DIFFERENCE ALGEBRAIC RELATIONS AMONG SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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Abstract We extend and apply the Galois theory of linear differential equations equipped with the action of an endomorphism. The Galois groups in this Galois theory are difference algebraic groups, and we use structure theorems for these groups to characterize the possible difference algebraic relations among solutions of linear differential equations. This yields tools to show that certain special functions are difference transcendent. One of our main results is a characterization of discrete integrability of linear differential equations with almost simple usual Galois group, based on a structure theorem for the Zariski dense difference algebraic subgroups of almost simple algebraic groups, which is a schematic version, in characteristic zero, of a result due to Z. Chatzidakis, E. Hrushovski, and Y. Peterzil.

1. Introduction

In [15], the authors developed a Galois theory for linear differential equations equipped with the action of an endomorphism σ. Such actions occur, for example, when considering linear differential equations depending on a parameter α, in which case the action of σ is determined by $\sigma(\alpha) = \alpha + 1$. In the p-adic analysis of differential equations one may choose σ to be a Frobenius lift.

The Galois groups in this Galois theory are linear difference algebraic groups, i.e., subgroups of the general linear group defined by equations which involve powers of σ. These Galois groups measure not only the algebraic relations among the solutions of a linear differential equation, but also the difference algebraic relations, i.e., the algebraic relations between all the transforms of the solutions under σ. For example, the difference algebraic relation $x J_{\alpha+2}(x) - 2(\alpha+1)J_{\alpha+1}(x) + x J_\alpha(x) = 0$ satisfied by the

Bessel function $J_\alpha(x)$ will be reflected by our new Galois group, whereas the usual (differential) Galois group of the Bessel equation does not take it into account.

In this article we prove structural results about difference algebraic groups to apply them to characterize the possible difference algebraic relations among various types of linear differential equation. Thereby we often obtain criteria to decide if certain functions are transformally (i.e., difference algebraically) independent.

This Galoisian approach to transformal independence is analogous to the approaches to differential independence of special functions via the Galois theories in [9, 20]. See [13] for a survey and [2] for a recent example.

The usual Galois theory of linear differential equations (see, e.g., [44] for an introduction) is aimed at studying the algebraic relations among the solutions of a linear differential system $\delta(y) = Ay$, $A \in K^{n \times n}$, over a differential field K, i.e., a field equipped with a derivation $\delta \colon K \to K$. In order to study the difference algebraic properties of the solutions, we enrich the situation with the action of a difference operator σ. Our base field K is a differential–difference field; i.e., it is equipped with a derivation $\delta \colon K \to K$ and an endomorphism $\sigma \colon K \to K$ such that δ and σ commute (up to a convenient factor). For example, K could be the field $\mathbb{C}(\alpha, x)$ with derivation $\delta = \frac{d}{dx}$ and endomorphism σ given by $\sigma(f(\alpha, x)) = f(\alpha + 1, x)$. One can also consider the $\delta\sigma$-field $K = \mathbb{C}(x)$ endowed with the derivation $\delta(x) = 1$ and the endomorphism $\sigma(x) = x^2$. To study the difference algebraic relations among the solutions we have to adjoin not only the solutions to the base field but also all their transforms under σ. As in the usual Galois theory of linear differential equations, it is crucial to do this without enlarging the δ-constants $K^\delta := \{a \in K \mid \delta(a) = 0\}$. A differential–difference field obtained in this way is called a σ-Picard–Vessiot extension of K for $\delta(y) = Ay$. The automorphism group of a σ-Picard–Vessiot extension is naturally a linear difference algebraic group over the difference field $k = K^\delta$, i.e., a subgroup of the general linear group $\mathrm{Gl}_{n,k}$ defined by algebraic difference equations. These groups play the role of Galois groups in the Galois theory developed in [15], and they encode valuable information about the difference algebraic relations among the solutions. For example, the difference–transcendence degree of a σ-Picard–Vessiot extension is the dimension (in the sense of difference algebra) of the corresponding Galois group.

The basic idea of this article is to apply, via the Galois theory from [15], structural results for difference algebraic groups to study the difference algebraic relations among the solutions of linear differential equations. We will show that often some understanding of the usual Galois group of $\delta(y) = Ay$ is already sufficient to obtain an explicit description of the possible difference algebraic relations among the solutions. For example, if A is a diagonal matrix, one knows that the usual Galois group is an algebraic subgroup of $\mathbf{G}_m^n$, and since the difference algebraic subgroups of $\mathbf{G}_m^n$ are well understood, a description of the possible difference algebraic relations among the solutions can be obtained.

Our explicit descriptions of the possible difference algebraic relations among the solutions often lead to criteria for transformal independence. For example, we prove the following statement (cf. Corollary 6.5).

Let L be a differential–difference field extension of K such that $L^\delta = K^\delta$ is algebraically closed and $\sigma \colon K \to K$ is surjective. For $Y \in \mathrm{Gl}_n(L)$ with $\delta(Y) = AY$ for some $A \in K^{n \times n}$, let $T \subset L$ be a transcendence basis of $K\langle Y \rangle | K$, and assume that the usual Galois group of $\delta(y) = Ay$ is an almost simple algebraic group. If, for every $d \geqslant 1$, the linear differential equation $\delta(B) + BA = \sigma^d(A)B$ over K has no non-zero solution B which is algebraic over K, then T is transformally independent over K; i.e., $T, \sigma(T), \ldots$ are algebraically independent over K.

Applying the above criterion to the Airy equation $\delta^2(y) - xy = 0$ yields the transformally independence of the Airy functions (see Corollary 6.10).

Let $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ be two $\mathbb{C}$-linearly independent solutions of the Airy equation. Then the functions $\mathrm{Ai}(x), \mathrm{Bi}(x), \mathrm{Bi}'(x), \mathrm{Ai}(x+1), \mathrm{Bi}(x+1),$ $\mathrm{Bi}'(x+1), \mathrm{Ai}(x+2), \ldots$ are algebraically independent over $\mathbb{C}(x)$.

Let us describe the content of the article in more detail. In § 2, we fix the notation and recall the basic definitions and results from the σ-Galois theory of linear differential equations developed in [15]. In § 3, we study three different classes of linear differential systems: first, systems of the form

$$\delta(y_1) = b_1, \ldots, \delta(y_n) = b_n,$$

where the usual Galois group is a subgroup of $\mathbf{G}_a^n$; second, systems of the form

$$\delta(y_1) = a_1 y_1, \ldots, \delta(y_n) = a_n y_n,$$

where the usual Galois group is a subgroup of $\mathbf{G}_m^n$; and finally an equation of the form

$$\delta(y) = ay + b,$$

where the usual Galois group is a subgroup of $\mathbf{G}_a \rtimes \mathbf{G}_m$.

In § 4, we investigate the inverse problem for difference algebraic subgroups of the additive group $\mathbf{G}_a$. The inverse problem for a continuous parameter was discussed in [51], where it is proved that having $\mathbf{G}_a$ or $\mathbf{G}_m$ as quotient is an obstruction for a differential algebraic group to be a parameterized Galois group in the sense of [9]. In this section, we show that, unlike the continuous parameter case, one can realize $\mathbf{G}_a$ as well as its non-reduced subgroups as a σ-Galois group.

In § 5, we introduce and study discrete integrability, also called discrete isomonodromy. A linear differential system $\delta(y) = Ay$ over a differential–difference field K is called σ^d-integrable for an integer $d \geqslant 0$ if there exists $B \in \mathrm{Gl}_n(K)$ such that $\delta(B) + BA = \sigma^d(A)B$. Note that σ-integrability is a necessary condition for the existence of a fundamental solution matrix Y of $\delta(y) = Ay$ with $\sigma(Y) = BY$ derived from the identity $\sigma(\delta(Y)) = \delta(\sigma(Y))$. The main result here is a characterization of σ^d-integrability via the σ-Galois group (Theorem 5.11): the differential system $\delta(y) = Ay$ is σ^d-integrable if and only if, up to extension of the constants and conjugation inside Gl_n, the associated σ-Galois group $G \leqslant \mathrm{Gl}_n$ satisfies $\sigma^d(g) = g$ for all $g \in G$. In § 5.2, we give examples of

σ^d-integrability as contiguity relations and Frobenius structures for p-adic differential equations.

Then, in § 6, we combine this characterization of σ^d-integrability with a classification result for the Zariski dense difference algebraic subgroups of almost simple algebraic groups to obtain a strong dichotomy for the possible difference algebraic relations among the solutions of linear differential systems $\delta(y) = Ay$, whose usual Galois group is an almost simple algebraic group. The philosophy of the result is that either the system is σ^d-integrable for some $d \geqslant 1$ or there are no (proper) difference algebraic relations among the solutions; i.e., all the difference algebraic relations can be derived from algebraic relations. We present two versions of this result, one for simple algebraic groups (Proposition 6.1) and the other for almost simple algebraic groups (Theorem 6.4).

The relevant structure results for difference algebraic groups have been collected in Appendix. Difference algebraic groups are the group objects in the category of difference varieties. A difference variety in the classical sense (see § 2.6 in [34]) is determined by its points in difference fields. There is a one-to-one correspondence between the difference subvarieties of affine n-space and the so-called *perfect* difference ideals in the difference polynomial ring in n difference variables (Theorem 2.6.4 in [34]). In this article we have to employ a more general notion of difference variety, which one might term a difference scheme. We consider points in arbitrary difference algebras. There is a one-to-one correspondence between the difference subschemes of affine n-space and *all* difference ideals in the difference polynomial ring in n difference variables. The difference schemes corresponding to the classical difference varieties we call perfectly σ-reduced.

Difference algebraic groups which are not perfectly σ-reduced occur rather frequently as σ-Galois groups. For example, if the system $\delta(y) = Ay$ has a solution z whose coordinates $z_1, \ldots, z_n$ are algebraic over K, then a σ-Picard–Vessiot extension L for $\delta(y) = Ay$ will contain all the $\sigma^j(z_i)$ ($j \geqslant 1$, $1 \leqslant i \leqslant n$). Unless σ is of a rather special nature, the field extension generated by all the $\sigma^j(z_i)$ will be an infinite algebraic extension of K. Thus the relative algebraic closure of K in L is of infinite degree over K. This shows that the σ-Galois group of $L|K$ is not perfectly σ-reduced, since a perfectly σ-reduced difference algebraic group can have only finitely many components. (cf. Lemma 6.4 in [15].)

The majority of the appendix is concerned with a classification result for the Zariski dense difference algebraic subgroups of almost simple algebraic groups. In [5], the authors proved this result for perfectly σ-reduced difference algebraic groups. We show that a naive generalization of [5] does not hold. Thus, we adapt their arguments to our schematic framework and introduce new techniques to obtain Theorems A.20 and A.25 as structure theorem. (See § A.4.2 for a detailed discussion). A further benefit of our version is that it is expected to be applicable (via the Galois theory in [41]) to linear difference equations as well.

2. Difference Galois theory of linear differential equations

In this section, we quickly recall some basic facts of difference/differential algebra as well as some very basic notions of difference algebraic geometry, mainly in the affine case. We recall also some fundamental results from [15] that we will need in what follows.

More precise notions will be recalled in the paper when needed. We largely use standard notation of difference and differential algebra, as can be found in [7, 29, 34].

2.1. Differential algebra

All rings considered in this work are commutative with identity and contain the field of rational numbers. In particular, all fields are of characteristic zero. A differential ring (or δ-ring for short) is a ring R together with a derivation $\delta : R \to R$. A δ-ring R is said to be δ-simple if it does not contain any proper δ-ideals, i.e, proper ideals invariant under the action of δ. The ring of δ-constants is $R^\delta = \{r \in R \mid \delta(r) = 0\}$. We will use more than once the following lemma on δ-simple δ-rings.

Lemma 2.1 (Lemma 2.3 in [15]). *Let R be a δ-simple δ-ring, $k := R^\delta$, and S a k-algebra, considered as a constant δ-algebra; i.e., $S^\delta = S$. Then $(R \otimes_k S)^\delta = S$ and the assignments $\mathfrak{a} \mapsto R \otimes_k \mathfrak{a}$ and $\mathfrak{b} \mapsto S \cap \mathfrak{b}$ define mutually inverse bijections between the set of ideals of S and the set of δ-ideals of $R \otimes_k S$. In particular, every δ-ideal $\mathfrak{b}$ of $R \otimes_k S$ is generated by $\mathfrak{b} \cap S$ as an ideal.*

2.2. Difference algebra

A difference ring (or σ-ring for short) is a ring R together with a ring endomorphism $\sigma : R \to R$. We do not assume that σ is an automorphism or injective. A σ-ring with σ injective is called σ-reduced. If σ is an automorphism, the σ-ring is called inversive. The ring of σ-constants is $R^\sigma = \{r \in R \mid \sigma(r) = r\}$.

A σ-ideal of a σ-ring R is an ideal $\mathfrak{a}$ of R such that $\sigma(\mathfrak{a}) \subset \mathfrak{a}$. A σ-ideal $\mathfrak{a} \subset R$ is called reflexive if $\sigma^{-1}(\mathfrak{a}) \subset \mathfrak{a}$, which is equivalent to saying that $R/\mathfrak{a}$ is σ-reduced. A σ-ideal $\mathfrak{q}$ of R is said to be σ-prime if it is prime and reflexive. Finally, a σ-ideal $\mathfrak{a}$ is called perfect if $\sigma^{\alpha_1}(r) \cdots \sigma^{\alpha_n}(r) \in \mathfrak{a}$ implies $r \in \mathfrak{a}$, for all $r \in R$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$. This is equivalent to saying that $R/\mathfrak{a}$ is perfectly σ-reduced; that is, if $f \in R/\mathfrak{a}$, $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$ and $\sigma^{\alpha_1}(f) \cdots \sigma^{\alpha_n}(f) = 0$, then $f = 0$.

Given two σ-rings (R, σ) and (R', σ'), a morphism $\psi : R \to R'$ of σ-rings is a morphism of rings such that $\psi\sigma = \sigma'\psi$. A σ-field is a field that is also a σ-ring. A σ-field extension is an extension of σ-fields such that the inclusion is also a morphism of σ-rings.

Let k be a σ-field and R a k-σ-algebra, i.e., a k-algebra such that the morphism $k \to R$ is a morphism of σ-rings. We say that R is σ-separable (respectively, perfectly σ-separable) over k if the zero ideal in $R \otimes_k k'$ is reflexive (respectively, perfect), for every σ-field extension k' of k.

If R is a k-σ-algebra over a σ-field k and B a subset of R, then $k\{B\}_\sigma$ denotes the smallest k-σ-subalgebra of R that contains B. If $R = k\{B\}_\sigma$ for some finite subset B of R, we say that R is finitely σ-generated over k. If $K|k$ is an extension of σ-fields and $B \subset K$, then $k\langle B\rangle_\sigma$ denotes the smallest σ-field extension of k inside K that contains B. The k-σ-algebra $k\{x\}_\sigma = k\{x_1, \ldots, x_n\}_\sigma$ of σ-polynomials over k in the σ-variables $x_1, \ldots, x_n$ is the polynomial ring over k in the countable set of algebraically independent variables $x_1, \ldots, x_n, \sigma(x_1), \ldots, \sigma(x_n), \ldots$, with an action of σ as suggested by the names of the variables.

Definition 2.2 (Definition 4.1.7 in [34]). Let $L|K$ be a σ-field extension. Elements $a_1, \ldots, a_n \in L$ are called *transformally (or σ-algebraically) independent over K* if the elements $a_1, \ldots, a_n, \sigma(a_1), \ldots, \sigma(a_n), \ldots$ are algebraically independent over K. Otherwise, they are called transformally dependent over K. A *σ-transcendence basis of L over K* is a maximal transformally independent over K subset of L. Any two σ-transcendence bases of $L|K$ have the same cardinality, and so we can define the *σ-transcendence degree of $L|K$*, or σ-trdeg$(L|K)$ for short, as the cardinality of any σ-transcendence basis of L over K.

2.3. Difference algebraic geometry

In [15], we work with the formalism of difference group schemes. They are defined as follows.

Definition 2.3. Let k be a σ-field. A *group k-σ-scheme* is a (covariant) functor G from the category of k-σ-algebras to the category of groups which is representable by a k-σ-algebra. That is, there exists a k-σ-algebra $k\{G\}$ such that $G \simeq \mathrm{Alg}_k^\sigma(k\{G\}, -)$, where Alg_k^σ stands for morphisms of k-σ-algebras. If $k\{G\}$ is finitely σ-generated over k, we say that G is a σ-algebraic group over k.

By a *σ-closed subgroup H of G* we mean a group k-σ-scheme H such that $H(S)$ is a subgroup of $G(S)$ for every k-σ-algebra S. We call H *normal* if $H(S)$ is a normal subgroup of $G(S)$ for every k-σ-algebra S.

As in the classical setting, the Yoneda lemma implies that the algebra $k\{G\}$ is a k-σ-Hopf algebra, i.e., a k-σ-algebra equipped with the structure of a Hopf algebra over k such that the Hopf algebra structure maps are morphisms of difference rings. It also follows immediately that the category of group k-σ-schemes is anti-equivalent to the category of k-σ-Hopf algebras. We are not giving further details on this point (see Appendices A.2 and A.8 in [15]).

A σ-closed subgroup H of G corresponds to a σ-Hopf-ideal $\mathbb{I}(H)$ of $k\{G\}$, i.e., a Hopf-ideal which is a difference ideal, which we will call the vanishing ideal of H inside G. Then $k\{H\} \cong k\{G\}/\mathbb{I}(H)$. Normal σ-closed subgroups of G correspond to normal σ-Hopf-ideals, i.e., σ-Hopf-ideals which are normal Hopf-ideals.

Difference algebraic properties of the vanishing ideal of a difference group scheme G translate into geometric properties of the difference group scheme G. For instance, the difference analog of irreducibility is that the zero ideal of $k\{G\}$ is σ-prime. The difference analog of reduced scheme is more subtle, and it admits two definitions. First of all, we say that a group k-σ-scheme G is perfectly σ-reduced if the zero ideal in $k\{G\}$ is perfect. Perfectly σ-reduced difference schemes correspond to difference varieties in the classical sense [7, 34] or in the model theoretic sense [4, 5, 21], ...), where it suffices to focus on the solution set of a system of difference equations with values in a sufficiently big field, i.e., a σ-closed field. In the present work, we encounter a broader class of difference schemes, so we need a weaker notion. Thus, we say that a group k-σ-scheme G is σ-reduced if the zero ideal in $k\{G\}$ is reflexive. If one requires that these two notions remain unchanged by base field extension, we say absolutely perfectly σ-reduced and

absolutely σ-reduced. Of course, absolutely (respectively, absolutely perfectly) σ-reduced difference group schemes are in one-to-one correspondence with (respectively, perfectly) σ-separable k-σ-Hopf algebras.

The forgetful functor from the category of k-σ-algebras to the category of k-algebras admits a right adjoint. This functor allows us to attach to any group k-scheme $\widetilde{G}$ over k, a group k-σ-scheme, denoted $[\sigma]_k \widetilde{G}$ (see § A.4 in [15]). Let us recall the construction. Let $d \geqslant 0$ be a positive integer. For any k-algebra R, we set $^{\sigma^d} R = R \otimes_k k$, where the tensor product is formed by using $\sigma^d : k \to k$ on the right-hand side. We consider $^{\sigma^d} R$ as k-algebra via the right factor. We define

$$R_d = R \otimes_k \,^\sigma R \otimes_k \cdots \otimes_k \,^{\sigma^d} R$$

and $[\sigma]_k R$ as the limit (i.e., the union) of the R_d ($d \geqslant 0$). Let $\widetilde{G} = \mathrm{Spec}(k[\widetilde{G}])$ be a group k-scheme. Then we denote by $^{\sigma^d} \widetilde{G}$ the group k-scheme represented by $^{\sigma^d} k[\widetilde{G}]$, i.e., the group k-scheme obtained from $\widetilde{G}$ by extension of scalar via $\sigma^d : k \to k$. We also let $\widetilde{G}_d = \widetilde{G} \times \,^\sigma \widetilde{G} \times \ldots \times \,^{\sigma^d} \widetilde{G}$ be the group k-scheme represented by $k[\widetilde{G}]_d$. Finally, we define *the group k-σ-scheme $[\sigma]_k \widetilde{G}$ associated with $\widetilde{G}$* as the group k-σ-scheme represented by $[\sigma]_k k[\widetilde{G}]$. Notice that, if S is a k-σ-algebra and $S^\sharp$ is the underlying k-algebra, then we have the following (see the end of § A.4 in [15]):

$$[\sigma]_k \widetilde{G}(S) = \widetilde{G}(S^\sharp). \tag{2.1}$$

By abuse of notation, we will say that H is a σ-closed subgroup of the algebraic group $\widetilde{G}$ to mean that H is a σ-closed subgroup of $[\sigma]_k \widetilde{G}$.

For instance, let $\widetilde{G} = \mathbf{G}_a$ be the additive group k-scheme whose k-Hopf algebra is $k[x]$, the k-algebra of polynomials over k in the variable x. Its comultiplication sends x to $1 \otimes x + x \otimes 1$. Then $[\sigma]_k \widetilde{G}$ is represented by the k-σ-Hopf algebra $k\{x\}_\sigma$, the k-σ-algebra of σ-polynomials over k in the σ-variable x. Its comultiplication sends $\sigma^i(x)$ to $1 \otimes \sigma^i(x) + \sigma^i(x) \otimes 1$. The closed subgroups of $\widetilde{G}$ are the trivial group and G itself, whereas the σ-closed subgroups of $[\sigma]_k \widetilde{G}$ are given by linear homogeneous σ-polynomials (see Theorem A.1).

On the other hand, if G is a group k-σ-scheme, one can define its dth-order Zariski closure $G[d]$ as follows (see Definition A.11 in [15]). Let $\widetilde{G}$ be a group scheme over k and G be a σ-closed subgroup of $\widetilde{G}$. We denote by $G^\sharp$ the group k-scheme defined by $G^\sharp(S) = \mathrm{Alg}_k(k\{G\}, S)$ for any k-algebra S, where Alg_k stands for morphism of k-algebra. For $d \geqslant 0$, we consider the group k-scheme $\widetilde{G}_d$ represented by $k[\widetilde{G}]_d$. Then the smallest closed subscheme $G[d]$ of $\widetilde{G}_d$ such that $G^\sharp \to \widetilde{G}_d$ factors through $G[d] \hookrightarrow \widetilde{G}_d$ is called the *dth-order Zariski closure* of G inside $\widetilde{G}$. If $\mathbb{I}(G)$ is the vanishing ideal of G inside $\widetilde{G}$, then the vanishing ideal of $G[d]$ inside $\widetilde{G}_d$ is nothing other than $\mathbb{I}(G) \cap k[\widetilde{G}]_d$.

2.4. σ-Picard–Vessiot extensions

A $\delta\sigma$-ring is a ring R that is simultaneously a δ-ring and a σ-ring, such that for some unit $\hbar \in R^\delta$ we have

$$\delta(\sigma(r)) = \hbar\sigma(\delta(r)), \quad \text{for all } r \in R. \tag{2.2}$$

We set $\hbar_0 = 1$ and $\hbar_j = \hbar\sigma(\hbar_{j-1})$, for all integers $j \geqslant 1$, so that $\delta(\sigma^j(r)) = \hbar_j \sigma^j(\delta(r))$, for all $r \in R$ and all non-negative integers j. The element $\hbar$ is understood to be part of

the data of a $\delta\sigma$-ring. So a morphism $\psi : R \to R'$ of $\delta\sigma$-rings is a morphism of rings such that $\psi\sigma = \sigma'\psi$, $\psi\delta = \delta'\psi$, and $\psi(\hbar) = \hbar'$. Note that condition (2.2) implies that R^δ is a σ-ring. The definitions of k-$\delta\sigma$-algebra, $\delta\sigma$-extension, and $\delta\sigma$-ideal are the intuitive ones. They are given in detail in [15].

Let K be a $\delta\sigma$-field and $A \in K^{n \times n}$. We consider the differential system $\delta(y) = Ay$. If R is a K-$\delta\sigma$-algebra, then a matrix $Y \in \mathrm{Gl}_n(R)$ such that $\delta(Y) = AY$ is called a fundamental solution matrix for $\delta(y) = Ay$. Notice that if $Y, Y' \in \mathrm{Gl}_n(R)$ are two fundamental solution matrices for $\delta(y) = Ay$ in some K-$\delta\sigma$-algebra then $\delta(Y^{-1}Y') = 0$. Thus, there exists $C \in \mathrm{Gl}_n(R^\delta)$ such that $Y' = YC$.

Definition 2.4. A σ-Picard–Vessiot extension of K for $\delta(y) = Ay$ is a $\delta\sigma$-field extension L of K, σ-generated by the entries of a fundamental solution matrix and without new δ-constants, i.e., such that $L^\delta = K^\delta$. A K-$\delta\sigma$-algebra R is called a σ-*Picard–Vessiot ring* *for* $\delta(y) = Ay$ if it is σ-generated by the entries of a fundamental solution matrix and the inverse of its determinant and it is δ-simple. A $\delta\sigma$-field extension L of K is called a σ-*Picard–Vessiot extension* if it is a σ-Picard–Vessiot extension for some differential equation $\delta(y) = Ay$ with $A \in K^{n \times n}$; similarly for σ-Picard–Vessiot rings.

We recall the following two fundamental results about σ-Picard–Vessiot extensions.

Proposition 2.5 (Corollary 1.13 in [15]). *Let K be a $\delta\sigma$-field and $A \in K^{n \times n}$. Assume that K^δ is an algebraically closed field. Then there exists a σ-Picard–Vessiot extension for $\delta(y) = Ay$ over K.*

Proposition 2.6 (Proposition 1.5 in [15]). *Let K be a $\delta\sigma$-field and $A \in K^{n \times n}$. If $L|K$ is a σ-Picard–Vessiot extension for $\delta(y) = Ay$, with fundamental solution matrix $Y \in \mathrm{Gl}_n(L)$, then $R := K\{Y, \frac{1}{\det(Y)}\}_\sigma$ is a σ-Picard–Vessiot ring for $\delta(y) = Ay$. Conversely, if R is a σ-Picard–Vessiot ring for $\delta(y) = Ay$ with $R^\delta = K^\delta$, then the field of fractions of R is a σ-Picard–Vessiot extension for $\delta(y) = Ay$.*

2.5. The σ-Galois group and its properties

If $R \subset S$ is an inclusion of $\delta\sigma$-rings, we denote by $\mathrm{Aut}^{\delta\sigma}(S|R)$ the automorphisms of S over R in the category of $\delta\sigma$-rings; i.e., the automorphisms are required to be the identity on R and to commute with δ and σ.

Definition 2.7. Let $L|K$ be a σ-Picard–Vessiot extension with σ-Picard–Vessiot ring $R \subset L$. Set $k = K^\delta$. We define σ-$\mathrm{Gal}(L|K)$ to be the functor from the category of k-σ-algebras to the category of groups given by

$$\sigma\text{-}\mathrm{Gal}(L|K)(S) := \mathrm{Aut}^{\delta\sigma}(R \otimes_k S | K \otimes_k S)$$

for every k-σ-algebra S. Notice that the action of δ on S is trivial; i.e., $\delta(r \otimes s) = \delta(r) \otimes s$ for $r \in R$ and $s \in S$. On morphisms, σ-$\mathrm{Gal}(L|K)$ is given by base extension. We call σ-$\mathrm{Gal}(L|K)$ the σ-*Galois group* of $L|K$.

Proposition 2.8 (Proposition 2.5 in [15]). *Let $L|K$ be a σ-Picard–Vessiot extension with σ-Picard–Vessiot ring $R \subset L$. Then $\sigma\text{-Gal}(L|K)$ is a σ-algebraic group over $k = K^\delta$. More precisely, $\sigma\text{-Gal}(L|K)$ is represented by the finitely σ-generated k-σ-algebra $(R \otimes_K R)^\delta$. The choice of matrices $A \in K^{n \times n}$ and $Y \in \text{Gl}_n(L)$ such that $L|K$ is a σ-Picard–Vessiot extension for $\delta(y) = Ay$ with fundamental solution matrix Y defines a σ-closed embedding*

$$\sigma\text{-Gal}(L|K) \hookrightarrow \text{Gl}_{n,k}$$

of σ-algebraic groups.

In the notation of the proposition above, we will identify $\sigma\text{-Gal}(L|K)$ with its image in $\text{Gl}_{n,k}$. For S a k-σ-algebra and $\tau \in \sigma\text{-Gal}(L|K)(S)$, we will usually denote by $[\tau]_Y$ the image under the above morphism, i.e., the matrix $[\tau]_Y$ in $\text{Gl}_n(S)$ such that $\tau(Y \otimes 1) = (Y \otimes 1)[\tau]_Y$. Another choice of fundamental solution matrix yields a conjugated representation of $\sigma\text{-Gal}(L|K)$ in $\text{Gl}_{n,k}$. Therefore, sometimes, we will consider $\sigma\text{-Gal}(L|K)$ as a σ-closed subgroup of $\text{Gl}_{n,k}$ without mentioning the fundamental solution matrix Y.

The σ-dimension $\sigma\text{-dim}_k(G)$ of a σ-algebraic group G over a σ-field k can be defined by tracking the growth rate of the algebraic dimension of the Zariski closures of G (inside some suitable algebraic group). We refer to Appendix A.7 in [15] for a precise definition (cf. [57, Definition 3.6]). As one may expect, for a group k-scheme $\widetilde{G}$, we have $\dim_k(\widetilde{G}) = \sigma\text{-dim}_k([\sigma]_k\widetilde{G})$.

Proposition 2.9 (Proposition 2.17 in [15]). *Let $L|K$ be a σ-Picard–Vessiot extension with σ-Galois group G and constant field $k = K^\delta$. Then*

$$\sigma\text{-trdeg}(L|K) = \sigma\text{-dim}_k(G).$$

Proposition 2.10 (Proposition 2.15 in [15]). *Let $L|K$ be a σ-Picard–Vessiot extension with σ-field of δ-constants $k = K^\delta$. Let $A \in K^{n \times n}$ and $Y \in \text{Gl}_n(L)$ such that $L|K$ is a σ-Picard–Vessiot extension for $\delta(y) = Ay$ with fundamental solution matrix Y. We consider the σ-Galois group G of $L|K$ as a σ-closed subgroup of $\text{Gl}_{n,k}$ via the embedding associated with the choice of A and Y. Set $L_0 = K(Y) \subset L$.*

Then $L_0|K$ is a (classical) Picard–Vessiot extension for the linear system $\delta(y) = Ay$. The (classical) Galois group of $L_0|K$ is naturally isomorphic to $G[0]$, the Zariski closure of G inside $\text{Gl}_{n,k}$.

2.6. Galois correspondence

In the notation of Proposition 2.10, let S be a k-σ-algebra, $\tau \in G(S)$, and $a \in L$. By definition, τ is an automorphism of $R \otimes_k S$. If we write $a = \frac{r_1}{r_2}$ with $r_1, r_2 \in R$, $r_2 \neq 0$, then we say that a is invariant under τ if and only if $\tau(r_1 \otimes 1) \cdot r_2 \otimes 1 = r_1 \otimes 1 \cdot \tau(r_2 \otimes 1)$ in $R \otimes_k S$.

If H is a subfunctor of G, we say that $a \in L$ is invariant under H if a is invariant under every element of $H(S) \subset G(S)$, for every k-σ-algebra S. The set of all elements in L, invariant under H, is denoted with L^H. Obviously L^H is an intermediate $\delta\sigma$-field of $L|K$.

If M is an intermediate $\delta\sigma$-field of $L|K$, then it is immediately clear from Definition 2.4 that $L|M$ is a σ-Picard–Vessiot extension with σ-Picard–Vessiot ring MR, the ring

compositum of M and R inside L. There is a natural σ-closed embedding $\sigma\text{-Gal}(L|M) \hookrightarrow \sigma\text{-Gal}(L|K)$ whose image consists of precisely those automorphisms that leave invariant every element of M.

Theorem 2.11 (σ-Galois correspondence; Theorem 3.2 in [15]). *Let $L|K$ be a σ-Picard–Vessiot extension with σ-Galois group $G = \sigma\text{-Gal}(L|K)$. Then there is an inclusion reversing bijection between the set of intermediate $\delta\sigma$-fields M of $L|K$ and the set of σ-closed subgroups H of G given by*

$$M \mapsto \sigma\text{-Gal}(L|M) \quad and \quad H \mapsto L^H.$$

Theorem 2.12 (Second fundamental theorem of σ-Galois theory; Theorem 3.3 in [15]). *Let $L|K$ be a σ-Picard–Vessiot extension with σ-Galois group G. Let $K \subset M \subset L$ be an intermediate $\delta\sigma$-field and $H \leqslant G$ a σ-closed subgroup of G such that M and H correspond to each other in the σ-Galois correspondence.*

Then M is a σ-Picard–Vessiot extension of K if and only if H is normal in G. If this is the case, the σ-Galois group of $M|K$ is the quotient G/H. (See Definition A.41 and Theorem A.43 in [15] for the definition and the existence of the quotient G/H in the category of group k-σ-schemes.)

3. Transformally dependent solutions of first-order linear differential equations

In this section, we study the existence of algebraic relations over a $\delta\sigma$-field K, satisfied by solutions of a differential equation with coefficients in K and their transforms of all order with respect to σ. We show that these relations reflect the structure of the σ-Galois group of the equation. More precisely, we use the σ-Galois correspondence and the classification of the σ-closed subgroups of $\mathbf{G}_a, \mathbf{G}_m$ and Gl_2 (see the appendix) to characterize the transformal algebraic relations satisfied by solutions of differential equations.

We focus on two special cases: the field $\mathbb{C}(x)$ equipped with the usual derivation and with either the shift operator $\sigma : x \mapsto x + 1$ or a q-difference operator $\sigma : x \mapsto qx$, for some $q \in \mathbb{C}$. Notice that these two operators cover all the possible non-trivial automorphisms of $\mathbb{C}(x)$, up to a Möbius transformation.

Analogous results for differential relations among solutions of linear difference equations are proved in § 3.1 of [20].

3.1. The additive case

As above, let K be a $\delta\sigma$-field, with $\delta\sigma = \hbar\sigma\delta$ and $\hbar \in k := K^\delta$. We remind the reader that $\hbar_j \in k$ is defined so that $\delta\sigma^j = \hbar_j\sigma^j\delta$ for any integer $j \geqslant 1$. It is convenient to set $\hbar_0 = 1$.

We consider a system of first-order inhomogeneous linear differential equations

$$\delta(y_1) = b_1, \ldots, \delta(y_n) = b_n, \quad \text{with } b_1, \ldots, b_n \in K. \tag{3.1}$$

We are going to prove a result on the transformal dependence of a set of solutions of (3.1). It can be considered as a σ-analog of the following theorem by Ostrowski [40].

Theorem 3.1. *Let M be a field of meromorphic functions, containing $\mathbb{C}$ and stable by the derivation $\delta = \frac{d}{dx}$. Let $z_1, \ldots, z_n$ be meromorphic functions such that $\delta(z_i) \in M$, for all $i = 1, \ldots, n$. Then $z_1, \ldots, z_n$ are algebraically dependent over M if and only if there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, not all equal to zero, such that $\sum_{i=1}^{n} \lambda_i z_i \in M$.*

The theorem below relies on the classification of σ-closed subgroups of $\mathbf{G}_a^n$ (see § A.1 below).

Theorem 3.2. *Let $L|K$ be a $\delta\sigma$-field extension containing a solution $z_1, \ldots, z_n \in L$ of (3.1) and having the property that $L^\delta = k(:= K^\delta)$. Then $z_1, \ldots, z_n$ are transformally dependent over K if and only if there exist a non-zero homogeneous linear σ-polynomial $\mathcal{L}(X_1, \ldots, X_n) \in k\{X_1, \ldots, X_n\}_\sigma$ and an element g of K such that $\mathcal{L}(b_1, \ldots, b_n) = \delta(g)$.*

Remark 3.3. Since $L^\delta = K^\delta$ and $\delta\sigma = \hbar\sigma\delta$, one can show that there exist a non-zero homogeneous linear σ-polynomial $\mathcal{L}(X_1, \ldots, X_n) \in k\{X_1, \ldots, X_n\}_\sigma$ and an element g of K such that $\mathcal{L}(b_1, \ldots, b_n) = \delta(g)$ if and only if there exists a non-zero homogeneous linear σ-polynomial $\widetilde{\mathcal{L}}(X_1, \ldots, X_n) \in k\{X_1, \ldots, X_n\}_\sigma$ such that $\widetilde{\mathcal{L}}(z_1, \ldots, z_n) \in K$ (We give a detailed proof of an analogous fact in Remark 3.11). This remark emphasizes the analogy between our result and Ostrowski's theorem.

Proof. Let us first assume that there exist $\mathcal{L}(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=0}^{s} \lambda_{i,j} \sigma^j(X_i) \in k\{X_1, \ldots, X_n\}_\sigma$ and $g \in K$ such that

$$\sum_{i=1}^{n} \sum_{j=0}^{s} \lambda_{i,j} \sigma^j(b_i) = \delta(g). \tag{3.2}$$

Then a direct calculation shows that

$$\delta\left(\sum_{i,j} \hbar_j^{-1} \lambda_{i,j} \sigma^j(z_i) - g\right) = 0.$$

Since $L^\delta = K^\delta$, we conclude that $\sum_{i,j} \hbar_j^{-1} \lambda_{i,j} \sigma^j(z_i) \in K$. Hence $z_1, \ldots, z_n$ are transformally dependent over K.

To prove the inverse implication, we consider the differential system

$$\delta(y) = Ay, \quad \text{where } A = \text{diag}\left(\begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & b_n \\ 0 & 0 \end{pmatrix}\right)$$

$$\in K^{2n \times 2n} \text{ is a diagonal block matrix,} \tag{3.3}$$

and its fundamental solution $Y := \text{diag}\left(\begin{pmatrix} 1 & z_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & z_n \\ 0 & 1 \end{pmatrix}\right) \in \text{Gl}_{2n}(L)$.

First of all, we construct the σ-Galois group of $\delta(y) = Ay$. Since $L^\delta = K^\delta$, the $\delta\sigma$-field $K\langle z_1, \ldots, z_n \rangle_\sigma \subset L$ is a σ-Picard–Vessiot extension for (3.3), and it is not restrictive to assume that $L = K\langle z_1, \ldots, z_n \rangle_\sigma$. The corresponding σ-Picard–Vessiot ring is $R = K\{z_1, \ldots, z_n\}_\sigma$. So let $G = \sigma\text{-Gal}(L|K)$ denote the σ-Galois group of $L|K$, seen as a σ-closed subgroup of $\text{Gl}_{2n,k}$ via the representation associated to Y.

We claim that G is a proper σ-closed subgroup of $\mathbf{G}_a^n$, where we have identified the vector group $\mathbf{G}_a^n$ with a σ-closed subgroup of $\mathrm{Gl}_{2n,k}$ via the embedding $\mathbf{G}_a^n \hookrightarrow \mathrm{Gl}_{2n,k}$ defined by

$$\begin{aligned}\mathbf{G}_a^n(S)(&= S^n) \longrightarrow \mathrm{Gl}_{2n}(S) \\ (c_1, \ldots, c_n) &\longmapsto \mathrm{diag}\left(\begin{pmatrix} 1 & c_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & c_n \\ 0 & 1 \end{pmatrix}\right),\end{aligned} \quad \text{for any } k\text{-}\sigma\text{-algebra } S.$$

Indeed, for any k-σ-algebra S and $\tau \in G(S)$, the automorphism τ commutes with δ, so that in $R \otimes_k S$ we have $\delta(\tau(z_i \otimes 1)) = b_i \otimes 1$, for all $i = 1, \ldots, n$, and hence $\delta(\tau(z_i \otimes 1) - z_i \otimes 1) = 0$. So $\tau(z_i \otimes 1) = z_i \otimes 1 + 1 \otimes c_i$ for some $c_i \in S$, which identifies G to a σ-closed subgroup of $\mathbf{G}_a^n$. Since $z_1, \ldots, z_n$ are transformally dependent over K; i.e., since $\sigma\text{-trdeg}(L|K) < n$, Proposition 2.9 implies that

$$\sigma\text{-}\dim_k(G) = \sigma\text{-trdeg}(L|K) < n = \sigma\text{-}\dim_k(\mathbf{G}_a^n),$$

and therefore that G is a proper σ-closed subgroup of $\mathbf{G}_a^n$.

The group G being a proper σ-closed subgroup of $\mathbf{G}_a^n$, there exists a non-zero linear σ-polynomial in n variables, say $\sum_{i,j} \mu_{i,j} \sigma^j(X_i) \in k\{X_1, \ldots, X_n\}_\sigma$ (see Theorem A.1 below), which belongs to the vanishing ideal of G inside $\mathbf{G}_a^n$. This means that, for any k-σ-algebra S and any $\tau = (c_1, \ldots, c_n) \in G(S)$, we must have $\sum_{i,j} \mu_{i,j} \sigma^j(c_i) = 0$. We set $g = \sum_{i,j} \mu_{i,j} \sigma^j(z_i) \in R$. Then, for any k-σ-algebra S and any $\tau = (c_1, \ldots, c_n) \in G(S)$, in $R \otimes_k S$ we have

$$\tau(g \otimes 1) = \sum_{i,j} \mu_{i,j} \sigma^j(z_i \otimes 1 + 1 \otimes c_i) = g \otimes 1.$$

The Galois correspondence (see Theorem 2.11 above) implies that $g \in K$. Taking the derivative of g, we find that $\sum_{i,j} \hbar_j \mu_{i,j} \sigma^j(b_i) = \delta(g)$. To conclude, it is enough to choose $\mathcal{L}(X_1, \ldots, X_n) = \sum_{i,j} \hbar_j \mu_{i,j} \sigma^j(X_i)$. $\qquad\qquad\square$

Remark 3.4. Theorem 3.2 can be deduced from a more algebraic version of Ostrowski's theorem, involving instead of differential fields of meromorphic functions a differential field extension L of K with $L^\delta = k$ (see § 2 in [28]). If L is a $\delta\sigma$-field extension of K, with $L^\delta = k$, containing solutions $z_1, \ldots, z_n$ of $\delta(y_i) = b_i$, it is enough to apply Ostrowski's theorem to $z_1, \ldots, z_n, \sigma(z_1), \ldots, \sigma(z_n), \ldots, \sigma^l(z_1), \ldots, \sigma^l(z_n)$, for some large enough integer l, to get Theorem 3.2. However, we want to point out that the existence of such a $\delta\sigma$-field extension L is a very strong hypothesis, which is a priori not guaranteed by classical Picard–Vessiot theory. Indeed, for the differential equations

$$\delta(y_{i,j}) = \hbar_j \sigma^j(b_i) y_{i,j} \tag{3.4}$$

with $i = 1, \ldots, n$ and $j = 1, \ldots, l$, classical Picard–Vessiot theory provide us with a δ-field L with $L^\delta = k$ containing the solutions of (3.4). However, nothing guarantees that L is also a σ-field. The strength of our approach is to show that, if one assumes k to be algebraically closed, one can solve any linear differential equation in a $\delta\sigma$-field L with $L^\delta = k$ (see Proposition 2.5). Moreover, we also prefer to give here a proof of Theorem 3.2 relying on the classification of the σ-closed subgroups of $\mathbf{G}_a$ and parameterized Galois correspondence, in preparation of the more complicated applications to come.

In the special case $K = \mathbb{C}(x)$, we consider the following two situations:

$$\delta = \frac{d}{dx} \quad \text{and} \quad \sigma(x) = x + 1; \tag{3.5}$$

$$\delta = \frac{d}{dx} \quad \text{and} \quad \sigma(x) = qx, \quad \text{where } q \in \mathbb{C} \setminus \{0, 1, \text{roots of unity}\}. \tag{3.6}$$

Notice that, in the first case, σ and δ commute, so $\hbar = 1$, while in the second case we have $\delta\sigma = q\sigma\delta$, and therefore $\hbar = q$. In both cases, $\hbar_j = \hbar^j$, for any integer $j \geqslant 0$.

Example 3.5. We consider the situation (3.5). Let L be the field of meromorphic function over $\mathbb{C} \setminus \mathbb{R}_{\leqslant 0}$, and let $z = \log(x) \in L$ be the principal branch of the logarithm. Since $L^\delta = \mathbb{C}$ and $\delta(z) = \frac{1}{x}$, the assumptions of Theorem 3.2 are verified. If z was transformally dependent over $\mathbb{C}(x)$, then there would exist a non-zero homogeneous linear σ-polynomial $\mathcal{L}(X) = \sum_{j=0}^{s} \lambda_j \sigma^j(X) \in \mathbb{C}\{X\}_\sigma$ and $g \in \mathbb{C}(x)$, such that

$$\sum_{j=0}^{s} \frac{\lambda_j}{x+j} = \delta(g).$$

Writing the fractional expansion of $\delta(g)$, we see that $\delta(g)$ can never satisfy such a relation. This means that $\log(x), \log(x+1), \ldots, \log(x+n), \ldots$ are algebraically independent over $\mathbb{C}(x)$.

This example illustrates the following more general criterion.

Corollary 3.6. *Let $b \in \mathbb{C}(x)$, and let $L|\mathbb{C}(x)$ be a $\delta\sigma$-field extension, with $L^\delta = \mathbb{C}$, containing a solution z of $\delta(y) = b$. Then, in the situation (3.5) (respectively, (3.6)), z is transformally dependent over $\mathbb{C}(x)$ if and only if b has no simple poles (respectively, b has no non-zero simple poles).*

Proof. It follows from Theorem 3.2 that z is transformally dependent over $\mathbb{C}(x)$ if and only if there exists a non-zero homogeneous linear σ-polynomial $\mathcal{L}(X) = \sum_{j=0}^{s} \lambda_j \sigma^j(X) \in \mathbb{C}\{X\}_\sigma$ and $g \in \mathbb{C}(x)$, such that $\mathcal{L}(b) = \delta(g)$. Writing the fractional expansion of $\delta(g)$, we find that z is transformally dependent over $\mathbb{C}(x)$ if and only if there exists a non-zero homogeneous linear σ-polynomial $\mathcal{L}(X) \in \mathbb{C}\{X\}_\sigma$, such that $\mathcal{L}(b)$ has no simple poles.

We consider the case (3.5). We claim that b has no simple poles if and only if there exists a non-zero homogeneous linear σ-polynomial $\mathcal{L}(X) = \sum_{j=0}^{s} \lambda_j \sigma^j(X) \in \mathbb{C}\{X\}_\sigma$ such that $\mathcal{L}(b)$ has no simple poles. In fact, one implication is trivial. So let us assume that such an $\mathcal{L}$ exists and that γ is a simple pole of b. Then, let n be the largest integer such that $\gamma - n$ is a simple pole of b. Then $\gamma - n - s$ is a simple pole of $\sum_{j=0}^{s} \lambda_j \sigma^j(b)$, contradicting the assumption.

In the situation (3.6), we are going to prove that b has no non-zero simple poles if and only if there exists a non-zero homogeneous linear σ-polynomial $\mathcal{L}(X) = \sum_{j=0}^{s} \lambda_j \sigma^j(X) \in \mathbb{C}\{X\}_\sigma$ such that $\mathcal{L}(b)$ has no simple poles. If b has no non-zero simple poles, then we can write b as the sum $\frac{c}{x} + g$, where $c \in \mathbb{C}$ and $g \in \mathbb{C}(x)$ has no simple poles. It follows that $q\sigma(b) - b = qg(qx) - g$ has no simple poles. The same reasoning as in case (3.5) proves the inverse implication. $\qquad\square$

The above corollary allows us to describe precisely which σ-closed subgroups of $\mathbf{G}_a$ occur as σ-Galois groups of an equation $\delta(y) = b$ in the situations (3.5) and (3.6), anticipating the content of § 4.

Corollary 3.7. *In the situation (3.5), let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = b$ with $b \in K = \mathbb{C}(x)$, and let G be the σ-Galois group of $L|K$.*

(i) *If b has no simple poles then G is trivial; i.e., $L = K$.*

(ii) *If b has a simple pole then $G = \mathbf{G}_a$.*

Proof. Since the proof is quite similar to the proof of the q-difference case (see Corollary 3.8 below), we refer to the q-difference case below for the proof.   □

Corollary 3.8. *In the situation (3.6), let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = b$ with $b \in K = \mathbb{C}(x)$, and let G be the σ-Galois group of $L|K$.*

(i) *If b has no simple poles then G is trivial.*

(ii) *If b has a simple pole at zero but no other simple poles then $G = \mathbf{G}_a^\sigma$; i.e., $G(S) = \{c \in S|\ \sigma(c) = c\} \leqslant \mathbf{G}_a(S)$ for every k-σ-algebra S.*

(iii) *If b has a simple pole distinct from zero then $G = \mathbf{G}_a$.*

Proof. Let $z \in L$ be a solution of $\delta(y) = b$. Assume that b has no simple poles. Then there exists a solution of $\delta(y) = b$ in K. Therefore $z \in K$ and $L = K\langle z \rangle_\sigma = K$. This proves (i).

If b has a simple pole in zero but no other simple poles, $q\sigma(b) - b = \delta(h)$ for some $h \in K$. Then $\delta(\sigma(z) - z) = \delta(h)$, which implies that $\sigma(z) - z \in K$. We identify G with a σ-closed subgroup of $\mathbf{G}_a$ via its action on z. Then, for any k-σ-algebra S and $\tau \in G(S)$, there exists $c_\tau \in S$ such that $\tau(z \otimes 1) = z \otimes 1 + 1 \otimes c_\tau$. Since $\sigma(z) - z \in K$, the element $\sigma(z \otimes 1) - z \otimes 1$ is left invariant by τ. This is equivalent to saying that $\sigma(c_\tau) = c_\tau$. It follows that $G \leqslant \mathbf{G}_a^\sigma$. Since $z \notin K$, we must have $G = \mathbf{G}_a^\sigma$ (as $\mathbf{G}_a^\sigma$ has no non-trivial σ-closed subgroups). This proves (ii).

If b has a simple pole distinct from zero, then z is transformally independent over K by Corollary 3.6. Therefore $G = \mathbf{G}_a$ in this case.   □

3.2. The multiplicative case

Inspired by Ostrowski's theorem 3.1 above, Kolchin proved the following statement (see [28], p. 1156).

Theorem 3.9. *Let $L|K$ be a δ-field extension, with $L^\delta = K^\delta$, and assume that $z_1, \ldots, z_n \in L^\times$ satisfy $\frac{\delta(z_1)}{z_1}, \ldots, \frac{\delta(z_n)}{z_n} \in K^\times$. Then $z_1, \ldots, z_n$ are algebraically dependent over K if and only if there exist $r_1, \ldots, r_n \in \mathbb{Z}$, not all equal to zero, such that $z_1^{r_1} \cdots z_n^{r_n} \in K$.*

We are going to prove a σ-analog of Kolchin's theorem.

Theorem 3.10. *Let $L|K$ be a $\delta\sigma$-field extension with $L^\delta = K^\delta$. Let $a_1, \ldots, a_n \in K^\times$ and $z_1, \ldots, z_n \in L^\times$ be such that*

$$\delta(z_i) = a_i z_i \quad for\ i = 1, \ldots, n.$$

Then $z_1, \ldots, z_n$ are transformally dependent over K if and only if there exist a non-zero homogeneous linear σ-polynomial $\mathcal{L}(X_1, \ldots, X_n) = \sum_{i=1}^{n} \sum_{j=0}^{s} r_{i,j} \sigma^j(X_i) \in \mathbb{Z}\{X_1, \ldots, X_n\}_\sigma$ and a non-zero element $g \in K$ such that

$$\sum_{i=1}^{n} \sum_{j=0}^{s} r_{i,j} \hbar_j \sigma^j(a_i) = \frac{\delta(g)}{g}. \tag{3.7}$$

Remark 3.11. During our proof of Theorem 3.10, we are going to show that (3.7) is equivalent to $\prod_{i,j} \sigma^j(z_i)^{r_{i,j}} \in K$, which makes clear that the theorem above is the analog of Kolchin's result. This equivalence only relies on the fact that L contains no new constants.

Proof. Assume that (3.7) is satisfied. Then the element

$$z := \frac{1}{g} \prod_{i,j} \sigma^j(z_i)^{r_{i,j}} \in L^\times$$

satisfies $\frac{\delta(z)}{z} = 0$. Since $L^\delta = K^\delta$, we have $z \in K$. Hence, we conclude that $z_1, \ldots, z_n$ are transformally dependent over K.

Conversely, assume that $z_1, \ldots, z_n$ are transformally dependent over K. The proof is quite similar to the proof of Theorem 3.2, and therefore we will skip some details. Since $L^\delta = k(:= K^\delta)$, the $\delta\sigma$-field $K\langle z_1, \ldots, z_n\rangle_\sigma$ is a σ-Picard–Vessiot extension for the differential system $\delta(y) = Ay$, where $A = \mathrm{diag}(a_1, \ldots, a_n)$. We can assume that $L = K\langle z_1, \ldots, z_n\rangle_\sigma$. Let $G = \sigma\text{-Gal}(L|K)$. Via the fundamental solution matrix $Y := \mathrm{diag}(z_1, \ldots, z_n)$, the σ-Galois group G is naturally a σ-closed subgroup of $\mathbf{G}_m^n$. Now, since $z_1, \ldots, z_n$ are transformally dependent over K, we find that $\sigma\text{-trdeg}(L|K) < n$. Proposition 2.9 allows us to conclude that

$$\sigma\text{-}\dim_k(G) = \sigma\text{-trdeg}(L|K) < n = \sigma\text{-}\dim_k(\mathbf{G}_m^n),$$

and hence that G is a proper σ-closed subgroup of $\mathbf{G}_m^n$. It follows from Theorem A.7 that there exists a non-trivial multiplicative function $\psi \in k\{\mathbf{G}_m^n\} = k\{x_1, \ldots, x_n, \frac{1}{x_1}, \ldots, \frac{1}{x_n}\}_\sigma$, of the form

$$\psi(x_1, \ldots, x_n) = x_1^{r_{1,0}} \cdots x_n^{r_{n,0}} \sigma(x_1)^{r_{1,1}} \cdots \sigma(x_n)^{r_{n,1}} \cdots \sigma^s(x_1)^{r_{1,s}} \cdots \sigma^s(x_n)^{r_{n,s}}, \tag{3.8}$$

where $s \in \mathbb{Z}_{\geqslant 0}$ and $r_{i,j} \in \mathbb{Z}$, for $1 \leqslant i \leqslant n$, $0 \leqslant j \leqslant s$, such that $\psi(c_1, \ldots, c_n) = 1$, for all k-σ-algebra S and all $\tau = (c_1, \ldots, c_n) \in G(S) \leqslant \mathbf{G}_m^n(S)$. Consider the non-zero element $g := \psi(z_1, \ldots, z_n) = \prod_{i,j} \sigma^j(z_i)^{r_{i,j}}$ of L. For any k-σ-algebra S and any $\tau = (c_1, \ldots, c_n) \in G(S) \subset \mathbf{G}_m^n(S)$, in the $\delta\sigma$-algebra $R \otimes_k S$, we have

$$\tau(g \otimes 1) = \psi(z_1 \otimes c_1, \ldots, z_n \otimes c_n) = \psi(z_1, \ldots, z_n) \otimes \psi(\tau) = g \otimes 1.$$

It follows from the Galois correspondence (see Theorem 2.11 above) that $g \in K$. Taking the logarithmic derivative of g, we find that

$$\frac{\delta(g)}{g} = \frac{\delta\left(\prod_{i,j} \sigma^j(z_i^{r_{i,j}})\right)}{\prod_{i,j} \sigma^j(z_i^{r_{i,j}})} = \sum_{i,j} r_{i,j} \hbar_j \sigma^j\left(\frac{\delta(z_i)}{z_i}\right) = \sum_{i,j} r_{i,j} \hbar_j \sigma^j(a_i). \qquad \square$$

As in the previous section, we are going to give a more detailed description of the case $K = \mathbb{C}(x)$, under assumptions (3.5) and (3.6). It is an elementary observation that a rational function $g \in \mathbb{C}(x)$ is the logarithmic derivative of a rational function if and only if it is of the form $g = \sum_{i=1}^{k} \frac{c_i}{x-\alpha_i}$ with $c_1 \ldots c_k \in \mathbb{Z}$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. Therefore, the following corollaries yield a very direct and easy to use criterion to decide whether or not a meromorphic solution of $\delta(y) = ay$, with $a \in \mathbb{C}(x)^{\times}$, is transformally dependent.

Corollary 3.12. *In the situation (3.5), let $L|\mathbb{C}(x)$ be a $\delta\sigma$-field extension, with $L^{\delta} = \mathbb{C}$, and assume that $z \in L^{\times}$ satisfies $\delta(z) = az$, with $a \in \mathbb{C}(x)^{\times}$. Then z is transformally dependent over $\mathbb{C}(x)$ if and only if there exist $P \in \mathbb{C}[x]$, $f \in \mathbb{C}(x)^{\times}$ and $N \in \mathbb{Z}^{\times}$ such that $a = P + \frac{1}{N} \frac{\delta(f)}{f}$.*

Proof. We remind the reader that Theorem 3.10 implies that the solution z is transformally dependent over $\mathbb{C}(x)$ if and only if there exist a non-zero homogenous σ-polynomial $\mathcal{L}(X) = \sum_{j=0}^{s} r_j \sigma^j(X) \in \mathbb{Z}\{X\}_\sigma$ and $g \in \mathbb{C}(x)^{\times}$ such that

$$\sum_{j=0}^{s} r_j \sigma^j(a) = \frac{\delta(g)}{g}. \tag{3.9}$$

Notice that the rational function $\frac{\delta(g)}{g}$ has only simple poles and that the residue at each pole is an integer.

We claim that if the solution z is transformally dependent over $\mathbb{C}(x)$ then a has no poles of order greater than or equal to 2; i.e., the rational function a has the form $P + \sum_{h=1}^{k} \frac{d_h}{x-\gamma_h}$, for some $P \in \mathbb{C}[x]$ and $\gamma_h, d_h \in \mathbb{C}$. Suppose that γ is a pole of maximal order $m \geqslant 2$ of a and that $\gamma - n$ is not a pole of order m of a for any positive integer n. Notice that $\gamma - j$ is a pole of order m of $\sigma^j(a)$, and hence $\gamma - s$ is a pole of order m of $\sum_{j=0}^{s} r_j \sigma^j(a) = \frac{\delta(g)}{g}$, which is impossible, proving the claim.

We have proven that $a = P + \sum_{h=1}^{k} \frac{d_h}{x-\gamma_h}$. To conclude that there exist $N \in \mathbb{Z}^{\times}$ and $f \in \mathbb{C}(x)^{\times}$ such that $\sum_{h=1}^{k} \frac{d_h}{x-\gamma_h} = \frac{1}{N} \frac{\delta(f)}{f}$, it suffices to show that $d_h \in \mathbb{Q}$, for all $h = 1, \ldots, k$. To this purpose, let us consider a set Γ of representatives of the classes of the poles of a modulo $\mathbb{Z}$, and write $\sum_{h=1}^{k} \frac{d_h}{x-\gamma_h}$ as

$$\sum_{\gamma \in \Gamma} \sum_{n \in \mathbb{Z}} \frac{d_n(\gamma)}{x - (\gamma - n)}. \tag{3.10}$$

Only a finite number of $d_n(\gamma) \in \mathbb{C}$ in the expression above are non-zero. We know from (3.9) that $\sum_{j=0}^{s} r_j \sigma^j(a)$ is the logarithmic derivative of a rational function, and hence

$$\sum_{j=0}^{s} r_j \sigma^j \left(\sum_{n \in \mathbb{Z}} \frac{d_n(\gamma)}{x - (\gamma - n)} \right) = \sum_{n \in \mathbb{Z}} \frac{\sum_{j=0}^{s} d_{n-j}(\gamma) r_j}{x - (\gamma - n)}, \quad \text{for any } \gamma \in \Gamma,$$

is also the logarithmic derivative of a rational fraction. Thus, for any $\gamma \in \Gamma$ and all $n \in \mathbb{Z}$ we have

$$\sum_{j=0}^{s} d_{n-j}(\gamma) r_j \in \mathbb{Z}. \tag{3.11}$$

For a fixed $\gamma \in \Gamma$, we denote by n_0 the smallest integer such that $d_{n_0}(\gamma) \neq 0$ and by j_0 the smallest integer such that $r_{j_0} \neq 0$. The linear relation (3.11) for $n = n_0 + j_0$ implies that $d_{n_0}(\gamma) r_{j_0} \in \mathbb{Z}$ and hence that $d_{n_0}(\gamma) \in \mathbb{Q}$. We assume recursively that $d_k(\gamma) \in \mathbb{Q}$ for all $n_0 \leqslant k \leqslant p$. Expression (3.11) for $n = p + 1 + j_0$ yields

$$d_{p+1}(\gamma) r_{j_0} + \sum_{j=1}^{s-j_0} d_{p+1-j}(\gamma) r_{j+j_0} \in \mathbb{Z},$$

which implies that $d_{p+1}(\gamma) r_{j_0} \in \mathbb{Z}$ and hence that $d_{p+1}(\gamma) \in \mathbb{Q}$. We have proved the forward direction of the corollary.

Conversely, let us assume that there exist $P \in \mathbb{C}[x]$, $f \in \mathbb{C}(x)^{\times}$ and $N \in \mathbb{Z}^{\times}$, such that $a = P + \frac{1}{N} \frac{\delta(f)}{f}$. Let d be the degree of P, and let us denote by $\mathbb{Q}[x]_{\leqslant d}$ the $\mathbb{Q}$-vector space of polynomials of degree less than or equal to d. It is a $\mathbb{Q}$-vector space of finite dimension, on which $\sigma : \mathbb{Q}[x]_{\leqslant d} \to \mathbb{Q}[x]_{\leqslant d}$, $f(x) \mapsto f(x+1)$ induces a $\mathbb{Q}$-linear bijection. By the Cayley–Hamilton theorem the characteristic polynomial of σ annihilates σ. So there exists a polynomial $\mu(X) = \sum_{j=0}^{s} r_j X^j \in \mathbb{Z}[X]$ with $r_s \neq 0$ such that $\sum_{j=0}^{s} r_j \sigma^j = 0$ on $\mathbb{Q}[x]_{\leqslant d}$. Thus $\sum_{j=0}^{s} r_j \sigma^j$ annihilates every monomial of P, and therefore $\sum_{j=0}^{s} r_j \sigma^j(P) = 0$. The following calculation allows us to conclude the proof:

$$N \sum_{j=0}^{s} r_j \sigma^j(a) = \sum_{j=0}^{s} r_j \frac{\delta(\sigma^j(f))}{\sigma^j(f)} = \frac{\delta\left(\prod_{j=0}^{s} \sigma^j(f^{r_j})\right)}{\prod_{j=0}^{s} \sigma^j(f^{r_j})}. \qquad \square$$

Case (3.6) is slightly more complicated than the previous one, since one has to distinguish the case when q is an algebraic or a transcendental number. We start with an example illustrating the problem.

Example 3.13. Consider the equation $\delta(y) = y$, with the notation (3.6). Since $\delta \circ \sigma = q\sigma \circ \delta$, Theorem 3.10 above says that $\exp(q^j x)$, for $j \geqslant 0$, are algebraically dependent if and only if there exist a non-zero polynomial $\sum_{j=0}^{s} r_j X^j \in \mathbb{Z}[X]$ and $g \in \mathbb{C}(x)$ such that $\sum_{j=0}^{s} q^j r_j = \delta(g)g^{-1}$. The latter holds if and only if $\delta(g)g^{-1} \in \mathbb{C}$, and therefore if and only if $\sum_{j=0}^{s} q^j r_j = 0$. We deduce that $\exp(q^j x)$, for $j \geqslant 0$, are algebraically dependent if and only if q is an algebraic number.

In general, we have the following.

Corollary 3.14. *In the case of (3.6), let $a \in \mathbb{C}(x)^{\times}$, and let $L|\mathbb{C}(x)$ be a $\delta\sigma$-field extension with $L^{\delta} = \mathbb{C}$, containing a non-zero solution z of $\delta(y) = ay$. We have the following.*

- *If q is a transcendental number then z is transformally dependent over $\mathbb{C}(x)$ if and only if there exist $f \in \mathbb{C}(x)^{\times}$, $N \in \mathbb{Z}^{\times}$, and $c \in \mathbb{C}$ such that $a = \frac{c}{x} + \frac{1}{N} \frac{\delta(f)}{f}$.*
- *If q is an algebraic number then z is transformally dependent over $\mathbb{C}(x)$ if and only if there exist $f \in \mathbb{C}(x)^{\times}$, $N \in \mathbb{Z}^{\times}$, $P \in \mathbb{C}[x]$, and $Q \in \mathbb{C}[\frac{1}{x}]$ such that $a = P + Q + \frac{1}{N} \frac{\delta(f)}{f}$.*

Proof. As in the previous corollary, Theorem 3.10 implies that the solution z is transformally dependent over $\mathbb{C}(x)$ if and only if there exist a non-zero homogeneous

linear σ-polynomial $\mathcal{L}(X) = \sum_{j=0}^{s} r_j \sigma^j(X) \in \mathbb{Z}\{X\}_\sigma$ and a non-zero element $g \in K$ such that

$$\sum_{j=0}^{s} q^j r_j \sigma^j(a) = \frac{\delta(g)}{g}. \tag{3.12}$$

Once more, notice that the rational function $\frac{\delta(g)}{g}$ has only simple poles and that the residue at each pole is an integer. Moreover, the fractional expansion of a is the sum of $P(x) \in \mathbb{C}[x]$ and some polar terms of order greater than or equal to 1.

We suppose that z is transformally dependent over $\mathbb{C}(x)$. Then the non-zero poles of a must be of order 1. In fact, suppose that there exists a pole $\gamma \neq 0$ of a, of order $m \geqslant 2$. We can choose γ such that $q^{-n}\gamma$ is not a pole of a of order m, for any integer $n \geqslant 1$. Then $\sum_{j=0}^{s} r_j q^j \sigma^j(a)$ has a pole of order m in $q^{-s}\gamma$, which is impossible because of (3.12). This implies that

$$a = P + Q + \sum_{h=1}^{k} \frac{d_h}{x - \gamma_h},$$

with $P \in \mathbb{C}[x]$, $Q \in \mathbb{C}[\frac{1}{x}]$, $\gamma_h \in \mathbb{C}^\times$, and $d_h \in \mathbb{C}$. A reasoning analogous to the one in Corollary 3.12 shows that $d_h \in \mathbb{Q}$ for $h = 1, \ldots, k$. So we have proved that $a = P + Q + \frac{1}{N}\frac{\delta(f)}{f}$, for some $f \in \mathbb{C}(x)^\times$ and $N \in \mathbb{Z}^\times$. If q is algebraic there is nothing more to prove. If q is transcendental, we have to show that there exists $c \in \mathbb{C}$ such that $P + Q = \frac{c}{x}$. If not, there exists $n \in \mathbb{Z}$ not equal to -1 such that x^n appears in $P + Q$. Tracking the coefficient of x^n in (3.12), we find that $\sum_{j=0}^{s} q^{j(1+n)} r_j = 0$, which contradicts the assumption on q. We have now proved the forward direction of the corollary both under the assumption that q is algebraic and that q is transcendental.

Now we assume that q is transcendental and that $a = \frac{c}{x} + \frac{1}{N}\frac{\delta(f)}{f}$, for some $c \in \mathbb{C}$, $N \in \mathbb{Z}^\times$ and $f \in \mathbb{C}(x)^\times$. Then

$$Nq\sigma(a) - Na = \frac{\delta(\sigma(f)/f)}{\sigma(f)/f},$$

which, by Theorem 3.10, implies that z is transformally dependent over $\mathbb{C}(x)$.

Finally, let q be an algebraic number, and suppose that there exist $f \in \mathbb{C}(x)^\times$, $N \in \mathbb{Z}^\times$, $P \in \mathbb{C}[x]$, and $Q \in \mathbb{C}[\frac{1}{x}]$ such that $a = P + Q + \frac{1}{N}\frac{\delta(f)}{f}$. Let d denote the maximum of the degree of P in x and of the degree of Q in $\frac{1}{x}$ and

$$\mathbb{Q}(q)[x^{\pm 1}]_{\leqslant d} := \mathbb{Q}(q) + \sum_{i=1}^{d} \mathbb{Q}(q)x^i + \sum_{i=1}^{d} \mathbb{Q}(q)\frac{1}{x^i}.$$

The set $\mathbb{Q}(q)[x^{\pm 1}]_{\leqslant d}$ is a finite-dimensional $\mathbb{Q}$-vector space, and the $\mathbb{Q}(q)$-linear map $q\sigma : f(x) \mapsto qf(qx)$ is a $\mathbb{Q}$-linear endomorphism of $\mathbb{Q}(q)[x^{\pm 1}]_{\leqslant d}$. By the Cayley–Hamilton theorem the characteristic polynomial of $q\sigma$ annihilates $q\sigma$. So there exists a polynomial $\mu(X) = \sum_{j=0}^{s} r_j X^j \in \mathbb{Z}[X]$ with $r_s \neq 0$ such that $\sum_{j=0}^{s} r_j q^j \sigma^j = 0$ on $\mathbb{Q}(q)[x^{\pm 1}]_{\leqslant d}$. Thus $\sum_{j=0}^{s} r_j q^j \sigma^j$ annihilates every monomial of $P + Q$, and therefore $\sum_{j=0}^{s} r_j q^j \sigma^j(P + Q) = 0$. So $\sum_{j=0}^{s} Nr_j q^j \sigma^j(a) = \frac{\delta(\prod_{j=0}^{s} \sigma^j(f^{r_j}))}{\prod_{j=0}^{s} \sigma^j(f^{r_j})}$. $\qquad\square$

3.3. An analog of a result of Ishizaki

In 1998, Ishizaki proved a result of differential independence of meromorphic solutions of order-1 linear inhomogeneous q-difference equations (see Theorem 1.2 in [23]).

Theorem 3.15. *Let $q \in \mathbb{C}$, with $|q| \neq 1$, and let $z(x)$ be a meromorphic function over $\mathbb{C}$, not in $\mathbb{C}(x)$, such that $z(qx) = a(x)z(x) + b(x)$, for some non-zero $a(x), b(x) \in \mathbb{C}(x)$. Then $z(x)$ is differentially transcendental over $\mathbb{C}(x)$; i.e., the meromorphic function $z(x)$ and all its derivatives are algebraically independent over $\mathbb{C}(x)$.*

A Galoisian proof of Ishizaki's result is given in Proposition 3.5 in [20]. In this section we are going to prove a similar result on transformal independence of solutions of differential equations of the form $\delta(y) = ay + b$. The ingredients of our proof are also difference analogs of the ones used in [20]. For a general difference operator σ, Propositions 3.17 and 3.20 give partial answers to the difference version of Ishizaki's result, whereas in the case of a q-difference operator with transcendental q, Corollary 3.21 together with Proposition 3.17 gives a complete solution.

First we prove a lemma which does not involve σ.

Lemma 3.16. *Let $L|K$ be an extension of δ-fields, and assume that $v \in L$ satisfies $\delta(v) = av$, for some $a \in K$. If the equation $\delta(y) - cy = b$, for $b, c \in K$, has a solution in $K(v)$, then it already has a solution in K.*

Proof. First assume that v is transcendental over K. Let $g \in K(v)$ be a solution of $\delta(y) - cy = b$, and let us write $g = \frac{P}{Q} + R$, where $P, Q, R \in K[v]$, $Q \neq 0$, and $\deg_v(P) < \deg_v(Q)$. Then $b = \delta(g) - cg = \frac{\widetilde{P}}{\widetilde{Q}} + \delta(R) - cR$, where $\widetilde{P}, \widetilde{Q} \in K[v]$ and $\deg_v(\widetilde{P}) < \deg_v(\widetilde{Q})$. Comparing the degree in v, we find that $\delta(R) - cR = b$. This last equality implies that the constant term $r_0 \in K$ of $R \in K[v]$, as a polynomial in v, satisfies $\delta(r_0) - cr_0 = b$.

Now assume that v is algebraic over K. Let N denote the normal closure of $K(v)|K$. The derivation $\delta \colon K(v) \to K(v)$ extends uniquely to a derivation $\delta \colon N \to N$. Moreover, $\delta \colon N \to N$ commutes with every automorphism of $N|K$. The operator $\mathcal{R} \colon N \to K$ given by $\mathcal{R}(d) = \frac{1}{[N:K]} \sum_\tau \tau(d)$, where τ ranges over all elements in the Galois group of $N|K$, is K-linear, the identity on K, and commutes with δ. Thus, if $z \in K(v) \subset M$ is a solution of $\delta(y) - cy = b$, then $\mathcal{R}(z) \in K$ is a solution of $\delta(y) - cy = b$. $\qquad\square$

Let K be a $\delta\sigma$-field, and let $k := K^\delta$. We consider a differential equation $\delta(y) = ay + b$, with $a, b \in K$, and the associated differential system

$$\delta \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{3.13}$$

By a σ-Picard–Vessiot extension $L|K$ for $\delta(y) = ay + b$, we mean a σ-Picard–Vessiot extension $L|K$ for (3.13). So let $Z := \begin{pmatrix} z_1 & z_2 \\ c_1 & c_2 \end{pmatrix}$ be a fundamental solution of (3.13) with coefficients in L. It follows directly from (3.13) and the fact that $L|K$ is a σ-Picard–Vessiot extension that $c_1, c_2 \in k$ and that $z_1 c_2 - z_2 c_1 \neq 0$. Then, when $c_1 c_2 \neq 0$, the matrix

$$\begin{pmatrix} z_1 c_1^{-1} - z_2 c_2^{-1} & z_2 c_2^{-1} \\ 0 & 1 \end{pmatrix}$$

is a fundamental solution matrix of (3.13). So we can consider a fundamental solution matrix of (3.13), with coefficients in L, of the form

$$Y := \begin{pmatrix} v & z \\ 0 & 1 \end{pmatrix}, \quad \text{with } \delta(v) = av, \ v \neq 0, \text{ and } \delta(z) = az + b. \tag{3.14}$$

(The case $c_1 c_2 = 0$ is similar and yields immediately to the existence of a fundamental solution matrix Y of the same form as the one above.) Notice that the $\delta\sigma$-field extension $K \langle v \rangle_\sigma \subset L$ does not depend on the particular choice of the non-zero solution $v \in L^\times$ of $\delta(v) = av$; in fact two such solutions coincide up to a multiplicative factor in $k \subset K$.

We consider $G := \sigma\text{-Gal}(L|K)$ as a σ-closed subgroup of $\text{Gl}_{2,k}$ via the fundamental solution matrix (3.14) (That is, we identify G with its image in $\text{Gl}_{2,k}$, via $\tau \mapsto [\tau]_Y$). The action of an element $\tau \in G(S)$ on $R \otimes_k S$, where $R = K\{v, v^{-1}, z\}_\sigma$ is the σ-Picard Vessiot ring of L, is given by $\tau(v \otimes 1) = v \otimes \alpha$ and $\tau(z \otimes 1) = v \otimes \beta + z \otimes 1$, for some $\alpha, \beta \in S$. Let $\mathbb{G}$ denote the algebraic subgroup of $\text{Gl}_{2,k}$ given by

$$\mathbb{G}(S) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \middle| \alpha \in S^\times, \beta \in S \right\}, \quad \text{for any } k\text{-}\sigma\text{-algebra } S.$$

For $\tau \in G(S) \leqslant \text{Gl}_2(S)$ we have $\tau(Y \otimes 1) = (Y \otimes 1)[\tau]_Y$. We see that $[\tau]_Y \in \mathbb{G}(S)$. Therefore G is contained in $\mathbb{G}$. As in § A.3, let $\mathbb{G}_u$ denote the algebraic subgroup of $\mathbb{G}$ given by

$$\mathbb{G}_u(S) = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \middle| \beta \in S \right\}, \quad \text{for any } k\text{-}\sigma\text{-algebra } S,$$

and set $G_u = G \cap \mathbb{G}_u$. Since $\tau(v \otimes 1) = v \otimes \alpha$, it is clear from the Galois correspondence that $G_u = \sigma\text{-Gal}(L|K \langle v \rangle_\sigma)$. Moreover, we have $\delta(\frac{z}{v}) = \frac{b}{v}$, so $L|K \langle v \rangle_\sigma$ is a σ-Picard–Vessiot extension for $\delta(y) = \frac{b}{v}$. The action of an element $\tau = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in G_u(S) = \sigma\text{-Gal}(L|K \langle v \rangle_\sigma)(S)$ on L is given by $\tau(\frac{z}{v} \otimes 1) = \frac{z}{v} \otimes 1 + 1 \otimes \beta$. The situation is summarized in the following picture:
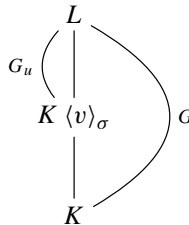
We remind the reader that, if G_u is properly contained in $\mathbb{G}_u$, it follows from Theorem A.9 that we only have two possible cases.

(i) There exists an integer $n \geqslant 0$ such that $\sigma^n(\beta) = 0$ for all $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in G_u(S)$ and all $k\text{-}\sigma$-algebras S.

(ii) There exist integers $n > m \geqslant 0$ such that $\sigma^n(\alpha) = \sigma^m(\alpha)$ for all $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in G(S)$ and all $k\text{-}\sigma$-algebras S.

Proposition 3.17. *Let $L|K$ be a $\delta\sigma$-field extension such that $k := K^\delta = L^\delta$. We fix $v, z \in L$, $v \neq 0$, such that $\delta(z) = az + b$ and $\delta(v) = av$, with $a, b \in K$. We assume that $z \notin K$ and K is inversive. If v is transformally independent over K, then z is transformally independent over K.*

Remark 3.18. Let us suppose that z is transformally dependent over K, and hence over $K\langle v \rangle_\sigma$. Since $\delta(\frac{z}{v}) = \frac{b}{v}$, Theorem 3.2 implies that there exist a non-zero homogeneous linear σ-polynomial $\sum_{j=0}^{s} \lambda_j \sigma^j(X) \in k\{X\}_\sigma$ and $g \in K\langle v \rangle_\sigma$ such that

$$\sum_{j=0}^{s} \lambda_j \sigma^j \left(\frac{b}{v} \right) = \delta(g).$$

The expression above, if non-trivial, provides a σ-relation for v, with coefficients in K, and proves that v is transformally dependent over K. The problem of this argument is that there is no reason for the expression above to be non-trivial, so it does not provide a proof of Proposition 3.17.

Proof. As we have already done in several proofs, we can assume that L is a σ-Picard–Vessiot extension for $\delta(y) = ay + b$. First assume that $G_u = \mathbb{G}_u$. Then Proposition 2.9 implies that $\sigma\text{-trdeg}(L|K\langle v \rangle_\sigma) = \sigma\text{-dim}(\mathbb{G}_u) = 1$. It follows that z is transformally independent over $K\langle v \rangle_\sigma$. In particular, z is transformally independent over K.

If G_u is a proper σ-closed subgroup of $\mathbb{G}_u$, we have to consider the two cases above. In the first case, we have

$$\tau \left(\sigma^n \left(\frac{z}{v} \otimes 1 \right) \right) = \sigma^n \left(\frac{z}{v} \otimes 1 + 1 \otimes \beta \right)$$
$$= \sigma^n \left(\frac{z}{v} \otimes 1 \right), \quad \text{for any } \tau = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in G_u(S) \text{ and any } k\text{-}\sigma\text{-algebra } S.$$

The Galois correspondence implies that $\sigma^n(\frac{z}{v}) \in K\langle v \rangle_\sigma$, and thus $\sigma^n(z) \in K\langle v \rangle_\sigma$. Since $z \notin K$ and K is inversive, we get that $\sigma^n(z) \notin K$. Since v is transformally independent over K, every element of the field $K\langle v \rangle_\sigma$ which does not belong to K is transformally independent over K. Hence z is transformally independent over K.

In the second case,

$$\tau \left(\frac{\sigma^n(v)}{\sigma^m(v)} \otimes 1 \right) = \frac{\sigma^n(v \otimes \alpha)}{\sigma^m(v \otimes \alpha)}$$
$$= \frac{\sigma^n(v)}{\sigma^m(v)} \otimes 1, \quad \text{for any } \tau = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in G(S) \text{ and any } k\text{-}\sigma\text{-algebra } S.$$

Therefore $\sigma^n(v) = f\sigma^m(v)$ for some $f \in K^\times$. This contradicts the assumption that v is transformally independent over K, so the second case cannot occur. $\qquad\square$

When we specialize Proposition 3.17 to $K = \mathbb{C}(x)$ endowed with the shift or the q-difference operator, we find the following.

Corollary 3.19. *Let $L|\mathbb{C}(x)$ be a $\delta\sigma$-field extension with $L^\delta = \mathbb{C}$, and assume that $z \in L$ satisfies $\delta(z) = az + b$ with $a, b \in \mathbb{C}(x)$, $a \neq 0$. If one of the following hypotheses is satisfied, then z is transformally independent over $\mathbb{C}(x)$.*

(i) *Assumption (3.5) and $a \neq P + \frac{1}{N}\frac{\delta(f)}{f}$ for any $P \in \mathbb{C}[x]$, $f \in \mathbb{C}(x)^\times$ and $N \in \mathbb{Z}^\times$.*

(ii) *Assumption (3.6), q is transcendental and $a \neq \frac{c}{x} + \frac{1}{N}\frac{\delta(f)}{f}$, for any $c \in \mathbb{C}$, $N \in \mathbb{Z}^\times$ and $f \in \mathbb{C}(x)^\times$.*

(iii) *Assumption (3.6), q is algebraic and $a \neq P + Q + \frac{1}{N}\frac{\delta(f)}{f}$, for any $P \in \mathbb{C}[x]$, $Q \in \mathbb{C}[\frac{1}{x}]$, $N \in \mathbb{Z}^\times$ and $f \in \mathbb{C}(x)^\times$.*

Proof. We can assume without loss of generality that there exists a non-zero $v \in L$ with $\delta(v) = av$. Indeed, since $L^\delta = \mathbb{C}$ is algebraically closed we can replace L with a σ-Picard–Vessiot extension for $\delta(y) = ay$ of L. It then suffices to combine Proposition 3.17 above with Corollary 3.12 and Corollary 3.14. $\qquad\square$

When v is transformally dependent, we have only a partial result.

Proposition 3.20. *Let K be a $\delta\sigma$-field, with $\sigma\delta = \delta\sigma$ and $k := K^\delta$. Let $L|K$ be a σ-Picard–Vessiot extension for the equation $\delta(y) = ay + b$, with $a, b \in K$. Let $z \in L$, with $z \notin K$, such that $\delta(z) = az + b$.*

If there exist $f \in K^\times$ and $\lambda \in k^\sigma$ such that $a = \lambda + \frac{\delta(f)}{f}$, then z is transformally dependent over K if and only if there exist a non-zero homogenous linear σ-polynomial $\mathcal{L}(X) \in k\{X\}_\sigma$ and an element $g \in K$ such that $\mathcal{L}(\frac{b}{f}) = \delta(g) - \lambda g$.

Proof. The assumption that there exist $f \in K^\times$ and $\lambda \in k^\sigma$ such that $a = \lambda + \frac{\delta(f)}{f}$ implies that any solution v of $\delta(y) = ay$ in L is transformally dependent over K. We fix a non-zero solution $v \neq 0$ of $\delta(y) = ay$ in L, and we set $\widetilde{v} = \frac{v}{f}$ and $\widetilde{z} = \frac{z}{f}$. Then $\widetilde{v}$ and $\widetilde{z}$ satisfy the differential equations

$$\delta(\widetilde{v}) = \left(a - \frac{\delta(f)}{f}\right)\widetilde{v} = \lambda\widetilde{v} \quad \text{and} \quad \delta(\widetilde{z}) = \lambda\widetilde{z} + \frac{b}{f}.$$

Since δ and σ commute to each other and $\sigma(\lambda) = \lambda$, we have that $\delta(\sigma(\widetilde{v})) = \sigma(\lambda)\sigma(\widetilde{v}) = \lambda\sigma(\widetilde{v})$. Therefore $\widetilde{v}$ and $\sigma(\widetilde{v})$ satisfy the same differential equation, and there exists $\mu \in k^\times$ such that $\sigma(\widetilde{v}) = \mu\widetilde{v}$. Hence $K\langle v\rangle_\sigma = K(v)$.

Assume that there exist a non-zero homogeneous linear σ-polynomial $\mathcal{L} \in k\{X\}_\sigma$ and $g \in K$ such that $\mathcal{L}(\frac{b}{f}) = \delta(g) - \lambda g$. Then

$$\delta(\mathcal{L}(\widetilde{z}) - g) = \mathcal{L}\left(\lambda\widetilde{z} + \frac{b}{f}\right) - \delta(g) = \lambda(\mathcal{L}(\widetilde{z}) - g).$$

Consequently, there exists $\kappa \in k$ such that $\mathcal{L}(\widetilde{z}) - g = \kappa\widetilde{v}$. Since $\widetilde{v}$ is transformally dependent over K, this shows that z is transformally dependent over K.

Conversely, assume that z is transformally dependent over K. Since $L|K\langle v\rangle_\sigma$ is a σ-Picard–Vessiot extension for $\delta(y) = \frac{b}{v}$ and $\frac{z}{v}$ is transformally dependent over $K\langle v\rangle_\sigma$, Theorem 3.2 implies that there exist a non-zero homogeneous linear σ-polynomial $\mathcal{L} \in k\{X\}_\sigma$ and $g \in K\langle v\rangle_\sigma$ such that $\mathcal{L}(\frac{b}{v}) = \delta(g)$. Since $\frac{b}{v} = \frac{b/f}{\widetilde{v}}$ and $\sigma(\widetilde{v}) = \mu\widetilde{v}$, there

exists a non-zero homogeneous linear σ-polynomial $\widetilde{\mathcal{L}} \in k\{X\}_\sigma$ such that $\widetilde{\mathcal{L}}(\frac{b}{f}) = \widetilde{v}\mathcal{L}(\frac{b}{v})$. Then

$$\widetilde{\mathcal{L}}\left(\frac{b}{f}\right) = \delta(g)\widetilde{v} = \delta(g\widetilde{v}) - \lambda g\widetilde{v}.$$

So the equation $\delta(y) - \lambda y = \widetilde{\mathcal{L}}(\frac{b}{f})$ has a solution in $K\langle v\rangle_\sigma = K(v)$. Now Lemma 3.16 provides an element $\widetilde{g} \in K$ such that $\widetilde{\mathcal{L}}(\frac{b}{f}) = \delta(\widetilde{g}) - \lambda\widetilde{g}$. □

In order to apply Proposition 3.20, we want σ and δ to commute: to ensure that this hypothesis is satisfied in the q-difference case, we take $\delta = x\frac{d}{dx}$, rather than $\delta = \frac{d}{dx}$. Moreover, we assume $q \in \mathbb{C}$ to be a transcendental number.

Notice that, if $\frac{dy}{dx} = by$, with $b \in \mathbb{C}(x)$, then $\delta(y) = xby$. Therefore a solution v of $\delta(y) = ay$, in a convenient extension of $\mathbb{C}(x)$, is transformally dependent over $\mathbb{C}(x)$ if and only if $a = c + \frac{1}{N}\frac{\delta(f)}{f}$ for some $c \in \mathbb{C}$, $N \in \mathbb{Z}^\times$ and $f \in \mathbb{C}(x)^\times$ (see Corollary 3.14).

Corollary 3.21. *Let* $L|\mathbb{C}(x)$ *be a* $\delta\sigma$*-field extension, with* $\delta = x\frac{d}{dx}$, $L^\delta = \mathbb{C}$, *and* $\sigma : f(x) \mapsto f(qx)$, *for some transcendental* $q \in \mathbb{C}$, *and assume that* $z \in L$, $z \notin \mathbb{C}(x)$, *satisfies* $\delta(z) = az + b$, *with* $a, b \in \mathbb{C}(x)$. *If* $a = c + \frac{1}{N}\frac{\delta(g)}{g}$ *for some* $c \in \mathbb{C}$, $N \in \mathbb{Z}^\times$, *and* $g \in \mathbb{C}(x)^\times$, *then there are two cases.*

(i) *If* $\frac{1}{N}\frac{\delta(g)}{g} = \frac{\delta(f)}{f}$, *for some* $f \in \mathbb{C}(x)^\times$, *then* z *is transformally dependent over* $\mathbb{C}(x)$ *if and only if there exist a non-zero homogeneous linear* σ*-polynomial* $\mathcal{L} \in \mathbb{C}\{X\}_\sigma$ *and* $h \in \mathbb{C}(x)$ *such that* $\mathcal{L}(\frac{b}{f}) = \delta(h) - ch$.

(ii) *If* $\frac{1}{N}\frac{\delta(g)}{g} \neq \frac{\delta(f)}{f}$, *for all* $f \in \mathbb{C}(x)^\times$, *then* z *is transformally dependent over* $\mathbb{C}(x)$ *if and only if there exists* $h \in \mathbb{C}(x)$ *such that* $b = \delta(h) - ah$.

Remark 3.22. For a general difference operator σ, we only have a partial answer to the question of transformal dependencies of solutions of $\delta(y) = ay + b$. On the other hand, for a q-difference operator with q transcendent, Corollary 3.21 together with Corollary 3.19(ii) describes completely the transformal relations of the solutions in terms of relations satisfied by the coefficients a and b.

Proof. As in the proof of Corollary 3.19, we can assume that there exists a non-zero $v \in L$ with $\delta(v) = av$. The first assertion follows immediately from Proposition 3.20 above. Let us prove the second one. Suppose that $a = c + \frac{1}{N}\frac{\delta(g)}{g}$, with $\frac{1}{N}\frac{\delta(g)}{g} \neq \frac{\delta(f)}{f}$, for all $f \in \mathbb{C}(x)^\times$, so that v is transformally dependent over $\mathbb{C}(x)$. If there exists $h \in \mathbb{C}(x)$, such that $b = \delta(h) - ah$, then $z - h$ is a solution of $\delta(y) = ay$. Hence there exists $\lambda \in \mathbb{C}$ such that $z - h = \lambda v$, which proves that z is transformally dependent over $\mathbb{C}(x)$.

Conversely, let us assume that z is transformally dependent over $K = \mathbb{C}(x)$. With no loss of generality, we can assume that $L = K\langle v, z\rangle_\sigma$ is a σ-Picard–Vessiot extension for $\delta(z) = az + b$. We refer to the discussion and the notation at the beginning of the section about the devissage of $G = \sigma\text{-Gal}(L|K)$. We remind the reader that the action of an element $\tau \in G(S)$ on $R \otimes_k S$, where $R = K\{v, v^{-1}, z\}_\sigma$ is the σ-Picard Vessiot ring of L and S is a k-σ-algebra, is given by $\tau(v \otimes 1) = v \otimes \alpha$ and $\tau(z \otimes 1) = v \otimes \beta + z \otimes 1$, for some $\alpha, \beta \in S$. Since $L|K\langle v\rangle_\sigma$ is a σ-Picard–Vessiot extension for $\delta(y) = \frac{b}{v}$ and $\frac{z}{v}$ is transformally

dependent over $K\langle v\rangle_\sigma$, we know that $G_u = G \cap \mathbb{G}_u$ is properly contained in $\mathbb{G}_u$. According to Theorem A.9 we have to study two cases: for all $\tau \in G(S)$, and all k-σ-algebras S, either there exists an integer $n \geqslant 0$ such that $\sigma^n(\beta) = 0$ or there exist integers $n > m \geqslant 0$ such that $\sigma^n(\alpha) = \sigma^m(\alpha)$.

In the first case, we have

$$
\tau\left(\sigma^n\left(\frac{z}{v} \otimes 1\right)\right) = \sigma^n\left(\frac{z}{v} \otimes 1 + 1 \otimes \beta\right)
$$
$$
= \sigma^n\left(\frac{z}{v} \otimes 1\right), \quad \text{for any } \tau \in G_u(S) \text{ and any } k\text{-}\sigma\text{-algebra } S.
$$

The Galois correspondence implies that $\sigma^n(\frac{z}{v}) \in K\langle v\rangle_\sigma$ and thus that $\sigma^n(z) \in K\langle v\rangle_\sigma$. This means that $\delta(y) - \sigma^n(a)y = \sigma^n(b)$ has a solution $\sigma^n(z) \in K\langle v\rangle_\sigma$. There exists a non-negative integer m such that $\sigma^n(z) \in K(v, \ldots, \sigma^m(v))$. Applying recursively Lemma 3.16, we find that there already exists a solution $\widetilde{h} \in K$ of $\delta(y) - \sigma^n(a)y = \sigma^n(b)$. So, if $h \in K$ with $\sigma^n(h) = \widetilde{h}$, then $\delta(h) - ah = b$.

Now consider the second case. Since K is inversive, we know that G is σ-reduced. So we can assume that $m = 0$ (see Proposition 4.3 in [15]). Since for any $\tau \in G(S)$ we have

$$
\tau\left(\frac{\sigma^n(v)}{v} \otimes 1\right) = \frac{\sigma^n(\tau(v \otimes 1))}{\tau(v \otimes 1)} = \frac{\sigma^n(v)}{v} \otimes 1,
$$

we conclude that $\sigma^n(v) = hv$ for some $h \in \mathbb{C}(x)^\times$. This implies that $\sigma^n(\frac{\delta(v)}{v}) = \frac{\delta(h)}{h} + \frac{\delta(v)}{v}$, which yields $\sigma^n(a) - a = \frac{\delta(h)}{h} = \frac{1}{N}\frac{\delta(\sigma^n(g)/g)}{\sigma^n(g)/g}$. So there exists $\mu \in \mathbb{C}^\times$ such that $\frac{\sigma^n(g)}{g} = \mu h^N$. It is now easy to see that this last equality implies that there exists $f \in \mathbb{C}(x)^\times$ with $g = f^N$. But then $\frac{1}{N}\frac{\delta(g)}{g} = \frac{\delta(f)}{f}$. So the second case cannot occur. $\qquad\square$

Remark 3.23. The existence of a non-zero homogeneous linear σ-polynomial $\mathcal{L} \in \mathbb{C}\{X\}_\sigma$ such that $\mathcal{L}(b) = \delta(y) - ay$ has a solution in $\mathbb{C}(x)$ does not imply that $b = \delta(y) - ay$ has a solution $\mathbb{C}(x)$, differently from the case of linear difference equations with a differential parameter. See Lemma 6.4 in [20]. Take for example $a = 1$ and $b = x$. Then $\frac{1}{q}\sigma(x) - x = 0$, but the differential equation $x\frac{d}{dx}y - y = x$ has no solution in $\mathbb{C}(x)$.

4. Inverse problem for σ-closed subgroups of $\mathbf{G}_a$

Let $k(x)$ be a differential field with $\delta(x) = 1$ and $k = k(x)^\delta$ algebraically closed. In the classical Galois theory of linear differential systems with coefficients in $k(x)$, the inverse problem consists in determining all the linear algebraic groups, that occur as Galois group of a linear differential system over $k(x)$. This question, which is also connected to the Riemann–Hilbert correspondence, has been addressed by many mathematicians, including recently J. Hartman, J. Kovacic, C. Mitschi, J.P. Ramis, and M.F. Singer. In [19], many references are given, and it is proved that any linear algebraic group over k is the Galois group of a linear differential system over $k(x)$.

Recently, Cassidy and Singer developed in [9] a parameterized Galois theory of linear differential systems which takes into account continuous actions on auxiliary parameters. For instance, if k above is a 'sufficiently large' $\frac{d}{dt}$-field extension of $\mathbb{C}(t)$, Cassidy and

Singer are able to attach to a differential system $\delta(y) = A(x, t)y$, a parameterized Galois group, which is a linear $\frac{d}{dt}$-differential algebraic group in the sense of Kolchin. In this setting, an inverse problem was also addressed. The first answers were given in [12] and [39]. Using transcendental descriptions via monodromy and Stokes matrices, the authors were able to show, for instance, that a linear $\frac{d}{dt}$-differential algebraic group occurs as parameterized Galois group in the sense of [9] if and only if it contains a Kolchin dense finitely generated subgroup. A purely algebraic characterization was then given in [51], where it is proved that a linear algebraic group G over k is a parameterized Galois group if and only if its identity component has no quotients isomorphic to $\mathbf{G}_a$ or $\mathbf{G}_m$. Further progress on the inverse (and the direct) problem for the parameterized Galois theory of linear differential systems has recently been made in [37] and [36]. Related results, dealing with more general Galois theories, can also be found in [33] and [38, 45].

The aim of the present section is to investigate some similar questions for a discrete action on the parameter. Since $\mathbf{G}_a$ appears as an obstruction in the continuous parameter case, we restrict our study of the inverse problem to σ-closed subgroups of $\mathbf{G}_a$. We show that one can realize $\mathbf{G}_a$ itself over some $k(x)$ and, thereby, that any of the σ-closed subgroups of $\mathbf{G}_a$ is a σ-Galois group over some $\delta\sigma$-field extension of $k(x)$, by σ-Galois correspondence.

In this section, we assume that we are in the following situation.

> ($\mathcal{H}_1$) Let K be a $\delta\sigma$-field, with $k = K^\delta$ algebraically closed and aperiodic with respect to σ; i.e., for any positive integer n, there exists $a \in k$ such that $\sigma^n(a) \neq a$. We consider a σ-Picard–Vessiot extension $L|K$, having σ-Galois group isomorphic to a σ-closed subgroup of the additive group $\mathbf{G}_a$.

In the first subsection, we give a decomposition theorem of such extensions (Proposition 4.3) into a 'σ-infinitesimal' and a perfectly σ-reduced part. Moreover, under additional assumptions on K, we show that L is the σ-Picard–Vessiot extension of some differential equation of the form $\delta(y) = g$. In the second subsection, we restrict ourselves to the case when $K = k(x)$, where $k(x)$ is a $\delta\sigma$-field extension of $\mathbb{C}(t, x)$ endowed with $\delta(x) = 1, \sigma(x) = x$ and $\delta(t) = 0, \sigma(t) = t + 1$. This situation can be seen as the discrete counterpart of the one studied in [51]. Assuming that k is linearly σ-closed, we show that any perfectly σ-reduced σ-closed subgroup of $\mathbf{G}_a$ is a σ-Galois group over $k(x)$ (see Proposition 4.6).

4.1. Some structure theorems

The next lemma is a first step in the description of the structure of the extension $L|K$.

Lemma 4.1. *Assume that we are in situation ($\mathcal{H}_1$). Then there exist $t_1, \ldots, t_s \in L$ such that $L = K\langle t_1, \ldots, t_s \rangle_\sigma$, and $\delta(t_1), \ldots, \delta(t_s) \in K$.*

Proof. We can suppose that $L = K\langle Y \rangle_\sigma$, where $Y \in \mathrm{Gl}_n(L)$ is a fundamental solution matrix of a linear differential system $\delta(y) = Ay$ of order n, with coefficients in K. We denote by $\sigma\text{-Gal}(L|K)[0]$ the Zariski closure of $\sigma\text{-Gal}(L|K)$ inside $\mathrm{Gl}_{n,k}$, and by $k[\sigma\text{-Gal}(L|K)[0]]$ its k-Hopf algebra.

Let G' be a σ-closed subgroup of $\mathbf{G}_a$, and let $\phi : G' \to \sigma\text{-Gal}(L|K)$ be an isomorphism of group k-σ-schemes (see § A.8 in [15]), so that

$$\phi^* : k\{\sigma\text{-Gal}(L|K)\} \to k\{G'\}$$

is an isomorphism of k-σ-Hopf algebras. We consider the σ-Picard–Vessiot ring $R = K\{Y, \frac{1}{\det(Y)}\}_\sigma$ inside L (see Proposition 2.6). Following Proposition 2.8 and [15, Lemma 1.8], we can identify $k\{\sigma\text{-Gal}(L|K)\}$ with $(R \otimes_K R)^\delta = k\{Z, \frac{1}{\det(Z)}\}_\sigma$, where $Z = (Y \otimes 1)^{-1}(1 \otimes Y) \in (R \otimes R)^\delta$. Moreover, $k\{G'\}$ is a quotient of the ring of σ-polynomials in one variable; i.e., it has the form $k\{\overline{x}\}_\sigma$. Therefore, there exists $r \in \mathbb{Z}_{\geqslant 0}$ such that $\phi^*(Z)$ is an element of the ring $k[\overline{x}, \sigma(\overline{x}), \ldots, \sigma^r(\overline{x})]$. This means in particular that the map ϕ^* injects $k[\sigma\text{-Gal}(L|K)[0]]$ in a sub-k-algebra of $k[\overline{x}, \sigma(\overline{x}), \ldots, \sigma^r(\overline{x})]$; i.e., $\sigma\text{-Gal}(L|K)[0]$ can be identified with a quotient of some $\mathbf{G}_a^r$. The quotients of an algebraic vector group are algebraic vector groups; hence there exists $s \in \mathbb{Z}_{>0}$ such that $\sigma\text{-Gal}(L|K)[0]$ is isomorphic to $\mathbf{G}_a^s$.

We have proved that $\sigma\text{-Gal}(L|K)[0] \cong \mathbf{G}_a^s$ is the differential Galois group of the Picard–Vessiot ring $R_0 := K[Y, \frac{1}{\det(Y)}] \subset R$, over K. Finally, $\mathbf{G}_a^s$ has a trivial first Galois cohomology group; therefore the $\sigma\text{-Gal}(L|K)[0]$-torsor R_0 is trivial (see Theorem 1.28 [44]). In other words, there exist $t_1, \ldots, t_s \in R_0$ such that

- $R_0 = K[t_1, \ldots, t_s]$,
- for all $\tau \in \sigma\text{-Gal}(L|K)[0](k) = \text{Aut}^\delta(R_0|K)$, we have $\tau(t_i) = t_i + c_i(\tau)$ with $c_i(\tau) \in k$.

For any $\tau \in \sigma\text{-Gal}(L|K)[0](k)$ and any $i = 1, \ldots, s$, we have $\tau(\delta(t_i)) = \delta(t_i + c_i(\tau)) = \delta(t_i)$. The differential Galois theory (see Proposition 1.34 in [44]) implies that $\delta(t_i) \in K$, for all $i = 1, \ldots, s$.

To summarize, we have found $t_1, \ldots, t_s \in L$ such that $K[Y, \frac{1}{\det Y}] = K[t_1, \ldots, t_s]$, and hence such that $L = K\langle t_1, \ldots, t_s \rangle_\sigma$, with $\delta(t_i) \in K$ for all $i = 1, \ldots, s$. $\qquad\square$

Now, we treat the case where the σ-Galois group is the whole $\mathbf{G}_a$.

Proposition 4.2. *Under assumption $(\mathcal{H}_1)$, suppose that $\sigma\text{-Gal}(L|K)$ is isomorphic to $\mathbf{G}_a$. Then there exists $\theta \in L$ such that $L = K\langle\theta\rangle_\sigma$ and $\delta(\theta) \in K$.*

Proof. It follows from Lemma 4.1 that we can find $t_1, \ldots, t_s \in L$ such that $L = K\langle t_1, \ldots, t_s \rangle_\sigma$, with $\delta(t_i) \in K$. Moreover, $R = K\{t_1, \ldots, t_s\}_\sigma$ is a σ-Picard–Vessiot ring of L. For every k-σ-algebra S and every $\tau \in \sigma\text{-Gal}(L|K)(S)$, there exist $c_1(\tau), \ldots, c_s(\tau) \in S^s$, such that in $R \otimes_k S$ we have $\tau(t_i \otimes 1) = t_i \otimes 1 + 1 \otimes c_i(\tau)$ for all $i = 1, \ldots, s$. This induces a σ-morphism $\phi : \sigma\text{-Gal}(L|K) \to \mathbf{G}_a^s$, such that $\phi(\tau) = (c_1(\tau), \ldots, c_s(\tau))$, for every k-σ-algebra S and every $\tau \in \sigma\text{-Gal}(L|K)(S)$. If $\tau \in \sigma\text{-Gal}(L|K)(S)$ is the identity on $t_1 \otimes 1, \ldots, t_s \otimes 1$, then it is the identity on $R \otimes S$. This proves that ϕ has a trivial kernel.

Now, we identify $\sigma\text{-Gal}(L|K)$ with $\mathbf{G}_a$ and $\tau \in \sigma\text{-Gal}(L|K)$ with its image, say $c(\tau)$, in $\mathbf{G}_a$. Then, since ϕ is a morphism of σ-algebraic groups from $\mathbf{G}_a$ to $\mathbf{G}_a^s$,[1] there exist

---

[1]In zero characteristic, morphisms of σ-algebraic groups from $\mathbf{G}_a$ to $\mathbf{G}_a$ are given by linear σ-polynomials. This can be proved as in the algebraic case [56, Theorem 8.4]. We omit this proof here.

homogeneous linear σ-operators $L_1, \ldots, L_s$, i.e., elements of the skew euclidean ring $k[\sigma]$, such that, for any k-σ-algebra S, we have

$$\phi : \mathbf{G}_a(S) \longrightarrow \mathbf{G}_a^s(S)$$
$$\tau \longmapsto (L_1(c(\tau)), \ldots, L_s(c(\tau))).$$

Since ϕ is injective, the σ-operators $L_1, \ldots, L_s$ do not annihilate simultaneously, and hence 1 belongs to the ideal $k[\sigma]L_1 + \cdots + k[\sigma]L_s$ of $k[\sigma]$. Therefore there exist $N_1, \ldots, N_s \in k[\sigma]$ such that $N_1 L_1 + \cdots N_s L_s = 1$ in $k[\sigma]$. We set $\theta := \sum_{i=1}^s N_i(t_i) \in L$. Then, for every k-σ-algebra S and every $\tau \in \sigma\text{-Gal}(L|K)(S)$, we have $\tau(\theta \otimes 1) = \sum_{i=1}^s N_i(t_i \otimes 1 + 1 \otimes L_i(c(\tau))) = \sum_{i=1}^s N_i(t_i \otimes 1) + 1 \otimes c(\tau) = \theta \otimes 1 + 1 \otimes c(\tau)$ in $R \otimes_k S$. This means that the only element of $\sigma\text{-Gal}(L|K)$ that fixes θ is the identity. By the σ-Galois correspondence, we finally get that $L = K\langle\theta\rangle_\sigma$. It follows from the construction itself that we have $\delta(\theta) \in K$. $\qquad\square$

The situation is more complicated when $\sigma\text{-Gal}(L|K)$ is isomorphic to a proper σ-closed subgroup of $\mathbf{G}_a$.

Proposition 4.3. *Under assumption* $(\mathcal{H}_1)$, *suppose that* $\sigma\text{-Gal}(L|K)$ *is isomorphic to a proper subgroup of* $\mathbf{G}_a$. *Then we have the following.*

(i) *There exist* $\theta \in L$ *and* $n \in \mathbb{Z}_{\geqslant 0}$, *such that* $\sigma^n(L) \subset K\langle\theta\rangle_\sigma$ *and* $\delta(\theta) \in K$.

(ii) *The* $\delta\sigma$-*field* $K\langle\theta\rangle_\sigma$ *is a* σ-*Picard–Vessiot extension of* K *for* $\delta(y) = \delta(\theta)$, *and we have a short exact sequence*

$$0 \to \sigma\text{-Gal}(L|K\langle\theta\rangle_\sigma) \to \sigma\text{-Gal}(L|K) \to \sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K) \to 0. \qquad (4.1)$$

Moreover, the group $\sigma\text{-Gal}(L|K\langle\theta\rangle_\sigma)$ *is* σ-*infinitesimal; i.e.,* $\sigma\text{-Gal}(L|K\langle\theta\rangle_\sigma)(S)$ *is trivial for any* σ-*reduced* k-σ-*algebra* S. *In addition, we can choose* θ *such that* $\sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K)$ *is perfectly* σ-*reduced.*

Remark 4.4. The previous proposition gives a decomposition of the σ-Picard–Vessiot extension $L|K$, with σ-Galois group isomorphic to a proper subgroup of $\mathbf{G}_a$, in a tower of σ-field extensions such that $K\langle\theta\rangle_\sigma|K$ is perfectly σ-separable and $L|K\langle\theta\rangle_\sigma$ is what we could call a 'purely σ-inseparable extension'.

Proof. Lemma 4.1 implies that there exist $t_1, \ldots, t_s \in L$ such that $L = K\langle t_1, \ldots, t_s\rangle_\sigma$, and $\delta(t_1), \ldots, \delta(t_s) \in K$. Notice that the assumption on $\sigma\text{-Gal}(L|K)$ and Proposition 2.9 imply that

$$\sigma\text{-trdeg}(L|K) = \sigma\text{-}\dim_k(\sigma\text{-Gal}(L|K)) = 0.$$

Therefore, it follows from Theorem 4.5.4 in [34] that there exist $c_1, \ldots, c_s \in k$ and $n \in \mathbb{Z}_{\geqslant 0}$ such that, if we take $\theta = \sum_{i=1}^s c_i t_i$, then $\sigma^n(L) \in K\langle\theta\rangle_\sigma$. By construction, $\delta(\theta) \in K$. This proves (i).

Notice that $K\langle\theta\rangle_\sigma^\delta \subset L^\delta = k$, which implies that $K\langle\theta\rangle_\sigma$ is a σ-Picard–Vessiot extension for $\delta(y) = \delta(\theta)$. The second fundamental theorem of the σ-Galois correspondence (see Theorem 2.12) gives immediately the exact sequence of group k-σ-schemes (4.1).

Now, we show that $\sigma\text{-Gal}(L|K\langle\theta\rangle_\sigma)$ is σ-infinitesimal. Let S be a σ-reduced k-σ-algebra, let R be the σ-Picard–Vessiot ring inside L, and let $\tau \in \sigma\text{-Gal}(L|K\langle\theta\rangle_\sigma)(S)$. Since $\sigma^n(L) \in$

$K\langle\theta\rangle_\sigma$, the action of τ on $\sigma^n(R)\otimes S$ is trivial. Since τ commutes with σ and σ is injective on S, we deduce that τ is the identity by looking at its action on a fundamental solution matrix for $L|K$.

It remains to prove that we can choose θ such that $\sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K)$ is perfectly σ-reduced. The $\delta\sigma$-field extension $K\langle\theta\rangle_\sigma|K$ is a σ-Picard–Vessiot extension for $\delta(y) = \delta(\theta)$, with fundamental solution matrix $\Theta := \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix}$. We can embed $\sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K)$ into $\mathbf{G}_a$ via Θ (we identify $\mathbf{G}_a$ with its image in $\text{Gl}_{2,k}$ via $a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$). If $\sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K) = \mathbf{G}_a$, we are done. Otherwise, Theorem A.1 implies that $\mathbb{I}(\sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K)) \subset k\{x\}_\sigma = k\{\mathbf{G}_a\}$ is generated, as σ-ideal, by a σ-polynomial of the form $\sum_{i=0}^s \lambda_i\sigma^i(x)$, for some $\lambda_0,\dots,\lambda_s \in k$ with $\lambda_s \neq 0$. Let $r = \min\{i = 0,\dots,s | \lambda_i \neq 0\}$ be minimal. We set $\theta_1 := \sigma^r(\theta)$. Then $\delta(\theta_1) \in K$ and $\sigma^{n+r}(L) \subset K\langle\theta_1\rangle_\sigma$. The σ-Galois correspondence for the σ-Picard–Vessiot extension $K\langle\theta\rangle_\sigma|K$ implies that $\sum_{i=0}^{s-r}\lambda_{i+r}\sigma^i(\theta_1)$ belongs to K. Using again the σ-Galois correspondence for the σ-Picard–Vessiot extension $K\langle\theta_1\rangle_\sigma|K$, we see that $\sum_{i=0}^{s-r}\lambda_{i+r}\sigma^i \in \mathbb{I}(\sigma\text{-Gal}(K\langle\theta_1\rangle_\sigma|K))$. The classification of the σ-closed subgroups of $\mathbf{G}_a$ (Corollary A.3) implies that $\mathbb{I}(\sigma\text{-Gal}(K\langle\theta_1\rangle_\sigma|K))$ is generated by a σ-polynomial of the form $\sum_{i=0}^l \mu_i\sigma^i$ with $\mu_0 \neq 0$ and that $\sigma\text{-Gal}(K\langle\theta_1\rangle_\sigma|K)$ is perfectly σ-reduced. $\qquad\square$

Now, we consider a more restrictive hypothesis on K; namely, we show that, if K is inversive, then only perfectly σ-reduced σ-closed subgroups of $\mathbf{G}_a$ can be realized as σ-Galois groups.

Proposition 4.5. *Assume that the extension $L|K$ verifies $(\mathcal{H}_1)$ and moreover that the σ-field K is inversive. Then $\sigma\text{-Gal}(L|K)$ is perfectly σ-reduced and there exists $\theta \in L$ such that*

- $\delta(\theta) \in K$, *and*
- L *is a σ-Picard–Vessiot extension for $\delta(y) = \delta(\theta)$.*

Proof. If $\sigma\text{-Gal}(L|K)$ is isomorphic to $\mathbf{G}_a$, then there is nothing to prove (see Proposition 4.2). If $\sigma\text{-Gal}(L|K)$ is isomorphic to a proper σ-closed subgroup of $\mathbf{G}_a$, then, by Proposition 4.3, there exist $\theta \in L$ and $n \in \mathbb{Z}_{\geqslant 0}$ such that $\delta(\theta) \in K$ and $\sigma^n(L) \in K\langle\theta\rangle_\sigma$. Moreover, we can choose n and θ such that $\sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K)$ is a perfectly σ-reduced proper σ-closed subgroup of $\mathbf{G}_a$.

To conclude, it suffices to prove that $K\langle\theta\rangle_\sigma$ is inversive. In fact, this immediately implies that $L = K\langle\theta\rangle_\sigma$. Embedding $\sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K)$ into $\mathbf{G}_a$ (via θ), we see that there exist $\lambda_0,\dots,\lambda_s \in k$ with $\lambda_0 \neq 0$ such that $\sum_{i=0}^s \lambda_i\sigma^i(x) \in k\{x\}_\sigma$ generates $\mathbb{I}(\sigma\text{-Gal}(K\langle\theta\rangle_\sigma|K))$. Using the σ-Galois correspondence, we see that $\sum_{i=0}^s \lambda_i\sigma^i(\theta) \in K$. Since K is inversive and $\lambda_0 \neq 0$, this implies that $K\langle\theta\rangle_\sigma$ is inversive and therefore that $L = K\langle\theta\rangle_\sigma$. $\qquad\square$

4.2. A discrete parameter inverse problem over $\mathbb{C}(x,t)$

We now restrict our attention to the following situation.

> $(\mathcal{H}_2)$ Let $\mathbb{C}(t)$ be the σ-field with $\sigma|_\mathbb{C} = id$ and $\sigma(t) = t+1$. Let k be an algebraically closed inversive linearly σ-closed σ-field extension of $\mathbb{C}(t)$. We now assume that $K = k(x)$ endowed with the derivation $\delta(x) = 1$, $\delta(c) = 0$ for

all $c \in k$ and the endomorphism extending the action of σ on k to K with $\sigma(x) = x$.

Since assumption $(\mathcal{H}_2)$ satisfies the hypothesis of Proposition 4.5, we know that only perfectly σ-reduced σ-closed subgroups of $\mathbf{G}_a$ can occur as σ-Galois groups for differential equations of the form $\delta(y) = g$ over K. The following proposition shows that any of these subgroups can appear.

Proposition 4.6. *Suppose that K verifies $(\mathcal{H}_2)$. Let G be a perfectly σ-reduced σ-closed subgroup of $\mathbf{G}_a$. Then there exists $g \in K$ such that $\delta(y) = g$ has σ-Galois group G.*

Proof. First of all, let us realize $\mathbf{G}_a$ itself. Let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = \frac{1}{x+t}$. Reasoning as in Corollary 3.7, we see that $\sigma\text{-Gal}(L|K) = \mathbf{G}_a$. Notice that the coefficients of the equation lie in $\mathbb{Q}(x, t)$.

Now, let G be a proper perfectly σ-reduced σ-closed subgroup of $\mathbf{G}_a$. Since G is perfectly σ-reduced, we know that there exist $\lambda_0, \ldots, \lambda_s \in k$, with $\lambda_0 \lambda_s \neq 0$, such that $\mathbb{I}(G) \subset k\{x\}_\sigma = k\{\mathbf{G}_a\}$ is generated by $\mathcal{L} := \sum_{i=0}^{s} \lambda_i \sigma^i(x)$. Since k is linearly σ-closed, we can find $c_1, \ldots, c_s \in k$, linearly independent over k^σ, whose k^σ-span is the group $G(k)$ (see Corollary A.6).

Let us consider the differential equation

$$\delta(y) = \sum_{i=1}^{s} \frac{c_i}{x+i} =: g.$$

Let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = g$, $z \in L$ be a solution, and $R = K\{z\}_\sigma$ its σ-Picard–Vessiot ring, in the sense of § 3.3. We have

$$\mathcal{L}(\delta(z)) = \mathcal{L}(g) = \sum_{i=1}^{s} \frac{\mathcal{L}(c_i)}{x+i} = 0,$$

so that $\delta(\mathcal{L}(z)) = 0$, and hence $\mathcal{L}(z) \in k$. We embed $\sigma\text{-Gal}(L|K)$ into $\mathbf{G}_a$, via its action on the solution z; i.e., for any k-σ-algebra S and any $\tau \in \sigma\text{-Gal}(L|K)(S)$, there exists $c(\tau) \in S$ such that $\tau(z \otimes 1) = z \otimes 1 + 1 \otimes c(\tau) \in R \otimes_k S$. We write $\tau(z) = z + c(\tau)$ to simplify the notation. Since $\mathcal{L}(z) \in k$, we have $\mathcal{L}(z) = \tau(\mathcal{L}(z)) = \mathcal{L}(\tau(z)) = \mathcal{L}(z) + \mathcal{L}(c(\tau))$, and hence $\mathcal{L}(c(\tau)) = 0$, which means that $\sigma\text{-Gal}(L|K) \subset G$. To conclude, we have to prove the inverse inclusion. Let $\mathcal{L}_1$ be a homogeneous linear σ-polynomial in $k\{x\}_\sigma$, generating the vanishing ideal of $\sigma\text{-Gal}(L|K)$ inside $\mathbf{G}_a$, and let z be a solution of $\delta(y) = g$, as above. By σ-Galois correspondence we find that $\mathcal{L}_1(z)$ belongs to K and, thus, that

$$\delta(\mathcal{L}_1(z)) = \mathcal{L}_1(\delta(z)) = \mathcal{L}_1(g) = \sum_{i=1}^{s} \frac{\mathcal{L}_1(c_i)}{x+i} \in \delta(K).$$

Reasoning as in Corollary 3.6, we find that $\mathcal{L}_1(c_i) = 0$, for all $i = 1, \ldots, s$. This implies that $\mathcal{L}_1(c) = 0$ for every $c \in G(S)$ and any k-σ-algebra S (Corollary A.6). Thus $G \subset \sigma\text{-Gal}(L|K)$. $\square$

5. Discrete integrability

The terminology of this section is borrowed from differential equations depending on a differential parameter. Indeed, Y. Sibuya proves the equivalence between the differential analog of the definition below, usually called integrability, and the notion of isomonodromy. See Theorem A.5.2.3 in [49].

Integrability has been studied from a Galoisian point of view in [9, 18, 20] for equations depending on differential parameters (see also [12, 39]). Here we consider the dependence on a difference parameter. We prove some results that are analogous to the ones in the papers above. However, our result on the descent of integrability is not inspired by any results in the previous literature and could actually be reproduced in the case of a differential parameter, improving some applications in the papers cited above.

5.1. Definition, first properties

First of all, we introduce the notion of σ^d-integrability that we are going to discuss in this section.

Definition 5.1. Let K be a $\delta\sigma$-field, $A \in K^{n\times n}$, for some positive integer n, and $d \in \mathbb{Z}_{>0}$. We say that $\delta(y) = Ay$ is σ^d-*integrable (over K)*, if there exists $B \in \mathrm{Gl}_n(K)$, such that

$$\begin{cases} \delta(y) = Ay \\ \sigma^d(y) = By \end{cases} \tag{5.1}$$

is compatible; i.e.,

$$\delta(B) + BA = \hbar_d \sigma^d(A)B, \tag{5.2}$$

where $\hbar_d = \hbar\sigma(\hbar)\cdots\sigma^{d-1}(\hbar)$.

The following proposition interprets the compatibility relation (5.2) in terms of solutions of the system (5.1).

Proposition 5.2. *Let K be a $\delta\sigma$-field, $\delta(y) = Ay$ be a linear differential equation with $A \in K^{n\times n}$, and L be a $\delta\sigma$-field extension of K.*

(i) *If there exist $B \in \mathrm{Gl}_n(K)$ and $Y \in \mathrm{Gl}_n(L)$ such that $\delta(Y) = AY$ and $\sigma^d(Y) = BY$ (i.e., Y is a fundamental solution of (5.1)), then B satisfies (5.2).*

(ii) *Conversely, assume that L is a σ-Picard–Vessiot extension for $\delta(y) = Ay$ such that $k = K^\delta$ is linearly σ^d-closed (see Definition A.4). If there exists a matrix $B \in \mathrm{Gl}_n(K)$ verifying (5.2), then there exists a fundamental solution $Y \in \mathrm{Gl}_n(L)$ of (5.1).*

Proof. For (i), observe that

$$\sigma^d(\delta(Y)) = \sigma^d(A)\sigma^d(Y) = \sigma^d(A)BY$$

and

$$\delta(\sigma^d(Y)) = \delta(BY) = \delta(B)Y + B\delta(Y) = (\delta(B) + BA)Y.$$

Because $\delta(\sigma^d(Y)) = \hbar_d\sigma^d(\delta(Y))$, this implies (5.2).

To prove (ii), fix $Y \in \mathrm{Gl}_n(L)$ with $\delta(Y) = AY$. Equation (5.2) implies that

$$\delta(\sigma^d(Y)) = \hbar_d \sigma^d(\delta(Y)) = \hbar_d \sigma^d(A)\sigma^d(Y) = (\delta(B)B^{-1} + BAB^{-1})\sigma^d(Y).$$

Hence

$$\delta(B^{-1}\sigma^d(Y)) = B^{-1}\delta(\sigma^d(Y)) - B^{-1}\delta(B)B^{-1}\sigma^d(Y) = AB^{-1}\sigma^d(Y).$$

We conclude that there exists $C \in \mathrm{Gl}_n(k)$ such that $B^{-1}\sigma^d(Y) = YC$. Since k is linearly σ^d-closed, there exists $D \in \mathrm{Gl}_n(k)$ such that $\sigma^d(D) = C^{-1}D$. The matrix $YD \in \mathrm{Gl}_n(L)$ is a fundamental solution of (5.1). □

Situations where the integer d is strictly greater than 1 have to be taken into consideration. Indeed, the following example shows that, for any prime integer p, one can construct a differential system which is σ^p-integrable and not σ^d-integrable for $d < p$.

Example 5.3. Let p be a prime integer, and let $K = \mathbb{C}(\alpha, x)$ be the $\delta\sigma$-field endowed with $\delta(x) = 1, \delta(\alpha) = 0$, and $\sigma(x) = x, \sigma(\alpha) = \alpha + \frac{1}{p}$. Let $d \in \mathbb{N}^\times$. The linear differential system $\delta(y) = \frac{\alpha}{1-x}y$ is σ^d-integrable if and only if there exists $b \in \mathbb{C}(\alpha, x)$ such that $\delta(b) = \frac{\frac{d}{p}}{1-x}b$. Since residues of logarithmic derivatives of rational functions must be integers, one easily sees that the differential system is σ^p-integrable and not σ^d-integrable for $d < p$.

Then a natural question is the existence of a bound for such an integer d, in terms of the coefficients of the matrix. One could hope to get this bound by taking a closer look at the local monodromies and their orbits under the action of σ. Nonetheless, in the applications to come, in order to prove that we have no transformal relations between the solutions and thus no σ^d-integrability, we will treat the integer d as a parameter.

5.2. Some examples

In this section, we show that σ^d-integrability is a situation that occurs rather frequently. Notice that, in p-adic differential equation theory, the σ^d-integrability with respect to an action of the lift of the Frobenius from the positive characteristic is called *strong Frobenius structure*. In such a context, R. Crew, in [8], has proved a weak form of the 'only if' part of Theorem 5.11 below.

5.2.1. Contiguity relations. The case of contiguity relations for hypergeometric equations is an example of the situation described in the previous definition.

Example 5.4. Let $\mathbb{C}(\alpha, x)$ be a rational function field in two variables, equipped with the derivation $\delta = \frac{d}{dx}$ and the $\mathbb{C}(x)$-linear automorphism σ such that $\sigma(\alpha) = \alpha + 1$. We consider the hypergeometric differential equation $\delta(y) = \frac{\alpha}{1-x}y$, whose solution is given by $\sum_{n \geqslant 0} \frac{(\alpha)_n}{n!}x^n$. It is easy to see that $\sigma(y) = \frac{1}{1-x}y$ and $\delta(y) = \frac{\alpha}{1-x}y$ are compatible, and that therefore $\delta(y) = \frac{\alpha}{1-x}y$ is σ-integrable.

Example 5.5. Bessel's equation

$$x^2\delta^2(y) + x\delta(y) + (x^2 - \alpha^2)y = 0$$

is a classical example of a differential equation which is σ-integrable with respect to the shift $\sigma : \alpha \mapsto \alpha + 1$. The well known contiguity relations satisfied by the Bessel functions J_α and Y_α can be written in matrix form as $\sigma(Y) = BY$, where

$$
B = \begin{pmatrix} \dfrac{\alpha}{x} & -1 \\ \dfrac{-\alpha(\alpha+1)}{x^2} + 1 & \dfrac{\alpha+1}{x} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} J_\alpha & Y_\alpha \\ J'_\alpha & Y'_\alpha \end{pmatrix}
$$

is a fundamental solution matrix. The σ-Galois group of Bessel's equation is Sl_2^σ. See Example 2.12 in [15].

In higher dimension, contiguity relations exist for non-resonant A-hypergeometric equations (see [16, Theorem 6.9.1]) as well as difference equations compatible with trigonometric KZ-differential equations (cf. [55]). It would be very interesting to understand these results in light of a generalization of our parameterized Galois theory.

5.2.2. Strong Frobenius structure of Dwork's exponential. Let k be a ultrametric field of characteristic zero, complete with respect to a discrete valuation, and let $\mathbb{F}_q$ be its residue field, of characteristic $p > 0$ and cardinality q. We denote by $|\,|$ the p-adic norm of k, normalized so that $|p| = p^{-1}$. The ring $\mathcal{E}_k^\dagger$ of all $f = \sum_{n\in\mathbb{Z}} a_n x^n$, with $a_n \in k$, such that

- there exists $\varepsilon > 0$, depending on f, such that, for any $1 < \varrho < 1 + \varepsilon$, we have $\lim_{n\to\pm\infty} |a_n| \varrho^n = 0$, and
- $\sup_n |a_n|$ is bounded,

is actually a Henselian field with residue field $\mathbb{F}_q((x))$.

We consider the base field $K = \mathcal{E}_k^\dagger$ endowed with the derivation $\delta = x\frac{d}{dx}$ and an action of a lifting F of the Frobenius automorphism of $\mathbb{F}_q$. Namely, we consider an endomorphism F of k such that $|F(a) - a^p| < 1$, for any $a \in k$, $|a| \leqslant 1$. We extend the action of F to K by setting $F(x) = x^p$, so that $F(\sum_{n\in\mathbb{Z}} a_n x^n) = \sum_{n\in\mathbb{Z}} F(a_n) x^{pn}$. We have

$$
\delta \circ F = pF \circ \delta.
$$

In the framework of p-adic differential equations (see [27] for an introduction to this topic), the F^d-integrability of a differential equation is called Frobenius structure (Definition 17.1.1 in [27]). This notion plays a crucial role in the study of p-adic differential equations since it is the analog of monodromy in the complex setting. The following examples illustrate how the F-Galois group distinguishes among differential equations the ones with Frobenius structure, unlike the 'usual' Galois group.

Let us suppose that there exists $\pi \in k$ such that $\pi^{p-1} = -p$. The Robba exponential $\exp(\pi x) := \sum_{n\geqslant 0} \frac{\pi^n x^n}{n!}$ is a solution of $\delta(y) = \pi x y$, has radius of convergence 1, and hence does not belong to K (see for instance [42, Theorem 2.1]). On the other hand, one can show that $\exp(\pi x)^p \in K$. This shows that the usual Galois group of $\delta(y) = \pi x y$ is the cyclic group of order p.

In this particular case, some sharp p-adic estimates show that $\frac{F(\exp(\pi x))}{\exp(\pi x)} \in K$ (see Ch. 2 in [17]). Therefore the σ-Galois group is μ_p^σ; cf. Example 2.14 in [15].

5.3. Order-1 differential equations with cyclic Galois group of prime order

In the appendix, we prove that any proper σ-closed subgroup of μ_p, the algebraic group of pth roots of unity, contains the σ-polynomial $\sigma^m(x) - \sigma^{m+d}(x) \in k\{x, x^{-1}\}_\sigma = k\{\mathbf{G}_m\}$ in its vanishing ideal inside $\mathbf{G}_m$, for convenient integers $d \geqslant 1$ and $m \geqslant 0$. (See Proposition A.8.) We are going to use this fact in the proof of the following proposition.

Proposition 5.6. *Let K be a $\delta\sigma$-field, and let $L|K$ be a σ-Picard–Vessiot extension for a differential equation $\delta(y) = ay$ with $a \in K^\times$. We assume that the Zariski closure of σ-$\mathrm{Gal}(L|K)$, i.e., the 'usual' Galois group of $\delta(y) = ay$, is μ_p, for some prime number p. If σ-$\mathrm{Gal}(L|K)$ is a proper subgroup of μ_p, there exist two integers $M \geqslant 0$ and $d \geqslant 1$ such that the system $\delta(y) = \hbar_m \sigma^m(a)y$ is σ^d-integrable for all $m \geqslant M$.*

Proof. Let $z \in L^\times$ be a non-zero solution of $\delta(y) = ay$, and let $R = K\{z, z^{-1}\}_\sigma$ be the σ-Picard–Vessiot ring of L. We identify σ-$\mathrm{Gal}(L|K)$ with a σ-closed subgroup of $\mathbf{G}_m$ via z. It follows from Proposition A.8 that there exist integers $d \geqslant 1$ and $M \geqslant 0$ such that $\sigma^{M+d}(x) - \sigma^M(x)$ belongs to $\mathbb{I}(\sigma$-$\mathrm{Gal}(L|K))$. Therefore $\sigma^{m+d}(x) - \sigma^m(x) \in \mathbb{I}(\sigma$-$\mathrm{Gal}(L|K))$ for any $m \geqslant M$. Now, for all k-σ-algebras S and for all $\tau \in \sigma$-$\mathrm{Gal}(L|K)(S)$, in $R \otimes_k S$ we have $\tau(z \otimes 1) = z \otimes c$, for some $c \in S$, with $\sigma^{m+d}(c) = \sigma^m(c)$. Therefore

$$\tau(\sigma^{m+d}(z)\sigma^m(z)^{-1} \otimes 1) = \sigma^{m+d}(z)\sigma^m(z)^{-1} \otimes \sigma^{m+d}(c)\sigma^m(c)^{-1} = \sigma^{m+d}(z)\sigma^m(z)^{-1} \otimes 1.$$

This means that $\sigma^{m+d}(z)\sigma^m(z)^{-1} \in K$ by Galois correspondence. Since $\sigma^m(z)$ is a solution of the equation $\delta(y) = \hbar_m \sigma^m(a)y$, we find that this system is σ^d-integrable for all $m \geqslant M$. $\qquad\square$

Example 5.7. In the notation introduced for the Dwork exponential, we can consider the differential equation $\delta(y) = \frac{y}{p}$. Its usual differential Galois group is μ_p, since the cyclic extension $K(x^{1/p})$ is generated by a solution of $\delta(y) = \frac{y}{p}$. We have $F(x^{1/p}) = x$, which obviously is F-integrable.

5.4. A descent result for σ^d-integrability

Our first concern is to show that 'being σ^d-integrable' descends along a σ-separable σ-field extension of the δ-constants (see § 2.2 for the definitions).

Let K be a $\delta\sigma$-field. We would like to extend the σ-field of δ-constants $k := K^\delta$ to a σ-field extension $\widetilde{k}$ of k. The ring $K \otimes_k \widetilde{k}$ is naturally a $\delta\sigma$-ring (where we consider $\widetilde{k}$ as δ-constant). Because k is relatively algebraically closed in K (Corollary to Proposition 4, Ch. II, § 4, p. 94 in [29]), we know that $K \otimes_k \widetilde{k}$ is an integral domain. Let

$$\widetilde{K} := \mathrm{Quot}(K \otimes_k \widetilde{k})$$

denote the quotient field of $K \otimes_k \widetilde{k}$. To ensure that σ is injective on $K \otimes_k \widetilde{k}$, and hence that it extends to $\widetilde{K}$, we have to assume that $\widetilde{k}$ is σ-separable over k. Note that, by Corollary A.14 in [15], this is automatic if k is inversive. Then $\widetilde{K}$ is naturally a $\delta\sigma$-field. Because $K \otimes_k \widetilde{k}$ is δ-simple by Lemma 2.1, it follows that $\widetilde{K}^\delta = \widetilde{k}$.

Proposition 5.8. *Let K be a $\delta\sigma$-field and $A \in K^{n \times n}$. Let $\widetilde{k}$ be a σ-separable σ-field extension of $k := K^\delta$ and $\widetilde{K} = \mathrm{Quot}(K \otimes_k \widetilde{k})$. Then the equation $\delta(y) = Ay$ is σ^d-integrable over K if and only if $\delta(y) = Ay$ is σ^d-integrable over $\widetilde{K}$.*

To prove the proposition above we need a lemma on the behavior of the vector space of solutions with respect to an extension of the constant field.

Lemma 5.9. *Let K be a δ-field and $A \in K^{n \times n}$. Let $\widetilde{k}$ be a (δ-constant) field extension of $k := K^\delta$, and let $\widetilde{K}$ denote the quotient field of $K \otimes_k \widetilde{k}$. Let us denote by $\widetilde{\mathbb{V}}$ (respectively, $\mathbb{V}$) the solution space of $\delta(y) = Ay$ in $\widetilde{K}^n$ (respectively, K^n). Then $\widetilde{\mathbb{V}} \simeq \mathbb{V} \otimes_k \widetilde{k}$.*

Proof. It is clear that the canonical map $\mathbb{V} \otimes_k \widetilde{k} \to \widetilde{\mathbb{V}}$ is injective. Indeed, if elements from $K^n \subset (K \otimes_k \widetilde{k})^n = K^n \otimes_k \widetilde{k}$ are linearly independent over k, they are also linearly independent over $\widetilde{k}$.

It remains to see that $\mathbb{V} \otimes_k \widetilde{k} \to \widetilde{\mathbb{V}}$ is surjective. Let $z \in \widetilde{\mathbb{V}} \subset \widetilde{K}^n$. We know from Lemma 2.1 that $S := K \otimes_k \widetilde{k}$ is a δ-simple δ-ring. We claim that $z \in S^n$. Set $\mathfrak{a} := \{r \in S \mid rz \in S^n\} \subset S$. Then $\mathfrak{a}$ is a non-zero ideal of S, and we will now show that $\mathfrak{a}$ is a δ-ideal of S. For $r \in \mathfrak{a}$ we have $\delta(rz) = r\delta(z) + \delta(r)z = rAz + \delta(r)z$. As $\delta(rz)$ and $rAz = Arz$ lie in S^n, it follows that $\delta(r)z$ lies in S^n. By the simplicity of S, we see that $1 \in \mathfrak{a}$; i.e., $z \in S^n$.

Now, let (λ_i) be a k-basis of $\widetilde{k}$ (which can be infinite, of course). We can write $z = \sum z_i \otimes \lambda_i$ for some $z_i \in K^n$. Then $\delta(z) = \sum \delta(z_i) \otimes \lambda_i$. On the other hand, $\delta(z) = Az = \sum Az_i \otimes \lambda_i$. Comparing the coefficients yields $\delta(z_i) = Az_i$ for all i. This shows that z lies in the image $\mathbb{V} \otimes_k \widetilde{k} \to \widetilde{\mathbb{V}}$. $\qquad\square$

Proof of Proposition 5.8. One implication is tautological, so let us prove the non-trivial one. Assume that $\delta(y) = Ay$ is σ^d-integrable over $\widetilde{K}$. This means that there exists $\widetilde{B} \in \mathrm{Gl}_n(\widetilde{K})$ such that $\delta(\widetilde{B}) + \widetilde{B}A = \hbar_d \sigma^d(A)\widetilde{B}$. The latter equation, when considered as an equation in $\widetilde{B}$, is a linear differential system over K of order n^2. Let $\mathbb{V} \subset K^{n \times n}$ (respectively, $\widetilde{\mathbb{V}} \subset \widetilde{K}^{n \times n}$) denote the corresponding solution vector space over k (respectively, over $\widetilde{k}$). We know from Lemma 5.9 that $\widetilde{\mathbb{V}} = \mathbb{V} \otimes_k \widetilde{k}$. We have to find a $B \in \mathbb{V}$ with non-zero determinant. Let $v_1, \ldots, v_m \in \widetilde{K}^{n \times n}$ be a k-basis of $\mathbb{V}$. Then $v_1, \ldots, v_m$ is also a $\widetilde{k}$-basis of $\widetilde{\mathbb{V}}$. It follows from the Wronskian lemma (Lemma 1.7, p. 7 in [44]) that $v_1, \ldots, v_m$ are also $\widetilde{K}$-linearly independent. With respect to the basis $v_1, \ldots, v_m$ of $\mathbb{V} \otimes_k \widetilde{K} \subset \widetilde{K}^{n \times n}$, the determinant $\det \colon \mathbb{V} \otimes_k \widetilde{K} \to \widetilde{K}$ is given by a polynomial (in m-variables) with coefficients in $\widetilde{K}$. Because the determinant does not vanish on all of $\widetilde{\mathbb{V}}$, this shows that the determinant cannot vanish on all of $\mathbb{V}$. So $\delta(y) = Ay$ is σ^d-integrable over K. $\qquad\square$

5.5. Galoisian characterization of σ^d-integrability

Let k be a σ-field, and let $d \geqslant 1$. For every k-σ-algebra S, we set

$$\mathrm{Gl}_{n,k}^{\sigma^d}(S) = \{g \in \mathrm{Gl}_n(S) \mid \sigma^d(g) = g\}.$$

Then $\mathrm{Gl}_{n,k}^{\sigma^d}$ is a σ-closed subgroup of $\mathrm{Gl}_{n,k}$. If $G \leqslant \mathrm{Gl}_{n,k}$ is a σ-closed subgroup and $h \in \mathrm{Gl}_n(k)$, then the σ-closed subgroup $hGh^{-1} \leqslant \mathrm{Gl}_{n,k}$ obtained from G by conjugation

with h is given by

$$(hGh^{-1})(S) = \{hgh^{-1} \mid g \in G(S)\} \leqslant \mathrm{Gl}_n(S),$$

for every k-σ-algebra S.

Definition 5.10. Let G be a σ-closed subgroup of $\mathrm{Gl}_{n,k}$. We call G a σ^d-*constant* subgroup of $\mathrm{Gl}_{n,k}$ if G is contained in $\mathrm{Gl}_{n,k}^{\sigma^d}$. If $\widetilde{k}$ is a σ-field extension of k, we say that G *is conjugate over* $\widetilde{k}$ *to a* σ^d-*constant group* if there exists $h \in \mathrm{Gl}_n(\widetilde{k})$ such that $hG_{\widetilde{k}}h^{-1} \leqslant \mathrm{Gl}_{n,\widetilde{k}}$ is σ^d-constant.

We prove a result on σ^d-integrability, which is analogous to Proposition 2.9 in [20], and § 1.2.1 in [14]. The statement below is more general than the cited results, because it relies on Proposition 5.8.

Theorem 5.11. *Let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = Ay$, with $A \in K^{n \times n}$. Then $\delta(y) = Ay$ is σ^d-integrable over K if and only if there exists a σ-separable σ-field extension $\widetilde{k}$ of $k := K^\delta$ such that the σ-Galois group σ-$\mathrm{Gal}(L|K)$ is conjugate over $\widetilde{k}$ to a σ^d-constant subgroup of $\mathrm{Gl}_{n,\widetilde{k}}$.*

First of all, we are going to prove one implication of Theorem 5.11 (see Lemma 5.15 below). To this purpose we need a few lemmas.

Lemma 5.12. *Let $L|K$ be a σ-Picard–Vessiot extension and $\widetilde{k}$ a σ-separable σ-field extension of $k = K^\delta$. Then $\widetilde{L} = \mathrm{Quot}(L \otimes_k \widetilde{k})$ is a σ-Picard–Vessiot extension of $\widetilde{K} = \mathrm{Quot}(K \otimes_k \widetilde{k})$ and the σ-Galois group $\widetilde{G}$ of $\widetilde{L}|\widetilde{K}$, is obtained from the σ-Galois group G of $L|K$ by base extension; i.e., $\widetilde{G} = G_{\widetilde{k}}$.*

Proof. As $\widetilde{L}^\delta = \widetilde{k} = \widetilde{K}^\delta$, it is clear that $\widetilde{L}|\widetilde{K}$ is a σ-Picard–Vessiot extension. Let $R \subset L$ (respectively, $\widetilde{R} \subset \widetilde{L}$) denote the σ-Picard–Vessiot rings of $L|K$ (respectively, of $\widetilde{L}|\widetilde{K}$). Then $\widetilde{R}$ is obtained from $R \otimes_k \widetilde{k}$ by localizing at the multiplicatively closed set of all non-zero divisors of $K \otimes_k \widetilde{k}$. It follows that, for every $\widetilde{k}$-σ-algebra S,

$$\begin{aligned} G_{\widetilde{k}}(S) &= \mathrm{Aut}^{\delta\sigma}(R \otimes_k S | K \otimes_k S) = \mathrm{Aut}^{\delta\sigma}((R \otimes_k \widetilde{k}) \otimes_{\widetilde{k}} S | (K \otimes_k \widetilde{k}) \otimes_{\widetilde{k}} S) \\ &= \mathrm{Aut}^{\delta\sigma}(\widetilde{R} \otimes_{\widetilde{k}} S | \widetilde{K} \otimes_{\widetilde{k}} S) = \widetilde{G}(S). \end{aligned}$$

This ends the proof. $\qquad\square$

Lemma 5.13. *Let k be a σ-field, $d \geqslant 1$, and $B \in \mathrm{Gl}_n(k)$. There exists a σ-separable σ-field extension $\widetilde{k}$ of k such that $\sigma^d(y) = By$ has a fundamental solution matrix in $\widetilde{k}$.*

Proof. Let $X_1, \ldots, X_d$ denote $n \times n$ matrices of indeterminates over k. Extend σ from k to $S := k[X_1, \ldots, X_d]$ by $\sigma(X_1) = X_2, \ldots, \sigma(X_{d-1}) = X_d, \sigma(X_d) = BX_1$. Obviously S is σ-separable over k. It follows from Lemma A.16 in [15] that the quotient field $\widetilde{k}$ of S is also σ-separable over k. Clearly $\sigma^d(X_1) = BX_1$. $\qquad\square$

Lemma 5.14. *Let k be a linearly σ-closed σ-field. Then k is linearly σ^d-closed for every $d \geqslant 1$.*

Proof. Assume that $d \geqslant 1$ and $B \in \mathrm{Gl}_n(k)$. We have to find a $Y \in \mathrm{Gl}_n(k)$ with $\sigma^d(Y) = BY$. Let $y_1, \ldots, y_d$ denote vectors of size n. The linear system of order nd given by

$$\sigma(y_1) = y_2, \ldots, \sigma(y_{d-1}) = y_d, \sigma(y_d) = By_1$$

has a fundamental solution matrix $Z \in \mathrm{Gl}_{nd}(k)$. Since the first n rows of Z are linearly independent, there exist n columns of Z such that the corresponding vectors of the first n entries are linearly independent. These form a fundamental solution matrix for $\sigma^d(y) = By$. $\qquad\square$

So we are ready to prove the first part of Theorem 5.11.

Lemma 5.15. *Let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = Ay$, with $A \in K^{n \times n}$. If $\delta(y) = Ay$ is σ^d-integrable over K, then, for a suitable σ-separable σ-field extension $\widetilde{k}$ of $k = K^{\delta}$, the σ-Galois group $\sigma\text{-}\mathrm{Gal}(L|K)$ is conjugate over $\widetilde{k}$ to a σ^d-constant subgroup of $\mathrm{Gl}_{n,\widetilde{k}}$. If k is linearly σ-closed, we can take $\widetilde{k}$ equal to k.*

Proof. Let $B \in \mathrm{Gl}_n(K)$ be such that the system

$$\begin{cases} \delta(y) = Ay \\ \sigma^d(y) = By \end{cases} \tag{5.3}$$

is integrable. Let us also fix a fundamental solution matrix $Y \in \mathrm{Gl}_n(L)$ for $\delta(y) = Ay$. The integrability condition (5.2) implies that

$$\begin{aligned} \delta((BY)^{-1}\sigma^d(Y)) &= \delta((BY)^{-1})\sigma^d(Y) + (BY)^{-1}\delta(\sigma^d(Y)) \\ &= -(BY)^{-1}\delta(BY)(BY)^{-1}\sigma^d(Y) + (BY)^{-1}\hbar_d\sigma^d(\delta(Y)) \\ &= -(BY)^{-1}((\delta(B)Y + BAY)(BY)^{-1}\sigma^d(Y) - \hbar_d\sigma^d(A)\sigma^d(Y)) \\ &= -(BY)^{-1}((\delta(B) + BA)B^{-1} - \hbar_d\sigma^d(A))\sigma^d(Y) = 0. \end{aligned}$$

Thus, there exists $D \in \mathrm{Gl}_n(k)$ such that

$$\sigma^d(Y) = BYD. \tag{5.4}$$

By Lemma 5.13 there exist a σ-separable σ-field extension $\widetilde{k}$ of k and a matrix $U \in \mathrm{Gl}_n(\widetilde{k})$ with $\sigma^d(U) = D^{-1}U$.

Set $\widetilde{K} = \mathrm{Quot}(K \otimes_k \widetilde{k})$ and $\widetilde{L} = \mathrm{Quot}(L \otimes_k \widetilde{k})$. We will now work in the σ-Picard–Vessiot extension $\widetilde{L}|\widetilde{K}$ (Lemma 5.12). The matrix $V := YU \in \mathrm{Gl}_n(\widetilde{L})$ is solution of (5.3); in fact,

$$\delta(V) = \delta(Y)U = AV$$

and

$$\sigma^d(V) = \sigma^d(Y)D^{-1}U = BYU = BV.$$

For $\tau \in \sigma\text{-}\mathrm{Gal}(\widetilde{L}|\widetilde{K})$ and S a $\widetilde{k}$-σ-algebra, let $[\tau]_V \in \mathrm{Gl}_n(S)$ be such that $\tau(V \otimes 1) = (V \otimes 1)[\tau]_V$. We have

$$\tau(\sigma^d(V \otimes 1)) = \tau(BV \otimes 1) = (BV \otimes 1)[\tau]_V$$

and

$$\tau(\sigma^d(V \otimes 1)) = \sigma^d(\tau(V \otimes 1)) = \sigma^d((V \otimes 1)[\tau]_V) = (BV \otimes 1)\sigma^d([\tau]_V).$$

We deduce that $[\tau]_V = \sigma^d([\tau]_V)$. Therefore, if we consider $\widetilde{G} := \sigma\text{-Gal}(\widetilde{L}|\widetilde{K})$ as a σ-closed subgroup of $\mathrm{Gl}_{n,\widetilde{k}}$ via the embedding associated with the choice of the fundamental solution matrix V, then $\widetilde{G}$ is a σ^d-constant subgroup of $\mathrm{Gl}_{n,\widetilde{k}}$. We consider the σ-Galois group $G := \sigma\text{-Gal}(L|K)$ as a σ-closed subgroup of $\mathrm{Gl}_{n,k}$ via the embedding associated with the choice of the fundamental solution matrix Y. Since $V = YU$, it is clear from Lemma 5.12 that $G_{\widetilde{k}}$ is conjugate to $\widetilde{G}$ via $U \in \mathrm{Gl}_n(\widetilde{k})$. $\square$

Remark 5.16. If k is linearly σ-closed, we can choose $U \in \mathrm{Gl}_n(k)$ and $\widetilde{k} = k$ by Lemma 5.14.

The following lemma is a crucial step in the proof of the inverse implication in Theorem 5.11.

Lemma 5.17. *Let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = Ay$, where $A \in K^{n \times n}$. If the σ-Galois group $\sigma\text{-Gal}(L|K)$ is conjugate (over $k = K^\delta$) to a σ^d-constant subgroup of $\mathrm{Gl}_{n,k}$, then $\delta(y) = Ay$ is σ^d-integrable over K.*

Proof. Since conjugation in $\mathrm{Gl}_n(k)$ corresponds to a change of fundamental solution matrix, we can find a fundamental solution matrix $Y \in \mathrm{Gl}_n(L)$ such that $G := \sigma\text{-Gal}(L|K)$ is a σ^d-constant subgroup of $\mathrm{Gl}_{n,k}$, with respect to the embedding $\tau \mapsto [\tau]_Y$ determined by Y. (As before, for a k-σ-algebra S and $\tau \in \sigma\text{-Gal}(L|K)(S)$ we denote by $[\tau]_Y \in \mathrm{Gl}_n(S)$ the matrix such that $\tau(Y \otimes 1) = Y \otimes 1[\tau]_Y$.) We set $B = \sigma^d(Y)Y^{-1} \in \mathrm{Gl}_n(L)$. The matrix B is an invariant of $\sigma\text{-Gal}(L|K)$; in fact, for every k-σ-algebra S and every $\tau \in \sigma\text{-Gal}(L|K)(S)$, we have

$$\begin{aligned}
\tau(B \otimes 1) &= \tau(\sigma^d(Y \otimes 1)(Y \otimes 1)^{-1}) = \sigma^d(\tau(Y \otimes 1))\tau(Y^{-1} \otimes 1) \\
&= \sigma^d(Y \otimes 1[\tau]_Y)[\tau]_Y^{-1}Y^{-1} \otimes 1 = \sigma^d(Y)Y^{-1} \otimes 1 = B \otimes 1,
\end{aligned}$$

since $\sigma^d([\tau]_Y) = [\tau]_Y$. It follows from the σ-Galois correspondence (Theorem 2.11) that $B \in \mathrm{Gl}_n(K)$. Since $Y \in \mathrm{Gl}_n(L)$ satisfies $\delta(Y) = AY$ and $\sigma^d(Y) = BY$, it follows easily from $\delta(\sigma^d(Y)) = \hbar_d \sigma^d(\delta(Y))$ that $\delta(y) = Ay$ is σ^d-integrable. $\square$

Finally, we are able to prove Theorem 5.11.

Proof of Theorem 5.11. Let $\widetilde{k}$ be a σ-separable σ-field extension of k such that $\sigma\text{-Gal}(L|K)$ is conjugate over $\widetilde{k}$ to a σ^d-constant group. Consider the σ-Picard–Vessiot extension $\widetilde{L}|\widetilde{K}$ obtained from $L|K$ by extending the δ-constants from k to $\widetilde{k}$ (Lemma 5.12). Since $\sigma\text{-Gal}(\widetilde{L}|\widetilde{K})$ is conjugate over $\widetilde{k} = \widetilde{K}^\delta$ to a σ^d-constant group, Lemma 5.17 implies that $\delta(y) = Ay$ is σ^d-integrable over $\widetilde{K}$. It follows from Proposition 5.8 that $\delta(y) = Ay$ is σ^d-integrable over K. The other implication is immediate from Lemma 5.15. $\square$

Corollary 5.18. *Let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = Ay$, with $A \in K^{n \times n}$. Assume that $k := K^\delta$ is linearly σ-closed. Then $\delta(y) = Ay$ is σ^d-integrable (over K) if and only if the σ-Galois group $\sigma\text{-Gal}(L|K)$ is conjugate (over k) to a σ^d-constant subgroup of $\mathrm{Gl}_{n,k}$.*

Proof. This is clear from Lemmas 5.15 and 5.17. $\square$

Remark 5.19. It may appear quite difficult to determine a priori, given a differential system $\delta(y) = Ay$, if this system is σ^d-integrable for some $d \geqslant 1$. As any d may occur (Example 5.3), there are in principle infinitely many compatibility conditions one has to check. In the next section, we will present some special cases where one is able to check the conditions by treating d as a parameter.

6. Discrete integrability in the case of almost simple Galois groups

We begin this section with the study of the discrete integrability of differential system having a simple classical Galois group. This is equivalent to asking that the σ-Galois group is a Zariski dense σ-closed subgroup of a simple algebraic group. This situation is very rigid, as proved in Theorem A.20, and the σ-Galois group has to be either equal to the whole simple algebraic group or conjugate to a σ^d-constant group for some positive integer d. Combined with our σ^d-integrability criteria, we find that either we have a very strong transformal relation between the solutions of our differential system, namely the σ^d-integrability, or we have no transformal relations at all. In Proposition 6.1 below we make this discussion precise.

Recall that a linear algebraic group H over a field k (i.e., an affine group scheme of finite type over k) is called *simple* if it is non-commutative, connected, and every normal closed subgroup is trivial. If H is non-commutative, connected, and every normal closed connected subgroup is trivial, then H is called *almost simple*. We say that H is *absolutely (almost) simple* if the base extension of H to the algebraic closure of k is (almost) simple.

Proposition 6.1. *Let K be an inversive $\delta\sigma$-field, $A \in K^{n \times n}$, and let $L|K$ be a σ-Picard–Vessiot extension for $\delta(y) = Ay$. We assume that the Zariski closure H of σ-$\mathrm{Gal}(L|K)$ inside $\mathrm{Gl}_{n,k}$ is an absolutely simple algebraic group of dimension $t \geqslant 1$ over $k = K^\delta$. Then the following statements are equivalent.*

(i) *σ-$\mathrm{Gal}(L|K)$ is a proper σ-closed subgroup of H.*

(ii) *The σ-transcendence degree of $L|K$ is strictly less than t.*

(iii) *There exists $d \in \mathbb{Z}_{>0}$ such that the system $\delta(y) = Ay$ is σ^d-integrable.*

Proof. Let $Y \in \mathrm{Gl}_n(L)$ be a fundamental solution matrix for $\delta(y) = Ay$ and let $G = \sigma$-$\mathrm{Gal}(L|K)$. Assumption (iii) implies the existence of matrices $B \in \mathrm{Gl}_n(K)$ and $D \in \mathrm{Gl}_n(k)$ such that $\sigma^d(Y) = BYD$. (See equation (5.4).) Therefore σ-$\mathrm{trdeg}(L|K) = 0 < t$, and we see that (iii) implies (ii).

We know by Proposition 2.9 that σ-$\mathrm{trdeg}(L|K) = \sigma$-$\dim_k(G)$. Therefore assumption (ii) means that we have σ-$\dim_k(G) = \sigma$-$\mathrm{trdeg}(L|K) < t$, and therefore G is a proper subgroup of H.

It remains to prove that (i) implies (iii). We first reduce to the case that $k = K^\delta$ is algebraically closed. Since K is inversive, $k = K^\delta$ is also inversive. Let $\widetilde{k}$ denote an algebraic closure of k, and extend σ from k to $\widetilde{k}$. Then $\widetilde{k}$ is automatically inversive. Thus $\widetilde{K} := \mathrm{Quot}(K \otimes_k \widetilde{k})$ is also inversive. We know from Lemma 5.12 that $\widetilde{L} = \mathrm{Quot}(K \otimes_k \widetilde{k})$ is a σ-Picard–Vessiot extension whose σ-Galois group $\widetilde{G}$ equals $G_{\widetilde{k}}$. It is easy to see

that the Zariski closure of $\widetilde{G}$ in $\mathrm{Gl}_{n,\widetilde{k}}$ equals $H_{\widetilde{k}}$. Since G is properly contained in H, $\widetilde{G}$ is also properly contained in $H_{\widetilde{k}}$. Thus, by assumption, it follows that $\delta(y) = Ay$ is σ^d-integrable over $\widetilde{K}$ for some convenient integer $d \geqslant 1$. But then Proposition 5.8 implies that $\delta(y) = Ay$ is σ^d-integrable over K.

So from now on we can assume that k is algebraically closed. Since K is inversive, Corollary 4.4 in [15] asserts that $G = \sigma\text{-}\mathrm{Gal}(L|K)$ is σ-reduced. Therefore, Theorem A.20 says that there exists a σ-separable σ-field extension $\widetilde{k}$ of k and an integer $d \geqslant 1$ such that G is conjugate in $\mathrm{Gl}_{n,\widetilde{k}}$ to a σ^d-constant group. Theorem 5.11 ends the proof. $\qquad\square$

The previous proposition allows us to determine very easily if the solutions of a differential system $\delta(y) = Ay$, with simple usual Galois group, are transformally independent or not. Indeed, if we prove that, for all $d \geqslant 0$, the differential system $\delta(y) + yA = \hbar_d \sigma^d(A)y$ has no rational solution $Y \in \mathrm{Gl}_n(K)$, there will be no (proper) transformal relations between the solutions of $\delta(y) = Ay$. Thus, we have reduced the question of the algebraic independence between the solutions and their successive transforms via σ, which is a non-linear problem, into a question of existence of rational solutions to a given *linear* differential system. This last question possesses many algorithmic answers, and we refer to § 4.1 of [44] for a detailed exposition of this subject.

However, in many situations of interest, the usual Galois group of a differential system is not simple but almost simple. See for example [3]. Unfortunately, as we show in the appendix, Proposition 6.1 fails, a priori, if the Zariski closure of $\sigma\text{-}\mathrm{Gal}(L|K)$ is only almost simple and not simple. This is mainly due to the fact that, contrary to the difference varieties studied in models of ACFA, the group k-σ-schemes associated to finite algebraic groups are not necessarily σ-constants.

We propose now two ways of getting rid of this problem. We first explore a naive less effective strategy to deal with the group $\mathrm{Sl}_{n,k}$ and then prove a more general statement for almost simple groups (see Theorem 6.4), based on a generalization of a result in [5] (see Theorem A.25). To conclude this section, we apply Theorem 6.4 to second-order differential equations, especially to the Airy equation.

We point out that Theorem A.25 on the classification of σ-closed subgroups of almost simple groups should also give applications of the theory developed in [41].

6.1. A naive approach to $\mathrm{Sl}_{n,k}$ through symmetric powers

Corollary 6.2 below yields to a characterization of the σ-algebraic relations satisfied by the solutions in terms of σ^d-integrability of the nth symmetric power of the initial differential system (see for instance Theorem 3.4 in [54] for a definition). Once again, this method allows us to translate the question of the existence of transformal relations among solutions of our initial differential system in terms of the much easier problem of existence of rational solutions to an auxiliary differential system. However, if the order of the initial system is n, the order of the auxiliary differential system will be less than or equal to $\binom{2n-1}{n}$. Computations may be then hard to manage.

Corollary 6.2. *Let K be an* inversive $\delta\sigma$*-field such that $k = K^\delta$ is algebraically closed,* $\mathcal{L}(y) = a_0 y + a_1 \delta(y) + \cdots + \delta^n(y)$ *a linear differential equation with coefficients in K, and*

$\mathcal{L}^{\circledS n}$ *the nth symmetric power of* $\mathcal{L}$. *We denote by*

$$A_{\mathcal{L}} = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{n-1} \end{pmatrix} \quad (\text{respectively, } A_{\mathcal{L}^{\circledS n}})$$

the companion matrix of $\mathcal{L}$ *(respectively, of* $\mathcal{L}^{\circledS n}$) *and by* $L|K$ *a* σ-*Picard–Vessiot extension for* $\delta(y) = A_{\mathcal{L}} y$. *We assume that the Zariski closure of* σ-$\mathrm{Gal}(L|K)$ *inside* $\mathrm{Gl}_{n,k}$ *is* $\mathrm{Sl}_{n,k}$. *Then the following statements are equivalent.*

(i) σ-$\mathrm{Gal}(L|K)$ *is a proper* σ-*closed subgroup of* $\mathrm{Sl}_{n,k}$.

(ii) *The* σ-*transcendence degree of* $L|K$ *is strictly less than* $n^2 - 1$.

(iii) *There exists* $d \in \mathbb{Z}_{>0}$ *such that the system* $\delta(y) = A_{\mathcal{L}^{\circledS n}} y$ *is* σ^d-*integrable.*

Proof. Let $y_1, \ldots, y_n \in L$ be n linearly independent solutions of $\mathcal{L}$ in L. By definition (see Theorem 3.4 in [54]), the symmetric power of order n of $\mathcal{L}$ is a linear differential equation with coefficients in K, whose solution space is spanned by the products of the form $y_{i_1} y_{i_2} \ldots y_{i_n}$, where $(i_1, \ldots, i_n)$ runs in $\{1, \ldots, n\}^n$. Let M be the $\delta\sigma$-field extension of K generated by the $y_{i_1} y_{i_2} \ldots y_{i_n}$. Then M is a σ-Picard–Vessiot extension for $\delta(y) = A_{\mathcal{L}^{\circledS n}} y$ over K. By Theorem 2.12, the group k-σ-scheme σ-$\mathrm{Gal}(L|M)$ is a normal subgroup of σ-$\mathrm{Gal}(L|K)$, and the quotient σ-$\mathrm{Gal}(L|K)/\sigma$-$\mathrm{Gal}(L|M)$ is isomorphic to σ-$\mathrm{Gal}(M|K)$. Since K is inversive, Corollary 4.4 in [15] implies that σ-$\mathrm{Gal}(L|K)$ and σ-$\mathrm{Gal}(M|K)$ are absolutely σ-reduced over k. The Zariski closure of σ-$\mathrm{Gal}(L|K)$ (respectively, σ-$\mathrm{Gal}(M|K)$) corresponds to the classical Galois group of $\delta(y) = A_{\mathcal{L}} y$ (respectively, $\delta(y) = A_{\mathcal{L}^{\circledS n}} y$), whose k-points are the K-δ-automorphism $\mathrm{Aut}^{\delta}(L_0|K)$ (respectively, $\mathrm{Aut}^{\delta}(M_0|K)$) of $L_0 = K\langle y_1, \ldots, y_n \rangle_{\delta}$ (respectively, $M_0 = K\langle y_{i_1} y_{i_2} \ldots y_{i_n} | (i_1, \ldots, i_n) \in \{1, \ldots, n\}^n \rangle_{\delta}$) (see Proposition 2.10). Since M_0 is a classical Picard–Vessiot extension, the usual Galois correspondence shows that $\mathrm{Aut}^{\delta}(L_0|M_0)$ is a normal algebraic subgroup of $\mathrm{Aut}^{\delta}(L_0|K) = \mathrm{Sl}_n(k)$. The center of $\mathrm{Aut}^{\delta}(L_0|K)$ is the cyclic group $\mu_n(k)$ of order n. Moreover, the action of $\lambda \in \mu_n(k)$ as an element of $\mathrm{Aut}^{\delta}(L_0|K)$ is determined by its action on the y_i, which is just the multiplication by λ. Then $\mu_n(k) \subset \mathrm{Aut}^{\delta}(L_0|K)$ fixes M_0. Thus, the algebraic group $\mathrm{Aut}^{\delta}(L_0|M_0)$ is a normal subgroup of $\mathrm{Sl}_n(k)$ containing $\mu_n(k)$. The group $\mathrm{Aut}^{\delta}(L_0|M_0)$ differs from $\mathrm{Sl}_n(k)$ because otherwise $M_0 = K$, and the y_i would be algebraically dependent over K. Since $\mu_n(k)$ is maximal among the normal subgroups of $\mathrm{Sl}_n(k)$, we find that $\mathrm{Aut}^{\delta}(L_0|M_0) = \mu_n(k)$. Then the Zariski closure of σ-$\mathrm{Gal}(M|K)$ is $\mathrm{PSl}_n(k)$, i.e., the projective special linear algebraic group over k.

Finally, the $\delta\sigma$-field extension $L|M$ is an algebraic field extension, and σ-$\mathrm{trdeg}(L|K) = \sigma$-$\mathrm{trdeg}(M|K)$ and σ-$\dim(\sigma$-$\mathrm{Gal}(L|K)) = \sigma$-$\dim(\sigma$-$\mathrm{Gal}(M|K))$. This shows that σ-$\mathrm{Gal}(L|K)$ is a proper σ-closed subgroup of $\mathrm{Sl}_{n,k}$ if and only if σ-$\mathrm{Gal}(M|K)$ is a proper σ-closed subgroup of $\mathrm{PSl}_{n,k}$. We conclude the proof by applying Proposition 6.1 to σ-$\mathrm{Gal}(M|K)$, a σ-reduced, Zariski dense σ-closed subgroup of the simple algebraic group $\mathrm{PSl}_{n,k}$. $\qquad\square$

As mentioned above, the third point in the equivalence of Corollary 6.2 may be difficult to test because of the order of the nth symmetric power of the initial differential equation. Therefore we propose below a more general method.

6.2. Characterization of σ-integrability for almost simple groups

For a differential system $\delta(y) = Ay$ with almost simple usual Galois group, we prove in Theorem 6.4 that the existence of transformal relations between the solutions in some σ-Picard–Vessiot extension L is equivalent to the σ^d-integrability of $\delta(y) = Ay$ over the relative algebraic closure K' of K inside L. That is, we extend Proposition 6.1 to the case of almost simple groups by allowing the σ^d-integrability to be algebraic and not only rational.

Unlike to the question of rational solutions, there is, to our knowledge, no effective method published to solve the problem of the existence of algebraic solutions of differential equations. However, in a private communication with the authors, M. Singer, relying on [50], gives an algorithm that starts with a linear differential equation $\mathcal{L}(y) = 0$ with coefficients in $\mathbb{C}(x)$, and produces a second linear differential equation whose solution space is spanned by the algebraic solutions of $\mathcal{L}$. Moreover, necessary conditions for the existence of algebraic solutions exist. We cite for instance the easy implication in the Grothendieck conjecture on p-curvatures; see [25]. In [6], the reader will also find some criteria for the existence of an algebraic solution based on residues.

We will need a simple lemma on extensions of σ-fields. Recall that a σ-ring R is called a σ-domain if it is an integral domain and $\sigma \colon R \to R$ is injective. Moreover, an extension of σ-fields $L|K$ is called σ-regular if $L \otimes_K \widetilde{K}$ is a σ-domain for every σ-field extension $\widetilde{K}$ of K.

Lemma 6.3. *Let $L|K$ be an extension of σ-fields such that K is inversive, and let K' denote the relative algebraic closure of K in L. Then $L|K'$ is σ-regular.*

Proof. By Lemma A.13(ii) in [15] it suffices to a find an inversive algebraically closed σ-field extension $\widetilde{K}$ of K' such that $L \otimes_{K'} \widetilde{K}$ is a σ-domain. Let $\widetilde{K}$ denote an algebraic closure of K containing K', and extend σ from K' to $\widetilde{K}$. Since K is inversive, $\widetilde{K}$ is also inversive. As the extension $L|K'$ is regular, $L \otimes_{K'} \widetilde{K}$ is a field, and consequently σ is automatically injective on $L \otimes_{K'} \widetilde{K}$. $\qquad\square$

Now, we are able to state our main theorem.

Theorem 6.4. *Let K be an inversive $\delta\sigma$-field, $\delta(y) = Ay$ a differential system with $A \in K^{n \times n}$, and $L|K$ a σ-Picard–Vessiot extension for $\delta(y) = Ay$. We assume that the Zariski closure H of σ-$\mathrm{Gal}(L|K)$ inside $\mathrm{Gl}_{n,k}$ is an absolutely almost simple algebraic group of dimension $t \geqslant 1$ over $k = K^\delta$. Let K' be the relative algebraic closure of K inside L. Then the following statements are equivalent.*

(i) *σ-$\mathrm{Gal}(L|K')$ is a proper σ-closed subgroup of H.*

(ii) *The σ-transcendence degree of $L|K$ is strictly less than t.*

(iii) *There exists $d \in \mathbb{Z}_{>0}$ such that the system $\delta(y) = Ay$ is σ^d-integrable over K'.*

Proof. As in the proof of Proposition 6.1, condition (iii) implies that σ-$\mathrm{trdeg}(L|K') = 0$. But then σ-$\mathrm{trdeg}(L|K) = \sigma$-$\mathrm{trdeg}(L|K') = 0 < t$. Thus (iii) implies (ii).

Condition (ii) implies that $\sigma\text{-}\dim(\sigma\text{-}\mathrm{Gal}(L|K)) = \sigma\text{-}\mathrm{trdeg}(L|K) < t$. Therefore $\sigma\text{-}\mathrm{Gal}(L|K)$ is properly contained in H. But then also $\sigma\text{-}\mathrm{Gal}(L|K') \leqslant \sigma\text{-}\mathrm{Gal}(L|K)$ is properly contained in H.

It remains to show that (i) implies (iii). Let us first show that the Zariski closure H' of $G' = \sigma\text{-}\mathrm{Gal}(L|K')$ in $\mathrm{Gl}_{n,k}$ equals H. Suppose that H' is properly contained in H. Then, since H is connected, the dimension t' of H' is strictly smaller than t. Let $Y \in \mathrm{Gl}_n(L)$ be a fundamental solution matrix for $\delta(y) = Ay$. Since $K'|K$ is algebraic, we arrive at the contradiction

$$\mathrm{trdeg}(K(Y)|K) = \mathrm{trdeg}(K'(Y)|K') = t' < t = \mathrm{trdeg}(K(Y)|K).$$

So $H = H'$, and condition (i) means that G' is properly contained in H'.

By Lemma 6.3 we know that $L|K'$ is σ-regular. By Proposition 4.3(iii) in [15] this implies that G' is absolutely σ-integral. To apply Theorem A.25 we need to go to the algebraic closure of k. So let $\widetilde{k}$ denote the algebraic closure of k, and extend σ from k to $\widetilde{k}$. Note that, since K is inversive, k and $\widetilde{k}$ are also inversive. Extending the δ-constant from k to $\widetilde{k}$ as in Lemma 5.12, we obtain σ-Picard–Vessiot extensions $\widetilde{K} \subset \widetilde{K}' \subset \widetilde{L}$ with field of δ-constants $\widetilde{k}$. Since G' is absolutely σ-integral, $\widetilde{G}' := \sigma\text{-}\mathrm{Gal}(\widetilde{L}|\widetilde{K}') = G'_{\widetilde{k}}$ is σ-integral. Since G' is properly contained in its Zariski closure $H = H'$, $\widetilde{G}'$ is also properly contained in its Zariski closure $H_{\widetilde{k}}$. We can thus apply Theorem A.25 to $\widetilde{G}' \leqslant \mathrm{Gl}_{n,\widetilde{k}}$ and combine it with Theorem 5.11 to find that $\delta(y) = Ay$ is σ^d-integrable over K' for a suitable integer $d \geqslant 1$. $\square$

Given a differential system $\delta(y) = Ay$ over a $\delta\sigma$-field K, it might be difficult to obtain information about the relative algebraic closure K' of K in a suitable σ-Picard–Vessiot extension. Nevertheless, Theorem 6.4 implies the following transformal independence criterion.

Corollary 6.5. *Let K be an inversive $\delta\sigma$-field, $A \in K^{n \times n}$, L a $\delta\sigma$-field extension of K with $L^\delta = K^\delta$, and $Y \in \mathrm{Gl}_n(L)$ with $\delta(Y) = AY$. Let T be a transcendence basis of $K(Y)$ over K, and assume that the (usual differential) Galois group of $K(Y)|K$ is an absolutely almost simple algebraic group. If, for every $d \geqslant 1$, the linear differential equation $\delta(B) + BA = \hbar_d \sigma^d(A)B$ over K has no non-zero solution B which is algebraic over K, then T is transformally independent over K.*

Proof. We may assume that $L = K\langle Y \rangle_\sigma$. So $L|K$ is a σ-Picard–Vessiot extension. Suppose that T is transformally dependent over K. Then $\sigma\text{-}\mathrm{trdeg}(L|K) < |T| = \mathrm{trdeg}(K(Y)|K)$. Thus condition (ii) of Theorem 6.4 is satisfied, and it follows that $\delta(y) = Ay$ is σ^d-integrable over the relative algebraic closure K' of K in L for some $d \geqslant 1$. This contradicts the assumption that $\delta(B) + BA = \hbar_d \sigma^d(A)B$ has no non-zero algebraic solutions. $\square$

6.3. The case of Sl_2

We now apply Theorem 6.4 to the almost simple algebraic group Sl_2.

Corollary 6.6. *Let $K = k(x)$ be a field of rational functions equipped with the derivation $\delta = \frac{d}{dx}$ and an automorphism σ commuting with δ, such that $k \subset \mathbb{C}$ is an algebraically*

closed inversive σ-field. We assume that the differential equation $\delta^2(y) - ry = 0$, with $r(x) \in K$, has (usual) Galois group $\mathrm{Sl}_2(k)$, and we denote by $L|K$ one of its σ-Picard–Vessiot extensions. Let K' be the relative algebraic closure of K in L. We have the following.

- If the σ-transcendence degree of $L|K$ is strictly less than 3, there exists $s \in \mathbb{Z}_{>0}$ such that the differential system

$$\begin{cases} \delta^2(b) + (\sigma^s(r) - r)b = 2\delta(d) \\ \delta^2(d) + (\sigma^s(r) - r)d = 2\sigma^s(r)\delta(b) + \delta(\sigma^s(r))b \end{cases} \tag{6.1}$$

has a non-zero algebraic solution $(b, d) \in (K')^2$.

- If we can find a solution $(b, d) \in (K')^2$ of (6.1), such that the matrix $B = \begin{pmatrix} d - \delta(b) & b \\ \sigma^s(r)b - \delta(d) & d \end{pmatrix}$ is invertible, then the σ-transcendence degree of $L|K$ is strictly less than 3.

Proof. The group $\mathrm{Sl}_{2,k}$ is an almost simple algebraic group of dimension 3. Since K is an inversive σ-field and k is algebraically closed,[2] we can apply Theorem 6.4 and obtain the equivalence between the following statements.

- The σ-transcendence degree of $L|K$ is strictly less than 3.

- There exist $B \in \mathrm{Gl}_n(K')$ and $s \in \mathbb{Z}_{>0}$ such that the system $\begin{cases} \delta(y) = Ay := \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix} y \\ \sigma^s(y) = By \end{cases}$ is compatible.

This means that A and B verify $\delta(B) + BA = \sigma^s(A)B$. If we write

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the integrability condition of order s is equivalent to the following system of differential equations:

$$\begin{cases} \delta(a) = c - rb \\ \delta(b) = d - a \\ \delta(c) = \sigma^s(r)a - rd \\ \delta(d) = \sigma^s(r)b - c. \end{cases}$$

Eliminating a and c, we obtain the differential system (6.1). This proves the first statement.

Suppose that we have a solution $(b, d) \in (K')^2$ such that the matrix B is invertible; then the system $\delta(y) = Ay$ is σ^s-integrable and therefore σ-trdeg$(L|K) < 3$. $\qquad \square$

Assuming that a second-order linear differential equation has usual Galois group $\mathrm{Sl}_2(k)$ is not a restrictive assumption, as we know from a famous paper of Kovacic [30, 31], which provides an algorithm to test whether a differential equation of order 2 with coefficients in $k(x)$ has Liouvillian solutions or not. Based on this property, the algorithm actually calculates the Galois group of the differential equation.

Let $K = k(x)$, where $k \subset \mathbb{C}$ is an algebraically closed field, and let $\delta = \frac{d}{dx}$, as above. Starting with a differential equation $\delta^2(y) + a\delta(y) + by = 0$ of order 2 with coefficients in

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[2]This assumption insures that the usual Galois group is well defined, i.e., unique up to isomorphism.

$K(x)$, one computes its normal form

$$\delta^2(z) - rz = 0, \tag{6.2}$$

by means of a change of unknown function $z = fy$, where f is a solution of $\delta(f) = \frac{a}{2}f$ and $r = b - \frac{1}{4}a^2 - \frac{1}{2}\delta(a)$. In the form (6.2), the existence of Liouvillian solutions is equivalent to finding solutions of the Riccati equation

$$u^2 - \delta(u) = r(x)$$

in $\bigcup_{m \in \mathbb{Z}_{\geqslant 0}} \mathbb{C}[x^{\frac{1}{m}}]$. Kovacic's algorithm says, in particular, the following.

Proposition 6.7. *If $\delta^2(y) - ry = 0$ has no Liouvillian solutions, then its (usual) Galois group is $\mathrm{Sl}_2(k)$.*

Remark 6.8. In [11] (see also [1]), the author gives a version of the Kovacic algorithm with differential parameters. The author gives, for instance, a characterization of the differential dependence with respect to the parameters $\underline{t} = (t_1, \ldots, t_k)$ of the solutions of $\delta^2(y) - r(x, \underline{t})y = 0$, in terms of the existence of rational solutions of auxiliary differential equations. A similar result should be true also in the present setting.

6.4. Airy's equation

For a differential system $\delta(y) = Ay$ with usual almost simple Galois group, Theorem 6.4 shows that the transformal relations satisfied by the solutions of $\delta(y) = Ay$ in some σ-Picard–Vessiot extension L are determined by the existence of solutions of an auxiliary linear differential system, lying in the relative algebraic closure K' of K inside L. The field K' is not easy to handle in general, but if the solutions in L are sufficiently regular, one can easily show that $K' = K$. Below, we illustrate this remark and combine it with Corollary 6.6 to characterize the transformal relations satisfied by the solutions of the Airy equation.

Lemma 6.9. *Let $\mathbb{C}(x)$ be the field of rational functions over $\mathbb{C}$, and let M be the field of meromorphic functions over $\mathbb{C}$. Then $\mathbb{C}(x)$ is relatively algebraically closed in M.*

Proof. Let $f \in M$ be algebraic over $\mathbb{C}(x)$. Consider the function field $E = \mathbb{C}(x)(f)$. It is a finite separable extension of the rational function field $\mathbb{C}(x)$. Then Corollary III.5.8 in [53] implies that either $E = \mathbb{C}(x)$ or $E|\mathbb{C}(x)$ is ramified. This is an easy consequence of the Riemann–Hurwitz genus formula. Since E is a subfield of M, and since a meromorphic function has no branch point, $E|\mathbb{C}(x)$ cannot be ramified, and then $f \in \mathbb{C}(x)$. □

We are now able to apply Corollary 6.6 to the case of the Airy equation

$$\delta^2(y) - xy = 0. \tag{6.3}$$

Notice that it has an irregular singularity at ∞, and that all the other points of $\mathbb{A}^1_{\mathbb{C}}$ are ordinary. This immediately implies that (6.3) admits a basis of solutions $A(x)$ and $B(x)$ in the field M of meromorphic functions over $\mathbb{C}$.

Corollary 6.10. *Let $\mathbb{C}(x)$ be the field of rational function over the complex numbers, equipped with the derivation $\delta = \frac{d}{dx}$ and the automorphism $\sigma : f(x) \mapsto f(x+1)$, and let M be the field of meromorphic functions over $\mathbb{C}$. In the notation above, let $L = \mathbb{C}(x)\langle A(x), B(x), \delta(A(x)), \delta(B(x))\rangle_\sigma \subset M$ be the σ-Picard–Vessiot extension for the Airy equation contained in M. Then $\sigma\text{-Gal}(L|\mathbb{C}(x))$ is equal to $\mathrm{Sl}_{2,\mathbb{C}}$, and the functions $A(x)$, $B(x)$ and $\delta(B(x))$ are transformally independent over $\mathbb{C}(x)$.*

Proof. By Example 4.2.9 in [26], the usual Galois group of the Airy equation is $\mathrm{Sl}_2(\mathbb{C})$. Moreover, by Lemma 6.9, the field $\mathbb{C}(x)$ is relatively algebraically closed in L.

We can then apply Corollary 6.6 to the case $K = \mathbb{C}(x)$, $\sigma(x) = x+1$, and $r(x) = x$. Then, if there exists $s \in \mathbb{Z}_{>0}$ such that the system (6.1) has a rational solution, we have to find $(b, d) \in \mathbb{C}(x)^2$ such that

$$\begin{cases} \delta^2(b) + sb = 2\delta(d) \\ \delta^2(d) + sd = 2(x+s)\delta(b) + b \end{cases}.$$

If we eliminate d from the previous system, we get

$$\delta^4(b) - (4x + 2s)\delta^2(b) - 6\delta(b) + s^2 b = 0. \tag{6.4}$$

Let us assume that this last equation has a non-zero solution $b_0 \in \mathbb{C}(x)$. It is easy to see that b_0 has no pole. In fact, if α was a pole of order r of b_0 then it would be a pole of order $r+4$ of $\delta^4(b_0) - (4x + 2s)\delta^2(b_0) - 6\delta(b_0) + s^2 b_0 = 0$, and this is impossible. Therefore b_0 must be a polynomial. Finally, if $b_0 \neq 0$, then $\delta^4(b_0) - (4x + 2s)\delta^2(b_0) - 6\delta(b_0) + s^2 b_0$ is a non-zero polynomial of the same degree than b_0, which is again impossible. So we conclude that $b_0 = 0$ and that the differential system above does not have any rational solution. Therefore $\sigma\text{-trdeg}(L|\mathbb{C}(x)) = 3$, which allows us to conclude. Since the relative algebraic closure of K in L equals K, condition (i) in Theorem 6.4 implies that $\sigma\text{-Gal}(L|K)$ is not a proper σ-closed subgroup of $\mathrm{Sl}_{2,\mathbb{C}}$. Therefore $\sigma\text{-Gal}(L|\mathbb{C}(x)) = \mathrm{Sl}_{2,\mathbb{C}}$. $\qquad\square$

Appendix A. Some classification results for σ-algebraic groups

In this appendix we have collected some results on the classification of σ-closed subgroups of certain algebraic groups. The results are typically of the following type: if G is a σ-closed subgroup of a certain algebraic group, then G is of a very special form. In the main body of the text, via the σ-Galois correspondence, this yields results of the following type: if there is a transformal relation between the solutions of a certain linear differential equation, then there is a transformal relation of a very special form.

We have made it a general assumption (in § 2.1) that all fields are of characteristic zero. However, in this appendix we do not make this assumption, but rather state the characteristic zero assumption whenever it is convenient.

A.1. The σ-closed subgroups of $\mathbf{G}_a^n$

Let k be a σ-field. As usual, we denote by $\mathbf{G}_a$ the additive algebraic group over k. We think of the vector group $\mathbf{G}_a^n$ as a σ-algebraic group; i.e., $\mathbf{G}_a^n(S) = S^n$ for any k-σ-algebra S. The σ-coordinate ring of $\mathbf{G}_a^n$ is $k\{\mathbf{G}_a^n\} = k\{x_1, \ldots, x_n\}_\sigma$. A linear, homogenous σ-polynomial $p = \sum_{i,j} \lambda_{i,j} \sigma^j(x_i) \in k\{x_1, \ldots, x_n\}_\sigma$ defines a σ-closed subgroup of $\mathbf{G}_a^n$. The following theorem provides a converse.

Theorem A.1. *Let k be a σ-field of characteristic zero, and let G be a σ-closed subgroup of $\mathbf{G}_a^n$. Then there exists a set F of linear homogeneous σ-polynomials such that*

$$G(S) = \{g \in \mathbf{G}_a^n(S)| \ p(g) = 0 \ \textit{for } p \in F\}$$

for any k-σ-algebra S.

Proof. For $d \in \mathbb{N}$, let $G[d]$ be the dth-order Zariski closure of G inside $\mathbf{G}_a^n$. By Lemma A.39 in [15], $G[d]$ is an algebraic subgroup of $(\mathbf{G}_a^n)_d = \mathbf{G}_a^{n(d+1)}$. The claim now follows from the fact that the algebraic subgroups of $\mathbf{G}_a^{n(d+1)}$ are defined by linear homogeneous polynomials. $\qquad\square$

Remark A.2. In positive characteristic, the situation is slightly more complicated because one has to take into account the Frobenius. For example, if k is a σ-field of characteristic $p > 0$, then

$$G(S) = \{g \in S| \ g + \sigma(g)^p = 0\} \leqslant \mathbf{G}_a(S),$$

for any k-σ-algebra S, defines a σ-closed subgroup G of $\mathbf{G}_a$.

Corollary A.3. *Let k be a σ-field of characteristic zero and G a σ-closed subgroup of $\mathbf{G}_a$, not equal to $\mathbf{G}_a$. Then there exists a non-zero, linear, homogenous σ-polynomial $p = a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_0x \in k\{x\}_\sigma$ such that*

$$G(S) = \{g \in S| \ p(g) = 0\}$$

for any k-σ-algebra S. Moreover, if $a_0 \neq 0$, then G is σ-integral; i.e., the σ-ideal $\mathbb{I}(G)$ is prime and reflexive.

Proof. By Theorem A.1, the vanishing ideal $\mathbb{I}(G) \subset k\{x\}_\sigma = k\{\mathbf{G}_a\}$ of G is generated as a difference ideal by linear, homogeneous σ-polynomials. Let $p \in \mathbb{I}(G)$ be a non-zero, linear, homogeneous σ-polynomial of minimal order. So

$$p = a_n\sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_0x \in k\{x\}_\sigma, \quad a_n \neq 0,$$

with n minimal. We have to show that $\mathbb{I}(G) = [p]$. Suppose there exists a non-zero, linear, homogeneous σ-polynomial in $\mathbb{I}(G)$ which does not lie in $[p]$. Among these σ-polynomials choose one, say q, of minimal order. If $q = b_m\sigma^m(x) + \cdots + b_0x$, $b_m \neq 0$, then necessarily $m \geqslant n$ and $q - b_m\sigma^{m-n}(a_n)^{-1}\sigma^{m-n}(p) \in \mathbb{I}(G)$ has order strictly less than m. So $q - b_m\sigma^{m-n}(a_n)^{-1}\sigma^{m-n}(p) \in [p]$. But then also $q \in [p]$, a contradiction.

Let us now assume that $a_0 \neq 0$. We have to show that G is σ-integral. The coordinate ring $k\{G\} = k\{x\}_\sigma/[p] \simeq k[\overline{x}, \sigma(\overline{x}), \ldots, \sigma^{n-1}(\overline{x})]$ is a polynomial ring in n variables over k. To see that σ is injective on $k\{G\}$ is suffices to note that $\sigma(\overline{x}), \sigma^2(\overline{x}), \ldots, \sigma^n(\overline{x}) = -\frac{1}{a_n}(a_0\overline{x} + \cdots + a_{n-1}\sigma^{n-1}(\overline{x}))$ are algebraically independent over k. $\qquad\square$

We recall the definition of linearly σ-closed and σ-closed fields.

Definition A.4 (cf. Def. 3.1, p. 1330 in [32]). A σ-field k is called *linearly σ-closed* if every linear system of difference equations over k has a fundamental solution matrix in k. That is, for every $B \in \mathrm{Gl}_n(k)$ there exists $Y \in \mathrm{Gl}_n(k)$ with $\sigma(Y) = BY$. We say that a σ-field k is *σ-closed*[3] if every system of difference polynomial equations with coefficients in k,

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[3] A σ-closed σ-field is also called a model of ACFA, in model theory language.

that has a solution in some σ-field extension of k, has a solution in k (see also § 1.1 in [5]).

Remark A.5. The reader familiar with the Galois theory of linear difference equations might be slightly puzzled by the above definition, because usually one is looking for fundamental solution matrices in finite products of fields, rather than fields [43]. The definition makes sense because we do not make any assumption on the constants of k. (The constants of a linearly σ-closed σ-field can be non-algebraically closed.) Note that a σ-closed σ-field is linearly σ-closed.

The difference Wronskian lemma [7, Lemma II, p. 271] implies that, if a σ-field k is linearly σ-closed, then any linear difference equation of order s, with coefficients in k,

$$a_0 y + a_1 \sigma(y) + \cdots + a_s \sigma^s(y) = 0,$$

with $a_0, \ldots, a_s \in k$, $a_0 a_s \neq 0$, has s solutions in k, linearly independent over k^σ. Therefore we have the following corollary.

Corollary A.6. *Let k be a linearly σ-closed σ-field of zero characteristic, and let G be a proper σ-closed subgroup of $\mathbf{G}_a$, such that $\mathcal{L} = \sum_{i=0}^s \lambda_i \sigma^i(x)$, $\lambda_0 \lambda_s \neq 0$, generates $\mathbb{I}(G)$ as a σ-ideal. Then $G(k)$ is a k^σ-vector space of dimension s, generated by a basis of solutions in k of the linear difference equation $\mathcal{L}(y) = 0$. Moreover, for any k-σ-algebra S, we have $G(S) = G(k) \otimes_{k^\sigma} S^\sigma$.*

A.2. The σ-closed subgroups of $\mathbf{G}_m^n$

By a multiplicative function on $\mathbf{G}_m^n$, we mean a σ-polynomial in $k\{\mathbf{G}_m^n\} = k\{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}_\sigma$ which is a product of monomials $\sigma^i(x_j)$. The structure of the σ-closed subgroups of $\mathbf{G}_m^n$ was already noted in Lemma A.40 in [15].

Theorem A.7. *Let k be a σ-field, and let G be a σ-closed subgroup of $\mathbf{G}_m^n$. Then there exists a set Ψ of multiplicative functions such that*

$$G(S) = \{g \in \mathbf{G}_m^n(S)| \ \psi(g) = 1 \text{ for } \psi \in \Psi\},$$

for any k-σ-algebra S.

In the following proposition, we show that, for certain σ-closed subgroup of $\mathbf{G}_m$, we can give a more precise description.

Proposition A.8. *Let k be a σ-field, p a prime number, $\mu_p \leqslant \mathbf{G}_m$ the group scheme of pth roots of unity, and G a proper σ-closed subgroup of μ_p. Then there exist two integers $m \geqslant 0$ and $d \geqslant 1$ such that*

$$G(S) \leqslant \{g \in \mu_p(S)| \ \sigma^{m+d}(g) = \sigma^m(g)\},$$

for any k-σ-algebra S.

Proof. Note that $\sigma^l(g)^p = 1$ for any integer $l \geqslant 0$, $g \in G(S) \subset S^\times$, and any k-σ-algebra S. Since G is properly contained in μ_p, it follows from Theorem A.7 that there exist $\alpha_0, \ldots, \alpha_l \in \{0, \ldots, p-1\}$, not all equal to zero, such that

$$g^{\alpha_0}\sigma(g)^{\alpha_1} \cdots \sigma^l(g)^{\alpha_l} = 1 \quad \text{for all } g \in G(S).$$

We may assume that $\alpha_l \neq 0$, so that there exist $a, b \in \mathbb{Z}$ such that $a\alpha_l + bp = 1$. Then

$$\sigma^l(g) = \sigma^l(g)^{1-bp} = \sigma^l(g)^{a\alpha_l} = g^{-a\alpha_0}\sigma(g)^{-a\alpha_1} \cdots \sigma^{l-1}(g)^{-a\alpha_{l-1}}.$$

Thus there exist $\beta_0, \ldots, \beta_{l-1} \in \{0, \ldots, p-1\}$ such that

$$\sigma^l(g) = g^{\beta_0}\sigma(g)^{\beta_1} \cdots \sigma^{l-1}(g)^{\beta_{l-1}}. \tag{A 1}$$

Applying σ to both sides of this identity and recursively substituting equation (A 1), we find that for every $m \geqslant l$ there exist $\beta_{0,m}, \ldots, \beta_{l-1,m} \in \{0, \ldots, p-1\}$ such that

$$\sigma^m(g) = g^{\beta_{0,m}}\sigma(g)^{\beta_{1,m}} \cdots \sigma^{l-1}(g)^{\beta_{l-1,m}}.$$

Since there are only finitely many possibilities for the exponents $\beta_{i,m}$, this shows that there must exist distinct integers $m_1, m_2 \geqslant l$ with $\sigma^{m_1}(g) = \sigma^{m_2}(g)$ for all $g \in G(S)$ and any k-σ-algebra S. $\qquad\square$

A.3. The σ-closed subgroups of $\mathbf{G}_a \rtimes \mathbf{G}_m$

Let k be a σ-field of characteristic zero, and let $\mathbb{G}$ denote the algebraic subgroup of $\mathrm{Gl}_{2,k}$ given by

$$\mathbb{G}(S) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \middle| \alpha \in S^\times, \beta \in S \right\} \leqslant \mathrm{Gl}_2(S),$$

for any k-algebra S. Notice that $\mathbb{G} \simeq \mathbf{G}_a \rtimes \mathbf{G}_m$. Let $\mathbb{G}_u$ denote the algebraic subgroup of $\mathbb{G}$ corresponding to $\mathbf{G}_a$; i.e.,

$$\mathbb{G}_u(S) = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \middle| \beta \in S \right\} \leqslant \mathrm{Gl}_2(S),$$

for any k-algebra S.

Theorem A.9. *Let G be a σ-closed subgroup of $\mathbb{G}$ such that the scheme theoretical intersection $G_u := G \cap \mathbb{G}_u$ is properly contained in $\mathbb{G}_u$. Then G has one of the following properties.*

(i) *There exists an integer $n \geqslant 0$ such that $\sigma^n(\beta) = 0$ for all $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in G_u(S)$ and all k-σ-algebras S.*

(ii) *There exist integers $n > m \geqslant 0$ such that $\sigma^n(\alpha) = \sigma^m(\alpha)$ for all $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in G(S)$ and all k-σ-algebras S.*

Proof. Since G_u is properly contained in $\mathbb{G}_u$, it follows from Corollary A.3 that there exists a non-zero, linear, homogeneous difference polynomial $p \in k\{x\}_\sigma$ such that

$$G_u(S) = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \in \mathbb{G}_u(S) \middle| p(\beta) = 0 \right\}$$

for any k-σ-algebra S. If p is a monomial we are in case (i). Otherwise, we may assume that p is of the form

$$p(x) = \sigma^n(x) + a_{n-1}\sigma^{n-1}(x) + \cdots + a_m\sigma^m(x),$$

where $n > m \geqslant 0$, $a_i \in k$ for $i = m, \ldots, n-1$ and $a_m \neq 0$. Let S be a k-σ-algebra and $g = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in G(S)$. We have to show that $\sigma^n(\alpha) = \sigma^m(\alpha)$. Note that, for $h = \begin{pmatrix} 1 & \beta_1 \\ 0 & 1 \end{pmatrix} \in G_u(S)$, we have $ghg^{-1} = \begin{pmatrix} 1 & \alpha\beta_1 \\ 0 & 1 \end{pmatrix} \in G_u(S)$. Therefore $p(\alpha\beta_1) = 0$ for all $\beta_1 \in S$ with $p(\beta_1) = 0$. Let

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \\ -a_m & -a_{m+1} & \cdots & \cdots & -a_{n-1} \end{pmatrix} \in \mathrm{Gl}_{n-m}(k).$$

Let $y_1, \ldots, y_{n-m}, \sigma(y_1), \ldots, \sigma(y_{n-m}), \ldots, \sigma^{n-1}(y_1), \ldots, \sigma^{n-1}(y_{n-m})$ be indeterminates over S, and abbreviate $y = (y_1, \ldots, y_{n-m})$. We extend the action of σ on S to an action on $S[y, \ldots, \sigma^{n-1}(y)]$ as suggested by the names of the variables and by setting $\sigma(\sigma^{n-1}(y_i)) = -a_{n-1}\sigma^{n-1}(y_i) - \cdots - a_m\sigma^m(y_i)$ for $i = 1, \ldots, n-m$. Then the matrix

$$Z = \begin{pmatrix} \sigma^m(y_1) & \cdots & \sigma^m(y_{n-m}) \\ \vdots & & \vdots \\ \sigma^{n-1}(y_1) & \cdots & \sigma^{n-1}(y_{n-m}) \end{pmatrix}$$

satisfies $\sigma(Z) = AZ$, which implies that $\sigma(\det(Z)) = \det(A)\det(Z)$. So we can extend the action of σ to $S' := S[y, \ldots, \sigma^{n-1}(y), \frac{1}{\det(Z)}]$. Since $p(y_i) = 0$, we have $p(\alpha y_i) = 0$ for $i = 1, \ldots, n-m$.

This implies that

$$\sigma\left(\begin{pmatrix} \sigma^m(\alpha) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \sigma^{n-2}(\alpha) & 0 \\ 0 & \cdots & 0 & \sigma^{n-1}(\alpha) \end{pmatrix} Z\right) = A \begin{pmatrix} \sigma^m(\alpha) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \sigma^{n-2}(\alpha) & 0 \\ 0 & \cdots & 0 & \sigma^{n-1}(\alpha) \end{pmatrix} Z.$$

Since two fundamental solution matrices for $\sigma(Y) = AY$ only differ by a constant matrix, there exists $C \in \mathrm{Gl}_{n-m}((S')^\sigma)$ such that $\begin{pmatrix} \sigma^m(\alpha) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \sigma^{n-2}(\alpha) & 0 \\ 0 & \cdots & 0 & \sigma^{n-1}(\alpha) \end{pmatrix} Z = ZC$. By applying the determinant to the previous equation, we find that $\sigma^m(\alpha) \cdots \sigma^{n-1}(\alpha) \in (S')^\sigma$, and thus

$$\sigma^{m+1}(\alpha) \cdots \sigma^n(\alpha) = \sigma^m(\alpha) \cdots \sigma^{n-1}(\alpha).$$

Since $\alpha \in S^\times$, we obtain $\sigma^m(\alpha) = \sigma^n(\alpha)$, as desired. $\qquad\qquad\square$

A.4. The Zariski dense σ-closed subgroups of almost simple algebraic groups

This section is devoted to the classification of the Zariski dense σ-closed subgroups of almost simple algebraic groups. In the framework of 'difference varieties', this classification has been achieved in Proposition 7.10, p. 309, in [5]. Since we have to deal with not necessarily perfectly σ-reduced difference group schemes, we need to obtain a schematic version of the result of Chatzidakis, Hrushovski, and Peterzil. In full generality, this schematic version fails to be true (see Remark A.22). But we succeed to obtain a satisfactory statement for Zariski dense σ-reduced subgroups of simple algebraic groups (see Theorem A.20) and Zariski dense σ-integral subgroups of almost simple algebraic groups (see Theorem A.25).

We start this section with a collection of technical lemmas on the compatibility of quotients, Zariski closure, and base extension in the category of group k-σ-schemes. This lemmas are only used in the almost simple case. Then we study the case of simple groups and conclude with the almost simple case.

A.4.1. Preliminary lemmas on quotients of group k-σ-schemes.
Let G be a group k-σ-scheme, and let $N \trianglelefteq G$ be a normal σ-closed subgroup. A quotient of G by N is a morphism $\pi : G \to G/N$ of a group k-σ-scheme which satisfies the universal property of quotients in the category of group k-σ-schemes. As shown in § A.9 of [15], quotients exist and are unique up to unique isomorphism. Moreover, we have $\ker(\pi) = N$, and $\pi^* : k\{G/N\} \to k\{G\}$ is injective.

Lemma A.10. *Let k be a σ-field. Let $\phi : G \to H$ be a morphism of group k-σ-schemes. Then the induced morphism $G/\ker(\phi) \to H$ is a σ-closed embedding.*

Proof. We will use the language of k-σ-Hopf algebras (Definition A.35 in [15]). If R is a Hopf algebra over k, we denote by R^+ the kernel of the counit $R \to k$.

Let $N = \ker(\phi) \trianglelefteq G$, and let $\phi^* : k\{H\} \to k\{G\}$ be the dual map. Note that $\mathbb{I}(N) = k\{G\}\phi^*(k\{H\}^+) \subset k\{G\}$. From Proposition A.42 and Theorem A.43 in [15] we know that $k\{G/N\}$ is the unique sub-Hopf algebra of $k\{G\}$ with the property that $k\{G\}k\{G/N\}^+ = \mathbb{I}(N)$. But $\phi^*(k\{H\})$ is also a sub-Hopf algebra of $k\{G\}$, and $k\{G\}\phi^*(k\{H\})^+ = k\{G\}\phi^*(k\{H\}^+) = \mathbb{I}(N)$. Therefore $k\{G/N\} = \phi^*(k\{H\})$. Thus $k\{H\} \to k\{G/N\}$ is surjective, and $G/N \to H$ is a σ-closed embedding. $\qquad\square$

Next, we want to show that the functorial construction $[\sigma]_k$ (see § 2 and § A.4 in the appendix in [15], for more details) is compatible with quotients. First, we prove the following lemma.

Lemma A.11. *Let k be a σ-field, and let $\phi : R \to S$ be a morphism of k-algebras. Let $[\sigma]_k \phi : [\sigma]_k R \to [\sigma]_k S$ be the morphism of k-σ-algebras deduced from ϕ by functoriality of $[\sigma]_k$. If ϕ is injective then $[\sigma]_k \phi$ is also injective.*

Proof. By construction, $[\sigma]_k \phi$ is the limit of $\phi_d : R_d \to S_d$ for $d \geqslant 0$. It is thus sufficient to prove that the ϕ_d are injective for all $d \geqslant 0$. We prove this by induction on d. For $d = 0$, it is true by assumption on ϕ. Assume that $\phi_d : R_d \to S_d$ is injective. We prove that $\phi_{d+1} : R_{d+1} \to S_{d+1}$ is injective by proving that it is the composition of the following

injective morphisms:

$$R_{d+1} = R_d \otimes_k {}^{\sigma^{d+1}} R \xrightarrow{\phi_d \otimes \mathrm{id}_{\sigma^{d+1} R}} S_d \otimes_k {}^{\sigma^{d+1}} R \xrightarrow{\mathrm{id}_{S_d} \otimes_k {}^{\sigma^{d+1}} \phi} S_d \otimes_k {}^{\sigma^{d+1}} S$$
$$= S_{d+1}.$$

Indeed the flatness of the tensor product over a field immediately implies that $\phi_d \otimes \mathrm{id}_{\sigma^{d+1} R}$ and $\mathrm{id}_{S_d} \otimes_k {}^{\sigma^{d+1}} \phi$ are injective. $\qquad \square$

Lemma A.12. *Let k be a σ-field. Let H be a group k-scheme, and let $N \trianglelefteq H$ be a normal closed subgroup. Then the group k-σ-scheme $[\sigma]_k(H/N)$ is the quotient of $[\sigma]_k H$ by $[\sigma]_k N$; i.e., $[\sigma]_k(H/N)$ and $[\sigma]_k H/[\sigma]_k N$ are canonically isomorphic.*

Proof. Since $p \colon H \to H/N$ is a quotient map, the dual morphism $p^* \colon k[H/N] \to k[H]$ is injective, and $\ker(p) = N$. Because of (2.1), for any k-σ-algebra S we have

$$\ker([\sigma]_k p \colon [\sigma]_k H(S) \to [\sigma]_k(H/N)(S)) = \ker(p \colon H(S^\sharp) \to (H/N)(S^\sharp))$$
$$= N(S^\sharp) = [\sigma]_k N(S),$$

where $S^\sharp$ is the underlying k-algebra of S. That is, $\ker([\sigma]_k p) = [\sigma]_k N$. Moreover, by Lemma A.11, $[\sigma]_k p^* \colon [\sigma]_k k[H/N] \to [\sigma]_k k[H]$ is injective. By Corollary A.44 in [15], $[\sigma]_k p \colon [\sigma]_k H \to [\sigma]_k(H/N)$ is the quotient of $[\sigma]_k H$ modulo $[\sigma]_k N$. $\qquad \square$

Remark A.13. *If there is no possible confusion, we write H/N instead of $[\sigma]_k(H/N)$.*

In the following lemma, we prove the compatibility between the dth-order Zariski closure and quotients.

Lemma A.14. *Let H be an algebraic group over k and $N \trianglelefteq H$ a normal closed subgroup of H. Let $i \colon G \to H$ be the inclusion of a σ-closed subgroup G of H into H, $G \cap N$ the schematic intersection of G with N. Let $p \colon H \to H/N$ and $\pi \colon G \to G/(G \cap N)$ be the quotient maps. Then the following hold.*

- *The morphism $p \circ i \colon G \to H/N$ factors through π into a σ-closed embedding $\iota \colon G/(G \cap N) \to H/N$.*

- *For $d \geqslant 0$, the dth-order Zariski closure $G/(G \cap N)[d]$ of $G/(G \cap N)$ inside H/N is a quotient of the dth-order Zariski closure $G[d]$ of G by $G[d] \cap N_d$. That is, the algebraic groups $G/(G \cap N)[d]$ and $G[d]/G[d] \cap N_d$ are canonically isomorphic over k.*

Proof. Since $\ker(p \circ i) = G \cap N$, Lemma A.10 implies that $p \circ i$ factors through π into a σ-closed embedding $\iota \colon G/(G \cap N) \to H/N$.

Let $d \geqslant 0$. First of all, we prove that $(H/N)_d$ is a quotient of H_d by N_d. Since $p \colon H \to H/N$ is a quotient map, its dual $p^* \colon k[H/N] \to k[H]$ is injective. Following the lines of the proof of Lemma A.11, we see that the dual map p_d^* of $p_d \colon H_d \to (H/N)_d$ is also injective. Moreover, $\ker(p_d)$ is exactly N_d. To see this, we remark that $H_d = H \times {}^\sigma H \times \ldots \times {}^{\sigma^d} H$, where ${}^{\sigma^i} H = H \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k)$ and the fiber product is taken with respect to the morphism induced from $\sigma^i \colon k \to k$. Moreover p_d acts on the ith component via

the morphism $^{\sigma^i}p$, obtained from p by extension of scalars $\sigma^i : k \to k$. This shows that $\ker(p_d) = N \times^\sigma N \times \ldots \times^{\sigma^d} N$. It follows that $p_d : H_d \to (H/N)_d$ is the quotient of H_d by N_d. (cf. [35, Theorem 7.8].) By construction of the dth-order Zariski closure, we have closed embeddings $G[d] \hookrightarrow H_d$ and $G/(G \cap N)[d] \hookrightarrow (H/N)_d$. Since $\ker(p_d) = N_d$, we find that the kernel of $G[d] \to G/(G \cap N)[d]$ is $G[d] \cap N_d$.

By construction of the dth-order Zariski closure, we have $k[G/(G \cap N)[d]] = \iota^*(k[H/N]_d)$, $k[G[d]] = i^*(k[H]_d)$, and $k[G/(G \cap N)[d]] \to k[G[d]]$ is the restriction of π^* to $k[G/(G \cap N)[d]]$. Since π^* is injective, the same holds for $k[G/(G \cap N)[d]] \to k[G[d]]$. We can then conclude that $G[d] \to G/(G \cap N)[d]$ is a quotient of $G[d]$ by $G[d] \cap N_d$. $\quad\square$

Lemma A.15. *Let G be a group k-σ-scheme, $N \trianglelefteq G$ be a normal σ-closed subgroup, and $k'|k$ be a σ-field extension. Let $\pi : G \to G/N$ be a quotient of G by N. Denote by $G_{k'}$, $N_{k'}$ and $(G/N)_{k'}$ the base extension to k' of G, N, and G/N. Then the group k'-σ-schemes $(G/N)_{k'}$ and $G_{k'}/N_{k'}$ are isomorphic.*

Proof. Since $\ker(\pi) = N$, for every k'-σ-algebra S, viewed as a k-σ-algebra S, the sequence $N(S) \to G(S) \to (G/N)(S)$ of abstract groups is exact. It follows that $\ker(\pi_{k'}) = N_{k'}$, where $\pi_{k'} : G_{k'} \to (G/N)_{k'}$ is the base extension of π. Moreover, since $k'|k$ is a faithfully flat extension and $\pi^* : k\{G/N\} \to k\{G\}$ is injective, we get that $\pi_{k'}^* : k'\{G/N\} \to k'\{G\}$ is injective. By Corollary A.44 in [15], we deduce that $(G/N)_{k'}$ is a quotient of $G_{k'}$ by $N_{k'}$. $\quad\square$

A.4.2. The Zariski dense σ-closed subgroups of a simple algebraic group.
In this section, we show that every σ-reduced σ-closed subgroup of $\mathrm{Gl}_{n,k}$ with simple Zariski closure is conjugate to a σ^d-constant subgroup of $\mathrm{Gl}_{n,k}$ for some $d \geqslant 1$. This is the crucial group theoretic input for the proof of Proposition 6.1. The result is originally due to Z. Chatzidakis, E. Hrushovski, and Y. Peterzil. A slightly simplified version of their result reads as follows.

Proposition A.16 (Prop. 7.10, p. 309 in [5]). *Let k be a σ-closed σ-field of characteristic zero, H an almost simple algebraic group over k, and let G be a Zariski dense definable (in the language of difference rings) subgroup of $H(k)$. Then either $G = H(k)$ or there exist an isomorphism $f : H \to H'$ of algebraic groups and an integer $d \geqslant 1$ such that some subgroup of $f(G)$ of finite index is conjugate to a subgroup of $H'(k^\sigma)$. If H is defined over the algebraic closure of k^σ, then we may take $H = H'$ and f to be conjugation by an element of $H(k)$.*

Unfortunately it seems impossible to apply the above proposition directly in the proof of Proposition 6.1 and Theorem 6.4. Our versions are Theorems A.20 and A.25 below. They differ in several aspects.

(i) We prefer to work over an inversive algebraically closed difference field of characteristic zero, rather than over a σ-closed σ-field.

(ii) We have to avoid the isomorphism f and the 'finite index'.

(iii) We are interested in the σ-closed subgroups of H, rather than in the definable subgroups of $H(k)$.

(iv) If H is simple, it suffices to assume that G is σ-reduced. If H is almost simple, we have to assume that G is σ-integral. See Remark A.22.

Because of these differences, we have chosen to include a full proof of our reformulation, even though our proof follows [5] very closely. The divergence in the formulations is mainly due to (iii). Since ACFA does not (fully) eliminate quantifiers, the definable subgroups, say of $\mathrm{Gl}_n(k)$, k a σ-closed σ-field, need not be defined by difference polynomials. For example, the group

$$\{g \in k^{\times} |\; \exists\, h \in k^{\times} : h^2 = g, \sigma(h) = h\} \leqslant \mathrm{Gl}_1(k)$$

does not correspond to a σ-closed subgroup of $\mathrm{Gl}_{1,k}$. On the other hand, the quantifier free definable subgroups of $\mathrm{Gl}_n(k)$, i.e., the subgroups of $\mathrm{Gl}_n(k)$ defined by difference polynomials in the matrix entries, only correspond to perfectly σ-reduced σ-closed subgroups of $\mathrm{Gl}_{n,k}$.

In this section we assume that the base field has characteristic zero. Note, however, that [5] also provides a version for positive characteristic. So let k be a field of characteristic zero. By an algebraic group over k we mean an affine group scheme of finite type over k. If H is an algebraic group over k and $\widetilde{k}$ a field extension of k, we denote by $H_{\widetilde{k}} = H \otimes_k \widetilde{k}$ the algebraic group over $\widetilde{k}$ obtained from H via base extension from k to $\widetilde{k}$. By a representation V of H we mean a finite dimensional k-vector space V together with a morphism $H \to \mathrm{Gl}(V)$ of algebraic groups over k. (Here, from a formal point of view, $\mathrm{Gl}(V)$ is the functor such that $\mathrm{Gl}(V)(S) = \mathrm{Gl}(V \otimes_k S)$ for every k-algebra S.)

It is well known that every simple algebraic group descends to $\mathbb{Q}$. To avoid the isomorphism f in the above proposition we require a slightly more precise statement. We thank Michael Singer for the proof of Lemma A.17 and Corollary A.18.

Lemma A.17. *Let $\mathcal{H}$ be a split and connected reductive algebraic group over $\mathbb{Q}$, and let k be an algebraically closed field of characteristic zero. If V is a representation of $H := \mathcal{H}_k$, then there exists a representation $\mathcal{V}$ of $\mathcal{H}$ such that $V \simeq \mathcal{V} \otimes_{\mathbb{Q}} k$, as representations of H.*

Proof. Since we are in characteristic zero, every representation of H decomposes as a direct sum of irreducible representations. Therefore, we can assume, without loss of generality, that V is irreducible. The irreducible representations of H are classified by the so-called dominant weights. See Part II, Ch. 2 of [24]. The claim now follows from Corollary 2.9, Part II, p. 203 in [24]. □

Corollary A.18. *Let k be an algebraically closed field of characteristic zero, and let $H \leqslant \mathrm{Gl}_{n,k}$ be a connected reductive algebraic group. Then there exists $g \in \mathrm{Gl}_n(k)$ such that $gHg^{-1} \leqslant \mathrm{Gl}_{n,k}$ is defined over $\mathbb{Q}$.*

Proof. Since k is algebraically closed, H is split, and it follows from the classification (see, e.g., Corollary 1.3, p. 411 in [10]) that $H = \mathcal{H}_k$ for some split and connected reductive algebraic group $\mathcal{H}$ over $\mathbb{Q}$. By Lemma A.17 the given representation $V := k^n$ of H is of the form $V = \mathcal{V} \otimes_{\mathbb{Q}} k$ for some representation $\mathcal{V}$ of $\mathcal{H}$. Since V is faithful, $\mathcal{V}$ is also faithful, and it follows that H is conjugate to $\mathcal{H}_k$. □

Let k be an algebraically closed field of characteristic zero. Recall that a connected non-commutative algebraic group H over k is called simple if every proper normal algebraic subgroup of H is trivial.

Proposition A.19. *Let k be an algebraically closed, inversive σ-field of characteristic zero. Let H be a simple algebraic group over k, and let G be a σ-reduced, Zariski dense, σ-closed subgroup of H, properly contained in H. Then there exist an integer $m \geqslant 1$ and an isomorphism $\varphi \colon H \to {}^{\sigma^m} H$ of algebraic groups such that*

$$G(S) = \{h \in H(S) \mid \sigma^m(h) = \varphi(h)\}$$

for every k-σ-algebra S.

Proof. Note that ${}^{\sigma^d} H$ is simple for every $d \geqslant 0$. (Indeed, since H descends to $\mathbb{Q} \subset k^\sigma$, H and ${}^{\sigma^d} H$ are (non-canonically) isomorphic as algebraic groups over k.) So $H_d = H \times {}^\sigma H \times \cdots \times {}^{\sigma^d} H$ is a product of simple algebraic groups. As G is Zariski dense in H, we have $G[0] = H_0 = H$. Since G is properly contained in H, we have $G[d] \subsetneq H_d$ for some $d \geqslant 1$. Let $m \geqslant 1$ denote the smallest integer such that $G[m] \subsetneq H_m$. Then $G[m-1] = H_{m-1}$. We will show that the projection $G[m] \to G[m-1] = H_{m-1}$ is an isomorphism (of algebraic groups).

Let $\pi \colon G[m] \to {}^{\sigma^m} H$ denote the projection onto the last factor of $H_m \supset G[m]$. Let $\mathbb{I}(G) \subset k\{H\}$ denote the defining ideal of G. Then

$$\mathbb{I}(G) \cap k[H_m] = \mathbb{I}(G[m]) \subset k[H_m] = k[H] \otimes_k \cdots \otimes_k k[{}^{\sigma^m} H]$$

defines $G[m] \leqslant H_m$. A non-zero element in the kernel of $\pi^* \colon k[{}^{\sigma^m} H] \to k[G[m]] = k[H_m]/\mathbb{I}(G[m])$ corresponds to an element of the form $1 \otimes \cdots \otimes 1 \otimes a \in \mathbb{I}(G[m]) \subset k[H] \otimes_k \cdots \otimes_k k[{}^{\sigma^m} H]$, with $a \in k[{}^{\sigma^m} H] \smallsetminus \{0\}$. As G is σ-reduced (i.e., $\mathbb{I}(G)$ is reflexive) and k is inversive, such an element would give rise to a non-zero element of $\mathbb{I}(G) \cap k[H_0] = \{0\}$. Thus π^* is injective; i.e., π is dominant.

Let $H' \leqslant {}^{\sigma^m} H$ denote the algebraic subgroup of ${}^{\sigma^m} H$ determined by $G[m] \cap (\{1\}^m \times {}^{\sigma^m} H) = \{1\}^m \times H' \leqslant H_m$. We will show that H' is normal in H. Because k is algebraically closed, it suffices to work with the k-rational points. Let $h' \in H'(k)$ and $h \in {}^{\sigma^m} H(k)$. Because π is dominant, $\pi(G[m](k)) = {}^{\sigma^m} H(k)$ (Proposition 2.2.5, (ii), p. 26, in [52]). Thus, there is a $g \in H_{m-1}(k)$ such that $(g, h) \in G[m](k)$. It follows that

$$(g, h)(1, h')(g, h)^{-1} = (1, hh'h^{-1}) \in G[m](k),$$

and so H' is normal in ${}^{\sigma^m} H$. Since ${}^{\sigma^m} H$ is simple, we have $H' = 1$ or $H' = {}^{\sigma^m} H$. Suppose that $H' = {}^{\sigma^m} H$. As $G[m] \to G[m-1] = H_{m-1}$ is dominant, this implies that $G[m] = H_m$, a contradiction. So $H' = 1$. This means that $G[m](k) \to G[m-1](k)$ is injective. Thus, $G[m](k) \to G[m-1](k)$ is actually bijective. Since we are in characteristic zero, this is enough to deduce that $G[m] \to G[m-1]$ is an isomorphism of algebraic groups (Exercise 5.3.5, p. 87 in [52]). Let

$$\varphi \colon H_{m-1} = G[m-1] \to G[m] \xrightarrow{\pi} {}^{\sigma^m} H$$

denote the composition of the inverse of this isomorphism with π. Then $G[m] \leqslant H_m$ is the graph of φ. That is,

$$G[m](k) = \{(h, \varphi(h)) \mid h \in H_{m-1}(k)\} \leqslant H_m(k).$$

Suppose that $H \times \{1\}^{m-1} \subset \ker(\varphi)$. Then

$$\varphi(1, h_1, \ldots, h_{m-1}) = \varphi(h_0, h_1, \ldots, h_{m-1}) = h_m \in {}^{\sigma^m} H(k)$$

for all $(h_0, \ldots, h_m) \in G[m](k) \leqslant H_m(k)$. For every $f \in k[{}^{\sigma^m} H]$, the map

$$(h_0, \ldots, h_m) \mapsto f(\varphi(1, h_1, \ldots, h_{m-1})) - f(h_m)$$

defines a regular function $\widetilde{f} \in k[H_m]$. Moreover, $\widetilde{f} \in \mathbb{I}(G[m]) \subset \mathbb{I}(G)$. We can choose $f \in k[{}^{\sigma^m} H]$ such that $\widetilde{f}$ is non-zero. Because G is σ-reduced and k is inversive, we can obtain from $\widetilde{f}$, by considering the inverse image of $\widetilde{f}$ under σ, a non-zero element of $\mathbb{I}(G[m-1]) = \mathbb{I}(G) \cap k[H_{m-1}]$. But this contradicts the minimality of m. Therefore $H \times \{1\}^{m-1} \nsubseteq \ker(\varphi)$. Since H is simple, we have $(H \times \{1\}^{m-1}) \cap \ker(\varphi) = \{1\}^m$. Thus, φ restricts to an isomorphism $\varphi \colon H = H \times \{1\}^{m-1} \to {}^{\sigma^m} H$ of algebraic groups.

Since the ${}^{\sigma^i} H$ are simple, a normal algebraic subgroup N of $H_{m-1} = H \times {}^{\sigma} H \times \cdots \times {}^{\sigma^{m-1}} H$ must be a product of some of the factors ${}^{\sigma^i} H$. (This is easy to see. For example, if $(h_0, \ldots, h_{m-1}) \in N(k)$ and $h_0 \neq 1$, then there exists $h'_0 \in H(k)$ with $h_0 h'_0 \neq h'_0 h_0$. Then $(h'_0, 1, \ldots, 1)(h_0, \ldots, h_{m-1})(h_0'^{-1}, 1, \ldots, 1)(h_0, \ldots, h_{m-1})^{-1} = (h'_0 h_0 h_0'^{-1} h_0^{-1}, 1, \ldots, 1)$ is a non-trivial element of $N(k) \cap (H(k) \times \{1\}^{m-1})$. It follows that $H \times \{1\}^{m-1} \subset N$.)

Because $\ker(\varphi)$ is a normal algebraic subgroup of H_{m-1}, we must have $\ker(\varphi) = \{1\} \times {}^{\sigma} H \times \cdots \times {}^{\sigma^{m-1}} H$. In summary, we find that

$$G[m](k) = \{(h_0, \ldots, h_{m-1}, \varphi(h_0)) \mid h_i \in {}^{\sigma^i} H(k)\} \leqslant H_m(k).$$

Because $k[G[m-1]] = k[G[m]] = k\{G\}$, the defining ideal $\mathbb{I}(G)$ of G is determined by $\mathbb{I}(G[m])$. Therefore

$$G(S) = \{h \in H(S) \mid \sigma^m(h) = \varphi(h)\}$$

for every k-σ-algebra S. $\qquad\qquad\square$

Recall (Definition 5.10) that a σ-closed subgroup G of $\mathrm{Gl}_{n,k}$ is called σ^d-constant if $\sigma^d(g) = g$ for every $g \in G(S) \leqslant \mathrm{Gl}_n(S)$ and all k-σ-algebras S. We say that G is conjugate to a σ^d-constant subgroup of $\mathrm{Gl}_{n,k}$ if there exists $u \in \mathrm{Gl}_n(k)$ such that uGu^{-1} is σ^d-constant.

Theorem A.20. *Let k be an algebraically closed, inversive σ-field of characteristic zero, and let G be a σ-reduced, σ-closed subgroup of $\mathrm{Gl}_{n,k}$. Assume that the Zariski closure of G in $\mathrm{Gl}_{n,k}$ is a simple algebraic group, properly containing G. Then there exist a σ-field extension $\widetilde{k}$ of k and an integer $d \geqslant 1$ such that $G_{\widetilde{k}}$ is conjugate to a σ^d-constant subgroup of $\mathrm{Gl}_{n,\widetilde{k}}$. If k is linearly σ-closed we can choose $\widetilde{k} = k$.*

Proof. Let $H \leqslant \mathrm{Gl}_{n,k}$ denote the Zariski closure of G. By Corollary A.18, there exist an algebraic group $\mathcal{H} \leqslant \mathrm{Gl}_{n,\mathbb{Q}}$ over $\mathbb{Q}$ and $g \in \mathrm{Gl}_n(k)$ such that $gHg^{-1} = \mathcal{H}_k \leqslant \mathrm{Gl}_{n,k}$. Therefore, we can assume without loss of generality that $H = \mathcal{H}_k$. Then, for every $d \geqslant 1$, the algebraic groups H and ${}^{\sigma^d} H$ are equal as subgroups of $\mathrm{Gl}_{n,k}$. So the φ in Proposition A.19 is an automorphism of H (and not merely an isomorphism).

The group $\mathrm{Inn}(H)$ of inner automorphisms of H is of finite index in the group $\mathrm{Aut}(H)$ of all automorphisms of H. (See Theorem 27.4, p. 166, in [22].) Therefore $\mathrm{Out}(H) :=$

$\mathrm{Aut}(H)/\mathrm{Inn}(H)$ is a finite group. If $\psi\colon H\to H$ is an automorphism of H, and $d\geqslant 0$ an integer, we denote by $^{\sigma^d}\psi\colon {}^{\sigma^d}H\to {}^{\sigma^d}H$ the automorphism of $^{\sigma^d}H$ obtained from ψ by extension of scalars via $\sigma^d\colon k\to k$. Since $^{\sigma^d}H=H$ (as subgroups of $\mathrm{Gl}_{n,k}$), we see that $^{\sigma^d}\psi$ is actually an automorphism of H. We obtain an action $\psi\mapsto {}^{\sigma}\psi$ of σ on $\mathrm{Aut}(H)$ and an induced action on $\mathrm{Out}(H)$. Since $\mathrm{Out}(H)$ is finite, there exists for every integer $m\geqslant 1$ and every element ψ of $\mathrm{Out}(H)$ an integer $e\geqslant 1$ such that $^{\sigma^{m(e-1)}}\psi\cdots{}^{\sigma^m}\psi\cdot\psi=1\in\mathrm{Out}(H)$. (For example, we can choose e as the product of the length of the orbit of ψ under $\psi\mapsto{}^{\sigma^m}\psi$ and the size of $\mathrm{Out}(H)$.) With m and φ as in Proposition A.19, we see that there exists an integer $e\geqslant 1$ such that $^{\sigma^{m(e-1)}}\varphi\cdots{}^{\sigma^m}\varphi\cdot\varphi$ is the conjugation with an element $h\in H(k)$.

Let S be a k-σ-algebra and $g\in H(S)$. Because $\sigma^m(g)=\varphi(g)$, we find that

$$\sigma^{me}(g)=\,^{\sigma^{m(e-1)}}\varphi(\cdots{}^{\sigma^m}\varphi(\varphi(g))\cdots)=hgh^{-1}.$$

Set $d=me$, and let $\widetilde{k}$ be a σ-field extension of k such that there exists $u\in\mathrm{Gl}_n(\widetilde{k})$ with $\sigma^d(u)=hu$. (For example, we can take $\widetilde{k}=k(X)$ with $\sigma(X)=hX$.) If k is linearly σ-closed we can choose $\widetilde{k}=k$ by Lemma 5.14.

Then, for every $\widetilde{k}$-σ-algebra S and $g\in G_{\widetilde{k}}(S)=G(S)$, we have

$$\sigma^d(u^{-1}gu)=\sigma^d(u)^{-1}\sigma^{me}(g)\sigma^d(u)=u^{-1}h^{-1}hgh^{-1}hu=u^{-1}gu.$$

Thus $u^{-1}G_{\widetilde{k}}u\leqslant\mathrm{Gl}_{n,\widetilde{k}}$ is σ^d-constant. $\qquad\square$

Remark A.21. The assumption that G is σ-reduced cannot be omitted. For example, let $H\leqslant\mathrm{Gl}_{n,k}$ be a simple algebraic group and H' a connected, non-trivial algebraic subgroup of H. Define a σ-closed subgroup G of H by

$$G(S)=\{h\in H(S)|\ \sigma(h)\in {}^{\sigma}H'(S)\}\leqslant H(S)$$

for every k-σ-algebra S. Then G is Zariski dense in H but not conjugate to a σ^d-constant group.

Remark A.22. The theorem does not extend to almost simple algebraic groups. For example, let G be a σ-closed subgroup of $\mathrm{Sl}_{n,k}$ containing the center Z of $\mathrm{Sl}_{n,k}$, given by $Z(S)=\{\lambda I|\ \lambda\in S^{\times},\lambda^n=1\}$ for every k-σ-algebra S. Suppose that there exist an integer $d\geqslant 1$ and a σ-field extension $\widetilde{k}$ of k, such that G is conjugate over $\widetilde{k}$ to a σ^d-constant subgroup of $\mathrm{Gl}_{n,\widetilde{k}}$. Since $Z_{\widetilde{k}}$ is in the center of $\mathrm{Gl}_{n,\widetilde{k}}$, this would imply that $Z_{\widetilde{k}}\leqslant\mathrm{Gl}_{n,\widetilde{k}}$ is σ^d-constant. Obviously this is not the case: despite the fact that $\sigma^n(\lambda)=\lambda$ for every λ in a σ-*field* extension of k with $\lambda^n=1$, this property is not retained if we choose λ from an arbitrary k-σ-algebra.

Remark A.23. In the notation of the proof of Theorem A.20, there exist a σ-field extension $\widetilde{k}$ of k and an element $u\in\mathrm{Gl}_n(\widetilde{k})$ such that $u^{-1}G_{\widetilde{k}}u\leqslant\mathrm{Gl}_{n,\widetilde{k}}$ is σ^d-constant. Assume that H is defined over $\mathbb{Q}$. One can show that a convenient choice of the σ-field extension $\widetilde{k}$ allows us to choose u in $H(\widetilde{k})$. In other words, there exists a σ-field extension $\widetilde{k}$ of k such that $G_{\widetilde{k}}$ is conjugate inside $H_{\widetilde{k}}$ to a σ^d-constant group.

Proof. Going back to the proof of Theorem A.20, we need to show that, for a given $h \in H(k)$, one can find a σ-field extension $\widetilde{k}$ of k and $u \in H(\widetilde{k})$ such that $\sigma^d(u) = hu$. Let $\widetilde{k}$ be the function field of $H^d = H \times \ldots \times H$, i.e., d copies of H. One can write $\widetilde{k} = k(X_1, \ldots, X_d)$, where X_i is an $n \times n$-matrix of coordinates on the ith copy of H. Since the vanishing ideal $\mathbb{I}(H) \subset k[X, \frac{1}{\det(X)}] = k[\mathrm{Gl}_n]$ is defined over $\mathbb{Q}$, the action of σ on the coefficients of the polynomial ring $k[X, \frac{1}{\det(X)}]$ stabilizes $\mathbb{I}(H)$. This allows us to define a structure of k-σ-field extension on $\widetilde{k}$ via

$$\sigma(X_1) = X_2, \sigma(X_2) = X_3, \ldots, \sigma(X_d) = hX_1.$$

Then X_1 is an element of $H(\widetilde{k})$ and satisfies $\sigma^d(X_1) = hX_1$ by construction. $\square$

A.4.3. The case of almost simple groups.   Let k be an algebraically closed field of characteristic zero. Recall that a connected non-commutative algebraic group H over k is called almost simple if every proper connected normal closed subgroup is trivial. In remark A.22, we have seen that a proper Zariski dense σ-closed subgroup of H is not necessarily σ^d-constant. However, the aim of this section is to show that Theorem A.20 still holds for a proper, σ-*integral*, Zariski dense σ-closed subgroup G of H.

We start with some remarks on almost simple algebraic groups, their center, and their adjoint group. Let H be an almost simple algebraic group. By Corollary 1.3, p. 411 in [10], the algebraic group H is defined over $\mathbb{Q}$, i.e., is equal to $\mathcal{H}_k$ for some split almost simple algebraic group $\mathcal{H}$ over $\mathbb{Q}$. Since we are in characteristic zero, the schematic center $\mathcal{N}$ of $\mathcal{H}$ is representable by a closed subscheme of $\mathcal{H}$ (see Exposé XI, 6.11, in [47]). Thus, we get that, in our situation, the formation of schematic center commutes with base extension. Precisely, $\mathcal{N}_k = N$. Now, by Exposé XXII, 4.3 in [48], the quotient $\mathcal{H}/\mathcal{N}$ is representable by an affine group scheme over $\mathbb{Q}$. Since formation of quotients commute with flat base extension (see Exposé VIIa, 4.6 of [46]), the k-scheme $(\mathcal{H}/\mathcal{N})_k$ is a quotient of H by N. In conclusion, given an almost simple algebraic group H over k, we see that its center N is defined over $\mathbb{Q}$ as well as the quotient map $p : H \to H/N$ and the adjoint group H/N.

Before proving our main result on the σ-closed subgroups of almost simple algebraic groups, we need one more lemma.

Lemma A.24. *Let k be an algebraically closed field of characteristic zero, H an almost simple algebraic group over k, and $N \trianglelefteq H$ the center of H. Let $i : G \to H$ be the inclusion of a proper σ-closed subgroup G of H into H. Let $p : H \to H/N$ and $\pi : G \to G/(G \cap N)$ be the quotient maps. Then the following hold.*

- *The morphism $p \circ i : G \to H/N$ factors through π into a σ-closed embedding $\iota : G/(G \cap N) \to H/N$.*

- *The σ-closed subgroup $G/(G \cap N)$ is a proper σ-closed subgroup of H/N.*

Proof. The first assertion comes from Lemma A.14. Suppose that $G/(G \cap N) = H/N$. Since G is a proper σ-closed subgroup of H, there exists $m \geqslant 1$ such that $G[m] \subsetneq H_m$. Now, by Lemma A.14, we have $G/(G \cap N)[m] = G[m]/G[m] \cap N_m$ and $(H/N)_m = H_m/N_m$. Then

$$\dim(G[m]/G[m] \cap N_m) = \dim(G[m]) - \dim(G[m] \cap N_m) \leqslant \dim(G[m])$$

and $\dim(H_m/N_m) = \dim(H_m) - \dim(N_m) = \dim(H_m)$ (since N_m is a finite algebraic group). Therefore

$$\dim(G[m]/G[m] \cap N_m) \leqslant \dim(G[m]) < \dim(H_m) = \dim(H_m/N_m),$$

which contradicts the assumption $G/(G \cap N) = H/N$. $\qquad\square$

Recall that a σ-algebraic group G is called σ-integral if $k\{G\}$ is a σ-domain, i.e., $k\{G\}$ is an integral domain and $\sigma \colon k\{G\} \to k\{G\}$ is injective.

Theorem A.25. *Let k be an algebraically closed, inversive σ-field of characteristic zero, and let G be a σ-integral, σ-closed subgroup of $\mathrm{Gl}_{n,k}$. Assume that the Zariski closure of G in $\mathrm{Gl}_{n,k}$ is an almost simple algebraic group, properly containing G. Then there exist a σ-field extension $\widetilde{k}$ of k and an integer $d \geqslant 1$ such that $G_{\widetilde{k}}$ is conjugate to a σ^d-constant subgroup of $\mathrm{Gl}_{n,\widetilde{k}}$.*

Proof. Let $H \leqslant \mathrm{Gl}_{n,k}$ denote the Zariski closure of G, $i \colon G \to H$ be the inclusion of G into H, N the center of H. Let $p \colon H \to H/N$ and $\pi \colon G \to G/(G \cap N)$ be the quotient maps. By Corollary A.18, we can assume that H is over $\mathbb{Q}$. By the discussion above, we get that the finite center N of H as well as the quotient map $p \colon H \to H/N$ are over $\mathbb{Q}$.

By Lemma A.14, the morphism $G \to H/N$ factors into a σ-closed embedding $\iota \colon G/(G \cap N) \to H/N$. Moreover, since G is Zariski dense in H, we get that $G/(G \cap N)$ is Zariski dense in H/N. By Lemma A.24, $G/G \cap N$ is a proper σ-closed subgroup of H/N. Finally, by Theorem A.43 in [15], one can identify the k-σ-Hopf algebra of $G/(G \cap N)$ with a sub-Hopf algebra of $k\{G\}$. Since $k\{G\}$ is σ-reduced, the same holds for $k\{G/(G \cap N)\}$.

Embed H/N into some $\mathrm{Gl}_{m,k}$ such that $H/N \leqslant \mathrm{Gl}_{m,k}$ is defined over $\mathbb{Q}$. We can then apply Theorem A.20 and Remark A.23 to the proper Zariski dense σ-closed subgroup $G/(G \cap N)$ of the simple algebraic group H/N. Thereby, there exist a σ-field extension $\tilde{k}$ of k, an integer $d \geqslant 1$, and $u \in H/N(\tilde{k})$ such that $u^{-1}(G/G \cap N)_{\tilde{k}} u$ is a σ^d-constant subgroup of $\mathrm{Gl}_{m,\tilde{k}}$. By Lemma A.15, we can exchange base field extension and quotients, which implies that $u^{-1} G_{\tilde{k}}/(G \cap N)_{\tilde{k}} u$ is a σ^d-constant subgroup of some $\mathrm{Gl}_{m,\tilde{k}}$.

Now, up to enlarging $\widetilde{k}$ to its algebraic closure, we can find an element v of $H(\tilde{k})$ such that $p(v) = u$. Since $p \colon H \to H/N$ is defined by polynomials over $\mathbb{Q}$, we have $p(\sigma^d(v)) = \sigma^d(u)$. Then, let $\phi \colon G_{\tilde{k}} \to \mathrm{Gl}_{n,\tilde{k}}$ be the σ-morphism defined by $\phi(g) = \sigma^d(v^{-1} g v)(v^{-1} g v)^{-1}$ for all $g \in G_{\tilde{k}}(S)$ and for every $\tilde{k}$-σ-algebra S. By the above, ϕ maps $G_{\tilde{k}}$ into $N_{\tilde{k}}$. Since k is algebraically closed and inversive, Corollary A.14 in [15] implies that $\tilde{k}\{G\} = k\{G\} \otimes_k \tilde{k}$ is also a σ-domain. Since N is a finite algebraic group, there exists a set of orthogonal idempotents $e_1, \ldots, e_r \in k[N]$, such that $k[N] = ke_1 \oplus \ldots \oplus ke_r$. Then $\tilde{k}\{N\}$ is generated as $\tilde{k}$-σ-algebra by $e_1, \ldots, e_r$. Since $\tilde{k}\{G\}$ is a domain, $e_1 + \cdots + e_r = 1$, and $e_i e_j = 0$ for $i \neq j$, there exists $1 \leqslant i_0 \leqslant r$ such that $\phi^*(e_{i_0}) = 1$ and $\phi^*(e_j) = 0$ for all $j \neq i_0$. This implies that $\phi^*(\tilde{k}\{N\}) \subset \tilde{k}$ and ϕ is a constant morphism. Considering the image of the identity element of $G_{\tilde{k}}$ under ϕ, we see that $v^{-1} G_{\tilde{k}} v$ is a σ^d-constant subgroup of $\mathrm{Gl}_{n,\tilde{k}}$. $\qquad\square$

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