Teaching Notes

Extending the parity proof that $\sqrt{2}$ is irrational

The following argument extends one of the usual proofs that $\sqrt{2}$ is irrational to show that $\sqrt{k^2 + 1}$ is irrational for all odd integers, k. Assume for contradiction that $\sqrt{k^2 + 1}$ is rational. Then $\sqrt{k^2 + 1} = k + \frac{b}{a}$ for integers a, b with $0 < b < a$ and the fraction $\frac{b}{a}$ in its lowest form. Squaring both sides leads to $1 = 2k\frac{b}{a} + \frac{b^2}{a^2}$ or $a^2 - b^2 = 2kab$ so that $(a - b)(a + b) = 2kab$ (*). Since $\frac{b}{a}$ is in its lowest form, we have a contradiction because either:

- a, b are both odd, in which case 4 divides the left-hand side of (*) but not the right-hand side (because k is odd);
- a, b have opposite parity, in which case the left-hand side of (*) is odd and the right-hand side even.

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Areas of images of singular 3 × 3 matrix transformations

For a 3×3 matrix, A, $|\det A|$ gives the volume of the image of the unit cube under transformation by A. For a singular matrix this is zero, but we can still ask: what is the area of the image of the unit cube? If the columns of A are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ then the image of the unit cube when A is singular lies in the plane spanned by $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and is the convex region given by $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c}$; $0 \leqslant \alpha, \beta, \gamma \leqslant 1$. As in Figure 1, this is a hexagon with pairs of parallel, equal sides represented by the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. It is made up of the three parallelograms labelled I, II, III spanned by the respective pairs of vectors $\mathbf{a}$, $\mathbf{b}$; $\mathbf{a}$, $\mathbf{c}$; $\mathbf{b}$, $\mathbf{c}$. The area of the hexagon is thus

$$|\mathbf{a} \times \mathbf{b}| + |\mathbf{a} \times \mathbf{c}| + |\mathbf{b} \times \mathbf{c}|. \qquad (1)$$

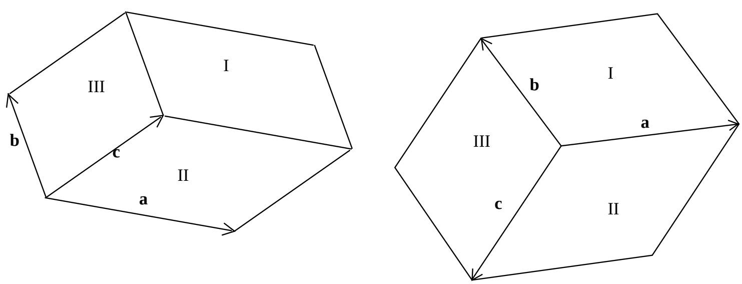

FIGURE 1: Possible images of the unit cube.

Evaluating the three vector products in (1) thus answers our original question, but it is worth making two further points in connection with (1).

- If we omit the modulus signs in (1), then $\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} + \mathbf{b} \times \mathbf{c}$ gives a normal vector to the plane containing $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, [1]. (It will be $\mathbf{0}$ if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are collinear.)

- Usually, calculating the matrix of cofactors of A is a waste of time if A is singular, but here we note that any row and non-zero column gives us all the information we need to evaluate (1).

  For example, if $A = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \end{pmatrix} = \begin{pmatrix} -2 & 1 & 2 \\ -1 & 2 & 3 \\ 1 & 4 & 5 \end{pmatrix}$, then the matrix of cofactors is $\begin{pmatrix} -2 & 8 & -6 \\ 3 & -12 & 9 \\ -1 & 4 & -3 \end{pmatrix}$. The columns are the (necessarily parallel) vectors $\mathbf{b} \times \mathbf{c}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times \mathbf{b}$ but to work out (1) all we need is the magnitude of the first column $\sqrt{(-2)^2 + 3^2 + (-1)^2} = \sqrt{14}$ and the size of the ratio between the column vectors $1 : 4 : 3$ from which (1) evaluates as $(1 + 4 + 3)\sqrt{14} = 8\sqrt{14}$.

Reference

1. A. Ellis-Davies, The equation of the plane π containing *ABC*, *Math. Gaz*. **89** (November 2005) p. 507.

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