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COHOMOLOGY OF FIBER-BUNCHED TWISTED COCYCLES OVER HYPERBOLIC SYSTEMS

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Abstract A twisted cocycle taking values on a Lie group G is a cocycle that is twisted by an automorphism of G in each step. In the case where $G = GL(d, \mathbb{R})$, we prove that if two Hölder continuous twisted cocycles satisfying the so-called fiber-bunching condition have the same periodic data then they are cohomologous.

Keywords: twisted cocycles; cohomology; hyperbolic systems; periodic points

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1. Introduction

Given a homeomorphism $f : M \to M$ acting on a compact metric space (M, d) and an automorphism $\alpha \in \mathrm{Aut}(G)$ of a topological group G, we say that the map $A_\alpha : \mathbb{Z} \times M \to G$ is an *α-twisted cocycle* over f if

$$A_\alpha^{m+n}(x) = A_\alpha^n(f^m(x))\alpha^n(A_\alpha^m(x)) \tag{1}$$

for all $x \in M$ and $m, n \in \mathbb{Z}$.

Two α-twisted cocycles A_α and B_α over f are said to be *α-cohomologous* whenever there exists a *transfer map* $P : M \to G$ satisfying

$$A_\alpha^n(x) = P(f^n(x))B_\alpha^n(x)\alpha^n(P(x))^{-1}$$

for every $x \in M$ and $n \in \mathbb{Z}$. Observe that in the case where $\alpha = \mathrm{Id}$ the notions of α-twisted cocycle and α-cohomology coincide with the 'standard' notions of cocycles and cohomology in dynamical systems [**11**].

Cohomology of twisted cocycles appears naturally in many problems in dynamics. For instance, any map $A : M \to G$ naturally generates an α-twisted cocycle A_α over f (see § 2.2). In this case, we can consider the *twisted skew-product* $F_{A,\alpha} : M \times G \to M \times G$ given by $F_{A,\alpha}(x, g) = (f(x), A(x)\alpha(g))$. Now, the problem of determining whether two twisted skew-products $F_{A,\alpha}$ and $F_{B,\alpha}$ are conjugated reduces to the problem of studying whether A_α and B_α are α-cohomologous. In fact, the map $U(x,g)=(x,P(x)g)$ conjugates $F_{A,\alpha}$ and $F_{B,\alpha}$ precisely when P is a transfer map for A_α and B_α. This observation applied to the case when $G=GL(d,\mathbb{R})$ is what motivates much of this note. Other applications also appear in the study of regularity of the transfer map for non-abelian cocycles over Anosov actions [**17**], in applications to the differentiable rigidity of Anosov diffeomorphisms [**8**] and in the study of local rigidity of higher-rank abelian partially hyperbolic actions [**7**]. For more applications, refer to § 4.6 of [**11**] and to [**12**].

In the present paper we are interested in describing necessary and sufficient conditions under which two α-twisted cocycles A_α and B_α are α-cohomologous whenever f is a hyperbolic map. In the case where $\alpha=\mathrm{Id}$ and G is an abelian group admitting a bi-invariant metric, a first criterion was given by Livšic in his seminal papers [**13**] and [**14**]. More precisely, he proved that A_{Id} and B_{Id} are Id-cohomologous if and only if

$$A^n_{\mathrm{Id}}(p)=B^n_{\mathrm{Id}}(p) \quad \text{for every } p\in \mathrm{Fix}(f^n).$$

Because of its many applications, in the case where $\alpha=\mathrm{Id}$, this criterion has been extended by many authors to many different settings, usually eliminating the assumptions that G is abelian and admits a bi-invariant metric; see, for instance, [**1**, **3**, **5**, **9**, **18**–**20**].

The case where α is not the identity, on the other hand, despite its many applications, has received much less attention. To the best of the author's knowledge, the best result in this setting is a theorem by Walkden [**22**], who obtained an analogous result to the original Livšic's theorem under the assumptions that G is a connected Lie group admitting a bi-invariant metric* and the automorphism α satisfies some 'growth' conditions. The objective of this paper is to extend the results of [**22**] to the case when $G=GL(d,\mathbb{R})$.

1.1. Main results

The main result of this work is the following theorem (see § 2 for precise definitions).

Theorem 1.1. *Let $f:M\to M$ be a Lipschitz continuous transitive hyperbolic homeomorphism on a compact metric space (M,d), let $A,B:M\to GL(d,\mathbb{R})$ be two ν-Hölder continuous maps and let $\alpha\in Aut(GL(d,\mathbb{R}))$ be an automorphism of $GL(d,\mathbb{R})$. Suppose that the twisted cocycles A_α and B_α are fiber-bunched. Moreover, suppose that they satisfy the periodic orbit condition*

$$A^n_\alpha(p)=B^n_\alpha(p), \quad \forall n\in\mathbb{Z},\ \forall p\in Fix(f^n). \tag{2}$$

Then, there exists a ν-Hölder continuous map $P:M\to GL(d,\mathbb{R})$ such that

$$A^n_\alpha(x)=P(f^n(x))B^n_\alpha(x)\alpha^n\left(P(x)\right)^{-1}, \quad \forall x\in M,\ \forall n\in\mathbb{Z}. \tag{3}$$

This result consists of a generalization of the main results of [**3**] and [**19**] to the case of twisted cocycles. In fact, the main results of those works can be obtained as corollaries of the previous one by taking $\alpha=\mathrm{Id}$. Moreover, this result generalizes the main result of

* In the case when $G=GL(d,\mathbb{R})$ the existence of the bi-invariant metric can be replaced by a *bounded distortion condition*. See comments after Theorem 1.1

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[22] in the case where $G = GL(d,\mathbb{R})$. Indeed, it was observed in [22, Remark 3.4] that in this case, instead asking for the group to admit a bi-invariant metric (recall that $GL(d,\mathbb{R})$ does not admit such a metric), one can assume some *bounded distortion condition* in the twisted cocycles. Roughly speaking, this condition asks for each of the terms on the left-hand side of (5) to be uniformly bounded. In particular, such a condition is much more restrictive than our fiber-bunching assumption.

One can easily see that the α-cohomology relation is an equivalence one over the space of α-twisted cocycles. In particular, as a simple consequence of the previous result, one can obtain a complete characterization of the cohomology classes in the twisted scenario in terms of the periodic data.

Corollary 1.2. *Let f, A, B and α be as in Theorem 1.1 and, moreover, suppose that A or B satisfies (5), (6) and (7) with $7\rho + 2\theta < \nu\lambda$. Then, there exists a ν-Hölder continuous map $Q : M \to GL(d,\mathbb{R})$ such that*

$$A^n_\alpha(p) = Q(p)B^n_\alpha(p)\alpha^n\left(Q(p)\right)^{-1}$$

for every $n \in \mathbb{Z}$ and $p \in \mathrm{Fix}(f^n)$ if and only if there exists a ν-Hölder continuous map $P : M \to GL(d,\mathbb{R})$ such that

$$A^n_\alpha(x) = P\big(f^n(x)\big)B^n_\alpha(x)\alpha^n\left(P(x)\right)^{-1}, \quad \forall x \in M,\ \forall n \in \mathbb{Z}.$$

Proof. One implication is trivial. Let us deduce the other one. Assume that B satisfies (5), (6) and (7) with $7\rho + 2\theta < \nu\lambda$. The case where A satisfies it is similar. Let us consider

$$\tilde{B}^n_\alpha(x) = Q(f^n(x))B^n_\alpha(x)\alpha^n(Q(x))^{-1}.$$

We start by observing that $(\tilde{B}^n_\alpha)_{n\in\mathbb{Z}}$ is an α-twisted cocycle over f. Indeed,

$$\begin{aligned}
\tilde{B}^{n+m}_\alpha(x) &= Q(f^{n+m}(x))B^{n+m}_\alpha(x)\alpha^{m+n}(Q(x))^{-1} \\
&= Q(f^{n+m}(x))B^n_\alpha(f^m(x))\alpha^n\left(B^m_\alpha(x)\right)\alpha^{m+n}(Q(x))^{-1} \\
&= Q(f^{n+m}(x))B^n_\alpha(f^m(x))\alpha^n\left(Q(f^m(x))^{-1}Q(f^m(x))\right)\alpha^n \\
&\quad \times \left(B^m_\alpha(x)\right)\alpha^{m+n}(Q(x))^{-1} \\
&= Q(f^{n+m}(x))B^n_\alpha(f^m(x))\alpha^n\left(Q(f^m(x))\right)^{-1}\alpha^n\left(Q(f^m(x))B^m_\alpha(x)\alpha^m(Q(x))^{-1}\right) \\
&= \tilde{B}^n_\alpha(f^m(x))\alpha^n\big(\tilde{B}^m_\alpha(x)\big).
\end{aligned}$$

Moreover, our hypothesis on B ensures that $\tilde{B}$ is fiber-bunched in the sense of § 2.5. Thus, since $A^n_\alpha(p) = \tilde{B}^n_\alpha(p)$ for every $p \in \mathrm{Fix}(f^n)$, the result follows by applying our main result to these two cocycles. □

Observe that the previous proof gives us no apparent 'meaningful' relation between the maps P and Q given in the statement of Corollary 1.2.

In order to prove our main result, we follow the approaches of [3], which in turn was inspired by [18, 20], and [19, 22]. The main idea consists of constructing *invariant holonomies*, which are a family of linear maps with good properties (see Proposition 3.1),

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and then using this family to explicitly construct the transfer map on a dense set under the additional assumption that f admits a fixed point. The next step consists of showing that, restricted to this dense set, the transfer map is ν-Hölder continuous and then extending it to the whole space. Finally, we explain how to eliminate the hypothesis of existence of a fixed point for f. The main difference between this proof and that of [**3**] is that the estimates here are much more involved owing to the presence of twisting. The overall strategy is the same. In particular, the last step of the proof is the same, mutatis mutandis, as in the untwisted case and so we only indicate how to proceed.

Throughout the paper we use the letter C as a generic notation for a positive constant that may change from line to line. Whenever necessary, we will explicitly mention the parameters on which C depends.

2. Preliminaries

Let (M, d) be a compact metric space, $f : M \to M$ a homeomorphism, G a Lie group and $A : M \to G$ a ν-Hölder continuous map.

2.1. Hyperbolic homeomorphisms

Given any $x \in M$ and $\varepsilon > 0$, define the *local stable* and *unstable sets* of x with respect to f by

$$W^s_\varepsilon(x) := \{y \in M : d(f^n(x), f^n(y)) \leq \varepsilon,\ \forall n \geq 0\},$$
$$W^u_\varepsilon(x) := \{y \in M : d(f^n(x), f^n(y)) \leq \varepsilon,\ \forall n \leq 0\},$$

respectively. Following [**2**], we introduce a definition.

Definition 2.1. A homeomorphism $f : M \to M$ is said to be *hyperbolic with local product structure* (or just *hyperbolic* for short) whenever there exist constants $C, \varepsilon, \lambda, \tau > 0$ such that the following conditions are satisfied:

- $d(f^n(y_1), f^n(y_2)) \leq Ce^{-\lambda n} d(y_1, y_2), \quad \forall x \in M, \forall y_1, y_2 \in W^s_\varepsilon(x), \forall n \geq 0;$
- $d(f^{-n}(y_1), f^{-n}(y_2)) \leq Ce^{-\lambda n} d(y_1, y_2), \quad \forall x \in M, \forall y_1, y_2 \in W^u_\varepsilon(x), \forall n \geq 0;$
- if $d(x, y) \leq \tau$, then $W^s_\varepsilon(x)$ and $W^u_\varepsilon(y)$ intersect in a unique point which is denoted by $[x, y]$ and depends continuously on x and y.

For such homeomorphisms, one can define the *stable* and *unstable sets* by

$$W^s(x) := \bigcup_{n\geq 0} f^{-n}\big(W^s_\varepsilon(f^n(x))\big) \quad \text{and} \quad W^u(x) := \bigcup_{n\geq 0} f^{n}\big(W^u_\varepsilon(f^{-n}(x))\big),$$

respectively.

Notice that subshifts of finite type and basic sets of axiom A diffeomorphisms are particular examples of hyperbolic homeomorphisms with local product structure (see, for instance, [**16**, Chapter IV,§ 9] for details).

2.2. Twisted cocycles

Let $\text{Aut}(G)$ denote the group of automorphisms of G, and let $\alpha \in \text{Aut}(G)$. A map $A_\alpha : \mathbb{Z} \times M \to G$ is said to be an *α-twisted cocycle* over f if

$$A_\alpha^{m+n}(x) = A_\alpha^n(f^m(x))\alpha^n(A_\alpha^m(x))$$

for all $x \in M$ and $m, n \in \mathbb{Z}$. With any map $A : M \to G$ we may associate an α-twisted cocycle over f by

$$A_\alpha^n(x) = \begin{cases} A(f^{n-1}(x))\alpha(A(f^{n-2}(x)))\ldots\alpha^{n-2}(A(f(x)))\alpha^{n-1}(A(x)) & \text{if } n > 0 \\ \text{Id} & \text{if } n = 0 \\ \alpha^n(A_\alpha^{-n}(f^n(x))^{-1}) & \text{if } n < 0 \end{cases}$$

for all $x \in M$. In this case we say that A *generates* the α-twisted cocycle A_α over f. Reciprocally, every α-twisted cocycle A_α is generated by $A = A_\alpha^1$. In what follows, for the sake of simplicity, we write A_α instead of A_α^1.

2.3. Cohomology of α-twisted cocycles

Given a ν-Hölder continuous map $B : M \to G$, we say that the α-twisted cocycles A_α and B_α generated by A and B over f, respectively, are *α-cohomologous* if there exists a ν-Hölder continuous map $P : M \to G$ such that

$$A_\alpha(x) = P(f(x))B_\alpha(x)\alpha(P(x))^{-1}$$

for every $x \in M$. It is easy to verify that this equation is equivalent to

$$A_\alpha^n(x) = P(f^n(x))B_\alpha^n(x)\alpha^n(P(x))^{-1}$$

for every $x \in M$ and $n \in \mathbb{Z}$. As already observed in the introduction, whenever $\alpha = \text{Id}$ we recover the usual notions of cocycles and cohomology [**3**, **19**].

2.4. Linear α-twisted cocycles

From now on we restrict ourselves to the case where $G = GL(d, \mathbb{R})$. In particular, by $A : M \to GL(d, \mathbb{R})$ being ν-Hölder continuous we mean that there exists a constant $C > 0$ such that

$$\| A(x) - A(y) \| \leq Cd(x, y)^\nu \tag{4}$$

for all $x, y \in M$, where $\| A \|$ denotes the operator norm of a matrix A, that is, $\| A \| = \sup\{\| Av \| / \| v \|;\ \| v \| \neq 0\}$.

Observe that examples of automorphisms of $GL(d, \mathbb{R})$ include $\alpha_L : GL(d, \mathbb{R}) \to GL(d, \mathbb{R})$ and $\alpha_i : GL(d, \mathbb{R}) \to GL(d, \mathbb{R})$ given by

$$\alpha_L(A) = LAL^{-1} \quad \text{and} \quad \alpha_i(A) = \left(A^T\right)^{-1},$$

where $L \in GL(d, \mathbb{R})$ is a fixed matrix and A^T denotes the transpose of A. For more on $\text{Aut}(GL(d, \mathbb{R}))$, refer to [**15**].

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2.5. Fiber-bunched α-twisted cocycles

We say that the α-twisted cocycle A_α generated by A over f is *fiber-bunched* if there are constants $C > 0$ and $\rho, \theta > 0$ with $5\rho + 2\theta < \nu\lambda$, where ν and λ are as in (4) and Definition 2.1, respectively, such that for every $n \in \mathbb{Z}$:

(i)
$$\|\alpha^{-n}(A^n_\alpha(x))\| \|\alpha^{-n}(A^n_\alpha(x)^{-1})\| < Ce^{\theta|n|} \tag{5}$$
for every $x \in M$;

(ii)
$$\|\alpha^n(T_1) - \alpha^n(T_2)\| \leq Ce^{\rho|n|}\|T_1 - T_2\| \tag{6}$$
for every $T_1, T_2 \in GL(d, \mathbb{R})$;

(iii)
$$\|\alpha^n(T)\| \leq Ce^{\rho|n|}\|T\| \tag{7}$$
for every $T \in GL(d, \mathbb{R})$.

Once again, it is easy to see that by taking $\alpha = \text{Id}$ we recover the 'standard' notion of fiber-bunched cocycles used, for instance, in [**2**, **3**, **6**, **19**].

Observe that if A and α are sufficiently close to the identity then the fiber-bunching condition is automatically satisfied. Other examples of α-twisted cocycles with $\alpha \neq \text{Id}$ satisfying the fiber-bunching condition are given, for instance, by taking $\alpha = \alpha_L$ as in the previous subsection with L close enough to Id and assuming the cocycle (A, f) is fiber-bunched in the standard sense of [**6**, **21**]. It is also worth noting that this fiber-bunching notion is related to the partial hyperbolicity of the map $F_{A,\alpha} : M \times G \to M \times G$ given by $F_{A,\alpha}(x, g) = (f(x), A(x)\alpha(g))$. Indeed, condition (5) says that the rates of expansion and contraction given by $F_{A,\alpha}$ along the G-direction are 'dominated' by the rates of expansion and contraction along the M-direction.

3. Invariant holonomies

In this section we introduce the notion of *invariant holonomies* for twisted cocycles. This is done by generalizing the notion introduced in [**6**, **21**] in the untwisted case. As in the untwisted scenario, these objects are fundamental to our proof.

Proposition 3.1. *Let $f : M \to M$ be a hyperbolic homeomorphism on a compact metric space (M, d), let $A : M \to GL(d, \mathbb{R})$ be a ν-Hölder map and let $\alpha \in Aut(G)$. Suppose that the twisted cocycle A_α generated by A and α over f is fiber-bunched. Then there exists a constant $C = C(A, \alpha, f) > 0$ such that, for any $x \in M$ and any $y, z \in W^s(x)$, the limit*
$$H^{s,A,\alpha}_{yz} := \lim_{n \to +\infty} \alpha^{-n}\big(A^n_\alpha(z)^{-1}A^n_\alpha(y)\big)$$
exists, and
$$\|H^{s,A,\alpha}_{yz} - Id\| \leq Cd(y, z)^\nu, \tag{8}$$
whenever $y, z \in W^s_\varepsilon(x)$, where the constant $\varepsilon > 0$ associated with f is given by Definition 2.1.

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On the other hand, if $y, z \in W^u(x)$, we can analogously define

$$H_{yz}^{u,A,\alpha} := \lim_{n\to+\infty} \alpha^n \left(A_\alpha^{-n}(z)^{-1} A_\alpha^{-n}(y)\right),$$

and the very same Hölder estimate holds for these maps when $y, z \in W_\varepsilon^u(x)$.

Finally, for every $x \in M$ and $* \in \{s, u\}$, it holds that

$$H_{yz}^{*,A,\alpha} = H_{xz}^{*,A,\alpha} H_{yx}^{*,A,\alpha},$$

and

$$H_{f^m(y)f^m(z)}^{*,A,\alpha} = A_\alpha^m(z)\alpha^m(H_{yz}^{*,A,\alpha})A_\alpha^m(y)^{-1},$$

for every $y, z \in W^*(x)$ and $m \in \mathbb{Z}$.

Definition 3.2. The maps $H^{s,A,\alpha}$ and $H^{u,A,\alpha}$ given by Proposition 3.1 are called *stable* and *unstable holonomies*, respectively.

It is worth noting that the main ideas beyond this concept, although not under this name, were present in [**22**] (see also [**18**, **20**] for the case $\alpha = Id$). On the other hand, the *construction* of these holonomies in that setting was greatly simplified owing to the existence of a bi-invariant metric. Similarly, the proof in the case $\alpha = \mathrm{Id}$ was also much simpler compared with ours, owing to the lack of twisting (see, for instance, Proposition 2.5 of [**21**]).

We will prove only the assertions about $H_{yz}^{s,A,\alpha}$ since the ones about $H_{yz}^{u,A,\alpha}$ are similar. We start with the following proposition.

Proposition 3.3. *Let $\delta > 0$ be such that $5\rho + 2\theta + \delta < \lambda\nu$. Then, there exists $C = C(A, \alpha, f, \delta) > 0$ such that*

$$\|\alpha^{-n}(A_\alpha^n(y))\| \cdot \|\alpha^{-n}(A_\alpha^n(x)^{-1})\| \leq Ce^{(4\rho+2\theta+\delta)n}$$

for all $y \in W_\varepsilon^s(x)$, $x \in M$ and $n \geq 0$.

In order to prove this proposition, we need a couple of auxiliary results.

Lemma 3.4. *Fix $x \in M$. There exists a family of norms $(\|\cdot\|_k)_{k\in\mathbb{N}}$ such that*

$$\frac{\max\left\{\|\alpha^{-k}(A(f^{k-1}(x)))v\|_k;\ \|v\|_{k-1} = 1\right\}}{\min\left\{\|\alpha^{-k}(A(f^{k-1}(x)))w\|_k;\ \|w\|_{k-1} = 1\right\}} \leq e^{2\theta+\delta}.$$

Moreover, there exists $C > 0$ depending only on A, α, f and δ so that

$$\|\cdot\| \leq \|\cdot\|_k \leq Ce^{2\rho k}\|\cdot\| \text{ for every } k \in \mathbb{N}. \tag{9}$$

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Proof. Fix $u_0 \in \mathbb{R}^d$ with $\|u_0\| = 1$, and for any $k \in \mathbb{Z}$ set

$$u_k = \frac{\alpha^{-k}(A(f^{k-1}(x)))u_{k-1}}{\|\alpha^{-k}(A(f^{k-1}(x)))u_{k-1}\|}.$$

Now, given $v \in \mathbb{R}^d$, define

$$\|v\|_k^2 = \sum_{m\in\mathbb{Z}} \frac{\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))v\|^2}{\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))u_k\|^2 \cdot e^{(2\theta+\delta)|m|}}. \tag{10}$$

We start by observing that from (7),

$$\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))\frac{v}{\|v\|}\|\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))u_k\|^{-1}$$

is smaller than or equal to

$$Ce^{2\rho k}\|\alpha^{-m}(A_\alpha^m(f^k(x)))\frac{v}{\|v\|}\|\|\alpha^{-m}(A_\alpha^m(f^k(x)))u_k\|^{-1}.$$

Thus, using our hypothesis (5) and the fact that $\|T\|^{-1} \leq \|T^{-1}\|$ for any $T \in GL(d,\mathbb{R})$, we get that the last quantity is smaller than or equal to $C^2e^{2\rho k}e^{\theta|m|}$. In particular,

$$\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))v\|\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))u_k\|^{-1} \leq C^2e^{2\rho k}e^{\theta|m|}\|v\|$$

for every $m \in \mathbb{Z}$ and thus

$$\|v\|_k^2 \leq \sum_{m\in\mathbb{Z}} \frac{\left(C^2e^{2\rho k}e^{\theta|m|}\|v\|\right)^2}{e^{(2\theta+\delta)|m|}} \leq \tilde{C}e^{4\rho k}\|v\|^2,$$

where $\tilde{C} = \sum_{m\in\mathbb{Z}} C^4e^{-\delta|m|} < \infty$. Consequently, series (10) converges and $\|\cdot\|_k$ is well defined. Moreover,

$$\|v\|_k \leq Ce^{2\rho k}\|v\|$$

for any $v \in \mathbb{R}^d$ and some $C > 0$ independent of k and x. Furthermore, recalling that $\alpha^{-k}(\mathrm{Id}) = \mathrm{Id}$ and $\|u_k\| = 1$, looking at the term of (10) when $m = 0$ it follows that $\|v\| \leq \|v\|_k$ for every $v \in \mathbb{R}^d$; this, combined with the previous observations, completes the proof of (9). In order to prove the other claim, we observe that

$$\begin{aligned}
\|\alpha^{-k}(A(f^{k-1}(x)))v\|_k^2 &= \sum_{m\in\mathbb{Z}} \frac{\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))\alpha^{-k}(A(f^{k-1}(x)))v\|^2}{\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))u_k\|^2 \cdot e^{(2\theta+\delta)|m|}} \\
&= \sum_{m\in\mathbb{Z}} \frac{\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))\alpha^{-k}(A(f^{k-1}(x)))v\|^2}{\|\alpha^{-m-k}(A_\alpha^m(f^k(x)))\left(\frac{\alpha^{-k}(A(f^{k-1}(x)))u_{k-1}}{\|\alpha^{-k}(A(f^{k-1}(x)))u_{k-1}\|}\right)\|^2 \cdot e^{(2\theta+\delta)|m|}} \\
&= \sum_{m\in\mathbb{Z}} \frac{\|\alpha^{-m-k}(A_\alpha^{m+1}(f^{k-1}(x)))v\|^2\|\alpha^{-k}(A(f^{k-1}(x)))u_{k-1}\|^2}{\|\alpha^{-m-k}(A_\alpha^{m+1}(f^{k-1}(x)))u_{k-1}\|^2 \cdot e^{(2\theta+\delta)|m|}} \\
&= \|\alpha^{-k}(A(f^{k-1}(x)))u_{k-1}\|^2 \cdot S(v),
\end{aligned}$$

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where

$$S(v) := \sum_{m\in\mathbb{Z}} \frac{\|\alpha^{-(m+1)-(k-1)}(A_\alpha^{m+1}(f^{k-1}(x)))v\|^2}{\|\alpha^{-(m+1)-(k-1)}(A_\alpha^{m+1}(f^{k-1}(x)))u_{k-1}\|^2 \cdot e^{(2\theta+\delta)|m|}}.$$

Now, since $|m+1| \geq |m|-1$, we get that $S(v) \leq e^{2\theta+\delta}\|v\|_{k-1}^2$. Similarly, since $|m+1| \leq |m|+1$, we get that $S(v) \geq e^{-(2\theta+\delta)}\|v\|_{k-1}^2$. Combining these facts with the previous observations, it follows that

$$\begin{aligned} e^{-(\theta+(\delta/2))}\|\alpha^{-k}(A(f^{k-1}(x)))u_{k-1}\|\|v\|_{k-1} &\leq \|\alpha^{-k}(A(f^{k-1}(x)))v\|_k \\ &\leq e^{\theta+(\delta/2)}\|\alpha^{-k}(A(f^{k-1}(x)))u_{k-1}\|\|v\|_{k-1} \end{aligned}$$

for any $v \in \mathbb{R}^d$. Thus, taking $v, w \in \mathbb{R}^d$ so that $\|v\|_{k-1} = \|w\|_{k-1} = 1$, it follows that

$$\begin{aligned} e^{-(2\theta+\delta)}\|\alpha^{-k}(A(f^{k-1}(x)))v\|_k &\leq \|\alpha^{-k}(A(f^{k-1}(x)))w\|_k \\ &\leq e^{2\theta+\delta}\|\alpha^{-k}(A(f^{k-1}(x)))v\|_k. \end{aligned}$$

Consequently,

$$\frac{\max\left\{\|\alpha^{-k}(A(f^{k-1}(x)))v\|_k;\ \|v\|_{k-1}=1\right\}}{\min\left\{\|\alpha^{-k}(A(f^{k-1}(x)))w\|_k;\ \|w\|_{k-1}=1\right\}} \leq e^{2\theta+\delta}$$

as claimed. □

Thus, defining the *k-norm* of an operator $T \in GL(d,\mathbb{R})$ with respect to the family of norms $(\|\cdot\|_k)_{k\in\mathbb{N}}$ by

$$\|T\|_k = \sup_{v\neq 0} \frac{\|Tv\|_k}{\|v\|_{k-1}},$$

a corollary follows easily from the previous lemma.

Corollary 3.5. *For any* $k \in \mathbb{N}$,

$$\|\alpha^{-k}(A(f^{k-1}(x)))^{-1}\|_k\|\alpha^{-k}(A(f^{k-1}(x)))\|_k \leq e^{2\theta+\delta}.$$

Proof of Proposition 3.3. Let $(\|\cdot\|_k)_{k\in\mathbb{Z}}$ be the family of norms given by Lemma 3.4. Recalling (6), (7) and (9), we start by observing that

$$\begin{aligned} \frac{\|\alpha^{-k}(A(f^{k-1}(y)))\|_k}{\|\alpha^{-k}(A(f^{k-1}(x)))\|_k} &\leq 1 + \frac{\left|\|\alpha^{-k}(A(f^{k-1}(y)))\|_k - \|\alpha^{-k}(A(f^{k-1}(x)))\|_k\right|}{\|\alpha^{-k}(A(f^{k-1}(x)))\|_k} \\ &\leq 1 + \frac{\|\alpha^{-k}(A(f^{k-1}(y))) - \alpha^{-k}(A(f^{k-1}(x)))\|_k}{\|\alpha^{-k}(A(f^{k-1}(x)))\|_k} \\ &\leq 1 + \frac{Ce^{2\rho k}\|\alpha^{-k}(A(f^{k-1}(y))) - \alpha^{-k}(A(f^{k-1}(x)))\|}{\|\alpha^{-k}(A(f^{k-1}(x)))\|} \\ &\leq 1 + \frac{Ce^{4\rho k}\|A(f^{k-1}(y)) - A(f^{k-1}(x))\|}{\|A(f^{k-1}(x))\|}. \end{aligned}$$

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Thus, since A is ν-Hölder and M is compact, and recalling Definition 2.1, it follows that

$$\frac{\|\alpha^{-k}(A(f^{k-1}(y)))\|_k}{\|\alpha^{-k}(A(f^{k-1}(x)))\|_k} \leq 1 + Ce^{(4\rho-\lambda\nu)k} d(x,y)^\nu.$$

Now, Corollary 3.5 gives us that for any $j \in \mathbb{N}$,

$$\|\alpha^{-j}(A(f^{j-1}(x)))^{-1}\|_j \leq \frac{e^{2\theta+\delta}}{\|\alpha^{-j}(A(f^{j-1}(x)))\|_j}.$$

Combining these two observations with the fact that

$$\begin{aligned}\|\alpha^{-k}(A^k_\alpha(x))^{-1}\|_k &= \|\alpha^{-1}(A(x))^{-1}\alpha^{-2}(A(f(x)))^{-1}\ldots\alpha^{-k}(A(f^{k-1}(x)))^{-1}\|_k \\ &\leq \|\alpha^{-1}(A(x))^{-1}\|_1 \|\alpha^{-2}(A(f(x)))^{-1}\|_2 \ldots \|\alpha^{-k}(A(f^{k-1}(x)))^{-1}\|_k,\end{aligned}$$

and similarly

$$\|\alpha^{-k}(A^k_\alpha(y))\|_k \leq \|\alpha^{-k}(A(f^{k-1}(y)))\|_k \ldots \|\alpha^{-2}(A(f(x)))\|_2 \|\alpha^{-1}(A(x))\|_1,$$

it follows that

$$\begin{aligned}\|\alpha^{-k}(A^k_\alpha(x))^{-1}\|_k \|\alpha^{-k}(A^k_\alpha(y))\|_k &\leq \frac{\|\alpha^{-1}(A(y))\|_1}{\|\alpha^{-1}(A(x))\|_1} e^{2\theta+\delta} \ldots \frac{\|\alpha^{-k}(A(f^{k-1}(y)))\|_k}{\|\alpha^{-k}(A(f^{k-1}(x)))\|_k} e^{2\theta+\delta} \\ &\leq e^{(2\theta+\delta)k} \prod_{j=1}^{k} (1 + Ce^{(4\rho-\lambda\nu)j} d(x,y)^\nu) \\ &\leq \tilde{C} e^{(2\theta+\delta)k},\end{aligned}$$

where $\tilde{C} = \prod_{j=1}^\infty (1 + CDe^{(4\rho-\lambda\nu)j}) < \infty$ and $D = \sup_{x,y\in M} d(x,y)^\nu$ (recall that $4\rho - \lambda\nu < 0$). Thus, since by (9) we have that $\|T\| \leq Ce^{2\rho k}\|T\|_k$ for every $T \in GL(d,\mathbb{R})$, it follows that

$$\|\alpha^{-k}(A^k_\alpha(x))^{-1}\| \|\alpha^{-k}(A^k_\alpha(y))\| \leq Ce^{(4\rho+2\theta+\delta)k}$$

for some constant C independent of x and y as claimed. □

We are now ready to prove the main proposition of this section.

Proof of Proposition 3.1. By taking forward iterates, we can assume that $y, z \in W^u_{\varepsilon/2}(x)$. In particular, $z \in W^s_\varepsilon(y)$. We are going to show that the sequence $(\alpha^{-n}(A^n_\alpha(z)^{-1}A^n_\alpha(y)))_n$ is a Cauchy sequence. In order to do this, we start by observing

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that for every $n \in \mathbb{N}$,

$$\|\alpha^{-(n+1)}\left(A_\alpha^{n+1}(z)^{-1}A_\alpha^{n+1}(y)\right) - \alpha^{-n}\left(A_\alpha^n(z)^{-1}A_\alpha^n(y)\right)\|$$

is equal to

$$\begin{aligned}&\|\alpha^{-n}\left(A_\alpha^n(z)^{-1}\right)\alpha^{-(n+1)}\left(A(f^n(z))^{-1}A(f^n(y))\right)\alpha^{-n}\left(A_\alpha^n(y)\right)\\ &\quad - \alpha^{-n}\left(A_\alpha^n(z)^{-1}\right)\alpha^{-n}\left(A_\alpha^n(y)\right)\|,\end{aligned}$$

which is smaller than or equal to

$$\|\alpha^{-n}\left(A_\alpha^n(z)^{-1}\right)\|\|\alpha^{-n}\left(A_\alpha^n(y)\right)\|\|\alpha^{-(n+1)}\left(A(f^n(z))^{-1}A(f^n(y))\right) - \mathrm{Id}\|.$$

From Proposition 3.3 it follows that the previous quantity is smaller than or equal to

$$Ce^{(4\rho+2\theta+\delta)n}\|\alpha^{-(n+1)}\left(A(f^n(z))^{-1}A(f^n(y))\right) - \mathrm{Id}\|.$$

Thus, since

$$\begin{aligned}&\|\alpha^{-(n+1)}\left(A(f^n(z))^{-1}A(f^n(y))\right) - \mathrm{Id}\|\\ &\quad = \|\alpha^{-(n+1)}\left(A(f^n(z))^{-1}A(f^n(y))\right) - \alpha^{-(n+1)}(\mathrm{Id})\|\\ &\quad \le Ce^{\rho(n+1)}\|A(f^n(z))^{-1}A(f^n(y)) - \mathrm{Id}\|\\ &\quad \le Ce^{\rho(n+1)}e^{-\nu\lambda n}d(z,y)^\nu\\ &\quad = Ce^{\rho}e^{(\rho-\nu\lambda)n}d(z,y)^\nu,\end{aligned}$$

we get that

$$\begin{aligned}&\|\alpha^{-(n+1)}\left(A_\alpha^{n+1}(z)^{-1}A_\alpha^{n+1}(y)\right) - \alpha^{-n}\left(A_\alpha^n(z)^{-1}A_\alpha^n(y)\right)\|\\ &\quad \le Ce^{(4\rho+2\theta+\delta)n}Ce^{\rho}e^{(\rho-\nu\lambda)n}d(z,y)^\nu\\ &\quad = Ce^{(5\rho+2\theta+\delta-\nu\lambda)n}d(z,y)^\nu.\end{aligned}$$

Therefore, since $5\rho+2\theta+\delta-\nu\lambda<0$, we get that the sequence $(\alpha^{-n}(A_\alpha^n(z)^{-1}A_\alpha^n(y)))_n$ is indeed a Cauchy sequence. Consequently,

$$H_{yz}^{s,A,\alpha} = \lim_{n\to+\infty}\alpha^{-n}\left(A_\alpha^n(z)^{-1}A_\alpha^n(y)\right)$$

exists and, moreover,

$$\|H_{yz}^{s,A,\alpha} - \mathrm{Id}\| \le Cd(y,z)^\nu,$$

whenever $y,z \in W_\varepsilon^s(x)$ as claimed.

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To prove the last claim, we start by observing that, on the one hand,

$$\alpha^{-n}\left(A_\alpha^n(z)^{-1}A_\alpha^n(y)\right)\xrightarrow{n\to\infty}H_{yz}^{s,A,\alpha}.$$

On the other hand,

$$\alpha^{-n}\big(A_\alpha^n(z)^{-1}A_\alpha^n(y)\big)$$

is equal to

$$\alpha^{-m}\left(A_\alpha^m(z)^{-1}\right)\alpha^{-m}\big(\alpha^{-(n-m)}\big(A_\alpha^{n-m}(f^m(z))^{-1}A_\alpha^{n-m}(f^m(y))\big)\big)\alpha^{-m}\left(A_\alpha^m(y)\right),$$

which converges to

$$\alpha^{-m}\big(A_\alpha^m(z)^{-1}\big)\alpha^{-m}\big(H_{f^m(y)f^m(z)}^{s,A,\alpha}\big)\alpha^{-m}\big(A_\alpha^m(y)\big)$$

as n goes to infinity. Combining these observations, we conclude that

$$H_{f^m(y)f^m(z)}^{s,A,\alpha}=A_\alpha^m(z)\alpha^m(H_{yz}^{s,A,\alpha})A_\alpha^m(y)^{-1},$$

as claimed. □

Remark 3.6. From the proof of Proposition 3.3, we can easily see that in order to get

$$\|\alpha^{-k}(A_\alpha^k(y))\|\cdot\|\alpha^{-k}(A_\alpha^k(x)^{-1})\|\le Ce^{(4\rho+2\theta+\delta)k}$$

for every $0\le k\le n$ we do not actually need $y\in W_\varepsilon^s(x)$. In fact, we only need x and y to satisfy $d(f^k(x),f^k(y))\le Ce^{-\gamma k}d(x,y)$ for every $0\le k\le n$ and some $\gamma\in(0,\lambda)$ satisfying $4\rho+\delta<\nu\gamma$. In this case, the constant C will depend on A, α, f, δ and γ. We will use this fact in the sequel.

The notions of fiber-bunching and invariant holonomies in the case where $\alpha=\mathrm{Id}$ have important roles in many subareas of dynamical systems and arise naturally in various different contexts (for instance, [**2**–**4**, **6**, **19**, **21**]). Therefore, Proposition 3.1 is also likely to have many applications and can be seen as interesting in itself.

In order to simplify the notation, in what follows, whenever α is fixed and there is no ambiguity, we simply write $H^{*,A}$ instead of $H^{*,A,\alpha}$, for $*=s,u$, to denote the stable and unstable holonomies associated with A_α.

4. Constructing the transfer map

In this section we 'explicitly' build the transfer map. The method we use is similar to that used in [**3**, **5**] and [**19**] in the untwisted setting and in [**22**] in the twisted one: using the invariant holonomies, we define the transfer map on a dense set; prove that restricted to this set, it is Hölder continuous; and then extend it to the closure to obtain the desired result.

Assume there exists $x\in M$ such that $f(x)=x$. For such a point, we write $W(x):=W^s(x)\cap W^u(x)$. We define $P\colon W(x)\to GL(d,\mathbb{R})$ by

$$P(y)=H_{xy}^{s,A}(H_{xy}^{s,B})^{-1}=H_{xy}^{s,A}H_{yx}^{s,B},$$

where $H^{s,A}$ and $H^{s,B}$ are the holonomy maps given by Proposition 3.1 associated with the twisted cocycles A_α and B_α, respectively.

Note that P satisfies

$$A^n_\alpha(y) = P(f^n(y))B^n_\alpha(y)\alpha^n(P(y)^{-1})$$

for every $y \in W(x)$ and every $n \in \mathbb{N}$. Indeed, using $f(x) = x$, Proposition 3.1 and the hypothesis on periodic points (2),

$$\begin{aligned} P(f^n(y)) &= H^{s,A}_{xf^n(y)} H^{s,B}_{f^n(y)x} = H^{s,A}_{f^n(x)f^n(y)} H^{s,B}_{f^n(y)f^n(x)} \\ &= A^n_\alpha(y)\alpha^n(H^{s,A}_{xy})A^n_\alpha(x)^{-1}B^n_\alpha(x)\alpha^n(H^{s,B}_{yx})B^n_\alpha(y)^{-1} \\ &= A^n_\alpha(y)\alpha^n(H^{s,A}_{xy}H^{s,B}_{yx})B^n_\alpha(y)^{-1} \\ &= A^n_\alpha(y)\alpha^n(P(y))B^n_\alpha(y)^{-1}, \end{aligned}$$

and thus

$$A^n_\alpha(y) = P(f^n(y))B^n_\alpha(y)\alpha^n(P(y)^{-1})$$

as claimed.

We will now show that P is ν-Hölder continuous. This will allow us to extend P to $\overline{W(x)} = M$ and thus to get the desired transfer map. The main 'ingredient' in the proof is the next lemma, which says that P can be interchangeably defined using stable or unstable holonomies. Its proof is similar to that of [**3**, Lemma 3]; we only present the full details of it because of its central role in our proof and also because the presence of twist makes some estimates a little more involved than in the untwisted case.

Lemma 4.1. *For every $y \in W(x)$,*

$$P(y) = H^{s,A}_{xy}H^{s,B}_{yx} = H^{u,A}_{xy}H^{u,B}_{yx}.$$

The following classical result (see, for instance, [**10**, Corollary 6.4.17]) will be used in the proof.

Lemma 4.2 (Anosov closing lemma). *Given $\gamma \in (0, \lambda)$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that if $z \in M$ satisfies $d(f^n(z), z) < \varepsilon_0$ then there exists a periodic point $p \in M$ such that $f^n(p) = p$ and*

$$d(f^j(z), f^j(p)) \leq Ce^{-\gamma \min\{j,n-j\}}d(f^n(z), z) \tag{11}$$

for every $j = 0, 1, \ldots, n$.

Proof of Lemma 4.1. Let $\delta > 0$ be such that $5\rho + 2\theta + \delta < \lambda\nu$ and $\gamma \in (0, \lambda)$ such that $5\rho + 2\theta + \delta < \gamma\nu$. Let $C > 0$ and $\varepsilon_0 > 0$ be given by the Anosov closing lemma associated with γ.

Fix an arbitrary point $y \in W(x)$. We begin by noticing that, as $y \in W(x)$, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$d(f^{-n}(y), f^n(y)) \leq Ce^{-\lambda(n-n_0)}.$$

In fact, this follows from the fact that, as $y \in W(x) = W^s(x) \cap W^u(x)$, there exists $n_0 \in \mathbb{N}$ such that $f^{n_0}(y) \in W^s_\varepsilon(x)$ and $f^{-n_0}(y) \in W^u_\varepsilon(x)$, and from the exponential convergence towards x in $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$.

Let $n_1 \geq n_0$ be such that, for all $n \geq n_1$, $d(f^n(y), f^{-n}(y)) < \varepsilon_0$. Thus, by the Anosov closing lemma, for every $n \geq n_1$ there exists a periodic point $p_n \in M$ with $f^{2n}(p_n) = p_n$ such that

$$d(f^j(f^{-n}(p_n)), f^j(f^{-n}(y))) \leq Ce^{-\gamma \min\{j, 2n-j\}} d(f^{-n}(y), f^n(y))$$

for every $j = 0, 1, \ldots, 2n$. Using the periodic orbit condition (2) and noticing that $f^{2n}(f^{-n}(p_n)) = f^{-n}(p_n)$, we get

$$A^{2n}_\alpha(f^{-n}(p_n)) = B^{2n}_\alpha(f^{-n}(p_n)),$$

which can be rewritten as

$$A^n_\alpha(p_n)\alpha^n\left(A^n_\alpha(f^{-n}(p_n))\right) = B^n_\alpha(p_n)\alpha^n\left(B^n_\alpha(f^{-n}(p_n))\right),$$

or, equivalently, as

$$\alpha^n\left(A^n_\alpha(f^{-n}(p_n))B^n_\alpha(f^{-n}(p_n))^{-1}\right) = A^n_\alpha(p_n)^{-1}B^n_\alpha(p_n).$$

Thus, observing that

$$A^n_\alpha(f^{-n}(p_n)) = \alpha^n\left(A^{-n}_\alpha(p_n)^{-1}\right),$$

and similarly

$$B^n_\alpha(f^{-n}(p_n))^{-1} = \alpha^n\left(B^{-n}_\alpha(p_n)\right),$$

we get

$$\alpha^n\left(A^{-n}_\alpha(p_n)^{-1}B^{-n}_\alpha(p_n)\right) = \alpha^{-n}\left(A^n_\alpha(p_n)^{-1}B^n_\alpha(p_n)\right). \tag{12}$$

Now we claim that

$$\|\alpha^{-n}\left(A^n_\alpha(y)^{-1}B^n_\alpha(y)\right) - \alpha^{-n}\left(A^n_\alpha(p_n)^{-1}B^n_\alpha(p_n)\right)\| \xrightarrow{n\to+\infty} 0 \tag{13}$$

and

$$\|\alpha^n\left(A^{-n}_\alpha(y)^{-1}B^{-n}_\alpha(y)\right) - \alpha^n\left(A^{-n}_\alpha(p_n)^{-1}B^{-n}_\alpha(p_n)\right)\| \xrightarrow{n\to+\infty} 0. \tag{14}$$

Consequently, it follows from (12) and our claim that

$$\|\alpha^{-n}\left(A^n_\alpha(y)^{-1}B^n_\alpha(y)\right) - \alpha^n\left(A^{-n}_\alpha(y)^{-1}B^{-n}_\alpha(y)\right)\| \xrightarrow{n\to+\infty} 0.$$

Thus, observing that

$$\alpha^{-n}\left(A^n_\alpha(y)^{-1}B^n_\alpha(y)\right) = \alpha^{-n}\left(A^n_\alpha(y)^{-1}A^n_\alpha(x)B^n_\alpha(x)^{-1}B^n_\alpha(y)\right) \xrightarrow{n\to+\infty} H^{s,A}_{xy}H^{s,B}_{yx},$$

and similarly

$$\alpha^n\left(A^{-n}_\alpha(y)^{-1}B^{-n}_\alpha(y)\right) \xrightarrow{n\to+\infty} H^{u,A}_{xy}H^{u,B}_{yx},$$

we conclude that

$$P(y) = H^{s,A}_{xy}H^{s,B}_{yx} = H^{u,A}_{xy}H^{u,B}_{yx}$$

as we wanted.

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So, in order to complete the proof, it remains to prove our claim. We shall only prove (13) since (14) is completely analogous.

We start observing that

$$\|\alpha^{-n}\left(A^n_\alpha(y)A^n_\alpha(p_n)^{-1}\right)-\mathrm{Id}\|$$

is smaller than or equal to

$$\sum_{j=0}^{n-1}\|\alpha^{-(n-j)}\left(A^{n-j}_\alpha(f^j(y))A^{n-j}_\alpha(f^j(p_n))^{-1}\right) \\ -\alpha^{-(n-j)}\left(A^{n-j-1}_\alpha(f^{j+1}(y))A^{n-j-1}_\alpha(f^{j+1}(p_n))^{-1}\right)\|,$$

which by the cocycle property (1) is equal to

$$\sum_{j=0}^{n-1}\|\alpha^{-(n-j)}\left(A^{n-j-1}_\alpha(f^{j+1}(y))\right)\alpha^{-1}\left(A(f^j(y))A(f^j(p_n))^{-1}\right) \\ \times\alpha^{-(n-j)}\left(A^{n-j-1}_\alpha(f^{j+1}(p_n))^{-1}\right) \\ -\alpha^{-(n-j)}\left(A^{n-j-1}_\alpha(f^{j+1}(y))A^{n-j-1}_\alpha(f^{j+1}(p_n))^{-1}\right)\|.$$

By the property of the norm, this last quantity is smaller than or equal to

$$\sum_{j=0}^{n-1}\|\alpha^{-(n-j)}\left(A^{n-j-1}_\alpha(f^{j+1}(y))\right)\|\|\alpha^{-(n-j)}\left(A^{n-j-1}_\alpha(f^{j+1}(p_n))^{-1}\right)\| \\ \cdot\|\alpha^{-1}\left(A(f^j(y))A(f^j(p_n))^{-1}\right)-\mathrm{Id}\|,$$

which in turn is smaller than or equal to

$$\sum_{j=0}^{n-1}C^2e^{2\rho}\|\alpha^{-(n-j-1)}\left(A^{n-j-1}_\alpha(f^{j+1}(y))\right)\|\|\alpha^{-(n-j-1)}\left(A^{n-j-1}_\alpha(f^{j+1}(p_n))^{-1}\right)\| \\ \cdot\|A(f^j(y))A(f^j(p_n))^{-1}-\mathrm{Id}\|.$$

Now, using Remark 3.6, the fact that A is ν-Hölder continuous, and property (11) given by the Anosov closing lemma, it follows that the previous quantity is smaller than or equal to

$$\sum_{j=0}^{n-1}C^2e^{2\rho}Ce^{(4\rho+2\theta+\delta)(n-j-1)}Ce^{-\gamma\nu(n-j-1)}d(f^{-n}(y),f^n(y))^\nu.$$

Recalling that $d(f^{-n}(y),f^n(y))\leq e^{-\lambda(n-n_0)}$ for every $n\geq n_0$ and $5\rho+2\theta+\delta<\gamma\nu$, it follows that

$$\|\alpha^{-n}\left(A^n_\alpha(y)A^n_\alpha(p_n)^{-1}\right)-\mathrm{Id}\|\leq Ce^{-\lambda\nu(n-n_0)} \tag{15}$$

for every $n\geq n_0$, for some constant $C>0$ independent of n and p_n. Similarly, we can prove that

$$\|\alpha^{-n}\left(B^n_\alpha(p_n)A^n_\alpha(y)^{-1}\right)-\mathrm{Id}\|\leq Ce^{-\lambda\nu(n-n_0)} \tag{16}$$

for every $n\geq n_0$.

Now, as there exists $N > 0$ so that $\|\alpha^{-n}(A^n_\alpha(y)^{-1}A^n_\alpha(x))\| < N$ and $\|\alpha^{-n}(B^n_\alpha(x)^{-1} B^n_\alpha(y))\| < N$ for every sufficiently large n, since these two quantities converge to $H^{s,A}_{xy}$ and $H^{s,B}_{yx}$, respectively; $\|\alpha^{-n}(A^n_\alpha(y)A^n_\alpha(p_n)^{-1})\| < N$ for every sufficiently large n by (15); $A^n_\alpha(x) = B^n_\alpha(x)$ since A_α and B_α satisfy periodic orbit condition (2) and $f(x) = x$; and using (5), (15) and (16), we get

$$\begin{aligned}
&\|\alpha^{-n}\big(A^n_\alpha(y)^{-1}B^n_\alpha(y)\big) - \alpha^{-n}\big(A^n_\alpha(p_n)^{-1}B^n_\alpha(p_n)\big)\| \\
&\quad = \|\alpha^{-n}\big(A^n_\alpha(y)^{-1}A^n_\alpha(x)\big)\alpha^{-n}\big(B^n_\alpha(x)^{-1}B^n_\alpha(y)\big) - \alpha^{-n}\big(A^n_\alpha(p_n)^{-1}B^n_\alpha(p_n)\big)\| \\
&\quad \le N^2\|\mathrm{Id} - \alpha^{-n}\big(A^n_\alpha(x)^{-1}A^n_\alpha(y)A^n_\alpha(p_n)^{-1}B^n_\alpha(p_n)B^n_\alpha(y)^{-1}B^n_\alpha(x)\big)\| \\
&\quad \le N^2\|\alpha^{-n}\big(A^n_\alpha(x)^{-1}\big)\|\|\alpha^{-n}\big(A^n_\alpha(x)\big)\|\|\mathrm{Id} - \alpha^{-n}\big(A^n_\alpha(y)A^n_\alpha(p_n)^{-1}B^n_\alpha(p_n)B^n_\alpha(y)^{-1}\big)\| \\
&\quad \le N^2Ce^{\theta n}\big(\|\alpha^{-n}\big(A^n_\alpha(y)A^n_\alpha(p_n)^{-1}\big)\|\|\alpha^{-n}\big(B^n_\alpha(p_n)B^n_\alpha(y)^{-1}\big) - \mathrm{Id}\| \\
&\qquad + \|\alpha^{-n}\big(A^n_\alpha(y)A^n_\alpha(p_n)^{-1}\big) - \mathrm{Id}\|\big) \\
&\quad \le N^2Ce^{\theta n}\big(NCe^{-\lambda\nu(n-n_0)} + Ce^{-\lambda\nu(n-n_0)}\big) \\
&\quad \le \tilde{C}e^{(\theta-\lambda)n},
\end{aligned}$$

for some $\tilde{C} > 0$ independent of n and p_n and $n \gg 0$. In particular,

$$\|\alpha^{-n}\left(A^n_\alpha(y)^{-1}B^n_\alpha(y)\right) - \alpha^{-n}\left(A^n_\alpha(p_n)^{-1}B^n_\alpha(p_n)\right)\| \xrightarrow{n\to+\infty} 0,$$

proving (13) and thus completing the proof of Lemma 4.1. □

Lemma 4.3. *P is ν-Hölder continuous on $W(x)$.*

Proof. The proof of this fact is analogous to the proof of [**3**, Lemma 4], and so we just summarize the main idea. Full details can be checked in the original work.

From Lemma 4.1 we know that P can be defined using both stable and unstable holonomies. By property (8) we get that restricted to local stable or unstable manifolds, P is Hölder continuous with an uniform Hölder constant. Now, since f has local product structure, points that are τ-close (where τ is as in Definition 2.1) can be connected via local stable and unstable manifolds. Putting all these facts together, we conclude that P is Hölder continuous on balls of radius τ. Finally, using the compactness of M, we conclude that P is Hölder continuous in $W(x)$. □

Therefore, we can extend $P : W(x) \to GL(d, \mathbb{R})$ to the closure of $W(x)$, that is, the whole space M. By continuity, such an extension clearly satisfies the cohomological equation (3), completing the proof of Theorem 1.1 in the case where f has a fixed point.

Now, following the argument given in § 5 of [**3**], mutatis mutandis, we eliminate the additional assumption about the existence of a fixed point for f and conclude the proof of Theorem 1.1.

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