f^2(M)}, \quad \text{for } M \gg 1,$$

since $Mf(M) \to 0$ as $M \to \infty$. Thus for $x \in \Omega_\varepsilon$ if

$$0 \leqslant \dot{M}(t) = \min\left\{\frac{1}{f(M)}, -\frac{1}{2Mf'(M)}\right\},$$

and using the previous estimate, we obtain

$$\mathscr{F}(V) = w_t - \Delta w - \frac{\lambda f(w)}{(\int_\Omega f(V)dx)^p}$$
$$= A + B - w_r\Delta d + \frac{\mu}{\varepsilon^2}f(w) - \frac{\lambda f(w)}{(\int_\Omega f(V)dx)^p}$$
$$\leqslant \frac{4f(w)}{f^2(M)} - \frac{\lambda f(w)}{(2|\partial\Omega|)^p f^p(M)\left(\frac{|\Omega|}{\sqrt{2}|\partial\Omega|} + 1\right)^p} \leqslant 0 \quad \text{for } M \gg 1,$$

since $p > 2$ and $f(M) \to 0$ as $M \to \infty$.

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Also $V(x,t) = u(x,t) = 0$ on the boundary $\partial\Omega$ and taking $V(x,0) \leqslant u_0(x)$, the function $V(x,t)$ is a lower solution to the problem (1.1). Hence $u(x,t) \geqslant V(x,t)$ for $M$ large enough (after some time at which $u$ is sufficiently large if $T = \infty$).

The rest of the proof is similar to that of Theorem 4.1. We omit it. □

We now consider the Dirichlet problem, in which we rewrite (1.1) as

$$\begin{aligned} u_t &= \Delta u + g(t)f(u), && x \in \Omega,\ t > 0, \\ u(x,t) &= 0, && x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), && x \in \Omega, \end{aligned}$$

where $g(t) = \lambda/(\int_\Omega f(u)dx)^p$.

We seek a formal asymptotic approximation for $u(x,t)$ near the blow-up time $T$, still taking $f$ to be decreasing and to satisfy $\int_0^\infty f(s)ds = 1$. Set $M(t) = \max_{x\in\Omega} u(x,t)$.

As in [11], we obtain that $\lim_{t\to T} g(t) = \infty$ and $u(x,t) \sim M$ except in some boundary layers near $\partial\Omega$. In the main core (outer) region we neglect $\Delta u$, so

$$\frac{dM}{dt} \sim g(t)f(M)$$

and significant contributions to the integral $\int_\Omega f(u)dx$ can come from the largest (core) region which has volume $\sim |\Omega|$ (contribution $\sim |\Omega| f(M)$) and from the boundary layers where $f$ is large, $f(u)$ is $O(1)$ where $u(x,t)$ is $O(1)$. If the boundary layers have volume $O(\delta)$, for some small $\delta$, then to obtain a balance involving $\Delta u$, either $\delta^{-2} = O(g)$ or $\delta^{-2} = O((T-t)^{-1})$, whichever is the larger, see [11].

Supposing that $g(t) \ll (T-t)^{-1}$ for $t \to T$ the contribution to the integral from the boundary layer is $O(\delta) = O(\sqrt{T-t})$, whereas

$$\int_\Omega f(u)dx = O(g(t)^{-1/p}) \gg (T-t)^{1/p} \geqslant \sqrt{T-t} \quad \text{as} \quad t \to T.$$

This suggests that the core dominates and

$$\int_\Omega f(u)dx \sim |\Omega| f(M).$$

Then

$$g(t) \sim \frac{\lambda}{|\Omega|^p f^p(M)}, \quad f(M) \sim \frac{1}{|\Omega|}\left(\frac{\lambda}{g}\right)^{1/p},$$

and

$$\frac{dM}{dt} \sim g(t)f(M) \sim \frac{1}{|\Omega|}\lambda^{1/p} g^{(p-1)/p} \ll (T-t)^{(1-p)/p} \quad \text{for} \quad t \to T.$$

This would indicate that $M$ is actually bounded as $t \to T$, contradicting the occurrence of blow-up.

Next we suppose that $g(t) = O\big((T-t)^{-1}\big)$ for $t \to T$. Since

$$|\Omega| f(M) \lesssim \int_\Omega f(u)dx = \left(\frac{\lambda}{g}\right)^{1/p},$$

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we must have $f(M) \leqslant O((T-t)^{1/p})$. Again,

$$\frac{dM}{dt} \sim g(t)f(M) \leqslant O\big((T-t)^{(1-p)/p}\big),$$

which contradicts the assumption of blow-up. There remains only one possibility:

$$g(t) \gg (T-t)^{-1} \quad \text{for} \quad t \to T.$$

The boundary layer has volume $O(g(t)^{-1/2}) \ll \sqrt{T-t}$, where $u(x,t)$ is $O(1)$ and $u_t$ is negligible compared to $\Delta u$. There has to be a balance between $\Delta u$ and $g(t)f(u)$, that is,

$$-\Delta u \sim g(t)f(u).$$

Without loss of generality, we assume that the hyperplane $\{x \in \mathbb{R}^n : x_1 = 1\}$ is tangent to $\Omega$ at $y_0$ ($y_0 = (1,0')$), and $\Omega$ lies in the half-space $\{x : x_1 < 1\}$. Writing $x_1 = 1 - g^{-1/2}y$ ($g^{-1/2}y \ll 1$) gives

$$\begin{aligned} -u_{yy}(y,0') &\sim f(u(y,0')), \quad && y > 0,\\ u(y,0') &= 0, && y = 0,\\ u(y,0')) \gg 1 &\gg u_y(y,0'), && y \gg 1. \end{aligned} \tag{4.3}$$

Multiplying both sides of (4.3) by $u_y(y,0')$ and integrating, we get

$$u_y^2(y,0') \sim 2F(u(y,0')),$$

where $F(u(y,0')) = \int_{u(y,0')}^{\infty} f(s)ds$. Integrating again gives $u(y,0') \sim U(y)$, where

$$\sqrt{2}y = \int_0^{U(y)} F^{-1/2}(s)ds. \tag{4.4}$$

Since $y_0$ is arbitrary, it follows from (4.4) that the boundary layers contribute to a total amount

$$\int_{d(x,\partial\Omega)\leqslant y/\sqrt{g}} f(u)dx \sim |\partial\Omega| \int_{x_1}^{1} f(u(x_1,0'))dx_1 \sim \frac{|\partial\Omega|}{\sqrt{g}} \int_0^\infty f(U(y))dy,$$

this is automatically of the correct size $g(t) = \lambda/(\int_\Omega f(u)dx)^p$. It should also be observed that

$$\int_0^\infty f(U(y))dy = U'(0).$$

Now look at the following steady problem:

$$w'' + \mu f(w) = 0, \quad -1 < x < 1; \quad w(\pm 1) = 0.$$

Set $M(\mu) = \max_{-1<x<1} w(x) = w(0)$ and $x = 1 - y/\sqrt{\mu}$, then

$$\frac{d^2w}{dy^2} + f(w) = 0, \quad w(0) = 0, \quad \frac{dw}{dy}|_{y=\sqrt{\mu}} = 0, \quad w(\sqrt{\mu}) = M.$$

From Lemma 2.5 (in case of $n = 1$), we have

$$\lim_{\mu\to\infty} -\frac{1}{\sqrt{\mu}}\frac{dw(1)}{dx} = \sqrt{2},$$

which implies that

$$\lim_{\mu\to\infty} \frac{dw(0)}{dy} = \sqrt{2},$$

and it appears that the problem in limit for large $\mu$ is the same as the asymptotic problem (4.3). Thus,

$$\int_0^\infty f(U(y))dy = U'(0) = \lim_{\mu\to\infty} \frac{dw(0)}{dy} = \sqrt{2}.$$

We deduce that the contribution to $\int_\Omega f(u)dx$ from the boundary layers $\sim\sqrt{2}|\partial\Omega|/\sqrt{g}$.

Now

$$\int_\Omega f(u)dx \sim |\Omega|f(M) + \sqrt{2}|\partial\Omega|/\sqrt{g}$$

and

$$g \sim \frac{\lambda}{(|\Omega|f(M) + \sqrt{2}|\partial\Omega|/\sqrt{g})^p} \quad \text{for} \quad t \to T \quad (g, M \to \infty).$$

We see that

$$\lambda^{1/p} \sim g^{1/p}(|\Omega|f(M) + \sqrt{2}|\partial\Omega|/\sqrt{g}) = |\Omega|f(M)g^{1/p} + \sqrt{2}|\partial\Omega|g^{(2-p)/(2p)},$$

i.e.,

(i) If $p = 2$, then $f(M) \sim \dfrac{\sqrt{\lambda} - \sqrt{2}|\partial\Omega|}{|\Omega|\sqrt{g}}$.

(ii) If $p > 2$, then $f(M) \sim \dfrac{1}{|\Omega|}\left(\dfrac{\lambda}{g}\right)^{1/p}$.

Therefore, in the core region $u(x,t) \sim M$, which satisfies

$$\frac{dM}{dt} \sim g(t)f(M) \sim \frac{\Lambda_1^2}{f(M)} \quad \text{if} \quad p = 2, \tag{4.5}$$

where $\Lambda_1 = (\sqrt{\lambda} - \sqrt{2}|\partial\Omega|)/|\Omega|$, and

$$\frac{dM}{dt} \sim g(t)f(M) \sim \frac{\Lambda_2}{f^{p-1}(M)} \quad \text{if} \quad p > 2, \tag{4.6}$$

where $\Lambda_2 = \lambda/|\Omega|^p$.

**Remark 4.3** By (4.5) and (4.6), we obtain that the significant contributions to integral $\int_\Omega f(u)dx$ come from the largest core region and the boundary layers where $f$ is large if $p = 2$, but the core dominates for $p > 2$.

Let us consider two examples.

*Example 1* Suppose $f(s)$ is decreasing, $\int_0^\infty f(s)ds = 1$, $f(s) \sim B/s^{1+b}$ as $s \to \infty$ for some positive constants $b$ and $B$.

For $p = 2$,

$$\frac{dM}{dt} \sim \frac{\Lambda_1^2}{f(M)},$$

which implies

$$M \sim \left(\frac{b\Lambda_1^2}{B}\right)^{-1/b} (T-t)^{-1/b}.$$

For $p > 2$,

$$\frac{dM}{dt} \sim \frac{\Lambda_2}{f^{p-1}(M)},$$

which follows that

$$M \sim \left(\frac{(1+b)(p-1)-1}{B^{p-1}}\Lambda_2\right)^{\frac{1}{1-(1+b)(p-1)}} (T-t)^{\frac{1}{1-(1+b)(p-1)}}.$$

*Example 2* $f(s) = e^{-s}$.

For $p = 2$,

$$\frac{dM}{dt} \sim \frac{\Lambda_1^2}{e^{-M}},$$

which implies

$$M \sim -\ln(T-t) - 2\ln \Lambda_1.$$

For $p > 2$,

$$\frac{dM}{dt} \sim \frac{\Lambda_2}{e^{(1-p)M}},$$

that is,

$$M \sim \frac{1}{1-p}\ln((p-1)\Lambda_2) + \frac{1}{1-p}\ln(T-t).$$

## 5 Discussion

We have considered the multi-dimensional problem

$$u_t = \Delta u + \frac{\lambda f(u)}{(\int_\Omega f(u)dx)^p}, \quad x \in \Omega,\ t > 0,$$

with a homogeneous Dirichlet boundary condition, which arises, for example, in the analytical study of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates [2, 12], in modelling the phenomena of Ohmic heating [1, 10, 11], in the investigation of the fully turbulent behaviour of a real flow, using invariant measures for Euler equation [3], and in the theory of gravitational equilibrium of polytropic stars [9].

We have seen that in a physically important case of $p = 2$ with $\Omega$ satisfying (H), the critical value of $\lambda$ for the non-local elliptic problem (2.1) is $\lambda^* = 2|\partial\Omega|^2$ in the sense that there exists at least one solution of (2.1) for $0 < \lambda < \lambda^*$ and no solution for $\lambda \geqslant \lambda^*$.

This critical value was suggested but was not verified in [1, Theorem 2.2]. However, for general domain without assumption (H), we are now unable to verify $\lambda^* = 2|\partial\Omega|^2$. Next, we saw that for $p = 2$, the solution $u$ of (1.1) is globally bounded if $0 < \lambda < \lambda^*$, $u$ is a global-in-time solution and $u \to \infty$ as $t \to \infty$ for all $x \in \Omega$ if $\lambda = \lambda^*$ and $u$ blows up globally in finite time if $\lambda > \lambda^*$.

We also proved that for $0 < p \leqslant 2$, $u$ is globally bounded for any $\lambda > 0$. For $p > 2$, which is also of practical significance, there exists a critical value $\lambda^*$ such that for $\lambda > \lambda^*$ or for any $0 < \lambda \leqslant \lambda^*$ and $u_0(x)$ sufficiently large, $u(x,t)$ blows up globally in finite time. We obtained some formal asymptotic estimates for the local behaviour of $u$ as it blows up for $p \geqslant 2$.

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