Field Unification in the Maxwell-Lorentz Theory with Absolute Space

Robert Rynasiewicz†

Although Trautman (1966) appears to give a unified-field treatment of electrodynamics in Newtonian spacetime, there are difficulties in cogently interpreting it as such in relation to the facts of electromagnetic and magneto-electric induction. Presented here is a covariant, nonunified field treatment of the Maxwell-Lorentz theory with absolute space. This dispels a worry in Earman (1989) as to whether there are any historically realistic examples in which absolute space plays an indispensable role. It also shows how Trautman's formulation can be rendered coherent, albeit at the cost of deunification, by reinterpreting the Maxwell tensor as a composite object involving, in part, elements from Newtonian spacetime.

1. Introduction. It's been said time and again that Maxwell's theory represents the first case in the history of physics of a unified field theory. If what is meant is that it has this status as formulated prior to Einstein's electrodynamics of moving bodies, then this strikes me as fundamentally misguided. For, at least according to my understanding of the history, the electric and magnetic fields in pre-relativistic electrodynamics characterize intrinsic, frame-independent states of the aether. To be sure, they are dynamically coupled, as Maxwell's equations indicate. But that is quite short of unification in the sense available in special relativity, where the electric and magnetic fields are no longer individually fundamental, but rather frame-dependent projections of the basic unified electromagnetic field, as represented by the Maxwell tensor. (Compare this with general relativity: The spacetime metric and the gravitational potentials are genuinely unified into a single field quantity $g_{\mu\nu}$. Einstein's field equations show how $g_{\mu\nu}$ and the stress energy tensor $T_{\mu\nu}$ are dynamically coupled. But we do not thereby think that $g_{\mu\nu}$ and $T_{\mu\nu}$ have been unified.)

†To contact the author, please write to: Department of Philosophy, Johns Hopkins University, Baltimore, MD 21218; e-mail: ryno@lorentz.phl.jhu.edu.

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Nonetheless, there is a fairly well-known formulation of classical electrodynamics in Newtonian spacetime in which the field equations are expressed directly in terms of the Maxwell tensor (Trautman 1966). So it would appear that there is indeed a coherent way of understanding pre-relativistic electrodynamics as a genuine instance of field unification. But, as suggested by Earman (1989), this formulation is not without problems, at least if it is supposed to make direct contact with the experimental facts of electromagnetic and magneto-electric induction. Earman draws the conclusion that there is no coherent formulation of classical electrodynamics that is both historically realistic and in which absolute space plays an indispensable role.

Understood straightforwardly and without qualification, this is an audacious conclusion. For one would have thought that the Maxwell-Lorentz version of electrodynamics as canonically formulated in Lorentz's *Versuch* (1895) is just such a formulation. How is it that we are brought to the brink of paradox? There is a weak reading of Earman's conclusion according to which it claims only that there is no such coherent formulation of classical electrodynamics that gives a genuinely unified treatment of the electromagnetic field. This less audacious conclusion (although it is still not without teeth!) does not push us to the brink. The Maxwell-Lorentz theory poses no threat of counterexample if it does not qualify as a unified field theory. This, however, poses a challenge in turn: Can Trautman's generally covariant treatment of Maxwell's theory in Newtonian spacetime be fixed accordingly? In either case, whether Earman is read weakly or strongly, we have the question: Is it possible to give a generally covariant formulation of the Maxwell-Lorentz theory in Newtonian spacetime in such a way that the electric and magnetic field quantities are spacelike vectors invariant under Galilean velocity boosts?

Here I will briefly sketch how this can be done. This will serve to refute the strong version of Earman's conclusion. By then showing how to derive Trautman's formulation from this covariant nonunified field formulation it will become clear that the so-called Maxwell tensor in Trautman's formulation is actually a hybrid object containing contributions not just from the classically conceived electric and magnetic fields, but also from various components of the background Newtonian spacetime. In short, it is not really the Maxwell tensor from relativistic electrodynamics, but a properly prerelativistic quantity that might more aptly be called the *Lorentz* tensor.

2. Trautman's Formulation. Trautman (1966) presents a four-dimensional generally covariant version of classical electrodynamics in Newtonian spacetime, which has since been widely adopted as its canonical formulation in the philosophical literature (Earman and Friedman 1973; Earman 1974; Friedman 1983). The geometric background consists of a manifold M diffeomorphic to $\mathbf{R}^4$ together with:

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- a flat symmetric affine connection ∇
- a covariantly constant one-form t_a which at each point serves to classify each vector X^a of the tangent space as space-like or time-like according to whether or not $t_a X^a = 0$
- a symmetric contravariant tensor h^{ab} of signature $+ + +0$ such that $\nabla_c h^{ab} = 0$ and $h^{ab} t_b = 0$ (this serves to induce at each point an inner product on the subspace of space-like vectors of the tangent space).

The one-form t_a suffices to foliate M into a family of E^3 hypersurfaces, which can then be rigged together by introducing a time-like vector field V^a (normalized so that $t_a V^a = 1$). Assuming $\nabla_b V^a = 0$, the integral curves of V^a can then be taken to represent the various points of the "stationary ether" or absolute space.

Taking the covariant Maxwell tensor F_{ab} as primitive, the source-free Maxwell equations assume a familiar covariant form:

$$\partial_{[a} F_{bc]} = 0$$

$$\nabla_b F^{ab} = 0$$

The term F^{ab} is obtained from F_{ab} through raising indices by repeated contraction with a contravariant tensor g^{ab} defined

$$g^{ab} =_{df} h^{ab} - V^a V^b / c^2,$$

where c is the velocity of light in vacuo. Explicitly,

$$\begin{aligned} F^{ab} &=_{df} \; g^{ac} g^{bd} F_{cd} \\ &= (h^{ac} - V^a V^c / c^2)(h^{bd} - V^b V^d / c^2) F_{cd}. \end{aligned}$$

The significance of g^{ab} is that its inverse g_{ab} is a Minkowski metric on M satisfying $\nabla_c g_{ab} = 0$. As Trautman points out, one can view the essential step taken by Einstein in 1905 to be that of denying any physical significance to V^a, t_a, and h^{ab} and instead taking only g_{ab} to have physical significance. This involves, of course, the historical fiction that Einstein already had the Maxwell tensor at his disposal.

3. Upstairs, Downstairs, chez Earman. One of the lessons Earman tries to drive home is, "There is no general argument . . . to the effect that absolute space is, ipso facto, metaphysically absurd; indeed . . . the acceptability of absolute space reduces to the contingent question of whether the world is such that the empirical adequacy of a theory of motion requires a distinguished inertial frame" (1989, 49). Late-nineteenth-century optics

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and electrodynamics would appear to provide a prima facie case. Although the aether (first purely optical, later electromagnetic) was initially conceived of as a material medium subject to Newton's laws of mechanics, by late century it was common to view it as "merely space equipped with certain physical properties" (Drude 1900, 420). This, at any rate, is the conception at the basis of Lorentz's version of Maxwell's theory. According to Earman, however,

> the resulting theory of classical electromagnetism is not free of internal troubles. It is worth working through the details in order to appreciate how difficult it is to construct an interesting and physically well motivated example where absolute space plays an indispensable role. (1989, 51)

The problem that Earman constructs takes its starting point from Trautman's formulation of nonrelativistic electrodynamics. In a relativistic spacetime, one gets used to raising and lowering tensor indices without giving thought to whether the tensor with raised indices represents the same physical quantity as that with lowered indices. The spacetime metric is a fundamental entity and induces a natural isomorphism. However, in a spacetime, such as Newtonian spacetime, in which there is no fundamental spacetime metric, there is no pre-existing natural isomorphism, and when indices are raised or lowered by multiplying by constructing quantities such as g^{ac} or its inverse g_{ab} and then contracting, there is no guarantee that the resulting object has the same physical significance. Thus, one needs to be clear at the outset whether one takes the "downstairs Maxwell tensor" or the "upstairs" Maxwell tensor as primitive. The problem that Earman then poses is that under the Galilean transformations, the resulting transformations of the "downstairs" and "upstairs" versions of the Maxwell tensor have classically conflicting physical interpretations, and the available contemporary experimental evidence provides as much justification for the one set of transformations as for the other.

More explicitly, take the "downstairs" $_*F_{ab}$ as primitive. Then the components of $_*F_{ab}$ in a coordinate system $\{x_i\}$ adapted to the stationary frame defined by V^a are by definition:

$$
{}_*F_{ij} = \begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix},
$$

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where the E_i's and B_i's are the electric and magnetic field strengths in the *x, y,* and *z* directions respectively. If $_*F_{ab}$ is to transform as a tensor, then, in ordinary 3-vector notation, the electric and magnetic field components $\vec{E'}$ and $\vec{B'}$ in coordinates $\{x_i'\}$ boosted by a Galilean transformation with velocity $\vec{v}$ must be (with $c = 1$):

$$\vec{E'} = \vec{E} + \vec{v} \times \vec{B} \tag{1}$$

$$\vec{B'} = \vec{B}. \tag{2}$$

Now consider taking the "upstairs" $\dagger F^{ab}$ as primitive. It's components in the aether frame coordinate system $\{x_i\}$ are by definition:

$$\dagger F^{ij} = \begin{pmatrix} 0 & B_z & -B_y & -E_x \\ -B_z & 0 & B_x & -E_y \\ B_y & -B_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{pmatrix}.$$

Again, assuming that $\dagger F^{ab}$ is a tensor quantity, this implies that the field components in the Galilean-boosted chart are given by

$$\vec{E'} = \vec{E} \tag{3}$$

$$\vec{B'} = \vec{B} - \vec{v} \times \vec{E}. \tag{4}$$

Hence classically, one appears to be forced to regard either the "upstairs" or the "downstairs" version of the Maxwell tensor as fundamental to the exclusion of the other. However, the phenomenon of Faraday induction suggests the electric field should transform according to equation (1), thus supporting the "downstairs" approach, while the "null results" of magneto-induction experiments such as those of Des Coudres (1889) and later Trouton (1902) and Trouton and Noble (1904) can be taken as evidence that the magnetic field should transform in accordance with equation (4). Earman concludes:

> Thus success does not greet the attempt to produce a version of classical electromagnetics in which absolute space plays an indispensable and coherent role, by imagining that **E** and **B** came to be recognized as field quantities in their own right and that optical experiments, such as that of Michelson and Morley, confirmed the law

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> of Galilean-velocity addition for light. These imaginings lead to two incompatible versions of electromagnetism, and to choose between them one needs further imaginings to the effect that either the Faraday or the magneto-induction experiments yielded non-standard results. At this point one loses contact with historical reality. . . .
>
> To summarize and repeat, absolute space in the sense of a distinguished reference frame is a suspect notion, not because armchair philosophical reflections reveal that it is somehow metaphysically absurd, but because it has no unproblematic instantiations in examples that are physically interesting and that conform even approximately to historical reality. (1989, 54–55)

Earman's line of reasoning is insightful insofar as it shows there is a problem to be overcome in producing a unified field version of the Maxwell-Lorentz theory in absolute space. But the stronger conclusion in the last quoted paragraph remains in doubt. For Lorentz did not pretend to give a unified theory of the electromagnetic field, at least in the sense that is on the table. The very idea of such had to await Einstein and Minkowski.

4. A Covariant Formulation of the Maxwell-Lorentz Theory in Newtonian Spacetime. Although equations such as (1) and (4) can be found in Lorentz's *Versuch* (1895) and subsequent writings (e.g., Lorentz 1904 and 1909), the quantities E' and B' are not introduced there as the components of the electric and magnetic fields in a uniformly moving frame, but merely as auxiliary expressions (given definitionally by these equations) which serve to simplify the manipulation of the field equations when dealing with moving systems (see Rynasiewicz 1988). Faraday and magnetic induction phenomena were not construed as indicating that the electric and magnetic field intensities are frame dependent. Rather the components of the field quantities were assumed to be invariant under Galilean boosts, and certain causal mechanisms, specifically the Lorentz force and the "compensation charge," were invoked to explain these induction phenomena.

However, what needs to be done in order to meet Earman's challenge fully is to provide a four-dimensional, generally covariant formulation of the Maxwell-Lorentz theory as understood by its inventor. To see how this can be done, it is heuristically advantageous (although slightly unfaithful historically) to start with the classical scalar potential φ and vector potential A^a, where the latter is assumed to be everywhere space-like, i.e., $A^a t_a = 0$. The electric field is obtained from the equation

$$E^a = -h^{ab}\nabla_b\varphi - V^b\nabla_b A^a.$$

For the magnetic field, we first define the tensor quantity

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$$B^{ab} = h^{ac}\nabla_c A^b - h^{bc}\nabla_c A^a.$$

The classical magnetic field strength can then be defined by contracting this with the natural three-dimensional volume element ϵ_{abc} associated with the simultaneity sheets of the spacetime, yielding the co-vector

$$B_a = \frac{1}{2}\epsilon_{abc}B^{bc}.$$

In what follows, however, it will be more convenient to work directly with the tensor representation B^{ab} of the magnetic field. At this point, though, the reader can verify that E^a and B^{ab} are both space-like and their components remain unchanged under a rotation-free Galilean boost, as required by the prerelativistic conception of the electric and magnetic fields.

For reasons of brevity, I'll simply state, rather than derive, the covariant form of the Maxwell-Lorentz field equations. For the first derivatives of the magnetic field, we have the pair:

$$V^c\nabla_c B^{ab} = h^{cb}\nabla_c E^a - h^{ca}\nabla_c E^b \tag{5}$$

$$h^{d[c}\nabla_d B^{ab]} = 0, \tag{6}$$

which when expressed in coordinates adapted to the "stationary" frame require[1]

$$-\frac{\partial \vec{B}}{\partial t} = \text{curl } \vec{E}$$

$$\text{div } \vec{B} = 0.$$

And, since all that is in question here is an existence proof, for simplicity I'll take the liberty of stating just the source free version of the other field equations:

$$V^b\nabla_b E^a = \nabla_b B^{ab} \tag{7}$$

$$\nabla_a E^a = 0. \tag{8}$$

Expressed in "stationary" coordinates, these yield

$$\frac{\partial \vec{E}}{\partial t} = \text{curl } \vec{B}$$

$$\text{div } \vec{E} = 0.$$

1. Throughout this discussion the velocity of light has been set equal to unity.

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Finally, the equation for the Lorentz force on a point mass with charge q is:

$$\mathcal{F}^a = q(E^a + B^{ab} h_{bc} U^c),$$

where h_{bc} is obtained by lowering indices on h^{bc} using Trautman's g_{ab}.

In order to appreciate the role played by absolute velocity in these equations, the reader is invited to write them in component form in a system of coordinates moving with a uniform velocity **p** through absolute space (the aether) and to compare with the equations given by Lorentz in chapter 2 of the *Versuch* for this case.[2]

5. Trautman Revisited. The equations given above are in fact formally identical to those given by Trautman under the appropriate definitions of the quantities F^{ab} and F_{ab}. First construct the tensor

$$E^{ab} = E^b V^a - E^a V^b.$$

The appropriate "upstairs" version of the Maxwell tensor is then obtained by

$$F^{ab} = B^{ab} + E^{ab}.$$

One can then use g_{ab} as defined by Trautman to lower indices to define the "downstairs" F_{ab}. Then, grinding out the details, Trautman's first equation is equivalent to the pair of equations (5) and (6), while his second to the pair (7) and (8).

But this should not be taken as an indication that there is anything preferred about the "upstairs" approach. Alternatively, one could proceed by constructing a "downstairs" counterpart of E^{ab} by

$$E_{ab} = E_a t_b - E_b t_a,$$

and then defining F_{ab} by

$$F_{ab} = B_{ab} + E_{ab},$$

where $B_{ab} = B^{cd} g_{ac} g_{bd}$.

What is edifying here is that the counterpart of the Maxwell tensor in prerelativistic electrodynamics explicitly contains components, not just of the classical electric and magnetic fields, but also of the Newtonian spacetime structure. For (at least) this reason I would prefer to call it the

2. These are his equations (Ia)–(IVa).

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Lorentz tensor. Its components in the aether rest frame agree only coextensively, not definitionally, with those of Earman's "upstairs" (respectively, "downstairs") version, of the electromagnetic field tensor. The asymmetry in the resulting component transformations in these two versions is a reflection of the roles played by V^a and t_a in the definition of the Lorentz tensor and their asymmetric properties under Galilean boosts.

6. Conclusion. I hope here to have achieved two goals. The first is to convince the reader that, contrary to the strong reading of Earman's conclusion—that there are no historically realistic examples from the history of physics in which absolute space plays a coherent and ineliminable role—the Maxwell-Lorentz theory of the late-nineteenth century is in fact such an example. The second is to show that, despite the undisputed validity of the argument for the weak reading of Earman's conclusion, there is a way to rescue Trautman's formulation of electrodynamics in Newtonian spacetime as a cogent nonrelativistic theory by appropriately reinterpreting the Maxwell tensor as a representation, not of a unified electromagnetic field, but as a composite entity constructed from the classical electric and magnetic fields together with objects from the Newtonian spacetime. Together they support the original intuition that Einstein's electrodynamics of moving bodies is the first instance in the history of physics of a genuine unified field theory.

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