OPTIMAL REINSURANCE FROM THE PERSPECTIVES OF BOTH AN INSURER AND A REINSURER

BY

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ABSTRACT

Optimal reinsurance from an insurer's point of view or from a reinsurer's point of view has been studied extensively in the literature. However, as two parties of a reinsurance contract, an insurer and a reinsurer have conflicting interests. An optimal form of reinsurance from one party's point of view may be not acceptable to the other party. In this paper, we study optimal reinsurance designs from the perspectives of both an insurer and a reinsurer and take into account both an insurer's aims and a reinsurer's goals in reinsurance contract designs. We develop optimal reinsurance contracts that minimize the convex combination of the Value-at-Risk (VaR) risk measures of the insurer's loss and the reinsurer's loss under two types of constraints, respectively. The constraints describe the interests of both the insurer and the reinsurer. With the first type of constraints, the insurer and the reinsurer each have their limit on the VaR of their own loss. With the second type of constraints, the insurer has a limit on the VaR of his loss while the reinsurer has a target on his profit from selling a reinsurance contract. For both types of constraints, we derive the optimal reinsurance forms in a wide class of reinsurance policies and under the expected value reinsurance premium principle. These optimal reinsurance forms are more complicated than the optimal reinsurance contracts from the perspective of one party only. The proposed models can also be reduced to the problems of minimizing the VaR of one party's loss under the constraints on the interests of both the insurer and the reinsurer.

KEYWORDS

Optimal reinsurance, value-at-risk, insurer's loss, reinsurer's profit, expected value reinsurance principle.

1. INTRODUCTION

In a one-period reinsurance model, the underlying (aggregate) loss faced by an insurer is assumed to be a non-negative random variable X. Under a reinsurance

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contract I, a reinsurer will cover the part of the loss X, denoted by $I(X)$, and the insurer needs to pay a reinsurance premium, denoted by P_I, to the reinsurer, where the function I is called the ceded loss function. Under the reinsurance contract I, $X - I(X) + P_I$ and $I(X) - P_I$ are the total losses faced by the insurer and the reinsurer, respectively. To avoid moral issues, a feasible reinsurance contract I should satisfy the following two conditions: (a) $I(x)$ is increasing in $x \in [0, \infty)$ with $I(0) = 0$; and (b) the 1-Lipschitz continuity, namely, $0 \leq I(y) - I(x) \leq y - x$ for any $0 \leq x \leq y < \infty$. Throughout this paper, we denote by $\mathcal{I}$ the set of all feasible reinsurance contracts satisfying these two conditions and define $(x)^+ = \max\{x, 0\}$, $x \wedge y = \min\{x, y\}$, and $x \vee y = \max\{x, y\}$. In addition, the term "increasing" means "non-decreasing" and "decreasing" means "non-increasing".

An optimal reinsurance design is obtained by determining a ceded loss function I^* from a set of feasible reinsurance contracts such that the contract I^* is optimal under certain optimization criteria. In the literature, optimal reinsurance from an insurer's point of view or from a reinsurer's point of view has been studied extensively. In particular, the classical results on optimal reinsurance from an insurer's perspective have shown under certain conditions that a stop-loss reinsurance is optimal for an insurer if the optimization criteria is to minimize the variance of the insurer's loss or to maximize the expected utility of the insurer's terminal wealth. These classical results have been extended in different ways. Indeed, the criterion of minimizing the variance of one party's loss or maximizing the expected utility of one party's terminal wealth has been generalized to the criterion of minimizing the risk measure of one party's loss, while the classical single-risk reinsurance models have been extended to multiple-risk reinsurance models, reinsurance models with default risk and other related models. Recent references on these generalizations and extensions can be found in Asimit *et al.* (2013a, 2013b), Balbás *et al.* (2009, 2011, 2015), Cai *et al.* (2013, 2014), Cai and Tan (2007), Cai and Wei (2012), Cheung (2010), Cheung *et al.* (2014a, 2014b), Chi (2012), Chi and Meng (2014), Chi and Tan (2011), Hürlimann (2011), and references therein.

As the two parties of a reinsurance contract, an insurer and a reinsurer have conflicting interests. An optimal form of reinsurance from one party's point of view may be not acceptable to the other party as pointed out by Borch (1969). Hence, a very interesting question is to take into consideration both an insurer's objectives and a reinsurer's goals in optimal reinsurance designs so that an optimal reinsurance form is acceptable to both parties. There are two general ways to consider both an insurer's objectives and a reinsurer's goals in an optimal reinsurance design. One way is to minimize or maximize an objective function that considers both an insurer's aims and a reinsurer's goals, and the other way is to minimize or maximize an objective function from one party's point of view under some constraints on the other party's goals and on the party's own objectives. Borch (1960) first addressed this issue by discussing the quota-share and stop-loss reinsurance contracts and deriving the optimal retention of these contracts under the optimization criterion of maximizing the product of the

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expected utility functions of the two parties' terminal wealth. Recently, Hürlimann (2011) has readdressed this issue by studying the combined quota-share and stop-loss contracts and obtaining the optimal retention of these contracts under the optimization criterion of minimizing the sum of the variances of the losses of the insurer and the reinsurer and several other related optimization criteria. Cai *et al.* (2013) proposed the optimization criteria of maximizing the joint survival probability and the joint profitable probability of the two parties and derived sufficient conditions for a reinsurance contract to be optimal in a wide class of reinsurance policies and under a general reinsurance premium principle. Using the results of Cai *et al.* (2013), Fang and Qu (2014) derived the optimal retentions of a combined quota-share and stop-loss reinsurance under the criterion of maximizing the joint survival probability of the two parties under the expected value reinsurance premium principle.

One of the main objectives for an insurer when buying a reinsurance is to control his risk, while one of the main goals for a reinsurer when selling a reinsurance is to make a profit. Of course, a reinsurer also worries about his own risk when selling a reinsurance contract and needs to control his risk as well. One of the important risk measures used in risk management is the VaR risk measure. In this paper, the VaR of a random variable or a risk is defined as follows:

Definition 1.1. *The* VaR *of a random variable Y at a risk level* $\alpha \in (0, 1)$ *is defined as* $\mathrm{VaR}_\alpha(Y) \triangleq \inf\{y \in \mathbb{R} : \mathbb{P}(Y > y) \le \alpha\} = S_Y^{-1}(\alpha)$, *where* $S_Y^{-1}(y)$ *is the generalized inverse function of the survival function* $S_Y(y)$ *of* Y. □

We point out with Definition 1.1 that $\mathrm{VaR}_\alpha(Y) \le y \Longleftrightarrow S_Y(y) \le \alpha$ and that $\mathrm{VaR}_\alpha(Y)$ is decreasing in $\alpha \in (0, 1)$. Roughly speaking, the VaR of a loss random variable Y is the maximum possible loss at the confidence level $1-\alpha$. In practice, the risk level $\alpha \in (0, 1)$ is a small value such as $\alpha = 0.01$ or $\alpha = 0.05$. In addition, we recall that VaR satisfies the following properties: (a) $\mathrm{VaR}_\alpha(X+c) = \mathrm{VaR}_\alpha(X) + c$ for any constant c; (b) $\mathrm{VaR}_\alpha(X + Y) = \mathrm{VaR}_\alpha(X) + \mathrm{VaR}_\alpha(Y)$ for any comonotonic random variables X and Y; (c) $\mathrm{VaR}_\alpha(X) \le \mathrm{VaR}_\alpha(Y)$ for any random variables $X \le Y$ and (d) $\mathrm{VaR}_\alpha(f(X)) = f(\mathrm{VaR}_\alpha(X))$ for any continuous and increasing function f.

When both the insurer and the reinsurer use VaR to measure their own risk, then from the insurer's perspective, the insurer prefers to buy a reinsurance contract that is a solution to the optimization problem

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_\alpha\left(X - I(X) + P_I\right). \tag{1}$$

However, from the reinsurer's point of view, the reinsurer likes to sell a reinsurance contract that is a solution to the optimization problem

$$\min_{I \in \mathcal{I}} \mathrm{VaR}_\beta\left(I(X) - P_I\right), \tag{2}$$

where α and β are the risk levels of the insurer and the reinsurer, respectively, for VaR. Optimal solutions to Problems (1) and (2) are different. Indeed, when the

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reinsurance premium P_I is determined by the expected value principle, namely $P_I = (1+\theta)\mathbb{E}[I(X)]$ with a positive risk loading factor $\theta > 0$, Cheung *et al.* (2014b) proved that the optimal reinsurance form for Problem (1) or for the insurer is

$$I_i^*(x) = (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))^+ - (x - \mathrm{VaR}_\alpha(X))^+.$$

Then, using the solution to Problem (1), it is easy to obtain that the optimal reinsurance form for Problem (2) or for the reinsurer is

$$I_r^*(x) = x - (x - \mathrm{VaR}_{\frac{1}{1+\theta}}(X))^+ + (x - \mathrm{VaR}_\beta(X))^+.$$

Obviously, in Problems (1) and (2), the optimal reinsurance form for one party is not optimal for the other. Indeed, the optimal contract minimizing the VaR of one party's loss may lead to an unacceptable large value for the VaR of the other party's loss.

In this paper, we study optimal reinsurance designs from the perspectives of both an insurer and a reinsurer and take into account both an insurer's aims and a reinsurer's goals in reinsurance contract designs. We assume both the insurer and the reinsurer use the VaR to measure their own loss and develop optimal reinsurance contracts that minimize the convex combination of the VaR risk measures of the insurer's loss and the reinsurer's loss under two types of constraints, respectively. One reason why this criterion was chosen is that it enables mathematically tractable solutions to our problem. This surely oversimplifies how the conflicting points of view of the reinsurer and insurer may be jointly analyzed in practice, as is the case with the criteria proposed by Borch (1960) and Hürlimann (2011), which were mentioned above. However, our convex combination has the advantage that we can study the effect of varying the relative importance of the interests of each party, and also recover the two individual points of view as extreme points. Furthermore, this criterion can lead to the following type of economical interpretation: assume the reinsurer is designing the contract to meet its objectives, but also wants to propose a contract that will be attractive to the insurer. As the weight given to the insurer's VaR increases, the reinsurer is able to see the effect of putting more and more importance on having a competitive contract, and can thus make a better informed decision when designing the contract.

The constraints describe the interests of both the insurer and the reinsurer. With the first type of constraints, the insurer and the reinsurer have their own limit on the VaR of their own loss. With the second type of constraints, the insurer has a limit on the VaR of his loss while the reinsurer has a target on his profit in selling a reinsurance contract. For both types of constraints, we derive the optimal reinsurance forms within a wide class of reinsurance policies and under the expected value reinsurance premium principle. These optimal reinsurance forms are more complicated than the optimal reinsurance contracts from the perspective of one party only. The proposed models can also be reduced to the problems of minimizing the VaR of one party's loss under the constraints on the interests of both the insurer and the reinsurer.

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To avoid tedious discussions and arguments, in this paper, we simply suppose that the survival function $S_X(x)$ of the underlying non-negative loss random X is continuous and decreasing on $[0, \infty)$ with $S_X(0) = 1$. Furthermore, we assume that the reinsurance premium is calculated by the expected value principle, namely, $P_I = (1+\theta)\mathbb{E}[I(X)]$, where $\theta > 0$.

The rest of the paper is organized as follows. In Section 2, we propose two reinsurance problems that take into consideration the interests of both an insurer and a reinsurer. The optimal solutions to the two problems are derived in Sections 3 and 4, respectively. Concluding remarks are given in Section 5. The proofs of all the results presented in this paper are given in the appendix.

2. Reinsurance models taking into account the interests of both an insurer and a reinsurer

Assume the insurer and the reinsurer use the VaR with risk levels $0 < \alpha < 1$ and $0 < \beta < 1$, respectively, to measure their own losses. Without reinsurance, the VaR of the insurer' loss is $\mathrm{VaR}_\alpha(X)$. With a reinsurance contract I, the VaR of the insurer's loss is $\mathrm{VaR}_\alpha(X - I(X) + P_I)$, and the insurer requires $\mathrm{VaR}_\alpha(X - I(X) + P_I) \le \mathrm{VaR}_\alpha(X)$. Furthermore, the insurer wants the VaR to be reduced to a tolerated value L_1 so that

$$\mathrm{VaR}_\alpha(X - I(X) + P_I) \le L_1, \tag{3}$$

where $L_1 > 0$ is the threshold representing the maximum VaR tolerated by the insurer after a reinsurance. Thus, it is reasonable to assume $L_1 \le \mathrm{VaR}_\alpha(X)$.

On the other hand, the reinsurer also worries about his loss in selling the contract I and wants to set a threshold $L_2 > 0$ for the VaR of his loss so that

$$\mathrm{VaR}_\beta(I(X) - P_I) \le L_2. \tag{4}$$

Note that $I(X) - X \le 0 \le P_I$. Thus, $I(X) - P_I \le X$ and $\mathrm{VaR}_\beta(I(X) - P_I) \le \mathrm{VaR}_\beta(X)$. Hence, it is reasonable to assume $L_2 \le \mathrm{VaR}_\beta(X)$.

As the seller of the reinsurance contract I, the reinsurer expects to make a profit, namely, to have $I(X) \le P_I$. Assume that the reinsurer wants to make a profit at least $L_3 \ge 0$ at a confidence level at least $0 < \gamma < 1$ in selling the reinsurance contract I, namely the profit target L_3 and the confidence level γ satisfy

$$\mathbb{P}(P_I - I(X) \ge L_3) = 1 - \mathbb{P}(I(X) > P_I - L_3) \ge \gamma. \tag{5}$$

To obtain feasible and applicable models for optimal reinsurance designs from the perspectives of both an insurer and a reinsurer, we have to make some assumptions on the relationships between the confidence level γ and each of the risk levels α and β, and the safety loading factor θ. In doing so, suppose $1 - \gamma \le \beta$. Then, $\mathrm{VaR}_\beta(I(X) - P_I) \le \mathrm{VaR}_{1-\gamma}(I(X) - P_I) \le -L_3 \le 0$, where the second inequality follows from (5). However, the risk level β is used

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to measure the maximum possible loss of the reinsurer. Thus, if $1-\gamma \leq \beta$, then the level β will lead to a non-positive VaR for his loss $I(X) - P_I$. Such a non-positive VaR cannot provide useful information for the reinsurer. Thus, we assume $\beta < 1-\gamma$. In addition, we assume $\alpha < 1-\gamma$ as well, since the risk levels α and β should be near in practice.

Furthermore, for a feasible contract $I \in \mathcal{I}$, note that $I(X)$ is a non-negative random variable and $P_I = (1+\theta)\mathbb{E}[I(X)]$, thus by Markov's inequality, it is easy to see $\mathbb{P}(I(X) > P_I) \leq 1/(1+\theta)$ or equivalently $\mathbb{P}(I(X) \leq P_I) \geq \theta/(1+\theta)$, which implies that the reinsurer will make a profit, namely, $I(X) \leq P_I$, with a probability at least $\theta/(1+\theta)$. Thus, it is reasonable to assume $\gamma > \theta/(1+\theta)$ since L_3 is the profit target or the minimum profit desire for the reinsurer to sell a reinsurance contract and only a very high confidence level γ is acceptable for the reinsurer. Note that $\gamma > \theta/(1+\theta)$ is equivalent to $1-\gamma < 1/(1+\theta)$. Hence, the assumptions of $\alpha < 1-\gamma$ and $\beta < 1-\gamma$ imply $\alpha < 1/(1+\theta)$ and $\beta < 1/(1+\theta)$, respectively.

Throughout the paper, we denote $a = \mathrm{VaR}_\alpha(X)$, $b = \mathrm{VaR}_\beta(X)$, $c = \mathrm{VaR}_{1-\gamma}(X)$ and $v_\theta = \mathrm{VaR}_{\frac{1}{1+\theta}}(X)$. Therefore, for any $I \in \mathcal{I}$, by the properties of the VaR, we have $\mathrm{VaR}_\alpha(X - I(X) + P_I) = a - I(a) + P_I$, $\mathrm{VaR}_\beta(I(X) - P_I) = I(b) - P_I$ and $\mathrm{VaR}_{1-\gamma}(I(X)) = I(c)$. It is easy to check that (3) is equivalent to $a - I(a) \leq L_1 - P_I$, (4) is equivalent to $I(b) \leq L_2 + P_I$ and (5) is equivalent to $I(c) \leq P_I - L_3$. Moreover, note that $\alpha \vee \beta < 1-\gamma < 1/(1+\theta)$ is equivalent to $v_\theta < c < a \wedge b$.

Thus, when the insurer and the reinsurer have the limits L_1 and L_2, respectively, on the VaRs of their own losses in a reinsurance contract, the set of the feasible reinsurance contracts acceptable by both the insurer and the reinsurer is

$$\mathcal{I}_1 \triangleq \{I \in \mathcal{I} :\ I(b) - L_2 \leq P_I \leq I(a) - a + L_1\}, \tag{6}$$

where $\mathcal{I}_1$ is obtained when the constraints (3) and (4) are imposed on $\mathcal{I}$.

Furthermore, when the insurer has the limit L_1 on the VaR of his loss and the reinsurer has the target L_3 on his profit in a reinsurance contract, the set of the feasible reinsurance contracts acceptable by both the insurer and the reinsurer is

$$\mathcal{I}_2 \triangleq \{I \in \mathcal{I} :\ I(c) + L_3 \leq P_I \leq I(a) - a + L_1\}, \tag{7}$$

where $\mathcal{I}_2$ is obtained when the constraints (3) and (5) are imposed on $\mathcal{I}$.

The desired sets $\mathcal{I}_1$ and $\mathcal{I}_2$ may be empty. We have to impose some restrictions on L_1, L_2 and L_3 so that $\mathcal{I}_1$ and $\mathcal{I}_2$ are non-empty. First, for any $I \in \mathcal{I}_1$, we have $L_1 + L_2 \geq a - I(a) + P_I + I(b) - P_I = a + I(b) - I(a)$. Furthermore, by the 1-Lipschitz continuity of I, we have $I(b) - I(a) \geq 0$ if $b > a$ and $I(b) - I(a) \geq b - a$ if $a > b$. Hence, $L_1 + L_2 \geq a \wedge b$. Moreover, we assume

$$v_\theta + (1+\theta)\int_{v_\theta}^{a} S_X(x)\mathrm{d}x \leq L_1. \tag{8}$$

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This condition will guarantee that $\mathcal{I}_1$ is non-empty as showed in Lemma 3.1.

Next, for any $I \in \mathcal{I}_2$, because $a \geq c$ and I is 1-Lipschitz continuous, we have $a + P_I - L_1 - P_I + L_3 \leq I(a) - I(c) \leq a - c$, and thus $c \leq L_1 - L_3$.

Furthermore, we assume

$$(1+\theta)\left(\int_0^{v_\theta} + \int_c^{\infty}\right) S_X(x)\mathrm{d}x - v_\theta \geq L_3. \tag{9}$$

The conditions (8) and (9) will guarantee $\mathcal{I}_2$ to be non-empty as proved in Lemma 4.1.

When $\mathcal{I}_i$, $i = 1, 2$, is the set of feasible reinsurance contracts acceptable by both the insurer and the reinsurer, from the insurer's perspective, an optimal reinsurance contract is a solution to the optimization problem of

$$\min_{I \in \mathcal{I}_i} \mathrm{VaR}_\alpha\left(X - I(X) + P_I\right), \tag{10}$$

while from the reinsurer's perspective, an optimal reinsurance contract is a solution to the optimization problem of

$$\min_{I \in \mathcal{I}_i} \mathrm{VaR}_\beta\left(I(X) - P_I\right). \tag{11}$$

Instead of solving Problems (10) and (11) separately, we consider the unified minimization problem of

$$\min_{I \in \mathcal{I}_i} V(I), \tag{12}$$

where the objective function

$$\begin{aligned} V(I) &\triangleq \lambda \mathrm{VaR}_\alpha\left(X - I(X) + P_I\right) + (1-\lambda)\mathrm{VaR}_\beta\left(I(X) - P_I\right) \\ &= \lambda a + (2\lambda - 1)P_I - \lambda I(a) + (1-\lambda)I(b) \end{aligned}$$

is the convex combination of the VaRs of the insurer's loss and the reinsurer's loss, with $\lambda \in [0, 1]$ a weighting factor. When $\lambda = 0$, $V(I) = \mathrm{VaR}_\beta\left(I(X) - P_I\right)$ and Problem (12) is reduced to Problem (11). When $\lambda = 1$, $V(I) = \mathrm{VaR}_\alpha\left(X - I(X) + P_I\right)$ and Problem (12) is reduced to Problem (10). Thus, Problems (10) and (11) can be viewed as special cases of Problem (12).

When $a = b$, the objective function $V(I)$ becomes

$$V(I) = \lambda a + (1 - 2\lambda)(I(a) - P_I) = (1-\lambda)a + (2\lambda - 1)(a - I(a) + P_I),$$

which implies that Problem (12) is reduced to either Problem (10) when $1/2 < \lambda \leq 1$ or Problem (11) when $0 \leq \lambda < 1/2$. However, these two problems are covered in Problem (12) by setting $\lambda = 1$ and $\lambda = 0$, respectively. Thus, we assume $a \neq b$.

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Furthermore, when $\lambda = 1/2$, the objective function $V(I)$ becomes

$$V(I) = \frac{a}{2} + \frac{1}{2}\left(I(b) - I(a)\right).$$

Thus, Problem (12) is reduced to $\min_{I\in\mathcal{I}_i}\{I(b) - I(a)\}$, $i = 1, 2$. Note that the 1-Lipschitz property of I implies that $0 \le I(b) - I(a) \le b - a$ for $a < b$ and $I(b) - I(a) \ge -(a - b)$ for $a > b$. Hence, $\min_{I\in\mathcal{I}_i}\{I(b) - I(a)\} = -(a - b)^+$. Thus, the optimal contract I^* to the problem of $\min_{I\in\mathcal{I}_i}\{I(b) - I(a)\}$ and hence to Problem (12) is any contract $I^* \in \mathcal{I}_i$ satisfying $I^*(a) - I^*(b) = (a - b)^+$. We will see in Remarks 3.1 and 4.1 that such optimal contracts I^* exist in $\mathcal{I}_i$ for $i = 1, 2$, and thus Problem (12) is solved for $\lambda = 1/2$. Hence, we assume $\lambda \neq 1/2$.

In summary, in the rest of this paper, we assume that the following conditions hold:

$$\begin{cases} \lambda \neq \frac{1}{2}, \ \ a \neq b, \ \ L_3 + c \le L_1 \le a, \ \ L_2 \le b, \\ 0 < v_\theta < c < a \wedge b \le L_1 + L_2, \\ \text{and the inequalities (8) and (9) hold.} \end{cases} \tag{13}$$

We point out that in Assumption (13), conditions $L_3 + c \le L_1$, $v_\theta < c < a \wedge b$ and the inequality (9) are actually not required for Problem (14) in Section 3.

Next, we will solve Problem (12) for $i = 1, 2$ in Sections 3 and 4, respectively.

3. Optimal reinsurance with constraints on the VaRs of both an insurer's loss and a reinsurer's loss

In this section, we will solve Problem (12) for $i = 1$, namely, to solve the minimization problem of

$$\min_{I\in\mathcal{I}_1} V(I). \tag{14}$$

In this problem, $V(I) = \lambda a + (2\lambda - 1)P_I - \lambda I(a) + (1 - \lambda)I(b)$ and $\mathcal{I}_1$ is the set of feasible reinsurance contracts acceptable by both the insurer and the reinsurer. The definition of $\mathcal{I}_1$ also describes the constraints on the VaRs of both an insurer's loss and a reinsurer's loss. A reinsurance contract I is said to be acceptable if $I \in \mathcal{I}_1$.

First, we introduce some notation. Define the two types of feasible contracts $I^m_{\xi_a,\xi_b}$ and $I^M_{\xi_a,\xi_b}$ in $\mathcal{I}$ for some pairs of (ξ_a, ξ_b) as follows:

1. If $a < b$, for each pair $(\xi_a, \xi_b) \in [0, a] \times [0, b]$ and $\xi_a \le \xi_b$, define

$$I^m_{\xi_a,\xi_b}(x) = (x - a + \xi_a)^+ - (x - a)^+ + (x - (b - \xi_b + \xi_a))^+ - (x - b)^+,$$

$$I^M_{\xi_a,\xi_b}(x) = x - (x - \xi_a)^+ + (x - a)^+ - (x - (a + \xi_b - \xi_a))^+ + (x - b)^+.$$

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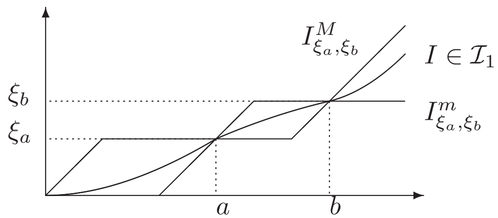

FIGURE 1: Relation between arbitrary $I \in \mathcal{I}_1$ and the pair $(I^m_{\xi_a,\xi_b}, I^M_{\xi_a,\xi_b})$ when $a < b$.

2. If $a > b$, for each pair $(\xi_a, \xi_b) \in [0, a] \times [0, b]$ and $\xi_a \geq \xi_b$, define

$$I^m_{\xi_a,\xi_b}(x) = (x - b + \xi_b)^+ - (x - b)^+ + (x - (a - \xi_a + \xi_b))^+ - (x - b)^+,$$
$$I^M_{\xi_a,\xi_b}(x) = x - (x - \xi_b)^+ + (x - b)^+ - (x - (b + \xi_a - \xi_b))^+ + (x - a)^+.$$

Since $I^m_{\xi_a,\xi_b}(0) = 0$ and $\lim_{x\to\infty} S_X(x) = 0$, we have

$$\begin{aligned} P_{I^m_{\xi_a,\xi_b}} &= (1+\theta)\mathbb{E}\left[I^m_{\xi_a,\xi_b}(X)\right] = (1+\theta)\int_0^\infty I^m_{\xi_a,\xi_b}(x)\mathrm{d}F_X(x) \\ &= -(1+\theta)\int_0^\infty I^m_{\xi_a,\xi_b}(x)\mathrm{d}S_X(x) = (1+\theta)\int_0^\infty S_X(x)\mathrm{d}I^m_{\xi_a,\xi_b}(x) \\ &= (1+\theta)\left(\int_{a\wedge b-\xi_a\wedge\xi_b}^{a\wedge b} + \int_{a\vee b-|\xi_b-\xi_a|}^{a\vee b}\right) S_X(x)\mathrm{d}x. \end{aligned}$$

Similarly, we have

$$\begin{aligned} P_{I^M_{\xi_a,\xi_b}} &= (1+\theta)\mathbb{E}\left[I^M_{\xi_a,\xi_b}(X)\right] \\ &= (1+\theta)\left(\int_0^{\xi_a\wedge\xi_b} + \int_{a\wedge b}^{a\wedge b+|\xi_b-\xi_a|} + \int_{a\vee b}^\infty\right) S_X(x)\mathrm{d}x. \end{aligned}$$

It is easy to verify that for any $I \in \mathcal{I}_1$ satisfying $I(a) = \xi_a$ and $I(b) = \xi_b$, we have $I^m_{\xi_a,\xi_b}(x) \leq I(x) \leq I^M_{\xi_a,\xi_b}(x)$ for all $x \geq 0$ as illustrated by Figure 1 and thus $P_{I^m_{\xi_a,\xi_b}} \leq P_I \leq P_{I^M_{\xi_a,\xi_b}}$.

Next, we define the set $\Xi_{a,b} \subset [0, a] \times [0, b]$ as follows:

a. When $a < b$, $\Xi_{a,b}$ is the set of all pairs (ξ_a, ξ_b) satisfying

$$\xi_a \leq \xi_b \leq \xi_a + b \wedge (L_1 + L_2) - a, \tag{15}$$

$$\xi_b - L_2 \leq P_{I^M_{\xi_a,\xi_b}} = (1+\theta)\left(\int_0^{\xi_a} + \int_a^{a+\xi_b-\xi_a} + \int_b^\infty\right) S_X(x)\mathrm{d}x, \tag{16}$$

$$L_1 - a + \xi_a \geq P_{I^m_{\xi_a,\xi_b}} = (1+\theta)\left(\int_{a-\xi_a}^a + \int_{b-\xi_b+\xi_a}^b\right) S_X(x)\mathrm{d}x. \tag{17}$$

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b. When $a > b$, $\Xi_{a,b}$ is the set of all pairs (ξ_a, ξ_b) satisfying

$$\xi_b + (a - L_1 - L_2)^+ \leq \xi_a \leq \xi_b + a - b, \tag{18}$$

$$\xi_b - L_2 \leq P_{I^M_{\xi_a,\xi_b}} = (1+\theta)\left(\int_0^{\xi_b} + \int_b^{b+\xi_a-\xi_b} + \int_a^{\infty}\right) S_X(x)\mathrm{d}x, \tag{19}$$

$$L_1 - a + \xi_a \geq P_{I^m_{\xi_a,\xi_b}} = (1+\theta)\left(\int_{b-\xi_b}^{b} + \int_{a-\xi_a+\xi_b}^{a}\right) S_X(x)\mathrm{d}x. \tag{20}$$

To solve Problem (14), we introduce the auxiliary functions g_1, g_2 and g_3 and discuss their properties in the following proposition.

Proposition 3.1. *a. Define* $g_1(\xi) \triangleq \xi - (1+\theta)\int_{a-\xi}^{a} S_X(x)\mathrm{d}x$ *for* $\xi \in [0, a]$. *Then,* g_1 *is continuous, increasing on* $[0, a - v_\theta)$*, strictly decreasing on* $(a - v_\theta, a]$*, and* $\max_{\xi\in[0,a]} g_1(\xi) = g_1(a - v_\theta)$.

b. Define $g_2(\xi) \triangleq \xi - (1+\theta)\left(\int_0^{\xi} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x$ *for* $\xi \in [0, a \wedge b]$. *Then,* g_2 *is continuous, strictly decreasing on* $[0, v_\theta)$*, increasing on* $(v_\theta, a \wedge b]$*, and* $\min_{\xi\in[0,a\wedge b]} g_2(\xi) = g_2(v_\theta)$.

c. Define $g_3(\xi) \triangleq \xi - (1+\theta)\int_{b-\xi}^{a} S_X(x)\mathrm{d}x$ *for* $\xi \in [0, b]$. *Then,* g_3 *is continuous, increasing on* $[0, b - v_\theta)$*, strictly decreasing on* $(b - v_\theta, b]$*, and* $\max_{\xi\in[0,b]} g_3(\xi) = g_3(b - v_\theta)$.

d. Assume $a < b$. *Then,* $g_2(\xi_a) < g_1(\xi_a)$ *for any* $\xi_a \in [0, a]$. *In addition, for any fixed* $\xi_a \in [0, a]$, $P_{I^M_{\xi_a,\xi_b}}$, $P_{I^m_{\xi_a,\xi_b}}$ *and* $\xi_b - P_{I^M_{\xi_a,\xi_b}}$ *are continuous and strictly increasing in* $\xi_b \in [\xi_a, \xi_a + b - a]$.

e. Assume $a > b$. *Then,* $g_2(\xi_b) < g_3(\xi_b)$ *for any* $\xi_b \in [0, b]$. *In addition, for any fixed* $\xi_b \in [0, b]$, $P_{I^M_{\xi_a,\xi_b}}$, $P_{I^m_{\xi_a,\xi_b}}$ *and* $\xi_a - P_{I^m_{\xi_a,\xi_b}}$ *are continuous and strictly increasing in* $\xi_a \in [\xi_b, \xi_b + a - b]$.

Lemma 3.1. *The following three statements are equivalent: (i) Inequality (8) holds. (ii)* $\mathcal{I}_1 \neq \emptyset$. *(iii)* $\Xi_{a,b} \neq \emptyset$. *In addition, (8) implies*

$$v_\theta - (1+\theta)\left(\int_0^{v_\theta} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x \leq L_2. \tag{21}$$

Lemma 3.2. *Problem (14) has the same minimal value as the minimization problem*

$$\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) \tag{22}$$

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in the sense that $\min_{I\in\mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b)$*, where,* $v(\xi_a, \xi_b) = \lambda a + (2\lambda - 1) P_{\xi_a,\xi_b} - \lambda\xi_a + (1-\lambda)\xi_b$ *and*

$$P_{\xi_a,\xi_b} \triangleq \begin{cases} (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}, & \text{if } 0 \le \lambda < \frac{1}{2}, \\ (\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}}, & \text{if } \frac{1}{2} < \lambda \le 1. \end{cases} \tag{23}$$

Moreover, let $(\xi_a^*, \xi_b^*) \in \Xi_{a,b}$ *be the minimizer of Problem (22). Then, a contract* I^* *of the form*

$$I^*(x) = (x-d_1)^+ - (x-(d_1+\xi_a^*\wedge\xi_b^*))^+ + (x-d_2)^+ - (x-(d_2+|\xi_b^*-\xi_a^*|))^+ + (x-d_3)^+ \tag{24}$$

for some $(d_1, d_2, d_3) \in [0,\ a\wedge b - \xi_a^*\wedge\xi_b^*] \times [a\wedge b,\ a\vee b - |\xi_b^* - \xi_a^*|] \times [a\vee b,\ \infty]$*, satisfying* $P_{I^*} = P_{\xi_a^*,\xi_b^*}$*, is the optimal solution to Problem (14).*

Lemma 3.2 reduces the infinite-dimensional optimization problem (14) to a two-dimensional optimization problem (22). In the following two theorems, we give the explicit expressions of $(\xi_a^*,\ \xi_b^*)$ and (d_1, d_2, d_3) for the optimal solution I^* presented in (24).

Theorem 3.1. *Suppose* $a < b$*, then Problem (22) has minimizer* (ξ_a^*, ξ_b^*) *with* $\xi_a^* = \xi_b^*$ *and the optimal solution to Problem (14), denoted by* I^**, is given as follows:*

a. In the case $0 \le \lambda < 1/2$*:*
  - *i. If* $g_1(v_\theta) \ge a - L_1$*, then* $\xi_a^* = v_\theta$ *and* $I^*(x) = (x-d_1)^+ - (x-d_1-v_\theta)^+ + (x-a)^+ + (x-d_3)^+$ *for some* $d_1 \in [0, a-v_\theta]$ *and* $d_3 \in [b,\infty]$ *such that* $P_{I^*} = v_\theta - (a-L_1)\vee g_2(v_\theta)$*.*
  - *ii. If* $g_1(v_\theta) < a - L_1$*, then there exists* $\xi_1 \in [v_\theta \wedge (a-v_\theta),\ v_\theta \vee (a-v_\theta)]$ *such that* $g_1(\xi_1) = a - L_1$*. Moreover,* $\xi_a^* = \xi_1$ *and* $I^*(x) = (x-a+\xi_1)^+ - (x-a)^+$*.*

b. In the case $1/2 < \lambda \le 1$*:*
  - *i. If* $g_2(a-v_\theta) \le L_2$*, then* $\xi_a^* = a - v_\theta$ *and* $I^*(x) = (x-d_1)^+ - (x-d_1-a+v_\theta)^+ + (x-a)^+ + (x-d_3)^+$ *for some* $d_1 \in [0, v_\theta]$ *and* $d_3 \in [b,\infty]$ *such that* $P_{I^*} = a - v_\theta - L_2 \wedge g_1(a-v_\theta)$*.*
  - *ii. If* $g_2(a-v_\theta) > L_2$*, then there exists* $\xi_2 \in [v_\theta \wedge (a-v_\theta),\ v_\theta \vee (a-v_\theta)]$ *such that* $g_2(\xi_2) = L_2$*. Moreover,* $\xi_a^* = \xi_2$ *and* $I^*(x) = x - (x-\xi_2)^+ + (x-b)^+$*.*

Theorem 3.2. *Suppose* $a > b$*, then Problem (22) has minimizer* (ξ_a^*, ξ_b^*) *with* $\xi_a^* = \xi_b^* + a - b$ *and the optimal solution to Problem (14), denoted by* I^**, is given as follows:*

a. In the case $0 \le \lambda < 1/2$*:*
  - *i. If* $g_3(v_\theta) \ge b - L_1$*, then* $\xi_b^* = v_\theta$ *and* $I^*(x) = (x-d_1)^+ - (x-d_1-v_\theta)^+ + (x-b)^+ - (x-a)^+ + (x-d_3)^+$ *for some* $d_1 \in [0, b-v_\theta]$ *and* $d_3 \in [a,\infty]$ *such that* $P_{I^*} = v_\theta - (b-L_1)\vee g_2(v_\theta)$*.*

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ii. *If* $g_3(v_\theta) < b - L_1$*, then there exists* $\xi_3 \in [v_\theta \wedge (b - v_\theta),\ v_\theta \vee (b - v_\theta)]$ *such that* $g_3(\xi_3) = b - L_1$*. Moreover,* $\xi_b^* = \xi_3$ *and* $I^*(x) = (x - b + \xi_3)^+ - (x - a)^+$.

b. *In the case of* $1/2 < \lambda \leq 1$*:*

i. *If* $g_2(b - v_\theta) \leq L_2$*, then* $\xi_b^* = b - v_\theta$ *and* $I^*(x) = (x - d_1)^+ - (x - d_1 - b + v_\theta)^+ + (x - b)^+ - (x - a)^+ + (x - d_3)^+$ *for some* $d_1 \in [0, v_\theta]$ *and* $d_3 \in [a, \infty]$ *such that* $P_{I^*} = b - v_\theta - L_2 \wedge g_3(b - v_\theta)$.

ii. *If* $g_2(b - v_\theta) > L_2$*, then there exists* $\xi_4 \in [v_\theta \wedge (b - v_\theta),\ v_\theta \vee (b - v_\theta)]$ *such that* $g_2(\xi_4) = L_2$*. Moreover,* $\xi_b^* = \xi_4$ *and* $I^*(x) = x - (x - \xi_4)^+ + (x - b)^+$.

Remark 3.1. *By the proofs of Theorems 3.1 and 3.2, we know that the optimal contracts* I^* *in Theorems 3.1 and 3.2 satisfy* $I^*(a) - I^*(b) = (a - b)^+$*, and hence the optimal solutions* I^* *in Theorems 3.1 and 3.2 are also the solutions to Problem (14) when* $\lambda = 1/2$. □

4. Optimal reinsurance with constraints on the VaR of an insurer's loss and a reinsurer's profit

In this section, we solve Problem (12) for $i = 2$, namely, we solve the minimization problem

$$\min_{I \in \mathcal{I}_2} V(I). \tag{25}$$

In this problem, $V(I) = \lambda a + (2\lambda - 1) P_I - \lambda I(a) + (1 - \lambda) I(b)$ and $\mathcal{I}_2$ is the set of feasible reinsurance contracts acceptable by both the insurer and the reinsurer. The definition of $\mathcal{I}_2$ also describes the constraints on the VaR of the insurer's loss and on the reinsurer's profit. A reinsurance contract I is said to be acceptable if $I \in \mathcal{I}_2$.

It is easy to check that for any given $(\xi_c, \xi_a, \xi_b) \in [0, c] \times [0, a] \times [0, b]$, if $I \in \mathcal{I}$ satisifes $I(c) = \xi_c$, $I(a) = \xi_a$, and $I(b) = \xi_b$, then $I^m_{\xi_c, \xi_a, \xi_b}(x) \leq I(x) \leq I^M_{\xi_c, \xi_a, \xi_b}(x)$ for all $x \geq 0$ and $P_{I^m_{\xi_c, \xi_a, \xi_b}} \leq P_I \leq P_{I^M_{\xi_c, \xi_a, \xi_b}}$, where

$$\begin{aligned} I^m_{\xi_c, \xi_a, \xi_b}(x) &\triangleq (x - c + \xi_c)^+ - (x - c)^+ + (x - (a \wedge b - \xi_a \wedge \xi_b + \xi_c))^+ \\ &\quad - (x - a \wedge b)^+ + (x - (a \vee b - |\xi_a - \xi_b|))^+ - (x - a \vee b)^+, \\ I^M_{\xi_c, \xi_a, \xi_b}(x) &\triangleq x - (x - \xi_c)^+ + (x - c)^+ - (x - (c + \xi_a \wedge \xi_b - \xi_c))^+ \\ &\quad + (x - a \wedge b)^+ - (x - (a \wedge b + |\xi_a - \xi_b|))^+ + (x - a \vee b)^+, \end{aligned}$$

are two feasible reinsurance contracts in $\mathcal{I}$.

To solve Problem (25), we introduce auxiliary functions h_i for $i = 1, \ldots, 7$, $A^M_{\xi_c}$, A_{ξ_c}, $A^m_{\xi_c}$, $B^M_{\xi_c}$, and $B^m_{\xi_c}$, and discuss their properties in the following three propositions.

Proposition 4.1. *Assume $a \neq b$.*

a. Define $h_1(\xi_c) \triangleq (1+\theta)\left(\int_0^{\xi_c} + \int_c^{\infty}\right) S_X(x)\mathrm{d}x - \xi_c$ for $\xi_c \in [0, c]$. Then, $h_1(\xi_c)$ is continuous, concave, strictly increasing on $[0, v_\theta)$, decreasing on $(v_\theta, c]$, and $\max_{\xi_c \in [0,\, c]} h_1(\xi_c) = h_1(v_\theta)$.

b. Define $h_2(\xi_c) \triangleq (1+\theta)\int_{c-\xi_c}^{a} S_X(x)\mathrm{d}x - \xi_c$ for $\xi_c \in [0, c]$. Then, $h_2(\xi_c)$ is continuous, convex, decreasing on $[0, c - v_\theta)$, strictly increasing on $(c - v_\theta, c]$, and $\min_{\xi_c \in [0,\, c]} h_2(\xi_c) = h_2(c - v_\theta)$. Moreover, $h_2(\xi_c) < h_1(\xi_c)$ for $\xi_c \in [0, c]$.

Proposition 4.2. *Assume $a < b$.*

a. Functions $P_{I^M_{\xi_c,\xi_a,\xi_b}}$, $P_{I^m_{\xi_c,\xi_a,\xi_b}}$, $\xi_b - P_{I^M_{\xi_c,\xi_a,\xi_b}}$ and $\xi_b - P_{I^m_{\xi_c,\xi_a,\xi_b}}$, are continuous and strictly increasing in $\xi_b \in [\xi_a, \xi_a + b - a]$.

b. Given $\xi_c \in [0, c]$, define $A^M_{\xi_c}(\xi_a) \triangleq P_{I^M_{\xi_c,\xi_a,\xi_a+b-a}}$ and $A^m_{\xi_c}(\xi_a) \triangleq P_{I^m_{\xi_c,\xi_a,\xi_a}}$, for $\xi_a \in [\xi_c, \xi_c + a - c]$, and $A_{\xi_c}(\xi_a) \triangleq P_{I^M_{\xi_c,\xi_a,\xi_a}}$, for $\xi_a \in [\xi_c, \xi_c + b - c]$. Then, all the functions $A^M_{\xi_c}(\xi_a)$, $A^m_{\xi_c}(\xi_a)$, $\xi_a - A^M_{\xi_c}(\xi_a)$ and $\xi_a - A^m_{\xi_c}(\xi_a)$ are continuous and strictly increasing in $\xi_a \in [\xi_c, \xi_c + a - c]$, and $A_{\xi_c}(\xi_a)$ and $\xi_a - A_{\xi_c}(\xi_a)$ are continuous and strictly increasing in $\xi_a \in [\xi_c, \xi_c + b - c]$.

c. Define $h_3(\xi_c) \triangleq A_{\xi_c}(\xi_c + a - c) - \xi_c$ for $\xi_c \in [0, c]$. Then, $h_3(\xi_c)$ is continuous, concave, strictly increasing on $[0, v_\theta)$, decreasing on $(v_\theta, c]$ and $\max_{\xi_c \in [0,\, c]} h_3(\xi_c) = h_3(v_\theta)$.

d. Define $h_4(\xi_c) \triangleq A^m_{\xi_c}(\xi_c + a - L_1 + L_3) - \xi_c$ for $\xi_c \in [0, c]$. Then, $h_4(\xi_c)$ is continuous, convex, decreasing on $[0, c - v_\theta)$, strictly increasing on $(c - v_\theta, c]$ and $\min_{\xi_c \in [0,\, c]} h_4(\xi_c) = h_4(c - v_\theta)$.

e. Define $h_5(\xi_c) \triangleq A_{\xi_c}(\xi_c + a - L_1 + L_3) - \xi_c$ for $\xi_c \in [0, c]$. Then, $h_5(\xi_c)$ is continuous, concave, strictly increasing on $[0, v_\theta)$, decreasing on $(v_\theta, c]$ and $\max_{\xi_c \in [0,\, c]} h_5(\xi_c) = h_5(v_\theta)$.

f. Given $\xi_c \in [0, c]$, it holds that $A^m_{\xi_c}(\xi_a) < A_{\xi_c}(\xi_a) < A^M_{\xi_c}(\xi_a)$ for $\xi_a \in [\xi_c, \xi_c + a - c]$. Furthermore, it holds that $h_4(\xi_c) < h_5(\xi_c) \leq h_3(\xi_c)$ for $\xi_c \in [0, c]$. In addition, $h_5(\xi_c) = h_3(\xi_c)$ if and only if $c = L_1 - L_3$.

Proposition 4.3. *Assume $a > b$.*

a. Functions $P_{I^M_{\xi_c,\xi_a,\xi_b}}$, $P_{I^m_{\xi_c,\xi_a,\xi_b}}$, $\xi_a - P_{I^M_{\xi_c,\xi_a,\xi_b}}$ and $\xi_a - P_{I^m_{\xi_c,\xi_a,\xi_b}}$, are continuous and strictly increasing in $\xi_a \in [\xi_b, \xi_b + a - b]$.

b. Given $\xi_c \in [0, c]$, define $B^M_{\xi_c}(\xi_b) \triangleq P_{I^M_{\xi_c,\xi_b+a-b,\xi_b}}$ and $B^m_{\xi_c}(\xi_b) \triangleq P_{I^m_{\xi_c,\xi_b+a-b,\xi_b}}$ for $\xi_b \in [\xi_c, \xi_c + b - c]$. Then, all the functions $B^M_{\xi_c}(\xi_b)$, $B^m_{\xi_c}(\xi_b)$, $\xi_b - B^M_{\xi_c}(\xi_b)$ and $\xi_b - B^m_{\xi_c}(\xi_b)$, are continuous and strictly increasing in $\xi_b \in [\xi_c, \xi_c + b - c]$.

c. Define $h_6(\xi_c) \triangleq B^m_{\xi_c}(\xi_c + (b - L_1 + L_3)^+) - \xi_c$ for $\xi_c \in [0, c]$. Then, $h_6(\xi_c)$ is continuous, convex, decreasing on $[0, c - v_\theta)$, strictly increasing on $(c - v_\theta, c]$ and $\min_{\xi_c \in [0,c]} h_6(\xi_c) = h_6(c - v_\theta)$.

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d. *Define* $h_7(\xi_c) \triangleq B^M_{\xi_c}(\xi_c + (b - L_1 + L_3)^+) - \xi_c$ *for* $\xi_c \in [0, c]$*. Then,* $h_7(\xi_c)$ *is continuous, concave, strictly increasing on* $[0, v_\theta)$*, decreasing on* $(v_\theta, c]$ *and* $\max_{\xi_c \in [0,c]} h_7(\xi_c) = h_7(v_\theta)$*.*

e. *Given* $\xi_c \in [0, c]$*, it holds that* $B^m_{\xi_c}(\xi_b) < B^M_{\xi_c}(\xi_b)$ *for* $\xi_b \in [\xi_c, \xi_c + b - c]$*. Furthermore, it holds that* $h_6(\xi_c) < h_7(\xi_c)$ *for* $\xi_c \in [0, c]$*.*

Furthermore, we need to define the following sets. Let $\Xi_{c,a,b}$ be the set of all $(\xi_c, \xi_a, \xi_b) \in [0, c] \times [0, a] \times [0, b]$ such that

$$\xi_c + (a \wedge b + L_3 - L_1)^+ \leq \xi_a \wedge \xi_b \leq \xi_a \vee \xi_b, \tag{26}$$

$$\xi_c + L_3 \leq P_{I^M_{\xi_c,\xi_a,\xi_b}}, \tag{27}$$

$$L_1 - a + \xi_a \geq P_{I^m_{\xi_c,\xi_a,\xi_b}}. \tag{28}$$

Let Ξ_c be the set of all $\xi_c \in [0, c]$ such that

$$L_3 + \xi_c \leq (1+\theta)\left(\int_0^{\xi_c} + \int_c^{\infty}\right) S_X(x)\mathrm{d}x, \tag{29}$$

$$L_1 - c + \xi_c \geq (1+\theta)\int_{c-\xi_c}^{a} S_X(x)\mathrm{d}x. \tag{30}$$

For each $\xi_c \in \Xi_c$, if $a < b$, then let Ξ_{a,ξ_c} be the set of all $\xi_a \in [\xi_c + a + L_3 - L_1,\ \xi_c + a - c]$ such that

$$\xi_c + L_3 \leq (1+\theta)\left(\int_0^{\xi_c} + \int_c^{c+\xi_a-\xi_c} + \int_a^{\infty}\right) S_X(x)\mathrm{d}x, \tag{31}$$

$$a - L_1 \leq \xi_a - (1+\theta)\left(\int_{c-\xi_c}^{c} + \int_{a-\xi_a+\xi_c}^{a}\right) S_X(x)\mathrm{d}x, \tag{32}$$

and if $b < a$, let Ξ_{b,ξ_c} be the set of all $\xi_b \in [\xi_c + (b + L_3 - L_1)^+,\ \xi_c + b - c]$ such that

$$\xi_c + L_3 \leq (1+\theta)\left(\int_0^{\xi_c} + \int_c^{c+\xi_b-\xi_c} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x, \tag{33}$$

$$b - L_1 \leq \xi_b - (1+\theta)\left(\int_{c-\xi_c}^{c} + \int_{b-\xi_b+\xi_c}^{a}\right) S_X(x)\mathrm{d}x. \tag{34}$$

If $a < b$, for each $(\xi_c, \xi_a) \in \Xi_c \times \Xi_{a,\xi_c}$, let Ξ_{b,ξ_c,ξ_a} be the set of all $\xi_b \in [\xi_a, \xi_a + b - a]$ such that $(\xi_c, \xi_a, \xi_b) \in \Xi_{c,a,b}$. If $a > b$, for each $(\xi_c, \xi_b) \in \Xi_c \times \Xi_{b,\xi_c}$, let Ξ_{a,ξ_c,ξ_b} be the set of all $\xi_a \in [\xi_b, \xi_b + a - b]$ such that $(\xi_c, \xi_a, \xi_b) \in \Xi_{c,a,b}$.

Proposition 4.4. *All the sets* Ξ_c*,* Ξ_{a,ξ_c}*,* Ξ_{b,ξ_c}*,* Ξ_{b,ξ_c,ξ_a} *and* Ξ_{a,ξ_c,ξ_b}*, are closed intervals and can be expressed as follows:*

a. *The set* $\Xi_c = [\xi_c^m,\ \xi_c^M]$ *for some* $0 \leq \xi_c^m \leq \xi_c^M \leq c$*.*

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b. *When $a < b$, given $\xi_c \in \Xi_c$, the set $\Xi_{a,\xi_c} = [\xi_a^m(\xi_c),\ \xi_a^M(\xi_c)]$ for some $\xi_c + a + L_3 - L_1 \le \xi_a^m(\xi_c) \le \xi_a^M(\xi_c) \le \xi_c + a - c$, and given $(\xi_c, \xi_a) \in \Xi_c \times \Xi_{a,\xi_c}$, the set $\Xi_{b,\xi_c,\xi_a} = [\xi_b^m(\xi_c, \xi_a),\ \xi_b^M(\xi_c, \xi_a)]$ for some $\xi_a \le \xi_b^m(\xi_c, \xi_a) \le \xi_b^M(\xi_c, \xi_a) \le \xi_a + b - a$.*

c. *When $a > b$, given $\xi_c \in \Xi_c$, the set $\Xi_{b,\xi_c} = [\xi_b^m(\xi_c),\ \xi_b^M(\xi_c)]$ for some $\xi_c + (b + L_3 - L_1)^+ \le \xi_b^m(\xi_c) \le \xi_b^M(\xi_c) \le \xi_c + b - c$, and given $(\xi_c, \xi_b) \in \Xi_c \times \Xi_{b,\xi_c}$, the set $\Xi_{a,\xi_c,\xi_b} = [\xi_a^m(\xi_c, \xi_b),\ \xi_a^M(\xi_c, \xi_b)]$ for some $\xi_b \le \xi_a^m(\xi_c, \xi_b) \le \xi_a^M(\xi_c, \xi_b) \le \xi_b + a - b$.*

Lemma 4.1. *The following three statements are equivalent: (i) Inequalities (8) and (9) hold. (ii) $\mathcal{I}_2 \neq \emptyset$. (iii) $\Xi_{c,a,b} \neq \emptyset$.*

Lemma 4.2. *Problem (25) has the same minimal value as the minimization problem*

$$\min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c, \xi_a, \xi_b) \tag{35}$$

in the sense that $\min_{I\in\mathcal{I}_2} V(I) = \min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c, \xi_a, \xi_b)$, where $w(\xi_c, \xi_a, \xi_b) = \lambda a + (2\lambda - 1)\, P_{\xi_c,\xi_a,\xi_b} - \lambda\xi_a + (1 - \lambda)\xi_b$ and

$$P_{\xi_c,\xi_a,\xi_b} \triangleq \begin{cases} (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}, & \text{if } 0 \le \lambda < \frac{1}{2}, \\ (\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}}, & \text{if } \frac{1}{2} < \lambda \le 1. \end{cases} \tag{36}$$

To solve the three-dimensional problem (35), we consider the following three-step minimization problem:

$$\begin{cases} \min_{\xi_c\in\Xi_c} \left(\min_{\xi_a\in\Xi_{a,\xi_c}} \left[\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c, \xi_a, \xi_b) \right] \right), & \text{if } a < b, \\ \min_{\xi_c\in\Xi_c} \left(\min_{\xi_b\in\Xi_{b,\xi_c}} \left[\min_{\xi_a\in\Xi_{a,\xi_c,\xi_b}} w(\xi_c, \xi_a, \xi_b) \right] \right), & \text{if } a > b. \end{cases} \tag{37}$$

In doing so, we define the minimizers of Problem (37) and the corresponding functions as follows:

For $a < b$, define $\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c, \xi_a, \xi_b) = w(\xi_c, \xi_a, \xi_b^*(\xi_c, \xi_a)) \triangleq w_2(\xi_c, \xi_a)$ and
$\min_{\xi_a\in\Xi_{a,\xi_c}} w_2(\xi_c, \xi_a) = w_2(\xi_c, \xi_a^*(\xi_c)) = w_1(\xi_c)$, where

$$\xi_b^*(\xi_c, \xi_a) \triangleq \arg\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c, \xi_a, \xi_b) \ \text{ and } \ \xi_a^*(\xi_c) \triangleq \arg\min_{\xi_a\in\Xi_{a,\xi_c}} w_2(\xi_c, \xi_a).$$

For $a > b$, denote $\min_{\xi_a\in\Xi_{a,\xi_c,\xi_b}} w(\xi_c, \xi_a, \xi_b) = w(\xi_c, \xi_a^*(\xi_c, \xi_b), \xi_b) \triangleq w_2(\xi_c, \xi_b)$ and $\min_{\xi_b\in\Xi_{b,\xi_c}} w_2(\xi_c, \xi_b) = w_2(\xi_c, \xi_b^*(\xi_c)) = w_1(\xi_c)$, where

$$\xi_a^*(\xi_c, \xi_b) \triangleq \arg\min_{\xi_a\in\Xi_{a,\xi_c,\xi_b}} w(\xi_c, \xi_a, \xi_b) \ \text{ and } \ \xi_b^*(\xi_c) \triangleq \arg\min_{\xi_b\in\Xi_{b,\xi_c}} w_2(\xi_c, \xi_b).$$

Moreover, denote $\min_{\xi_c\in\Xi_c} w_1(\xi_c) \triangleq w_1(\xi_c^*)$, where $\xi_c^* \triangleq \arg\min_{\xi_c\in\Xi_c} w_1(\xi_c)$. In addition, for $a < b$, denote $\xi_a^* \triangleq \xi_a^*(\xi_c^*)$ and $\xi_b^* \triangleq \xi_b^*(\xi_c^*, \xi_a^*)$. For $a > b$, denote $\xi_b^* \triangleq \xi_b^*(\xi_c^*)$ and $\xi_a^* \triangleq \xi_a^*(\xi_c^*, \xi_b^*)$.

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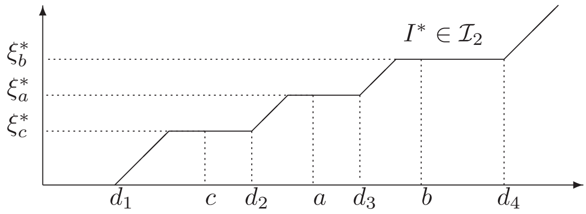

FIGURE 2: Optimal form of the contract (38) from Theorem 4.1 when $a < b$.

Lemma 4.3. *The three-step minimization problem (37) is well-defined in the sense that the minimizer for each step exists. In particular, the minimizers of Problem (37) can be expressed as follows:*

a. If $a < b$ *and* $0 \le \lambda < \frac{1}{2}$, *then* $\xi_c^* = \xi_c^m \vee \left(v_\theta \wedge \xi_c^M\right)$, $\xi_a^* = \sup\left\{\xi_a \in \Xi_{a,\xi_c^*} : A_{\xi_c^*}(\xi_a) < \xi_c^* + L_3\right\}$ *and* $\xi_b^* = \xi_b^m(\xi_c^*, \xi_a^*)$.

b. If $a < b$ *and* $\frac{1}{2} < \lambda \le 1$, *then* $\xi_c^* = \xi_{L_3,h_2} \vee \xi_{L_3,h_3}$, $\xi_a^* = \xi_c^* + a - c$, *and* $\xi_b^* = \xi_b^m(\xi_c^*, \xi_a^*)$, *where* $\xi_{L_3,h_2} = \sup\{\xi_c \in [0, c - v_\theta] : h_2(\xi_c) \ge L_3\}$ *and* $\xi_{L_3,h_3} = \sup\{\xi_c \in [0, v_\theta] : h_3(\xi_c) \le L_3\}$.

c. If $a > b$ *and* $0 \le \lambda < \frac{1}{2}$, *then* $\xi_c^* = \xi_c^m \vee \left(v_\theta \wedge \xi_c^M\right)$, $\xi_b^* = \xi_b^m(\xi_c^*)$ *and* $\xi_a^* = \xi_b^* + a - b$.

d. If $a > b$ *and* $\frac{1}{2} < \lambda \le 1$, *then* $\xi_c^* = \xi_c^m \vee \left[(c - v_\theta) \wedge \xi_c^M\right]$, $\xi_b^* = \xi_c^* + b - c$ *and* $\xi_a^* = \xi_c^* + a - c$.

Theorem 4.1. *A contract* I^* *of the form*

$$I^*(x) = (x - d_1)^+ - (x - d_1 - \xi_c^*)^+ + (x - d_2)^+ - (x - (d_2 + \xi_a^* \wedge \xi_b^* - \xi_c^*))^+ + (x - d_3)^+ - (x - (d_3 + \left|\xi_b^* - \xi_a^*\right|))^+ + (x - d_4)^+ \tag{38}$$

for some $(d_1, d_2, d_3, d_4) \in [0,\ c - \xi_c^*] \times [c,\ a \wedge b - \xi_a^* \wedge \xi_b^* + \xi_c^*] \times [a \wedge b,\ a \vee b - |\xi_a^* - \xi_b^*|] \times [a \vee b,\ \infty]$, *satisfying* $P_{I^*} = P_{\xi_c^*, \xi_a^*, \xi_b^*}$, *is an optimal solution to Problem (25).*

Remark 4.1. *Figure 2 illustrates the optimal form* (38) *in the case of* $a < b$.

By the proof of Theorem 4.1, we know that the optimal contract I^* *in Theorem 4.1 satisfies* $I^*(a) - I^*(b) = (a - b)^+$, *and hence the optimal solution* I^* *in Theorem 4.1 is also the solution to Problem (25) when* $\lambda = 1/2$. □

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Next, we will derive the explicit expressions of the parameters in the optimal solution I^* given in Theorem 4.1 in the following four corollaries.

Corollary 4.1. *Suppose $a < b$ and $0 \le \lambda < 1/2$ and let I^* be the optimal solution to Problem (25).*

a. In the case $h_2(v_\theta) \le L_1 - c$:
   i. If $L_3 \le h_4(v_\theta)$, then $I^*(x) = (x-c+v_\theta)^+ - (x-c)^+ + (x-a+\xi_{a,0}-v_\theta)^+ - (x-a)^+$, where $\xi_{a,0}$ is the solution to the equation of $P_{I^*} = \xi_{a,0} + L_1 - a$.
   ii. If $h_4(v_\theta) < L_3 \le h_5(v_\theta)$, then $I^*(x) = (x-d_1^*)^+ - (x-d_1^*-v_\theta)^+ + (x-d_2^*)^+ - (x-d_2^*-(a-L_1+L_3))^+ + (x-d_3^*)^+$, where $(d_1^*, d_2^*, d_3^*) \in [0,\ c-v_\theta] \times [c,\ L_1-L_3] \times [b,\ \infty]$ is the solution to the equation of $P_{I^*} = v_\theta + L_3$.
   iii. If $h_5(v_\theta) < L_3$, then $I^*(x) = x - (x-v_\theta)^+ + (x-c)^+ - (x-c-\xi_{a,1}+v_\theta)^+ + (x-b)^+$, where $\xi_{a,1}$ is the solution to the equation of $P_{I^*} = v_\theta + L_3$.

b. In the case $h_2(v_\theta) > L_1 - c$, then we have $I^*(x) = (x-c+\xi_{L_1-c,h_2})^+ - (x-a)^+$, where $\xi_{L_1-c,h_2} = \inf\{\xi_c \in [v_\theta \wedge (c-v_\theta),\ v_\theta \vee (c-v_\theta)] : h_2(\xi_c) = L_1 - c\}$.

Corollary 4.2. *Suppose $a < b$ and $1/2 < \lambda \le 1$ and let I^* be the optimal solution to Problem (25).*

a. If $L_3 \le h_2(0)$, then $I^*(x) = (x-c+\xi_{L_3,h_2})^+ - (x-a)^+$, where $\xi_{L_3,h_2} = \sup\{\xi_c \in [0, c-v_\theta] : h_2(\xi_c) \ge L_3\}$.

b. If $h_2(0) < L_3 < h_3(0)$, then $I^*(x) = (x-c)^+ - (x-a)^+ + (x-d^*)^+$, where $d^* \in [b, \infty]$ satisifes $P_{I^*} = L_3$.

c. If $h_3(0) \le L_3$, then $I^*(x) = x - (x-\xi_{L_3,h_3})^+ + (x-c)^+ - (x-(c+\xi_b^*-\xi_{L_3,h_3}))^+ + (x-b)^+$, where $\xi_{L_3,h_3} = \sup\{\xi_c \in [0, v_\theta] : h_3(\xi_c) \le L_3\}$ and $\xi_b^* \in [\xi_{L_3,h_3}+a-c,\ \xi_{L_3,h_3}+b-c]$ satisfies $P_{I^*} = \xi_{L_3,h_3} + L_3$.

Corollary 4.3. *Suppose $a > b$ and $0 \le \lambda < 1/2$ and let I^* be the optimal solution to Problem (25).*

a. In the case $h_2(v_\theta) \le L_1 - c$:
   i. If $(b+L_3-L_1)^+ + L_1 - b < h_6(v_\theta)$, then $I^*(x) = (x-c+v_\theta)^+ - (x-c)^+ + (x-(b-\xi_{b,0}+v_\theta))^+ - (x-a)^+$, where $\xi_{b,0} \in [v_\theta+(b+L_3-L_1)^+,\ v_\theta+b-c]$ is the solution to the equation of $P_{I^*} = \xi_{b,0} - b + L_1$.
   ii. If $h_6(v_\theta) \le (b+L_3-L_1)^+ + L_1 - b < h_7(v_\theta)$, then $I^*(x) = (x-d_1^*)^+ - (x-d_1^*-v_\theta)^+ + (x-c)^+ - (x-c-(b+L_3-L_1)^+)^+ + (x-b)^+ - (x-a)^+ + (x-d_2^*)^+$, where $(d_1^*, d_2^*) \in [0,\ c-v_\theta] \times [a, \infty]$ is the solution to the equation of $P_{I^*} = v_\theta + L_3 \vee (L_1 - b)$.
   iii. If $L_3 < h_7(v_\theta) \le (b+L_3-L_1)^+ + L_1 - b$, then $I^*(x) = x - (x-v_\theta)^+ + (x-c)^+ - (x-c-(b+L_3-L_1)^+)^+ + (x-b)^+$.
   iv. If $h_7(v_\theta) \le L_3$, then $I^*(x) = x - (x-v_\theta)^+ + (x-c)^+ - (x-(c+\xi_{b,1}-v_\theta))^+ + (x-b)^+$, where $\xi_{b,1} \in [v_\theta+(b+L_3-L_1)^+,\ v_\theta+b-c]$ is the solution to the equation of $P_{I^*} = v_\theta + L_3$.

b. In the case $h_2(v_\theta) > L_1 - c$, then we have $I^*(x) = (x-c+\xi_{L_1-c,h_2})^+ - (x-a)^+$, where $\xi_{L_1-c,h_2} = \inf\{\xi_c \in [v_\theta \wedge (c-v_\theta),\ v_\theta \vee (c-v_\theta)] : h_2(\xi_c) = L_1 - c\}$.

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Corollary 4.4. *Suppose* $a > b$ *and* $1/2 < \lambda \leq 1$ *and let* I^* *be the optimal solution to Problem (25).*

a. If $h_1(c - v_\theta) < L_3$*, then* $I^*(x) = x - (x - \xi_{L_3, h_1})^+ + (x - c)^+$*, where* $\xi_{L_3, h_1} = \sup\{\xi_c \in [v_\theta \wedge (c - v_\theta), v_\theta \vee (c - v_\theta)] : h_1(\xi_c) = L_3\}$.
b. If $L_3 \leq h_1(c - v_\theta)$*, then* $I^*(x) = (x - d_1^*)^+ - (x - d_1^* - c + v_\theta)^+ + (x - c)^+ - (x - a)^+ + (x - d_2^*)^+$*, where* $(d_1^*, d_2^*) \in [0, v_\theta] \times [a, \infty]$ *is the solution to the equation of* $P_{I^*} = c - v_\theta + L_3 \vee h_2(c - v_\theta)$.

5. Conclusions

In this paper, we describe feasible reinsurance contracts that are acceptable to both an insurer and a reinsurer and explore optimal reinsurance contracts which take into account both an insurer's aims and a reinsurer's goals. The models and problems proposed in this paper are interesting in theory and applications. As showed in this paper, solving the proposed problems and finding the optimal reinsurance contracts from the perspective of both an insurer and a reinsurer are challenging jobs. The optimal reinsurance contracts from the perspectives of both an insurer and a reinsurer are more complicated than the optimal reinsurance contracts from one party's point of view only. The models and problems proposed in this paper can be explored further in different ways such as replacing the VaR by other risk measures and accommodating other demands of an insurer and a reinsurer in the study of optimal reinsurance designs.

As mentioned in the introduction, the criterion used in this paper is arguably oversimplifying how the two parties' conflicting interests should jointly be analyzed in practice. For future work, we plan to consider more general approaches to perform this type of analysis. For instance, we could use a two-step procedure where the party that is designing the contract performs a first optimization step based on its own criterion, with a choice of constraints such that there exist multiple optimal solutions. One particular solution among those would then be selected using a secondary criterion representing the other party's objectives.

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APPENDIX A

Proof of Proposition 3.1. We only prove (a) and (d). Other results of Proposition 3.1 can be proved similarly and are omitted.

a. It is easy to see that $g_1(\xi)$ is continuous in $\xi \in [0, a]$. Since $\alpha < 1/(1+\theta)$, we have that $g_1'(\xi) = 1 - (1+\theta)S_X(a-\xi)$ is non-negative for $\xi \in [0, a - v_\theta)$ and is negative for $\xi \in (a - v_\theta, a]$. Hence, the desired results hold.

d. Suppose $a < b$, note that $g_2(\xi_a) = \xi_a - P_{I^M_{\xi_a,\xi_a}} < \xi_a - P_{I^m_{\xi_a,\xi_a}} = g_1(\xi_a)$ for any $\xi_a \in [0, a]$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, $\xi_a \le \xi_b$ by (15) and it is obvious that $P_{I^M_{\xi_a,\xi_b}}$ and $P_{I^m_{\xi_a,\xi_b}}$ are continuous and strictly increasing in $\xi_b \in [0, b]$. For any (ξ_a, ξ_1) and $(\xi_a, \xi_2) \in \Xi_{a,b}$ with $\xi_1 < \xi_2$, we have $0 \le P_{I^M_{\xi_a,\xi_2}} - P_{I^M_{\xi_a,\xi_1}} = (1+\theta)\int_{a+\xi_1-\xi_a}^{a+\xi_2-\xi_a} S_X(x)\mathrm{d}x \le (1+\theta)(\xi_2 - \xi_1)S_X(a) \le (1+\theta)\alpha\,(\xi_2 - \xi_1) < \xi_2 - \xi_1$, where the third inequality follows from $S_X(x) \le \alpha$ for any $x \ge \mathrm{VaR}_\alpha(X) = a$. Therefore, $\xi_b - P_{I^M_{\xi_a,\xi_b}}$ is continuous and strictly increasing in $\xi_b \in [\xi_a, \xi_a + b - a]$. ■

Proof of Lemma 3.1. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

(i) ⇒ (ii). Suppose (8) holds, namely $g_1(a - v_\theta) \ge a - L_1$. Since $g_1(0) = 0 \le a - L_1$ and g_1 is continuous and increasing on $[0, a - v_\theta]$, there exists $\xi_a \in [0, a - v_\theta]$ such that $g_1(\xi_a) = a - L_1$, and moreover, $g_1(\xi_a) = a - L_1 \le L_2$. Consider the contract $I(x) = (x - a + \xi_a)^+ - (x-a)^+ \in \mathcal{I}$, it is easy to check that $I(a) = I(b) = \xi_a$ and $P_I = (1+\theta)\mathbb{E}[I(X)] = \xi_a - g_1(\xi_a) = \xi_a - a + L_1$. This contract I is acceptable, namely $I \in \mathcal{I}_1$, because the contract I satisfies $a - I(a) + P_I = a - \xi_a + \xi_a - a + L_1 = L_1$, and $I(b) - P_I = \xi_a - (\xi_a - a + L_1) = a - L_1 \le L_2$. Thus, $\mathcal{I}_1 \ne \emptyset$.

Meanwhile, by Proposition 3.1(b) and (d), we know that $g_1(\xi_a) = a - L_1 \le L_2$ implies that $g_2(v_\theta) \le g_2(\xi_a) < g_1(\xi_a) \le L_2$, namely (21) holds. Thus, (8) implies (21).

(ii) ⇒ (iii). Suppose $\mathcal{I}_1 \ne \emptyset$. For any $I \in \mathcal{I}_1$, denote $\xi_a = I(a)$ and $\xi_b = I(b)$. We are going to check that (ξ_a, ξ_b) satisfies (15), (16) and (17). Since $I \in \mathcal{I}_1$, we have

$$\xi_b - L_2 \le P_I \le \xi_a + L_1 - a. \tag{39}$$

Furthermore, the 1-Lipschitz property of I implies $\xi_a \le \xi_b \le \xi_a + b - a$. Hence, (15) holds. Moreover, it is easy to see that $I^m_{\xi_a,\xi_b}(x) \le I(x) \le I^M_{\xi_a,\xi_b}(x)$ for all $x \ge 0$, and thus

$$P_{I^m_{\xi_a,\xi_b}} \le P_I \le P_{I^M_{\xi_a,\xi_b}}. \tag{40}$$

From (39) and (40), we have $\xi_b - L_2 \le P_{I^M_{\xi_a,\xi_b}}$ and $P_{I^m_{\xi_a,\xi_b}} \le \xi_a + L_1 - a$, namely (16) and (17) hold. Therefore, $(\xi_a, \xi_b) \in \Xi_{a,b}$ and thus $\Xi_{a,b} \ne \emptyset$.

(iii) ⇒ (i). Suppose $\Xi_{a,b} \ne \emptyset$. For any $(\xi_a, \xi_b) \in \Xi_{a,b}$, we have

$$\begin{aligned} a - L_1 &\le \xi_a - (1+\theta)\left(\int_{a-\xi_a}^{a} + \int_{b-\xi_b+\xi_a}^{b}\right) S_X(x)\mathrm{d}x \le \xi_a - (1+\theta)\int_{a-\xi_a}^{a} S_X(x)\mathrm{d}x = g_1(\xi_a) \\ &\le a - v_\theta - (1+\theta)\int_{v_\theta}^{a} S_X(x)\mathrm{d}x = g_1(a - v_\theta), \end{aligned}$$

where the first inequality is from (17) and the last one is due to the fact that g_1 is increasing on $[0, a - v_\theta]$. Thus, (8) holds. ■

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Proof of Lemma 3.2. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, by (15), we have $\xi_b - L_2 \le \xi_a + b \wedge (L_1 + L_2) - L_2 - a \le \xi_a + L_1 - a$, which, together with (16), implies $\xi_b - L_2 \le (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}$. Hence, by (17) and $P_{I^m_{\xi_a,\xi_b}} \le P_{I^M_{\xi_a,\xi_b}}$, we have $(\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}} \le (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}$. Therefore, by the definition of P_{ξ_a,ξ_b} given in (23), we have

$$(\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}} \le P_{\xi_a,\xi_b} \le (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}. \tag{41}$$

It is easy to check that any contract with the form of

$$I(x) = (x - d_1)^+ - (x - d_1 - \xi_a)^+ + (x - d_2)^+ - (x - d_2 - \xi_b + \xi_a)^+ + (x - d_3)^+, \tag{42}$$

for some $(d_1, d_2, d_3) \in [0,\ a - \xi_a] \times [a,\ b - \xi_b + \xi_a] \times [b,\ \infty]$, satisfies $I \in \mathcal{I}$, $I(a) = \xi_a$, $I(b) = \xi_b$, and $I^m_{\xi_a,\xi_b}(x) \le I(x) \le I^M_{\xi_a,\xi_b}(x)$ for all $x \ge 0$. Thus, $P_{I^m_{\xi_a,\xi_b}} \le P_I \le P_{I^M_{\xi_a,\xi_b}}$. In particular, when $d_1 = a - \xi_a$, $d_2 = b - \xi_b + \xi_a$, and $d_3 = \infty$, the form (42) is reduced to $I^m_{\xi_a,\xi_b}$. When $d_1 = 0$, $d_2 = a$, and $d_3 = b$, the form (42) is reduced to $I^M_{\xi_a,\xi_b}$. For the contract I of the form (42), its premium

$$P_I = (1+\theta)\,\mathbb{E}[I(X)] = (1+\theta)\left(\int_{d_1}^{d_1+\xi_a} + \int_{d_2}^{d_2+\xi_b-\xi_a} + \int_{d_3}^{\infty}\right) S_X(x)\mathrm{d}x$$

can be viewed as a function of (d_1, d_2, d_3). Obviously, the premium $P_I = P_I(d_1, d_2, d_3)$ is a real-valued continuous function on $[0, a - \xi_a] \times [a, b - \xi_b + \xi_a] \times [b, \infty]$. Since $[0, a - \xi_a] \times [a, b - \xi_b + \xi_a] \times [b, \infty]$ is a connected set, the image of $P_I(d_1, d_2, d_3)$ is also a connected set. Thus,

$$\{P_I = (1+\theta)\,\mathbb{E}[I(X)] : \ I \text{ has the expression (42)}\} = \left[P_{I^m_{\xi_a,\xi_b}},\ P_{I^M_{\xi_a,\xi_b}}\right].$$

For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, note that $P_{\xi_a,\xi_b} \in [P_{I^m_{\xi_a,\xi_b}},\ P_{I^M_{\xi_a,\xi_b}}]$, thus there exists $I \in \mathcal{I}$ with the expression (42) such that $P_I = P_{\xi_a,\xi_b}$, and moreover, such $I \in \mathcal{I}_1$ due to (41).

The existence of the minimizer (ξ_a^*, ξ_b^*) of Problem (22) will be demonstrated in the proof of Theorems 3.1 and 3.2. Since $(\xi_a^*, \xi_b^*) \in \Xi_{a,b}$, by the above arguments, there exists $I^* \in \mathcal{I}_1$ of the form (24) such that $I^*(a) = \xi_a^*$, $I^*(b) = \xi_b^*$, and $P_{I^*} = P_{\xi_a^*,\xi_b^*}$. It can be easily checked that $V(I^*) = v(\xi_a^*, \xi_b^*)$. Meanwhile, for any $I \in \mathcal{I}_1$, we have $(I(a), I(b)) \in \Xi_{a,b}$ by the proof of Lemma 3.1 for (ii) $\Rightarrow$ (iii). From (23), we have $P_I \le P_{I(a),\,I(b)}$ when $0 \le \lambda < 1/2$, and $P_I \ge P_{I(a),\,I(b)}$ when $1/2 < \lambda \le 1$. Therefore, $(2\lambda - 1)\,P_I \ge (2\lambda - 1)\,P_{I(a),\,I(b)}$ and

$$V(I) = \lambda a + (2\lambda - 1)\,P_I - \lambda I(a) + (1-\lambda)I(b) \ge v(I(a), I(b)) \ge \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b), \tag{43}$$

which implies that $\min_{I\in\mathcal{I}_1} V(I) \ge \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) = v(\xi_a^*, \xi_b^*) = V(I^*) \ge \min_{I\in\mathcal{I}_1} V(I)$. Hence, $\min_{I\in\mathcal{I}_1} V(I) = V(I^*)$ and I^* is the optimal solution to Problem (14). Therefore, a contract I^* of the form (24) for some $(d_1, d_2, d_3) \in [0,\ a - \xi_a^*] \times [a,\ b - \xi_b^* + \xi_a^*] \times [b,\ \infty]$, satisfying $I^*(a) = \xi_a^*$, $I^*(b) = \xi_b^*$, $P_{I^*} = P_{\xi_a^*,\xi_b^*}$, is the optimal solution to Problem (14). ■

Proof of Theorem 3.1. Assume $a < b$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, we have $\xi_a \le \xi_b$ by (15), $\xi_b - P_{I^M_{\xi_a,\xi_b}} \le L_2$ by (16), and $P_{I^m_{\xi_a,\xi_b}} \le L_1 - a + \xi_a$ by (17). Since $\xi_b - P_{I^M_{\xi_a,\xi_b}}$ and $P_{I^m_{\xi_a,\xi_b}}$ are strictly increasing in $\xi_b \in [\xi_a, \xi_a + b - a]$ by Proposition 3.1(d), we have $\xi_a - P_{I^M_{\xi_a,\xi_a}} \le \xi_b - P_{I^M_{\xi_a,\xi_b}} \le L_2$

and $P_{I^m_{\xi_a,\xi_a}} \leq P_{I^m_{\xi_a,\xi_b}} \leq L_1 - a + \xi_a$. Thus, $(\xi_a, \xi_a) \in \Xi_{a,b}$. From (16) and (17), we know that $(\xi_a, \xi_a) \in \Xi_{a,b}$ is equivalent to

$$g_2(\xi_a) \leq L_2 \text{ and } a - L_1 \leq g_1(\xi_a). \tag{44}$$

a. Consider the case $0 \leq \lambda < \frac{1}{2}$. By Lemma 3.2, $\min_{I \in \mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b) \in \Xi_{a,b}} v(\xi_a, \xi_b)$, where $v(\xi_a, \xi_b) = \lambda a + (2\lambda - 1) P_{\xi_a,\xi_b} - \lambda \xi_a + (1-\lambda)\xi_b$ and $P_{\xi_a,\xi_b} = (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, since $\xi_a \leq \xi_b$ and $\xi_a - P_{I^M_{\xi_a,\xi_a}} \leq \xi_b - P_{I^M_{\xi_a,\xi_b}}$, together with the definition of P_{ξ_a,ξ_b} given by (23) and the facts that $-(x \wedge y) = (-x) \vee (-y)$ and $kz + k(x \vee y) = k[(z + x) \vee (z + y)]$ for $k > 0$, we have

$$\begin{aligned} v(\xi_a, \xi_b) &= \lambda a - \lambda \xi_a + (1-\lambda)\xi_b - (1-2\lambda)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}\right] \\ &= \lambda a - \lambda \xi_a + \lambda \xi_b + (1-2\lambda)\left[(\xi_b - L_1 + a - \xi_a) \vee \left(\xi_b - P_{I^M_{\xi_a,\xi_b}}\right)\right] \\ &\geq \lambda a + (1-2\lambda)\left[(a - L_1) \vee \left(\xi_a - P_{I^M_{\xi_a,\xi_a}}\right)\right] \\ &= (1-\lambda)a - (1-2\lambda)L_1 + (1-2\lambda)\left[g_2(\xi_a) - (a - L_1)\right]^+ = v(\xi_a, \xi_a). \end{aligned}$$

Hence, $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) \geq \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a)$, and since $(\xi_a, \xi_a) \in \Xi_{a,b}$, we have

$$\begin{aligned} \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) &= \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a) \\ &= (1-\lambda)a - (1-2\lambda)L_1 + (1-2\lambda) \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} \left[g_2(\xi_a) - (a - L_1)\right]^+ \\ &= (1-\lambda)a - (1-2\lambda)L_1 + (1-2\lambda)\left[\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a) - (a - L_1)\right]^+. \end{aligned}$$

Note that $P_{I^M_{\xi_a,\xi_a}} = \xi_a - g_2(\xi_a)$ and then

$$P_{\xi_a,\xi_a} = (\xi_a - a + L_1) \wedge P_{I^M_{\xi_a,\xi_a}} = \xi_a - (a - L_1) \vee g_2(\xi_a). \tag{45}$$

i. If $g_1(v_\theta) \geq a - L_1$, note that $g_2(v_\theta) \leq L_2$ by (21), thus $\xi_a = v_\theta$ satisfies condition (44), namely $(v_\theta, v_\theta) \in \Xi_{a,b}$. In this case,

$$\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a) \geq \min_{\xi_a \in [0,a]} g_2(\xi_a) = g_2(v_\theta) \geq \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a),$$

where the equality holds due to Proposition 3.1(b). Therefore, $\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a) = g_2(v_\theta)$ and $(\xi_a^*, \xi_b^*) = (v_\theta, v_\theta)$. It implies that $P_{\xi_a^*,\xi_b^*} = P_{v_\theta,v_\theta} = v_\theta - (a - L_1) \vee g_2(v_\theta)$ from (45), and

$$\begin{aligned} \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a) &= v(v_\theta, v_\theta) \\ &= (1-\lambda)a - (1-2\lambda)L_1 + (1-2\lambda)\left[g_2(v_\theta) - (a - L_1)\right]^+. \end{aligned}$$

By Lemma 3.2, a contract I^* of the form (24) satisfying $I^*(a) = v_\theta$, $I^*(b) = v_\theta$, and $P_{I^*} = P_{v_\theta,v_\theta}$, is the optimal solution to Problem (14). Note that $\xi_a^* = \xi_b^*$. Thus, $I^*(x) = (x - d_1)^+ - (x - d_1 - v_\theta)^+ + (x - d_3)^+$ for some $d_1 \in [0, a - v_\theta]$ and $d_3 \in [b, \infty]$ such that $P_{I^*} = v_\theta - (a - L_1) \vee g_2(v_\theta)$ is the optimal solution to Problem (14).

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ii. If $g_1(v_\theta) < a-L_1$, note that $g_1(a-v_\theta) \geq a-L_1$ from (8), thus there exists $\xi_1 \in [v_\theta \wedge (a-v_\theta),\ v_\theta \vee (a-v_\theta)]$ such that $g_1(\xi_1) = a-L_1$ due to the continuity and monotonicity of g_1 on this interval. From Proposition 3.1(d), we know $g_2(\xi) < g_1(\xi)$ for any $\xi \in [0, a]$. In particular, $g_2(\xi_1) < g_1(\xi_1) = a - L_1 \leq L_2$ and thus ξ_1 satisfies condition (44), namely $(\xi_1, \xi_1) \in \Xi_{a,b}$. For any $(\xi_a, \xi_a) \in \Xi_{a,b}$, we have $[g_2(\xi_1) - (a - L_1)]^+ = 0 \leq [g_2(\xi_a) - (a - L_1)]^+$. Then,

$$\left[\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a) - (a - L_1)\right]^+ = 0 = [g_2(\xi_1) - (a - L_1)]^+ ,$$

and $\xi_a^* = \xi_1$. In this case, we have $P_{\xi_a^*,\xi_b^*} = P_{\xi_1,\xi_1} = \xi_1 - (a - L_1) \vee g_2(\xi_1) = \xi_1 - a + L_1$ and $\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a) = v(\xi_1, \xi_1) = (1-\lambda)a - (1-2\lambda)L_1$. Therefore, the optimal contract of the form (24) is reduced to $I^*(x) = (x-a+\xi_1)^+ - (x-a)^+$ with $d_1 = a - \xi_a$ and $d_3 = \infty$ because the contract I^* satisfies $I^*(a) = I^*(b) = \xi_1$ and $P_{I^*} = \xi_1 - g_1(\xi_1) = \xi_1 - a + L_1 = P_{\xi_1,\xi_1}$.

b. For the case $\frac{1}{2} < \lambda \leq 1$. By Lemma 3.2, we have $\min_{I\in\mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b)$, where $v(\xi_a, \xi_b) = \lambda a + (2\lambda-1)P_{\xi_a,\xi_b} - \lambda\xi_a + (1-\lambda)\xi_b$ and $P_{\xi_a,\xi_b} = (\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}}$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, since $\xi_a \leq \xi_b$ and $P_{I^m_{\xi_a,\xi_a}} \leq P_{I^m_{\xi_a,\xi_b}}$, we have

$$\begin{aligned}
v(\xi_a, \xi_b) &= \lambda a - \lambda\xi_a + (1-\lambda)\xi_b + (2\lambda-1)\left[(\xi_b - L_2) \vee P_{I^m_{\xi_a,\xi_b}}\right] \\
&\geq \lambda a - \lambda\xi_a + (1-\lambda)\xi_a + (2\lambda-1)\left[(\xi_a - L_2) \vee P_{I^m_{\xi_a,\xi_a}}\right] \\
&= v(\xi_a, \xi_a) = \lambda a + (1-2\lambda)L_2 + (2\lambda-1)\left[(\xi_a - L_2) \vee P_{I^m_{\xi_a,\xi_a}} - (\xi_a - L_2)\right] \\
&= \lambda a + (1-2\lambda)L_2 + (2\lambda-1)\left[P_{I^m_{\xi_a,\xi_a}} - (\xi_a - L_2)\right]^+ \\
&= \lambda a + (1-2\lambda)L_2 + (2\lambda-1)\left[L_2 - g_1(\xi_a)\right]^+ .
\end{aligned}$$

Hence, $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) \geq \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a)$, and since $(\xi_a, \xi_a) \in \Xi_{a,b}$, we have

$$\begin{aligned}
\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b) &= \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a) \\
&= \lambda a + (1-2\lambda)L_2 + (2\lambda-1) \min_{(\xi_a,\xi_a)\in\Xi_{a,b}} [L_2 - g_1(\xi_a)]^+ \\
&= \lambda a + (1-2\lambda)L_2 + (2\lambda-1)\left[L_2 - \max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_1(\xi_a)\right]^+ .
\end{aligned}$$

Note that $P_{I^m_{\xi_a,\xi_a}} = \xi_a - g_1(\xi_a)$ and then

$$P_{\xi_a,\xi_a} = (\xi_a - L_2) \vee P_{I^m_{\xi_a,\xi_a}} = \xi_a - L_2 \wedge g_1(\xi_a), \tag{46}$$

i. If $g_2(a - v_\theta) \leq L_2$, note that $a - L_1 \leq g_1(a - v_\theta)$ by (8), thus $\xi_a = a - v_\theta$ satisfies condition (44), namely $(a - v_\theta,\ a - v_\theta) \in \Xi_{a,b} \subset [0, a] \times [0, b]$. In this case,

$$\max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_1(\xi_a) \leq \max_{\xi_a\in[0,a]} g_1(\xi_a) = g_1(a - v_\theta) \leq \max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_1(\xi_a),$$

where the equality holds due to Proposition 3.1(a). Therefore, $\max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_1(\xi_a) = g_1(a - v_\theta)$ and $(\xi_a^*, \xi_b^*) = (a - v_\theta,\ a - v_\theta)$. It implies that $P_{\xi_a^*,\xi_b^*} = P_{a-v_\theta,a-v_\theta} =$

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$a - v_\theta - L_2 \wedge g_1(a - v_\theta)$ due to (46), and $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_a) = v(a - v_\theta, a - v_\theta) = \lambda a - (2\lambda - 1)\,[g_1(a - v_\theta) \wedge L_2]$.

By Lemma 3.2, a contract I^* of the form (24) satisfying $I^*(a) = a - v_\theta$, $I^*(b) = a - v_\theta$, and $P_{I^*} = P_{a-v_\theta,a-v_\theta}$, is the optimal solution to Problem (14). Note that $\xi_a^* = \xi_b^* = a - v_\theta$. Thus, $I^*(x) = (x - d_1)^+ - (x - d_1 - a + v_\theta)^+ + (x - d_3)^+$ for any $d_1 \in [0, v_\theta]$ and $d_3 \in [b, \infty]$ such that $P_{I^*} = a - v_\theta - L_2 \wedge g_1(a - v_\theta)$ is the optimal solution to Problem (14).

ii. If $g_2(a - v_\theta) > L_2$, note that $L_2 \geq g_2(v_\theta)$ by (21), thus there exists $\xi_2 \in [v_\theta \wedge (a - v_\theta),\ v_\theta \vee (a - v_\theta)]$ such that $L_2 = g_2(\xi_2)$ due to the continuity and monotonicity of g_2 as showed in Proposition 3.1(b). Moreover, $(\xi_2, \xi_2) \in \Xi_{a,b}$ from the observation $a - L_1 \leq L_2 = g_2(\xi_2) < g_1(\xi_2)$. For any $(\xi_a, \xi_a) \in \Xi_{a,b}$, we have $[L_2 - g_1(\xi_2)]^+ = 0 \leq [L_2 - g_1(\xi_a)]^+$. Thus,

$$\left[L_2 - \max_{(\xi_a,\xi_a)\in\Xi_{a,b}} g_2(\xi_a)\right]^+ = 0 = [L_2 - g_1(\xi_a)]^+,$$

and $\xi_a^* = \xi_2$. In this case, we have $P_{\xi_a^*,\xi_a^*} = P_{\xi_2,\xi_2} = \xi_2 - L_2 \wedge g_1(\xi_2) = \xi_2 - L_2$ due to (46), and $\min_{(\xi_a,\xi_a)\in\Xi_{a,b}} v(\xi_a, \xi_a) = v(\xi_2, \xi_2) = \lambda a + (1 - 2\lambda)L_2$. Therefore, the optimal contract of the form (24) is reduced to $I^*(x) = x - (x - \xi_2)^+ + (x - b)^+$ with $d_1 = 0$ and $d_3 = b$ because the contract I^* satisfies $I^*(a) = I^*(b) = \xi_2$ and $P_{I^*} = \xi_2 - g_2(\xi_2) = \xi_2 - L_2$.

■

Proof of Theorem 3.2. Assume $b < a$. For each $(\xi_a, \xi_b) \in \Xi_{a,b}$, we have $\xi_b \leq \xi_a \leq \xi_b + a - b$ by (18), $\xi_b - L_2 \leq P_{I^M_{\xi_a,\xi_b}}$ by (19), and $a - L_1 \leq \xi_a - P_{I^m_{\xi_a,\xi_b}}$ by (20). Since $P_{I^M_{\xi_a,\xi_b}}$ and $\xi_a - P_{I^m_{\xi_a,\xi_b}}$ are continuous and strictly increasing in $\xi_a \in [0, a]$ by Proposition 3.1(e), we have $\xi_b - L_2 \leq P_{I^M_{\xi_a,\xi_b}} \leq P_{I^M_{\xi_b+a-b,\xi_b}}$ and $a - L_1 \leq \xi_a - P_{I^m_{\xi_a,\xi_b}} \leq \xi_b + a - b - P_{I^m_{\xi_b+a-b,\xi_b}}$. Thus, $(\xi_b + a - b,\ \xi_b) \in \Xi_{a,b}$. By (19) and (20), we know that $(\xi_b + a - b, \xi_b) \in \Xi_{a,b}$ is equivalent to

$$g_2(\xi_b) \leq L_2 \ \text{ and } \ g_3(\xi_b) \geq b - L_1. \tag{47}$$

a. Consider the case $0 \leq \lambda < \frac{1}{2}$. By Lemma 3.2, $\min_{I\in\mathcal{I}_1} V(I) = \min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a, \xi_b)$, where $v(\xi_a, \xi_b) = \lambda a + (2\lambda - 1)P_{\xi_a,\xi_b} - \lambda\xi_a + (1 - \lambda)\xi_b$ and $P_{\xi_a,\xi_b} = (L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}$. For $(\xi_a, \xi_b) \in \Xi_{a,b}$, since $\xi_a \leq \xi_b + a - b$ and $P_{I^M_{\xi_a,\xi_b}} \leq P_{I^M_{\xi_b+a-b,\xi_b}}$, we have

$$\begin{aligned}
v(\xi_a, \xi_b) &= \lambda a - \lambda\xi_a + (1 - \lambda)\xi_b - (1 - 2\lambda)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_a,\xi_b}}\right] \\
&\geq \lambda a - \lambda(\xi_b + a - b) + (1 - \lambda)\xi_b - (1 - 2\lambda)\left[(L_1 - b + \xi_b) \wedge P_{I^M_{\xi_b+a-b,\xi_b}}\right] \\
&= v(\xi_b + a - b, \xi_b) \\
&= \lambda b + (1 - 2\lambda)(b - L_1) - (1 - 2\lambda)\left[(L_1 - b + \xi_b) \wedge P_{I^M_{\xi_b+a-b,\xi_b}} - (L_1 - b + \xi_b)\right] \\
&= (1 - \lambda)b - (1 - 2\lambda)L_1 + (1 - 2\lambda)\left[L_1 - b + \xi_b - P_{I^M_{\xi_b+a-b,\xi_b}}\right]^+ \\
&= (1 - \lambda)b - (1 - 2\lambda)L_1 + (1 - 2\lambda)\,[g_2(\xi_b) - (b - L_1)]^+.
\end{aligned}$$

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Hence, $\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b) = \min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b,\xi_b)$, and since $(\xi_b+a-b,\xi_b)\in\Xi_{a,b}$, we have

$$\begin{aligned}\min_{(\xi_a,\xi_b)\in\Xi_{a,b}} v(\xi_a,\xi_b) &= \min_{(\xi_b+a-b,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b,\xi_b)\\ &= (1-\lambda)b-(1-2\lambda)L_1+(1-2\lambda)\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}}[g_2(\xi_b)-(b-L_1)]^+\\ &= (1-\lambda)b-(1-2\lambda)L_1+(1-2\lambda)\left[\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b)-(b-L_1)\right]^+.\end{aligned}$$

Note that $P_{I^M_{\xi_b+a-b,\xi_b}} = \xi_b - g_2(\xi_b)$ and then

$$P_{\xi_b+a-b,\xi_b} = (\xi_b+a-b-a+L_1)\wedge P_{I^M_{\xi_b+a-b,\xi_b}} = \xi_b-(b-L_1)\vee g_2(\xi_b), \tag{48}$$

i. If $g_3(v_\theta)\geq b-L_1$, note that $g_2(v_\theta)\leq L_2$ by (21), thus $\xi_b=v_\theta$ satisfies condition (47), namely $(v_\theta+a-b,v_\theta)\in\Xi_{a,b}$. In this case,

$$\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b) \geq \min_{\xi_b\in[0,b]} g_2(\xi_b) = g_2(v_\theta) \geq \min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b),$$

where the equality holds due to Proposition 3.1(b). Therefore, $\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b) = g_2(v_\theta)$ and $(\xi_a^*,\xi_b^*)=(v_\theta+a-b,v_\theta)$. It implies that $P_{\xi_a^*,\xi_b^*} = P_{v_\theta+a-b,v_\theta} = v_\theta-(b-L_1)\vee g_2(v_\theta)$ due to (48), and

$$\begin{aligned}\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b,\xi_b) &= v(v_\theta+a-b,v_\theta)\\ &= (1-\lambda)b-(1-2\lambda)L_1+(1-2\lambda)\left[g_2(v_\theta)-(b-L_1)\right]^+.\end{aligned}$$

By Lemma 3.2, a contract I^* of the form (24) satisfying $I^*(a)=v_\theta+a-b$, $I^*(b)=v_\theta$ and $P_{I^*}=P_{v_\theta+a-b,v_\theta}$, is the optimal solution to Problem (14). In this case, note that $\xi_a^*=v_\theta+a-b$ and $\xi_b^*=v_\theta$. It implies that the range for d_2 given in (24) is reduced to a single point set, that is $d_2\in[b,\ a-I^*(a)+I^*(b)]=\{b\}$ and then, $d_2=b$. Hence, the optimal solution to Problem (14) is reduced to $I^*(x)=(x-d_1)-(x-d_1-v_\theta)^+ +(x-b)^+-(x-a)^+ +(x-d_3)^+$ for some $d_1\in[0,b-v_\theta]$ and $d_3\in[a,\infty]$ such that $P_{I^*}=v_\theta-(b-L_1)\vee g_2(v_\theta)$.

ii. If $g_3(v_\theta)<b-L_1$, note that $g_3(b-v_\theta)\geq b-L_1$ by (8) and g_3 is continuous on $[0,b]$, thus there exists $\xi_3\in[v_\theta\wedge(b-v_\theta),\ v_\theta\vee(b-v_\theta)]$ such that $g_3(\xi_3)=b-L_1$. From Proposition 3.1(e), we known that $g_2(\xi)<g_3(\xi)$ for all $\xi\in[0,b]$. In particular, $g_2(\xi_3)<g_3(\xi_3)=b-L_1\leq L_2$ and then ξ_3 satisfies condition (47), namely $(\xi_3+a-b,\xi_3)\in\Xi_{a,b}$. For any $(\xi_b+a-c,\xi_b)\in\Xi_{a,b}$, we have $[g_2(\xi_3)-(b-L_1)]^+=0\leq[g_2(\xi_b)-(b-L_1)]^+$. Then,

$$\left[\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} g_2(\xi_b)-(b-L_1)\right]^+ = 0 = \left[g_2(\xi_3)-(b-L_1)\right]^+,$$

and $\xi_b^*=\xi_3$. In this case, we have $P_{\xi_a^*,\xi_b^*}=P_{\xi_3+a-b,\xi_3}=\xi_3-(b-L_1)\vee g_2(\xi_3)=\xi_3-b+L_1$ due to (48) and

$$\min_{(\xi_b+a-b,\,\xi_b)\in\Xi_{a,b}} v(\xi_b+a-b,\xi_b) = v(\xi_3+a-b,\xi_3) = (1-\lambda)b-(1-2\lambda)L_1.$$

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The optimal contract of the form (24) is reduced to $I^*(x) = (x - b + \xi_3)^+ - (x - a)^+$ with $d_1 = b - \xi_3$, $d_2 = b$, and $d_3 = \infty$ because the contract I^* satisfies $I^*(a) = \xi_3 + a - b$, $I^*(b) = \xi_3$ and $P_{I^*} = \xi_3 - g_3(\xi_3) = \xi_3 - b + L_1$.

b. Consider the case $\frac{1}{2} < \lambda \leq 1$. By Lemma 3.2, $\min_{I \in \mathcal{I}_1} V(I) = \min_{(\xi_a, \xi_b) \in \Xi_{a,b}} v(\xi_a, \xi_b)$, where $v(\xi_a, \xi_b) = \lambda a - \lambda \xi_a + (1-\lambda)\xi_b + (2\lambda - 1) P_{\xi_a, \xi_b}$ and $P_{\xi_a, \xi_b} = (\xi_b - L_2) \vee P_{I^m_{\xi_a, \xi_b}}$. For any $(\xi_a, \xi_b) \in \Xi_{a,b}$, it is easy to check that $\xi_a \leq \xi_b + a - b$ and $P_{I^m_{\xi_a, \xi_b}} - \xi_a \geq P_{I^m_{\xi_b + a - b, \xi_b}} - (\xi_b + a - b)$, thus

$$\begin{aligned} v(\xi_a, \xi_b) &= \lambda a + (1-\lambda)(\xi_b - \xi_a) + (2\lambda - 1)\left[(\xi_b - L_2 - \xi_a) \vee \left(P_{I^m_{\xi_a, \xi_b}} - \xi_a\right)\right] \\ &\geq \lambda a + (1-\lambda)(b - a) + (2\lambda - 1)\left[(b - a - L_2) \vee \left(P_{I^m_{\xi_b + a - b, \xi_b}} - (\xi_b + a - b)\right)\right] \\ &= v(\xi_b + a - b, \xi_b) = \lambda b + (2\lambda - 1)\left[(-L_2) \vee \left(P_{I^m_{\xi_b + a - b, \xi_b}} - \xi_b\right)\right] \\ &= \lambda b + (1 - 2\lambda) L_2 + (2\lambda - 1)\left[P_{I^m_{\xi_b + a - b, \xi_b}} - (\xi_b - L_2)\right]^+ \\ &= \lambda b + (1 - 2\lambda) L_2 + (2\lambda - 1)\left[L_2 - g_3(\xi_b)\right]^+ . \end{aligned}$$

Hence, $\min_{(\xi_a, \xi_b) \in \Xi_{a,b}} v(\xi_a, \xi_b) \geq \min_{(\xi_b + a - b, \xi_b) \in \Xi_{a,b}} v(\xi_b + a - b, \xi_b)$, and since $(\xi_b + a - b, \xi_b) \in \Xi_{a,b}$, we have

$$\begin{aligned} \min_{(\xi_a, \xi_b) \in \Xi_{a,b}} v(\xi_a, \xi_b) &= \min_{(\xi_b + a - b, \xi_b) \in \Xi_{a,b}} v(\xi_b + a - b, \xi_b) \\ &= \lambda b + (1 - 2\lambda) L_2 + (2\lambda - 1) \min_{(\xi_b + a - b,\, \xi_b) \in \Xi_{a,b}} \left[L_2 - g_3(\xi_b)\right]^+ \\ &= \lambda b + (1 - 2\lambda) L_2 + (2\lambda - 1)\left[L_2 - \max_{(\xi_b + a - b,\, \xi_b) \in \Xi_{a,b}} g_3(\xi_b)\right]^+ . \end{aligned}$$

Note that $P_{I^m_{\xi_b + a - b, \xi_b}} = \xi_b - g_3(\xi_b)$ and then

$$P_{\xi_b + a - b, \xi_b} = (\xi_b - L_2) \vee P_{I^m_{\xi_b + a - b, \xi_b}} = \xi_b - L_2 \wedge g_3(\xi_b), \tag{49}$$

i. If $g_2(b - v_\theta) \leq L_2$, note that $g_3(b - v_\theta) \geq b - L_1$ by (8), thus $\xi_b = b - v_\theta$ satisfies condition (47), namely $(a - v_\theta, b - v_\theta) \in \Xi_{a,b}$. It implies that

$$\max_{(\xi_b + a - b,\, \xi_b) \in \Xi_{a,b}} g_3(\xi_b) \leq \max_{\xi_b \in [0,b]} g_3(\xi_b) = g_3(b - v_\theta) \leq \max_{(\xi_b + a - b,\, \xi_b) \in \Xi_{a,b}} g_3(\xi_b),$$

where the equality holds due to Proposition 3.1(c). Therefore, we obtain that $\max_{(\xi_b + a - b,\, \xi_b) \in \Xi_{a,b}} g_3(\xi_b) = g_3(b - v_\theta)$ and $(\xi_a^*, \xi_b^*) = (a - v_\theta, b - v_\theta)$. It implies that $P_{\xi_a^*, \xi_b^*} = P_{a - v_\theta, b - v_\theta} = b - v_\theta - g_3(b - v_\theta) \wedge L_2$ due to (49), and

$$\min_{(\xi_b + a - b,\, \xi_b) \in \Xi_{a,b}} v(\xi_b + a - b, \xi_b) = \lambda b + (1 - 2\lambda) L_2 + (2\lambda - 1)\left[L_2 - g_3(b - v_\theta)\right]^+ .$$

By Lemma 3.2, a contract I^* of the form (24) satisfying $I^*(a) = a - v_\theta$, $I^*(b) = b - v_\theta$, and $P_{I^*} = P_{a - v_\theta, b - v_\theta}$, is the optimal solution to Problem (14). Note that in this case, $\xi_a^* = a - v_\theta$ and $\xi_b^* = b - v_\theta$. Hence, $d_2 \in [b, a - I^*(a) + I^*(b)] = \{b\}$ and thus $d_2 = b$. Therefore, the optimal solution I^* is reduced to $I^*(x) = (x - d_1)^+ - (x - d_1 - b + v_\theta)^+ + (x - b)^+ - (x - a)^+ + (x - d_3)^+$ for some $d_1 \in [0, v_\theta]$ and $d_3 \in [a, \infty]$ such that $P_{I^*} = b - v_\theta - L_2 \wedge g_3(b - v_\theta)$.

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ii. If $g_2(b-v_\theta) > L_2$, note that $g_2(v_\theta) \le L_2$ due to (21), thus there exists $\xi_4 \in [v_\theta \wedge (b-v_\theta),\ v_\theta \vee (b-v_\theta)]$ such that $g_2(\xi_4) = L_2$ due to the continuity and monotonicity of g_2. Since $a - L_1 \le L_2 = g_2(\xi_4) < g_3(\xi_4)$, we have that ξ_4 satisfies (47), namely $(\xi_4 + a - b,\ \xi_4) \in \Xi_{a,b}$. For all $(\xi_b + a - b,\ \xi_b) \in \Xi_{a,b}$, we have $[L_2 - g_3(\xi_4)]^+ = 0 \le [L_2 - g_3(\xi_b)]^+$. Thus,

$$\left[L_2 - \max_{(\xi_b+a-b,\ \xi_b)\in\Xi_{a,b}} g_3(\xi_b)\right]^+ = 0 = [L_2 - g_3(\xi_4)]^+ ,$$

and $\xi_b^* = \xi_4$. In this case, we have $P_{\xi_a^*,\xi_b^*} = P_{\xi_4+a-b,\xi_4} = \xi_4 - L_2 \wedge g_3(\xi_4) = \xi_4 - L_2$ due to (49), and $\min_{(\xi_b+a-b,\ \xi_b)\in\Xi_{a,b}} v(\xi_b + a - b, \xi_b) = v(\xi_4 + a - b, \xi_4) = \lambda b + (1-2\lambda)L_2$. Therefore, the optimal contract of the form (24) is reduced to $I^*(x) = x - (x-\xi_4)^+ + (x-b)^+$ because the contract f^* satisfies $I^*(a) = \xi_4 + a - b$, $I^*(b) = \xi_4$, and $P_{I^*} = \xi_4 - g_2(\xi_4) = \xi_4 - L_2 = P_{\xi_4+a-b,\xi_4}$.

■

Proof of Proposition 4.1. **a.** Obviously, $h_1(\xi_c) = (1+\theta)(\int_0^{\xi_c} + \int_c^\infty) S_X(x)\mathrm{d}x - \xi_c$ is continuous and differentiable with $h_1'(\xi_c) = (1+\theta)S_X(\xi_c) - 1$. Since $h_1'(\xi_c)$ is decreasing in ξ_c, we obtain that $h_1(\xi_c)$ is a concave function of ξ_c. For any $0 \le \xi_c < v_\theta$, we have $S_X(\xi_c) > \frac{1}{1+\theta}$, where $v_\theta = \mathrm{VaR}_{\frac{1}{1+\theta}}(X) = \inf\left\{x \ge 0 : S_X(x) \le \frac{1}{1+\theta}\right\}$. Thus, $h_1'(\xi_c) = (1+\theta)S_X(\xi_c) - 1 > 0$ for any $0 \le \xi_c < v_\theta$, and $h_1(\xi_c)$ is strictly increasing on $[0, v_\theta)$. For any $c \ge \xi_c > v_\theta$, we have $S_X(\xi_c) \le \frac{1}{1+\theta}$. Thus, $h_1'(\xi_c) = (1+\theta)S_X(\xi_c) - 1 \le 0$ for any $c \ge \xi_c > v_\theta$, and $h_1(\xi_c)$ is decreasing on $(v_\theta, c]$. Hence, $\max_{\xi_c\in[0,c]} h_1(\xi_c) = h_1(v_\theta)$.
b. Obviously, $h_2(\xi_c) = (1+\theta)\int_{c-\xi_c}^a S_X(x)\mathrm{d}x - \xi_c$ is continuous and differentiable with $h_2'(\xi_c) = (1+\theta)S_X(c-\xi_c) - 1$. For $\xi_c < c - v_\theta$, we have $c - \xi_c > v_\theta$ and $S_X(c-\xi_c) \le \frac{1}{1+\theta}$. For $\xi_c > c - v_\theta$, we have $c - \xi_c < v_\theta$ and $S_X(c-\xi_c) > \frac{1}{1+\theta}$. Thus, $h_2(\xi_c)$ is decreasing on $[0, c-v_\theta)$, strictly increasing on $(c-v_\theta, c]$ and $\min_{\xi_c\in[0,c]} h_2(\xi_c) = h_2(c-v_\theta)$. Since $c < a$ and $S_X(x)$ is continuous and decreasing in $x \ge 0$, we have, for $\xi_c \in [0, c]$,

$$\begin{aligned}
h_1(\xi_c) - h_2(\xi_c) &= (1+\theta)\left(\int_0^{\xi_c} + \int_c^a + \int_a^\infty\right) S_X(x)\mathrm{d}x - (1+\theta)\left(\int_{c-\xi_c}^c + \int_c^a\right) S_X(x)\mathrm{d}x \\
&= (1+\theta)\left(\int_0^{\xi_c} - \int_{c-\xi_c}^c + \int_a^\infty\right) S_X(x)\mathrm{d}x \\
&= (1+\theta)\int_0^{\xi_c} [S_X(x) - S_X(x+c-\xi_c)]\,\mathrm{d}x + (1+\theta)\int_a^\infty S_X(x)\mathrm{d}x > 0,
\end{aligned}$$

where $S_X(x) \ge S_X(x+c-\xi_c)$ and $S_X(a) = \alpha > 0$.

■

Proof of Proposition 4.2. We prove (b) for the function A_{ξ_c} only. The proofs for all the other functions and results in (a)–(f) can be obtained using similar arguments and are omitted.

(b) Clearly, $A_{\xi_c}(\xi_a) = P_{I^M_{\xi_c,\xi_a,\xi_a}} = (1+\theta)\left(\int_0^{\xi_c} + \int_c^{c+\xi_a-\xi_c} + \int_b^\infty\right) S_X(x)\mathrm{d}x$ is continuous and strictly increasing in ξ_a with $A_{\xi_c}'(\xi_a) = (1+\theta)S_X(c+\xi_a-\xi_c) > 0$. Note that $S_X(c+\xi_a-\xi_c) \le S_X(c) = 1-\gamma < \frac{1}{1+\theta}$ and $\frac{\mathrm{d}}{\mathrm{d}\xi_a}\left[\xi_a - A_{\xi_c}(\xi_a)\right] = 1 - (1+\theta)S_X(c+\xi_a-\xi_c) > 0$. Thus, $\xi_a - A_{\xi_c}(\xi_a)$ is continuous and strictly increasing in $\xi_a \in [\xi_c, \xi_c + b - c]$. ■

Proof of Proposition 4.3. The proof of this proposition is similar to the proof of Propositions 4.1 and 4.2 and is omitted. ■

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Proof of Proposition 4.4. (a) Note that $\Xi_c \subset [0, c]$. If $\Xi_c = [0, c]$, then $\xi_c^m = 0$, $\xi_c^M = c$ and the proof is done. Now, assume $\Xi_c \neq [0, c]$. From (29) and (30), $\xi_c \in \Xi_c$ is equivalent to $h_1(\xi_c) \geq L_3$ and $h_2(\xi_c) \leq L_1 - c$. From Proposition 4.1(a) and (b), we have that h_1 is concave and h_2 is convex on $[0, c]$. Denote $\xi_c^m = \inf \Xi_c$ and $\xi_c^M = \sup \Xi_c$. Then, $0 \leq \xi_c^m \leq \xi_c^M \leq c$ because $\Xi_c \subset [0, c]$. There exists a sequence $\{x_n\}_{n=1}^{\infty} \subset \Xi_c$ such that $x_n \to \xi_c^m$ as $n \to \infty$. For each n, we have $h_1(x_n) \geq L_3$ and $h_2(x_n) \leq L_1 - c$ because $x_n \in \Xi_c$. By continuity of h_1 and h_2, $h_1(\xi_c^m) = \lim_{n\to\infty} h_1(x_n) \geq L_3$ and $h_2(\xi_c^m) = \lim_{n\to\infty} h_2(x_n) \leq L_1 - c$ and thus, $\xi_c^m \in \Xi_c$. Using a similar argument, we can prove $\xi_c^M \in \Xi_c$. For any $\xi_c \in (\xi_c^m, \xi_c^M)$, there exists $\Delta \in (0, 1)$ such that $\xi_c = \Delta\xi_c^m + (1 - \Delta)\xi_c^M$. It is easy to see that $\xi_c \in \Xi_c$ because $h_1(\xi_c) = h_1(\Delta\xi_c^m + (1 - \Delta)\xi_c^M) \geq \Delta h_1(\xi_c^m) + (1 - \Delta)h_1(\xi_c^M) \geq \Delta L_3 + (1 - \Delta)L_3 = L_3$ from the concavity of h_1; and $h_2(\xi_c) = h_2(\Delta\xi_c^m + (1-\Delta)\xi_c^M) \leq \Delta h_2(\xi_c^m) + (1-\Delta)h_2(\xi_c^M) \leq \Delta(L_1 - c) + (1 - \Delta)(L_1 - c) = L_1 - c$ from the convexity of h_2. Therefore, $\Xi_c = [\xi_c^m, \xi_c^M] \subset [0, c]$.

The proofs of (b) and (c) are similar to (a) and are omitted. ■

Proof of Lemma 4.1. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

(i) ⇒ (ii). Suppose (8) and (9) hold, which are equivalent to $h_2(c - v_\theta) \leq L_1 - c$ and $L_3 \leq h_1(v_\theta)$, respectively. We will prove $\mathcal{I}_2 \neq \emptyset$ by considering the following two cases:

Case 1: If $h_2(0) \vee h_2(c) \geq L_1 - c$, by the continuity of h_2 and (8), there exists $\xi_c \in [0, c]$ such that $h_2(\xi_c) = L_1 - c$, and thus $L_3 \leq L_1 - c = h_2(\xi_c) < h_1(\xi_c)$. Consider the contract $I(x) = (x - c + \xi_c)^+ - (x - a)^+ \in \mathcal{I}$. It is easy to check that $I(c) = \xi_c$, $I(a) = \xi_c + a - c$, and $P_I = h_2(\xi_c) + \xi_c = L_1 - c + \xi_c$. Thus, $I \in \mathcal{I}_2$ since $a - I(a) + P_I = a - (\xi_c + a - c) + L_1 - c + \xi_c = L_1$ and $P_I - I(c) = L_1 - c \geq L_3$.

Case 2: If $h_2(0) \vee h_2(c) < L_1 - c$, then $h_2(\xi_c) \leq L_1 - c$ for all $\xi_c \in [0, c]$, and in particular, $h_2(v_\theta) \leq L_1 - c$. Note that $L_3 \leq h_1(v_\theta)$ by (9), we have

$$v_\theta - c + L_1 \geq h_2(v_\theta) + v_\theta = P_{I^m_{v_\theta,\, v_\theta+a-c}} \quad \text{and} \quad v_\theta + L_3 \leq h_1(v_\theta) + v_\theta = P_{I^M_{v_\theta,\, v_\theta+a-c}},$$

where $I^m_{v_\theta,\, v_\theta+a-c}(x) = (x - c + v_\theta)^+ - (x - a)^+$ and $I^M_{v_\theta,\, v_\theta+a-c}(x) = x - (x - v_\theta)^+ + (x - c)^+$ for all $x \geq 0$. Since $c \leq L_1 - L_3$, we have $v_\theta + L_3 \leq v_\theta - c + L_1$. Note that $P_{I^m_{v_\theta,v_\theta+a-c}} \leq P_{I^M_{v_\theta,v_\theta+a-c}}$, and thus $(v_\theta + L_3) \vee P_{I^m_{v_\theta,v_\theta+a-c}} \leq (v_\theta - c + L_1) \wedge P_{I^M_{v_\theta,v_\theta+a-c}}$. Using similar arguments to those used in the proof of Lemma 3.2, we know that as a function of $(d_1, d_2) \in [0, c - v_\theta] \times [a, \infty]$, $P_I = P_I(d_1, d_2) = (1 + \theta)\left(\int_{d_1}^{d_1+v_\theta} + \int_c^{d_2}\right) S_X(x)\mathrm{d}x$ can take all its intermediate values in the interval $[P_{I^m_{v_\theta,\, v_\theta+a-c}}, P_{I^M_{v_\theta,\, v_\theta+a-c}}]$. Thus, there exists $(d_1, d_2) \in [0, c - v_\theta] \times [a, \infty]$ such that $P_I(d_1, d_2) = (v_\theta + L_3) \vee P_{I^m_{v_\theta,\, v_\theta+a-c}}$. Consider the contract $I(x) = (x - d_1)^+ - (x - d_1 - v_\theta)^+ + (x - c)^+ - (x - d_2)^+$, it is easy to check that $I(c) = v_\theta$, $I(a) = v_\theta + a - c$, and $P_I = P_I(d_1, d_2) = (v_\theta + L_3) \vee P_{I^m_{v_\theta,\, v_\theta+a-c}}$. Thus, $I(c) + L_3 = v_\theta + L_3 \leq P_I \leq v_\theta - c + L_1 = I(a) - a + L_1$ and $I \in \mathcal{I}_2$.

Therefore, by combining **Cases 1 and 2**, we get $\mathcal{I}_2 \neq \emptyset$.

(ii) ⇒ (iii). Suppose $\mathcal{I}_2 \neq \emptyset$. For any $I \in \mathcal{I}_2$, denote $\xi_c = I(c)$, $\xi_a = I(a)$ and $\xi_b = I(b)$. Note that for $a < b$ and $I \in \mathcal{I}_2$, we have $\xi_a \leq \xi_b$ and $\xi_c + L_3 \leq P_I \leq \xi_a - a + L_1$, and thus (26) holds. It is easy to check that $I^m_{\xi_c,\xi_a,\xi_b}(x) \leq I(x) \leq I^M_{\xi_c,\xi_a,\xi_b}(x)$ for all $x \geq 0$ and thus $P_{I^m_{\xi_c,\xi_a,\xi_b}} \leq P_I \leq P_{I^M_{\xi_c,\xi_a,\xi_b}}$. Moreover, we get $(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}} \leq P_I \leq (\xi_a - a + L_1) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}$ and it implies that (27) and (28) hold for (ξ_c, ξ_a, ξ_b). By its definition, $(\xi_c, \xi_a, \xi_b) \in \Xi_{c,a,b}$ and then $\Xi_{c,a,b} \neq \emptyset$.

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(iii) $\Rightarrow$ (i). Suppose $\Xi_{c,a,b} \neq \emptyset$. From (27), we get

$$L_3 \leq (1+\theta)\left(\int_0^{\xi_c} + \int_c^{c+\xi_a-\xi_c} + \int_a^{a+\xi_b-\xi_a} + \int_b^{\infty}\right) S_X(x)\mathrm{d}x - \xi_c$$
$$\leq (1+\theta)\left(\int_0^{\xi_c} + \int_c^{\infty}\right) S_X(x)\mathrm{d}x - \xi_c = h_1(\xi_c) \leq h_1(v_\theta).$$

Thus, (9) holds. From (28) and the fact that $\xi_a - A^m_{\xi_c}(\xi_a)$ is increasing in ξ_a, we get

$$a - L_1 \leq \xi_a - (1+\theta)\left(\int_{c-\xi_c}^{c} + \int_{a-\xi_a+\xi_c}^{a} + \int_{b-\xi_b+\xi_a}^{b}\right) S_X(x)\mathrm{d}x$$
$$\leq \xi_a - (1+\theta)\left(\int_{c-\xi_c}^{c} + \int_{a-\xi_a+\xi_c}^{a}\right) S_X(x)\mathrm{d}x = \xi_a - A^m_{\xi_c}(\xi_a)$$
$$\leq \xi_c + a - c - A^m_{\xi_c}(\xi_c + a - c) = a - c - h_2(\xi_c) \leq a - c - h_2(c - v_\theta),$$

where $h_2(\xi_c) = A^m_{\xi_c}(\xi_c + a - c) - \xi_c$. Thus, (8) holds. ■

Proof of Lemma 4.2. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

For any $I \in \mathcal{I}_2$, from the proof of Lemma 4.1 for (ii) $\Rightarrow$ (iii), we have $(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}} \leq P_I \leq (\xi_a - a + L_1) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}$, where $(\xi_c, \xi_a, \xi_b) = (I(c), I(a), I(b)) \in \Xi_{c,a,b}$. By the definition (36) of P_{ξ_c,ξ_a,ξ_b}, it is easy to check $P_I \leq P_{I(c),I(a),I(b)}$ for $0 \leq \lambda < 1/2$ and $P_I \geq P_{I(c),I(a),I(b)}$ for $1/2 < \lambda \leq 1$. Therefore, we have $(2\lambda - 1)P_I \geq (2\lambda - 1)P_{I(c),I(a),I(b)}$, and

$$V(I) = \lambda a + (2\lambda - 1)P_I - \lambda I(a) + (1-\lambda)I(b)$$
$$\geq \lambda a + (2\lambda - 1)P_{I(c),I(a),I(b)} - \lambda I(a) + (1-\lambda)I(b) = w(I(c), I(a), I(b)).$$

Thus, $\min_{I\in\mathcal{I}_2} V(I) \geq \min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c, \xi_a, \xi_b)$.

On the contrary, for any $(\xi_c, \xi_a, \xi_b) \in \Xi_{c,a,b}$, using similar arguments to those used in the proof of Lemma 3.2, we know that there exists $I \in \mathcal{I}$ such that $P_I = P_{\xi_c,\xi_a,\xi_b}$, $I(c) = \xi_c$, $I(a) = \xi_a$ and $I(b) = \xi_b$. Thus, I satisfies $\xi_c + L_3 \leq P_I \leq \xi_a + L_1 - a$, namely $I \in \mathcal{I}_2$ and $V(I) = w(\xi_c, \xi_a, \xi_b)$. It implies that $\min_{I\in\mathcal{I}_2} V(I) \leq \min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c, \xi_a, \xi_b)$. Thus, $\min_{I\in\mathcal{I}_2} V(I) = \min_{(\xi_c,\xi_a,\xi_b)\in\Xi_{c,a,b}} w(\xi_c, \xi_a, \xi_b)$. ■

Proof of Lemma 4.3. **(a)** Assume $a < b$ and $0 \leq \lambda < 1/2$. For any $(\xi_c, \xi_a) \in \Xi_c \times \Xi_{a,\xi_c}$ where $\Xi_c = [\xi_c^m, \xi_c^M]$ and $\Xi_{a,\xi_c} = [\xi_a^m(\xi_c), \xi_a^M(\xi_c)]$, in the first step, we solve the problem of $\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c, \xi_a, \xi_b)$, where $\Xi_{b,\xi_c,\xi_a} = [\xi_b^m(\xi_c, \xi_a), \xi_b^M(\xi_c, \xi_a)]$. By Lemma 4.2, we have

$$w(\xi_c, \xi_a, \xi_b) = \lambda a + (2\lambda - 1)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b}}\right] - \lambda\xi_a + (1-\lambda)\xi_b$$
$$= \lambda a - \lambda\xi_a + \lambda\xi_b + (1 - 2\lambda)\left[(\xi_b - L_1 + a - \xi_a) \vee \left(\xi_b - P_{I^M_{\xi_c,\xi_a,\xi_b}}\right)\right],$$

thus $w(\xi_c, \xi_a, \xi_b)$ inherits the increment in $\xi_b \in \Xi_{b,\xi_c,\xi_a}$ from the function $\xi_b - P_{I^M_{\xi_c,\xi_a,\xi_b}}$ by Proposition 4.2(a). Therefore, the minimizer of $\min_{\xi_b\in\Xi_{b,\xi_c,\xi_a}} w(\xi_c, \xi_a, \xi_b)$, is the left-end point $\xi_b^*(\xi_c, \xi_a) = \xi_b^m(\xi_c, \xi_a)$ of the set Ξ_{b,ξ_c,ξ_a}.

In the second step, we solve the problem of $\min\limits_{\xi_a\in\Xi_{a,\xi_c}} w(\xi_c, \xi_a, \xi_b^m(\xi_c, \xi_a)) = \min\limits_{\xi_a\in\Xi_{a,\xi_c}} w_2(\xi_c, \xi_a)$. In doing so, consider the supremum of the set $\{\xi_a \in \Xi_{a,\xi_c} : A_{\xi_c}(\xi_a) <$

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$\xi_c + L_3\}$, denoted by

$$\xi_{a,\xi_c} = \sup\left\{\xi_a \in \Xi_{a,\xi_c} : A_{\xi_c}(\xi_a) < \xi_c + L_3\right\}. \tag{50}$$

By convention, the supremum (50) is defined as the left-end point $\xi_a^m(\xi_c)$ of the set Ξ_{a,ξ_c} if the set $\left\{\xi_a \in \Xi_{a,\xi_c} : A_{\xi_c}(\xi_a) < \xi_c + L_3\right\}$ is empty. Note that $A_{\xi_c}(\xi_a)$ is continuous and strictly increasing in ξ_a, thus there are three possible scenarios for the supremum (50). First of all, if $\xi_c + L_3 \le A_{\xi_c}(\xi_a^m(\xi_c))$, then $\xi_{a,\xi_c} = \xi_a^m(\xi_c)$. Second, if $A_{\xi_c}(\xi_a^m(\xi_c)) < \xi_c + L_3 < A_{\xi_c}(\xi_a^M(\xi_c))$, then $\xi_a^m(\xi_c) < \xi_{a,\xi_c} < \xi_a^M(\xi_c)$ and $A_{\xi_c}(\xi_{a,\xi_c}) = \xi_c + L_3$. The last scenario is that if $A_{\xi_c}(\xi_a^M(\xi_c)) \le \xi_c + L_3$, then $\xi_{a,\xi_c} = \xi_a^M(\xi_c)$. In the following, we discuss the properties of the function $w_2(\xi_c, \xi_a)$ in the second scenario, that is to assume $A_{\xi_c}(\xi_a^m(\xi_c)) < \xi_c + L_3 < A_{\xi_c}(\xi_a^M(\xi_c))$.

Case a.1. For $\xi_a^m(\xi_c) \le \xi_a \le \xi_{a,\xi_c}$, we have $A_{\xi_c}(\xi_a) \le \xi_c + L_3$, and then $P_{I^M_{\xi_c,\xi_a,\xi_a}} = A_{\xi_c}(\xi_a) \le \xi_c + L_3$. By (31), we have $P_{I^M_{\xi_c,\xi_a,\xi_a+b-a}} \ge \xi_c + L_3$. Since $P_{I^M_{\xi_c,\xi_a,\xi_b}}$ is continuous and strictly increasing in ξ_b, we know that the equation $P_{I^M_{\xi_c,\xi_a,\xi_b}} = \xi_c + L_3$ has a unique solution $\xi_{b,0} \in [\xi_a, \xi_a + b - a]$, namely, (27) is satisfied by $(\xi_c, \xi_a, \xi_{b,0})$. Meanwhile, (28) is satisfied by $(\xi_c, \xi_a, \xi_{b,0})$ because $P_{I^m_{\xi_c,\xi_a,\xi_{b,0}}} \le P_{I^M_{\xi_c,\xi_a,\xi_{b,0}}} = \xi_c + L_3 \le L_1 - a + \xi_a$. Thus, $(\xi_c, \xi_a, \xi_{b,0}) \in \Xi_{c,a,b}$ and $\xi_{b,0} \in \Xi_{b,\xi_c,\xi_a}$. For any $\xi_b < \xi_{b,0}$, because $P_{I^M_{\xi_c,\xi_a,\xi_b}} < \xi_c + L_3$, namely, (27) is not satisfied, we have that $(\xi_c, \xi_a, \xi_b) \notin \Xi_{c,a,b}$ and then $\xi_b \notin \Xi_{b,\xi_c,\xi_a}$. Therefore, $\xi_b^m(\xi_c, \xi_a) = \xi_{b,0}$ and $P_{I^M_{\xi_c,\xi_a,\xi_b^m(\xi_c,\xi_a)}} = \xi_c + L_3$. Now, for any ξ_1 and ξ_2 such that $\xi_a^m(\xi_c) \le \xi_1 < \xi_2 \le \xi_{a,\xi_c}$, we have that $\xi_b^m(\xi_c, \xi_i)$ satisfies $P_{I^M_{\xi_c,\xi_i,\xi_b^m(\xi_c,\xi_i)}} = \xi_c + L_3$, for $i = 1, 2$. Then, the equation $P_{I^M_{\xi_c,\xi_1,\xi_b^m(\xi_c,\xi_1)}} = \xi_c + L_3 = P_{I^M_{\xi_c,\xi_2,\xi_b^m(\xi_c,\xi_2)}}$ implies that

$$\int_{a+\xi_b^m(\xi_c,\xi_2)-\xi_2}^{a+\xi_b^m(\xi_c,\xi_1)-\xi_1} S_X(x)\mathrm{d}x = \int_{c+\xi_1-\xi_c}^{c+\xi_2-\xi_c} S_X(x)\mathrm{d}x > 0.$$

Since $S_X(x)$ is positive and decreasing in x, we have $a + \xi_b^m(\xi_c, \xi_1) - \xi_1 - (a + \xi_b^m(\xi_c, \xi_2) - \xi_2) \ge c + \xi_2 - \xi_c - (c + \xi_1 - \xi_c)$ and thus $\xi_b^m(\xi_c, \xi_1) \ge \xi_b^m(\xi_c, \xi_2)$. Moreover, $\xi_b^m(\xi_c, \xi_2) \to \xi_b^m(\xi_c, \xi_1)$ as $\xi_2 \to \xi_1$. Therefore, $\xi_b^m(\xi_c, \xi_a)$ is continuous and decreasing in $\xi_a \in [\xi_a^m(\xi_c), \xi_{a,\xi_c}]$. Since $\xi_b^*(\xi_c, \xi_a) = \xi_b^m(\xi_c, \xi_a)$ and $P_{I^M_{\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)}} = \xi_c + L_3 \le L_1 - a + \xi_a$, we have

$$\begin{aligned} w_2(\xi_c, \xi_a) &= \lambda a + (2\lambda - 1)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)}}\right] - \lambda\xi_a + (1-\lambda)\xi_b^*(\xi_c, \xi_a) \\ &= \lambda a + (2\lambda - 1)(\xi_c + L_3) - \lambda\xi_a + (1-\lambda)\xi_b^*(\xi_c, \xi_a) \end{aligned}$$

is continuous and decreasing in $\xi_a \in [\xi_a^m(\xi_c), \xi_{a,\xi_c}]$. In particular, when $\xi_a = \xi_{a,\xi_c}$, it is easy to check that the equation $A_{\xi_c}(\xi_{a,\xi_c}) = \xi_c + L_3$ implies that $\xi_b^m(\xi_c, \xi_{a,\xi_c}) = \xi_{a,\xi_c}$, and $w_2(\xi_c, \xi_{a,\xi_c}) = \lambda a + (1 - 2\lambda)\left(\xi_{a,\xi_c} - (\xi_c + L_3)\right)$.

Case a.2. For $\xi_{a,\xi_c} < \xi_a \le \xi_a^M(\xi_c)$, we have that $A_{\xi_c}(\xi_a) > \xi_c + L_3$, then (27) is satisfied by (ξ_c, ξ_a, ξ_a). Since $\xi_a \in \Xi_{a,\xi_c}$, (32) implies that (ξ_c, ξ_a, ξ_a) satisfies (28). Thus, $(\xi_c, \xi_a, \xi_a) \in \Xi_{c,a,b}$. It implies that $\xi_a \in \Xi_{b,\xi_c,\xi_a}$ and then $\xi_b^m(\xi_c, \xi_a) = \xi_a$. We have $\xi_b^*(\xi_c, \xi_a) = \xi_a$ and

$$\begin{aligned} w_2(\xi_c, \xi_a) &= \lambda a + (2\lambda - 1)\left[(L_1 - a + \xi_a) \wedge P_{I^M_{\xi_c,\xi_a,\xi_b^*(\xi_c,\xi_a)}}\right] - \lambda\xi_a + (1-\lambda)\xi_b^*(\xi_c, \xi_a) \\ &= \lambda a + (1 - 2\lambda)\left[(a - L_1) \vee \left(\xi_a - A_{\xi_c}(\xi_a)\right)\right], \end{aligned}$$

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which inherits the continuity and increment in ξ_a form the function $\xi_a - A_{\xi_c}(\xi_a)$ by Proposition 4.2(b). Note that $\xi_c + L_3 \le \xi_{a,\xi_c} - a + L_1$ and then

$$\begin{aligned}\lim_{\xi_a \downarrow \xi_{a,\xi_c}} w_2(\xi_c, \xi_a) &= \lambda a + (1-2\lambda)\left[(a - L_1) \vee \left(\xi_{a,\xi_c} - A_{\xi_c}(\xi_{a,\xi_c})\right)\right] \\ &= \lambda a + (1-2\lambda)\left[(a - L_1) \vee \left(\xi_{a,\xi_c} - (\xi_c + L_3)\right)\right] \\ &= \lambda a + (1-2\lambda)\left(\xi_{a,\xi_c} - (\xi_c + L_3)\right) = w_2(\xi_c, \xi_{a,\xi_c}).\end{aligned}$$

By combining **Cases a.1** and **Case a.2**, we obtain that, when $A_{\xi_c}(\xi_a^m(\xi_c)) < \xi_c + L_3 < A_{\xi_c}(\xi_a^M(\xi_c))$, the function $w_2(\xi_c, \xi_a)$ is continuous in $\xi_a \in \Xi_{a,\xi_c}$ and minimized at the point $\xi_a^*(\xi_c) = \xi_{a,\xi_c}$. If $\xi_c + L_3 \le A_{\xi_c}(\xi_a^m(\xi_c))$, by using the same arguments in **Case a.2**, we have that $w_2(\xi_c, \xi_a)$ is continuous in $\xi_a \in \Xi_{a,\xi_c}$ and minimized at the point $\xi_a^*(\xi_c) = \xi_{a,\xi_c} = \xi_a^m(\xi_c)$. If $A_{\xi_c}(\xi_a^M(\xi_c)) \le \xi_c + L_3$, by using the same arguments in **Case a.1**, we have that $w_2(\xi_c, \xi_a)$ is continuous in $\xi_a \in \Xi_{a,\xi_c}$ and minimized at the point $\xi_a^*(\xi_c) = \xi_{a,\xi_c} = \xi_a^M(\xi_c)$. In short, we conclude that $w_2(\xi_c, \xi_a)$ is continuous in $\xi_a \in \Xi_{a,\xi_c}$ and minimized at the point $\xi_a^*(\xi_c) = \xi_{a,\xi_c}$.

In the last step, we solve the problem of $\min_{\xi_c \in \Xi_c} w_2(\xi_c, \xi_a^*(\xi_c)) = \min_{\xi_c \in \Xi_c} w_1(\xi_c)$. Note that, for each $\xi_c \in \Xi_c$, $\xi_a^M(\xi_c) = \xi_c + a - c$ and $\xi_a^m(\xi_c) \ge \xi_c + L_3 + a - L_1$. By Proposition 4.2(b), we know that $\xi_a - A_{\xi_c}(\xi_a)$ and $\xi_a - A^m_{\xi_c}(\xi_a)$ are both continuous and strictly increasing in ξ_a. Consider the following two cases:

Case a.i. If $L_3 \le h_5(\xi_c)$, namely $\xi_c + L_3 \le A_{\xi_c}(\xi_c + L_3 + a - L_1)$, note that $\xi_a^m(\xi_c) \ge \xi_c + L_3 + a - L_1$ and $A_{\xi_c}(\xi_a)$ is increasing on Ξ_{a,ξ_c}, thus $A_{\xi_c}(\xi_a^m(\xi_c)) \ge A_{\xi_c}(\xi_c + L_3 + a - L_1) \ge \xi_c + L_3$. It implies that the set $\left\{\xi_a \in \Xi_{a,\xi_c} : A_{\xi_c}(\xi_a) < \xi_c + L_3\right\}$ is empty. Thus, we have $\xi_a^*(\xi_c) = \xi_a^m(\xi_c)$ and $A_{\xi_c}(\xi_a^*(\xi_c)) \ge \xi_c + L_3$. From the arguments in **Case a.2**, we have $\xi_b^*(\xi_c, \xi_a^*(\xi_c)) = \xi_a^*(\xi_c)$ and

$$w_2(\xi_c, \xi_a) = \lambda a + (1-2\lambda)\left[(a - L_1) \vee \left(\xi_a^*(\xi_c) - A_{\xi_c}(\xi_a^*(\xi_c))\right)\right].$$

Suppose $A_{\xi_c}(\xi_a^*(\xi_c)) < \xi_a^*(\xi_c) + L_1 - a$, then $\xi_a^*(\xi_c) > A_{\xi_c}(\xi_a^*(\xi_c)) - L_1 + a \ge \xi_c + L_3 + a - L_1$ and $\xi_a^*(\xi_c) - A^m_{\xi_c}(\xi_a^*(\xi_c)) \ge \xi_a^*(\xi_c) - A_{\xi_c}(\xi_a^*(\xi_c)) > a - L_1$. Note that $\xi_a - A^m_{\xi_c}(\xi_a)$ is continuous and increasing in $\xi_a \in \Xi_{a,\xi_c}$, then there exists $\xi \in [\xi_c + L_3 + a - L_1, \xi_a^*(\xi_c))$ such that $\xi - A^m_{\xi_c}(\xi) > a - L_1$, which implies that ξ satisfies (31). Moreover, ξ satisfies (32) because $\xi_c + L_3 \le A_{\xi_c}(\xi_c + L_3 + a - L_1) \le A_{\xi_c}(\xi) \le A^M_{\xi_c}(\xi)$. Conditions (31) and (32) imply $\xi \in \Xi_{a,\xi_c}$, namely $\xi \ge \xi_a^m(\xi_c)$, which contradicts the fact that $\xi < \xi_a^*(\xi_c) = \xi_a^m(\xi_c)$. Therefore, $A_{\xi_c}(\xi_a^*(\xi_c)) \ge \xi_a^*(\xi_c) + L_1 - a$ and $w_2(\xi_c, \xi_a) = \lambda a + (1-2\lambda)(a - L_1)$ is a constant function.

Case a.ii. If $L_3 > h_5(\xi_c)$, namely $\xi_c + L_3 > A_{\xi_c}(\xi_c + a - L_1 + L_3)$. Since $A_{\xi_c}(\xi_c + b - c) \ge \xi_c + L_3$, by (29), and the fact that $A_{\xi_c}(\xi_a)$ is continuous and strictly increasing in $\xi_a \in \Xi_{a,\xi_c}$, we see that there exists $\xi_{a,1} \in [\xi_c + a - L_1 + L_3, \xi_c + b - c]$, which is the unique solution to the equation of $A_{\xi_c}(\xi_{a,1}) = L_3 + \xi_c$. Thus, $\xi_a^*(\xi_c) = \xi_{a,1} \wedge \xi_a^M(\xi_c) = \xi_{a,1} \wedge (\xi_c + a - c) \le \xi_{a,1}$ and $A_{\xi_c}(\xi_a^*(\xi_c)) \le A_{\xi_c}(\xi_{a,1}) = \xi_c + L_3$. Consider the contract $I(x) = x - (x - \xi_c)^+ + (x - c)^+ - (x - (c + \xi_{a,1} - \xi_c))^+ + (x - b)^+$, it is easy to check that $I(c) = \xi_c$, $I(a) = (\xi_c + a - c) \wedge \xi_{a,1} = \xi_a^*(\xi_c)$, $I(b) = \xi_{a,1}$, and $P_I = A_{\xi_c}(\xi_{a,1}) = \xi_c + L_3$. Since $I(c) + L_3 = \xi_c + L_3 = P_I = (\xi_c + L_3 + a - L_1) - a + L_1 \le I(a) - a + L_1$, we have $I \in \mathcal{I}_2$ and $(\xi_c, (\xi_c + a - c) \wedge \xi_{a,1}, \xi_{a,1}) \in \Xi_{c,a,b}$. For any $\xi_b < \xi_{a,1}$, we have $(\xi_c, (\xi_c + a - c) \wedge \xi_{a,1}, \xi_b) \notin \Xi_{c,a,b}$ because either (26) is invalid when $\xi_{a,1} < \xi_c + a - c$, or (27) is invalid when $\xi_{a,1} \ge \xi_c + a - c$ from the observation that $P_{I^M_{\xi_c, \xi_c + a - c, \xi_b}} = A_{\xi_c}(\xi_b) < A_{\xi_c}(\xi_{a,1}) = \xi_c + L_3$. It implies that $\xi_b \notin \Xi_{b,\xi_c,\xi_a^*(\xi_c)}$ for any $\xi_b < \xi_{a,1}$ and then $\xi_b^*(\xi_c, \xi_a^*(\xi_c)) = \xi_b^m(\xi_c, \xi_a^*(\xi_c)) = \xi_{a,1}$. It is easy to check that,

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$P_{I^M_{\xi_c,(\xi_c+a-c)\wedge\xi_{a,1},\xi_{a,1}}} = A_{\xi_c}(\xi_{a,1}) = \xi_c + L_3 \le \xi_{a,1} - a + L_1$. Thus,

$$\begin{aligned} w_1(\xi_c) &= \lambda a - \lambda\xi_a^*(\xi_c) + (1-\lambda)\xi_b^*(\xi_c, \xi_a^*(\xi_c)) \\ &\quad + (2\lambda - 1)\left[(L_1 - a + \xi_a^*(\xi_c)) \wedge P_{I^M_{\xi_c,\xi_a^*(\xi_c),\xi_b^*(\xi_c,\xi_a^*(\xi_c))}}\right] \\ &= \lambda a - \lambda((\xi_c + a - c) \wedge \xi_{a,1}) + (1-\lambda)\xi_{a,1} + (2\lambda - 1)(\xi_c + L_3) \end{aligned}$$

and it has derivative on the set $\Xi_c \setminus \left\{\xi_c : \ A_{\xi_c}(\xi_c + a - c) = L_3 + \xi_c + a - c\right\}$ with

$$\begin{aligned} w_1'(\xi_c) &= [1 - \lambda - \lambda\,\mathbb{I}(\xi_{a,1} < \xi_c + a - c)]\left(\frac{\mathrm{d}}{\mathrm{d}\xi_c}\xi_{a,1} - 1\right) \\ &= \frac{1}{S_X(c + \xi_{a,1} - \xi_c)}\left(\frac{1}{1+\theta} - S_X(\xi_c)\right)[1 - \lambda - \lambda\,\mathbb{I}(\xi_{a,1} < \xi_c + a - c)], \end{aligned}$$

where $\frac{\mathrm{d}}{\mathrm{d}\xi_c}\xi_{a,1} = 1 + \left[\frac{1}{1+\theta} - S_X(\xi_c)\right] / S_X(c + \xi_{a,1} - \xi_c)$ since $\xi_{a,1}$ satisfies the equation $\xi_c + L_3 = A_{\xi_c}(\xi_{a,1})$. Note that $w_1'(\xi_c) \le 0 \iff \frac{1}{1+\theta} \le S_X(\xi_c) \iff \xi_c \le v_\theta$.

By combining **Case a.i** and **Case a.ii**, we obtain that $w_1'(\xi_c) \le 0$ when $\xi_c \le v_\theta$ and $w_1'(\xi_c) \ge 0$ when $\xi_c > v_\theta$. Therefore, $\xi_c^* = \xi_c^m \vee \left(v_\theta \wedge \xi_c^M\right)$.

The proofs of **(b)** and **(c)** are similar to **(a)** and omitted.

(d) Assume $b < a$ and $1/2 < \lambda \le 1$. By Lemma 4.2, we have

$$\begin{aligned} w(\xi_c, \xi_b, \xi_a) &= \lambda a - \lambda\xi_a + (1-\lambda)\xi_b + (2\lambda - 1)\left[(\xi_c + L_3) \vee P_{I^m_{\xi_c,\xi_a,\xi_b}}\right] \\ &= \lambda a + (1-\lambda)\xi_b - (1-\lambda)\xi_a + (2\lambda - 1)\left[(\xi_c + L_3 - \xi_a) \vee \left(P_{I^m_{\xi_c,\xi_a,\xi_b}} - \xi_a\right)\right]. \end{aligned}$$

Thus, $w(\xi_c, \xi_a, \xi_b)$ is continuous and decreasing in ξ_a due to the properties of $P_{I^m_{\xi_c,\xi_a,\xi_b}} - \xi_a$ given in Proposition 4.3(a). Hence, we have $\xi_a^*(\xi_c, \xi_b) = \xi_a^M(\xi_c, \xi_b) = \xi_b + a - b$ and

$$\begin{aligned} w_2(\xi_c, \xi_b) =& w(\xi_c, \xi_a^*(\xi_c, \xi_b), \xi_b) = \lambda a - \lambda(\xi_b + a - b) + (1-\lambda)\xi_b \\ &+ (2\lambda - 1)\left[(\xi_c + L_3) \vee B^m_{\xi_c}(\xi_b)\right] \\ =& \lambda b + (2\lambda - 1)\left[(\xi_c + L_3 - \xi_b) \vee \left(B^m_{\xi_c}(\xi_b) - \xi_b\right)\right]. \end{aligned}$$

Thus, $w_2(\xi_c, \xi_b)$ is continuous and decreasing in ξ_b due to the properties of $B^m_{\xi_c}(\xi_b) - \xi_b$ given in Proposition 4.3(b). It implies $\xi_b^*(\xi_c) = \xi_b^M(\xi_c) = \xi_c + b - c$ and thus

$$\begin{aligned} \min_{\xi_c \in \Xi_c} w_1(\xi_c) &= \min_{\xi_c \in \Xi_c} w_2(\xi_c, \xi_b^*(\xi_c)) \\ &= \min_{\xi_c \in \Xi_c}\left\{\lambda b + (2\lambda - 1)\left[(c + L_3 - b) \vee \left(B^m_{\xi_c}(\xi_c + b - c) - (\xi_c + b - c)\right)\right]\right\} \\ &= \min_{\xi_c \in \Xi_c}\{(1-\lambda)b + (2\lambda - 1)c + (2\lambda - 1)[L_3 \vee h_2(\xi_c)]\} \\ &= (1-\lambda)b + (2\lambda - 1)c + (2\lambda - 1)\left[L_3 \vee \min_{\xi_c \in \Xi_c} h_2(\xi_c)\right]. \end{aligned}$$

Since $h_2(\xi_c)$ is continuous, decreasing on $[0, c - v_\theta)$, and increasing on $(c - v_\theta, c]$, we obtain that $w_1(\xi_c)$ is continuous and $\xi_c^* = \xi_c^m \vee \left[(c - v_\theta) \wedge \xi_c^M\right]$. ∎

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Proof of Theorem 4.1. We assume $a < b$. The proof for the case of $a > b$ is similar to the case of $a < b$ and is omitted.

For $a < b$, we have $\xi_b^* = \xi_b^*(\xi_c^*, \xi_a^*) \in \Xi_{b,\xi_c^*,\xi_a^*}$. Note that Ξ_{b,ξ_c^*,ξ_a^*} is the set of all $\xi_b \in [\xi_a^*, \xi_a^* + b - a]$ such that $(\xi_c^*, \xi_a^*, \xi_b) \in \Xi_{c,a,b}$, thus $\left(\xi_c^*, \xi_a^*, \xi_b^*\right) \in \Xi_{c,a,b}$. It is easy to check that any contract I of the form

$$I(x) = (x - d_1)^+ - (x - d_1 - \xi_c^*)^+ + (x - d_2)^+ - (x - (d_2 + \xi_a^* - \xi_c^*))^+ \\ + (x - d_3)^+ - (x - (d_3 + \xi_b^* - \xi_a^*))^+ + (x - d_4)^+ \tag{51}$$

for some $(d_1, d_2, d_3, d_4) \in [0, c - \xi_c^*] \times [c, a - \xi_a^* + \xi_c^*] \times [a, b - \xi_b^* + \xi_a^*] \times [b, \infty]$, satisfies $I \in \mathcal{I}$, $I(c) = \xi_c^*$, $I(a) = \xi_a^*$, $I(b) = \xi_b^*$ and $I^m_{\xi_c^*,\xi_a^*,\xi_b^*}(x) \le I(x) \le I^M_{\xi_c^*,\xi_a^*,\xi_b^*}(x)$ for all $x \ge 0$. For I of the form (51), its premium is given by

$$P_I = P(d_1, d_2, d_3, d_4) = (1 + \theta) \left(\int_{d_1}^{d_1 + \xi_c^*} + \int_{d_2}^{d_2 + \xi_a^* - \xi_c^*} + \int_{d_3}^{d_3 + \xi_b^* - \xi_a^*} + \int_{d_4}^{\infty} \right) S_X(x) \mathrm{d}x,$$

which is a real-valued continuous function of (d_1, d_2, d_3, d_4). Thus, $\{P_I : I \text{ has expression (51)}\} = [P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}}, P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}}]$. By (36), we have $P_{I^m_{\xi_c^*,\xi_a^*,\xi_b^*}} \le P_{\xi_c^*,\xi_a^*,\xi_b^*} \le P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}}$ and $\xi_c^* + L_3 \le P_{\xi_c^*,\xi_a^*,\xi_b^*} \le \xi_a^* - a + L_1$. Therefore, there exists $I^* \in \mathcal{I}_2$ such that $I^*(c) = \xi_c^*$, $I^*(a) = \xi_a^*$, $I^*(b) = \xi_b^*$, and $P_{I^*} = P_{\xi_c^*,\xi_a^*,\xi_b^*}$. For any $I \in \mathcal{I}_2$, denote $\xi_c = I(c)$, $\xi_a = I(a)$, and $\xi_b = I(b)$, then

$$\begin{aligned} V(I^*) &= \lambda a + (2\lambda - 1) P_{\xi_c^*,\xi_a^*,\xi_b^*} - \lambda \xi_a^* + (1 - \lambda)\xi_b^* = w(\xi_c^*, \xi_a^*, \xi_b^*) \\ &= w(\xi_c^*, \xi_a^*(\xi_c^*), \xi_b^*(\xi_c^*, \xi_a^*(\xi_c^*))) = w_2(\xi_c^*, \xi_a^*(\xi_c^*)) = w_1\left(\xi_c^*\right) = \min_{\xi \in \Xi_c} w_1(\xi) \\ &\le w_1(\xi_c) = \min_{\xi \in \Xi_{a,\xi_c}} w_2(\xi_c, \xi) \le w_2(\xi_c, \xi_a) = \min_{\xi \in \Xi_{b,\xi_c,\xi_a}} w(\xi_c, \xi_a, \xi) \\ &\le w(\xi_c, \xi_a, \xi_b) = \lambda a + (2\lambda - 1) P_{\xi_c,\xi_a,\xi_b} - \lambda \xi_a + (1 - \lambda)\xi_b \le V(I), \end{aligned}$$

where the last inequality is from the proof of Lemma 4.2. Therefore, a contract I^* of the form (38) for some $(d_1, d_2, d_3, d_4) \in [0,\ c - \xi_c^*] \times [c,\ a - \xi_a^* + \xi_c^*] \times [a,\ b - \xi_b^* + \xi_a^*] \times [b,\ \infty]$, satisfying $I^*(c) = \xi_c^*$, $I^*(a) = \xi_a^*$, $I^*(b) = \xi_b^*$ and $P_{I^*} = P_{\xi_c^*,\xi_a^*,\xi_b^*}$, is an optimal solution to Problem (25). ■

Proof of Corollary 4.1. Suppose $a < b$ and $0 \le \lambda < 1/2$. By (29) and (30), we have that $\xi_c \in \Xi_c$ is equivalent to $h_1(\xi_c) \ge L_3$ and $h_2(\xi_c) \le L_1 - c$. Note that (8) implies $h_2(c - v_\theta) \le L_1 - c$ while (9) implies $h_1(v_\theta) \ge L_3$.

a. Assume $h_2(v_\theta) \le L_1 - c$. Note that $h_1(v_\theta) \ge L_3$, thus we have $v_\theta \in \Xi_c$. By Lemma 4.3(a), we get $\xi_c^* = v_\theta$. It follows that $\xi_a^* = \xi_a^*(v_\theta) = \sup\left\{\xi_a \in \Xi_{a,v_\theta} : A_{v_\theta}(\xi_a) < v_\theta + L_3\right\}$, $\xi_b^* = \xi_b^*(v_\theta, \xi_a^*) = \xi_b^m(v_\theta, \xi_a^*)$, and $P_{\xi_c^*,\xi_a^*,\xi_b^*} = (L_1 - a + \xi_a^*) \wedge P_{I^M_{\xi_c^*,\xi_a^*,\xi_b^*}}$.

i. If $L_3 \le h_4(v_\theta)$, note that $h_4(v_\theta) < h_5(v_\theta)$, thus $v_\theta + L_3 \le A^m_{v_\theta}(v_\theta + a - L_1 + L_3) < A_{v_\theta}(v_\theta + a - L_1 + L_3)$. From **Case a.i** in the proof of Lemma 4.3, we have $\xi_b^* = \xi_a^* = \xi_a^m(v_\theta)$. We will specify the value of $\xi_a^m(v_\theta)$. Since $(v_\theta + a - L_1 + L_3) + L_1 - a \le A^m_{v_\theta}(v_\theta + a - L_1 + L_3)$ from $L_3 \le h_4(v_\theta)$, $(v_\theta + a - c) + L_1 - a \ge A^m_{v_\theta}(v_\theta + a - c)$ from $h_2(v_\theta) \le L_1 - c$, and $A^m_{v_\theta}(\xi_a)$ is continuous and strictly increasing in $\xi_a \in \Xi_{a,v_\theta}$, there exists $\xi_{a,0} \in [v_\theta + L_3 + a - L_1,\ v_\theta + a - c]$, which is the unique solution to the equation of $\xi_{a,0} + L_1 - a = A^m_{v_\theta}(\xi_{a,0})$. Hence, $\xi_{a,0}$ satisfies (32) for $\xi_c = v_\theta$. Meanwhile, $\xi_{a,0}$

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satisfies (31) for $\xi_c = v_\theta$ because $v_\theta + L_3 \le \xi_{a,0} + L_1 - a = A^m_{v_\theta}(\xi_{a,0}) < A^M_{v_\theta}(\xi_{a,0})$. Thus, $\xi_{a,0} \in \Xi_{a,v_\theta}$. For any $\xi_a < \xi_{a,0}$, since $\xi_a - A^m_{v_\theta}(\xi_a)$ is strictly increasing, we know that $\xi_a - A^m_{v_\theta}(\xi_a) < \xi_{a,0} - A^m_{v_\theta}(\xi_{a,0}) = a - L_3$. It implies that (32) with $\xi_c = v_\theta$ is not satisfied by ξ_a, and then $\xi_a \notin \Xi_{a,v_\theta}$. Therefore, $\xi^m_a(v_\theta) = \xi_{a,0}$. It follows that $\xi^*_b = \xi^*_a = \xi_{a,0}$ and $P_{\xi^*_c,\xi^*_a,\xi^*_b} = (\xi_{a,0} + L_1 - a) \wedge P_{I^M_{v_\theta,\xi_{a,0},\xi_{a,0}}} = (\xi_{a,0} + L_1 - a) \wedge A_{v_\theta}(\xi_{a,0}) = \xi_{a,0} + L_1 - a$. Hence, $I^*(x) = (x - c + v_\theta)^+ - (x-c)^+ + (x - (a - \xi_{a,0} + v_\theta))^+ - (x-a)^+$ since it is easy to check that $I^*(c) = v_\theta$, $I^*(a) = I^*(b) = \xi_{a,0}$, and $P_{I^*} = A^m_{v_\theta}(\xi_{a,0}) = \xi_{a,0} + L_1 - a$. Thus, I^* is the optimal contract by Theorem 4.1.

ii. If $h_4(v_\theta) < L_3 \le h_5(v_\theta)$, which means $A^m_{v_\theta}(v_\theta + a - L_1 + L_3) < (v_\theta + L_3 + a - L_1) + L_1 - a = v_\theta + L_3 \le A_{v_\theta}(v_\theta + a - L_1 + L_3)$, then $v_\theta + L_3 + a - L_1$ satisfies (31) and (32) for $\xi_c = v_\theta$. It implies $v_\theta + L_3 + a - L_1 \in \Xi_{a,v_\theta}$, where $\Xi_{a,v_\theta} \subset [v_\theta + L_3 + a - L_1, a]$ by its definition, and thus, $\xi^m_a(v_\theta) = v_\theta + L_3 + a - L_1$. From **Case a.i** in the proof of Lemma 4.3, we have $\xi^*_b = \xi^*_a = \xi^m_a(v_\theta) = v_\theta + L_3 + a - L_1$ and $P_{\xi^*_c,\xi^*_a,\xi^*_b} = (\xi^*_a + L_1 - a) \wedge P_{I^M_{v_\theta,\xi^*_a,\xi^*_a}} = (\xi^*_a + L_1 - a) \wedge A_{v_\theta}(v_\theta + a - L_1 + L_3) = v_\theta + L_3$. As a function of $(d_1, d_2, d_3) \in [0, c - v_\theta] \times [c, L_1 - L_3] \times [b, \infty]$,

$$P_I = P_I(d_1, d_2, d_3) = (1+\theta)\left(\int_{d_1}^{d_1+v_\theta} + \int_{d_2}^{d_2+a+L_3-L_1} + \int_{d_3}^{\infty}\right) S_X(x)\mathrm{d}x$$

can take all values on $[P_I(c - v_\theta, L_1 - L_3, \infty), P_I(0, c, b)]$. Since $P_I(c - v_\theta, L_1 - L_3, \infty) = h_4(v_\theta) + v_\theta < L_3 + v_\theta \le h_5(v_\theta) + v_\theta = P_I(0, c, b)$, there exists $(d^*_1, d^*_2, d^*_3) \in [0, c - v_\theta] \times [c, L_1 - L_3] \times [b, \infty]$ such that $P_I(d^*_1, d^*_2, d^*_3) = v_\theta + L_3$. Therefore,

$$\begin{aligned} I^*(x) = &(x - d^*_1)^+ - (x - d^*_1 - v_\theta)^+ + (x - d^*_2)^+ \\ &- (x - d^*_2 - (a - L_1 + L_3))^+ + (x - d^*_3)^+ \end{aligned}$$

because it satisfies $I^*(x) = \xi^*_x$ for $x = c, a, b$ and $P_{I^*} = P_I(d^*_1, d^*_2, d^*_3) = v_\theta + L_3$.

iii. If $h_5(v_\theta) < L_3$, by the arguments in **Case a.ii** in the proof of Lemma 4.3, we know that there exists $\xi_{a,1} \in [v_\theta + L_3 + a - L_1, v_\theta + b - c]$ such that $A_{v_\theta}(\xi_{a,1}) = v_\theta + L_3$ and $(\xi^*_c, \xi^*_a, \xi^*_b) = (v_\theta, (v_\theta + a - c) \wedge \xi_{a,1}, \xi_{a,1})$. It implies that $P_{\xi^*_c,\xi^*_a,\xi^*_b} = (\xi^*_a - a + L_1) \wedge P_{I^M_{\xi^*_c,\xi^*_a,\xi^*_b}} = (\xi^*_a - a + L_1) \wedge A_{v_\theta}(\xi_{a,1}) = v_\theta + L_3$. Hence, $I^*(x) = x - (x - v_\theta)^+ + (x - c)^+ - (x - (c + \xi_{a,1} - v_\theta))^+ + (x - b)^+$ since it easy to check that $I^*(x) = \xi^*_x$ for $x = c, a, b$ and $P_{I^*} = A_{v_\theta}(\xi_{a,1}) = v_\theta + L_3$.

b. Assume $h_2(v_\theta) > L_1 - c$. Note that $h_2(c - v_\theta) \le L_1 - c$ and h_2 is continuous and monotone on $[v_\theta \wedge (c - v_\theta), v_\theta \vee (c - v_\theta)]$, thus the equation $h_2(\xi_c) = L_1 - c$ has solutions on $[v_\theta \wedge (c - v_\theta), v_\theta \vee (c - v_\theta)]$. Denote

$$\xi_{L_1-c,h_2} = \inf\{\xi_c \in [v_\theta \wedge (c - v_\theta), v_\theta \vee (c - v_\theta)] : h_2(\xi_c) = L_1 - c\}. \tag{52}$$

Notice that $L_3 \le L_1 - c = h_2(\xi_{L_1-c,h_2}) < h_1(\xi_{L_1-c,h_2})$ implies that (29) and (30) are satisfied by ξ_{L_1-c,h_2} and thus $\xi_{L_1-c,h_2} \in \Xi_c$. Suppose $v_\theta < c - v_\theta$, then $v_\theta \le \xi_{L_1-c,h_2} \le c - v_\theta$. For any $\xi_c < \xi_{L_1-c,h_2}$, we have $h_2(\xi_c) > L_1 - c$ because h_2 is decreasing on $[0, c - v_\theta]$. It implies that $\xi_c \notin \Xi_c$ because it does not satisfy (30). Thus, $\xi_{L_1-c,h_2} = \xi^m_c$ and moreover, $v_\theta \le \xi_{L_1-c,h_2} = \xi^m_c \le \xi^M_c$. By Lemma 4.3(a), we have $\xi^*_c = \xi^m_c \vee (v_\theta \wedge \xi^M_c) = \xi_{L_1-c,h_2}$. In the other case of $v_\theta \ge c - v_\theta$, we have $h_2(\xi_c) > h_2(\xi_{L_1-c,h_2}) = L_1 - c$ for any $\xi_c > \xi_{L_1-c,h_2}$ because h_2 is strictly increasing on $[c - v_\theta, c]$. It implies that $\xi_c \notin \Xi_c$ because it does not satisfy (30). Thus, $\xi^M_c = \xi_{L_1-c,h_2} \le v_\theta$. By Lemma 4.3(a), we have $\xi^*_c = \xi^m_c \vee (v_\theta \wedge \xi^M_c) = \xi_{L_1-c,h_2}$. Therefore, in both of the two cases, $\xi^*_c = \xi_{L_1-c,h_2}$. Note that the equation

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$h_2(\xi_{L_1-c,h_2}) = L_1 - c$ can be rewritten as $A^m_{\xi^*_c}(\xi^*_c + a - c) = (\xi^*_c + a - c) + L_1 - a$. Since the function $\xi_a - A^m_{\xi^*_c}(\xi_c)$ is strictly increasing in ξ_a, for any $\xi_a < \xi^*_c + a - c$, we have $\xi_a - A^m_{\xi^*_c}(\xi_c) < (\xi^*_c + a - c) - A^m_{\xi^*_c}(\xi^*_c + a - c) = a - L_1$, which means that (32) with $\xi_c = \xi^*_c$ are not satisfied by ξ_a. Thus, $\Xi_{a,\xi^*_c} = \{\xi^*_c + a - c\}$ is a single point set. It is easy to check that $\xi^*_b = \xi^m_b(\xi^*_c, \xi^*_a) = \xi^*_a = \xi^*_c + a - c$ and $P_{\xi^*_c,\xi^*_a,\xi^*_b} = (\xi^*_a - a + L_1) \wedge P_{I^M_{\xi^*_c,\xi^*_a,\xi^*_b}} = \xi^*_a - a + L_1$, where $P_{I^M_{\xi^*_c,\xi^*_a,\xi^*_b}} = A_{\xi^*_c}(\xi^*_a) \geq A^m_{\xi^*_c}(\xi^*_a) = h_2(\xi^*_c) + \xi^*_c = L_1 - c + \xi^*_c = \xi^*_a - a + L_1$. The contract $I^*(x) = (x - c + \xi_{L_1-c,h_2})^+ - (x-a)^+$ is the optimal one because it satisfies $I^*(x) = \xi^*_x$ for $x = c, a, b$ and $P_{I^*} = h_2(\xi^*_c) - \xi^*_c = \xi^*_c - c + L_1$.

■

Proofs of Corollaries 4.2 and 4.3 are similar to Corollary 4.1 and are omitted.

Proof of Corollary 4.4. Suppose $b < a$ and $1/2 < \lambda \leq 1$. By Lemma 4.3(d), we have $\xi^*_c = \xi^m_c \vee [(c - v_\theta) \wedge \xi^M_c]$, $\xi^*_b = \xi^*_c + b - c$, $\xi^*_a = \xi^*_c + a - c$ and $P_{\xi^*_c,\xi^*_a,\xi^*_b} = (\xi^*_c + L_3) \vee P_{I^m_{\xi^*_c,\xi^*_a,\xi^*_b}} = (\xi^*_c + L_3) \vee (h_2(\xi^*_c) + \xi^*_c)$. By (29) and (30), we know that $c - v_\theta \in \Xi_c$ is equivalent to $h_1(c - v_\theta) \geq L_3$ and $h_2(c - v_\theta) \leq L_1 - c$. Note that $h_2(c - v_\theta) \leq L_1 - c$ by (9).

a. If $h_1(c - v_\theta) < L_3$, then $c - v_\theta \notin \Xi_c$. Furthermore, note that $h_1(v_\theta) \geq L_3$ by (9) and h_1 is continuous and monotone on $[v_\theta \wedge (c - v_\theta),\ v_\theta \vee (c - v_\theta)]$, thus the equation $h_1(\xi_c) = L_3$ has solutions on $[v_\theta \wedge (c - v_\theta),\ v_\theta \vee (c - v_\theta)]$. Denote

$$\xi_{L_3,h_1} = \sup\{\xi_c \in [v_\theta \wedge (c - v_\theta),\ v_\theta \vee (c - v_\theta)] : h_1(\xi_c) = L_3\}. \tag{53}$$

Then, we have $h_1(\xi_{L_3,h_1}) = L_3$. Moreover, $h_2(\xi_{L_3,h_1}) \leq h_1(\xi_{L_3,h_1}) = L_3 \leq L_1 - c$. Thus, $\xi_{L_3,h_1} \in \Xi_c$. Suppose $v_\theta \leq \xi_{L_3,h_1} \leq c - v_\theta$, since h_1 is decreasing on $[v_\theta, c - v_\theta]$, we have $h_1(\xi_c) < L_3$, for any $\xi_c > \xi_{L_3,h_1}$, namely ξ_c does not satisfy (29) and $\xi_c \notin \Xi_c$. It implies that $\xi^M_c = \xi_{L_3,h_1} \leq c - v_\theta$ and thus $\xi^*_c = \xi_{L_3,h_1}$. Suppose $c - v_\theta \leq \xi_{L_3,h_1} \leq v_\theta$, since h_1 is strictly increasing on $[c - v_\theta, v_\theta]$, we have $h_1(\xi_c) < h_1(\xi_{L_3,h_1}) = L_3$ for any $\xi_c < \xi_{L_3,h_1}$, namely ξ_c does not satisfy (29) and $\xi_c \notin \Xi_c$. We also conclude that $\xi^*_c = \xi^m_c = \xi_{L_3,h_1}$. Moreover, $P_{\xi^*_c,\xi^*_a,\xi^*_b} = (\xi_{L_3,h_1} + L_3) \vee (h_2(\xi_{L_3,h_1}) + \xi_{L_3,h_1}) = L_3 + \xi_{L_3,h_1}$, where $h_2(\xi_{L_3,h_1}) \leq h_1(\xi_{L_3,h_1}) = \xi_{L_3,h_1}$. The optimal contract is $I^*(x) = x - (x - \xi_{L_3,h_1})^+ + (x-c)^+$ because it satisfies $I^*(x) = \xi^*_x$ for $x = c, a, b$ and $P_{I^*} = \xi_{L_3,h_1} + h_1(\xi_{L_3,h_1}) = \xi_{L_3,h_1} + L_3$.

b. If $h_1(c - v_\theta) \geq L_3$ which means $c - v_\theta \in \Xi_c$, then $\xi^*_c = c - v_\theta$, $\xi^*_b = b - v_\theta$, $\xi^*_a = a - v_\theta$, and $P_{\xi^*_c,\xi^*_a,\xi^*_b} = c - v_\theta + L_3 \vee h_2(c - v_\theta)$. As a function of $(d_1, d_2) \in [0, v_\theta] \times [a, \infty]$, $P_I = P_I(d_1, d_2) = (1+\theta)\left(\int_{d_1}^{d_1+c-v_\theta} + \int_c^a + \int_{d_2}^\infty\right) S_X(x)\mathrm{d}x$ is continuous and can take all the values on $[P_I(0,a),\ P_I(v_\theta, \infty)]$. Note that $h_1(c - v_\theta) \geq L_3$ by (29) and $h_1(\xi_c) \geq h_2(\xi_c)$ for all $\xi_c \in [0, c]$, then $P_I(0, a) = h_1(c - v_\theta) + c - v_\theta \geq L_3 \vee h_2(c - v_\theta) + c - v_\theta$. Together with $P_I(v_\theta, \infty) = h_2(c - v_\theta) + c - v_\theta \leq L_3 \vee h_2(c - v_\theta) + c - v_\theta$, we know that there exists $(d^*_1, d^*_2) \in [0, v_\theta] \times [a, \infty]$ such that $P_I(d^*_1, d^*_2) = P_{\xi^*_c,\xi^*_a,\xi^*_b}$. The optimal contract is $I^*(x) = (x - d^*_1)^+ - (x - d^*_1 - c + v_\theta)^+ + (x-c)^+ - (x-a)^+ + (x - d^*_2)^+$ because it satisfies $I^*(x) = \xi^*_x$ for $x = c, a, b$ and $P_{I^*} = P_I(d^*_1, d^*_2) = L_3 \vee h_2(c - v_\theta) + c - v_\theta$.

■

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