

A REMARK ON TAIL DISTRIBUTIONS OF PARTITION RANK AND CRANK

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Abstract

We examine the tail distributions of integer partition ranks and cranks by investigating tail moments, which are analogous to the positive moments introduced by Andrews *et al.* [‘The odd moments of ranks and cranks’, *J. Combin. Theory Ser. A* **120**(1) (2013), 77–91].

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1. Introduction and statement of results

Dyson’s rank [7] and the crank of Andrews and Garvan [3] were introduced to explain Ramanujan’s integer partition function congruences. A partition of a nonnegative integer n is a nonincreasing sequence of positive integers whose sum is n . The partition function $p(n)$ is defined as the number of partitions of n . The rank of a partition is the largest part minus the number of parts [7] and the crank of a partition is either the largest part, if 1 does not occur as a part, or the difference between the number of parts larger than the number of 1’s and the number of 1’s, if 1 does occur [3]. The generating functions for these two partition statistics are typical examples of mock Jacobi forms and Jacobi forms. Studying how their distributions differ is one of the main themes in the theory of partitions. To this end, Atkin and Garvan [4] introduced moments of partition ranks and cranks and Andrews *et al.* [1] introduced the positive moments. Define $M(m, n)$ (respectively $N(m, n)$) to be the number of partitions of n with crank (respectively rank) m . We define the partial moments

$$M_{k,\ell}^+(n) := \sum_{m=0}^{\ell} m^k M(m, n) \quad \text{and} \quad N_{k,\ell}^+(n) := \sum_{m=0}^{\ell} m^k N(m, n),$$

where the cases with $\ell \geq n - 1$ correspond to the ordinary positive moments. While proving the Andrews–Dyson–Rhoades conjecture on the spt-crank [2], Chen *et al.* [6] showed that

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$$N_{0,\ell}^+(n) \geq M_{0,\ell}^+(n) \tag{1.1}$$

holds for all nonnegative integers ℓ and positive integers n . (They actually proved a slightly different form of (1.1).) On the other hand, Andrews *et al.* proved that

$$M_{k,n-1}^+(n) > N_{k,n-1}^+(n) \tag{1.2}$$

for all positive integers k and n . Taken together, (1.1) and (1.2) imply that the tail distribution of crank is thicker than that of rank as $\sum_m N(m, n) = \sum_m M(m, n) = p(n)$. Therefore, we can expect for a fixed ℓ ,

$$N_{1,\ell}^+(n) > M_{1,\ell}^+(n)$$

holds for large enough n . However, as the first crank moment is always larger than the rank moment, the above inequality cannot hold for small integers n . The situation is different at the tail, and the goal of this note is a closer investigation of the tail distributions of ranks and cranks. To this end, we now define tail moments for nonnegative integers k and r by

$$M_{k,r}^t(n) := \sum_{m \geq r} m^k M(m, n) \quad \text{and} \quad N_{k,r}^t(n) := \sum_{m \geq r} m^k N(m, n).$$

If $r = 0$, they reduce to the usual positive moments. Our first result is the following theorem.

THEOREM 1.1. *For a positive integer r and all integers $n \geq r$,*

$$M_{1,r}^t(n) > N_{1,r}^t(n).$$

REMARK 1.2. We expect that the same inequality holds for the higher moments, but as our motivation is comparing tail distributions of crank and rank, we will be content with the first moment here.

Andrews *et al.* [1] proved that $\text{ospt}(n) = M_{1,0}^t(n) - N_{1,0}^t(n)$ counts certain kinds of strings along the partitions of n . In the light of the significance of the $\text{ospt}(n)$ function and Theorem 1.1, it is natural to study $\text{ospt}_r^t(n) = M_{1,r}^t(n) - N_{1,r}^t(n)$, which also illuminates how the tail distributions of cranks and ranks differ.

THEOREM 1.3. *For a nonnegative integer r ,*

$$\text{ospt}_r^t(n) \sim \left(\frac{1}{4} - \frac{\pi^2}{96n} r(r-1) \right) p(n).$$

2. Proofs of results

We first note that for a nonnegative integer m ,

$$\begin{aligned} \sum_{n=0}^{\infty} M(m, n)q^n &= \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2+mn} (1 - q^n), \\ \sum_{n=0}^{\infty} N(m, n)q^n &= \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+mn} (1 - q^n), \end{aligned}$$

where $(q)_\infty = \prod_{n=1}^\infty (1 - q^n)$. Therefore, we deduce that

$$\begin{aligned} C_{1,r}^t(q) &= \sum_{n \geq 0} M_{1,r}^t(n)q^n \\ &= \frac{1}{(q)_\infty} \sum_{n \geq 1} (-1)^{n-1} \frac{q^{n(n-1)/2+rn}(r - (r-1)q^n)}{1 - q^n} \\ &= rC_{0,r}^t(q) + \frac{1}{(q)_\infty} \sum_{n \geq 1} (-1)^{n-1} \frac{q^{n(n+1)/2+rn}}{1 - q^n}, \end{aligned}$$

where $C_{0,r}^t(q)$ is the generating function for $M_{0,r}^t(n)$. Similarly,

$$\begin{aligned} \mathcal{R}_{1,r}^t(q) &= \sum_{n \geq 0} N_{1,r}^t(n)q^n \\ &= \frac{1}{(q)_\infty} \sum_{n \geq 1} (-1)^{n-1} \frac{q^{n(3n-1)/2+rn}(r - (r-1)q^n)}{1 - q^n} \\ &= r\mathcal{R}_{0,r}^t(q) + \frac{1}{(q)_\infty} \sum_{n \geq 1} (-1)^{n-1} \frac{q^{n(3n+1)/2+rn}}{1 - q^n}, \end{aligned}$$

where $\mathcal{R}_{0,r}^t(q)$ is the generating function for $N_{0,r}^t(n)$. Hence,

$$C_{1,r}^t(q) - \mathcal{R}_{1,r}^t(q) = r(C_{0,r}^t(q) - \mathcal{R}_{0,r}^t(q)) + \frac{1}{(q)_\infty} \sum_{n \geq 1} (-1)^{n-1} q^{n(n+1)/2+rn} \frac{1 - q^{n^2}}{1 - q^n}.$$

By the result (1.1) of Chen *et al.* [6], we know that $C_{0,r}^t(q) - \mathcal{R}_{0,r}^t(q)$ has nonnegative coefficients, since the 0th partial rank moment is always larger than or equal to that of the crank. Denote the second sum as $F(q)$, which can be regarded as the difference between shifted moments. This is because

$$\begin{aligned} C_{1,r}^t(q) - rC_{0,r}^t(q) &= \sum_{n \geq 0} \left(\sum_{m > r} (m - r)M(m, n) \right) q^n, \\ \mathcal{R}_{1,r}^t(q) - r\mathcal{R}_{0,r}^t(q) &= \sum_{n \geq 0} \left(\sum_{m > r} (m - r)N(m, n) \right) q^n. \end{aligned}$$

Following Andrews *et al.* [1], we define

$$f_{j,m}(q) = \sum_{n=j}^\infty (-1)^{n+1} q^{\binom{n}{2} + jn + mn}.$$

The following theorem played the key role for showing that $M_{k,0}^t(n) - N_{k,0}^t(n)$ is positive.

THEOREM 2.1 [1, Theorem 12]. For $i, m \geq 0$,

$$f_{2i+1,m}(q) + f_{2i+2,m}(q) = \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+7i+5j+2mj+2mi+m+2} (1 - q^{4i+m+2})(1 - q^{4i+2j+m+3}) + \sum_{j=0}^{\infty} q^{6i^2+8ij+2j^2+5i+3j+2mj+2mi+m+1} (1 - q^{2i+1})(1 - q^{4i+2j+m+2}).$$

To employ the above theorem, we rewrite

$$F(q) = \frac{1}{(q)_{\infty}} \sum_{j=1}^{\infty} f_{j,r}.$$

Clearly, the coefficients of $F(q)$ are nonnegative. For their positivity, we note that $(q)_{\infty}^{-1}(f_{1,r} + f_{2,r})$ is equal to

$$\frac{1}{(q)_{\infty}} \left(\sum_{j=0}^{\infty} q^{2j^2+5j+2rj+r+2} (1 - q^{r+2})(1 - q^{2j+r+3}) + \sum_{j=0}^{\infty} q^{2j^2+3j+2rj+r+1} (1 - q)(1 - q^{2j+r+2}) \right).$$

In the first sum, when $j = 0$, we have the term

$$q^{r+2} \prod_{\substack{k \geq 1 \\ k \neq r+2, r+3}} \frac{1}{1 - q^k},$$

and the n th coefficient of $F(q)$ is positive if $n \geq r + 2$ because we have $(1 - q)^{-1}$ in the product. Using a similar argument on the second sum implies that the $(r + 1)$ th coefficient of $F(q)$ is positive. From these terms, we see that $F(q) = O(q^{r+1})$. To prove that $M_{1,r}^t(r) > N_{1,r}^t(r)$, we note that

$$C_{0,r}^t(q) = \frac{1}{(q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n-1} q^{n(n-1)/2+rn}}{1 - q^n} = \frac{1}{(q)_{\infty}} (q^r + O(q^{r+1})),$$

while

$$R_{0,r}^t(q) = \frac{1}{(q)_{\infty}} \sum_{n \geq 1} \frac{(-1)^{n-1} q^{n(3n-1)/2+rn}}{1 - q^n} = \frac{1}{(q)_{\infty}} (q^{r+1} + O(q^{r+2})),$$

which implies that $M_{1,r}^t(r) - N_{1,r}^t(r) = r$. This completes the proof of Theorem 1.1.

REMARK 2.2. By employing the argument in Andrews *et al.* [1], we can think of $F(q)$ as the generating function which counts certain strings along the partitions of n .

Now we turn to the proof of the asymptotic formula. Recall that

$$\text{ospt}_r^t(n) = \text{ospt}(n) - \sum_{j=1}^{r-1} j(M(j, n) - N(j, n)),$$

where $\text{ospt}(n) = M'_{1,0}(n) - N'_{1,0}(n)$. From [8, Corollary 2.1],

$$M(m, n) \sim \frac{1}{\sqrt{2\pi}} \left[\frac{1}{4} \left(\frac{\pi}{\sqrt{6n}} \right)^{5/2} I_{-5/2} \left(\pi \sqrt{\frac{2n}{3}} \right) + \frac{5}{96} \left(\frac{\pi}{\sqrt{6n}} \right)^{7/2} I_{-7/2} \left(\pi \sqrt{\frac{2n}{3}} \right) \right],$$

$$N(m, n) \sim \frac{1}{\sqrt{2\pi}} \left[\frac{1}{4} \left(\frac{\pi}{\sqrt{6n}} \right)^{5/2} I_{-5/2} \left(\pi \sqrt{\frac{2n}{3}} \right) + \frac{17}{96} \left(\frac{\pi}{\sqrt{6n}} \right)^{7/2} I_{-7/2} \left(\pi \sqrt{\frac{2n}{3}} \right) \right],$$

where $I_s(z)$ is the modified Bessel function of the second kind. Moreover, Bringmann and Mahlburg [5, Theorem 1.4] showed that

$$\text{ospt}(n) \sim \frac{1}{4} p(n).$$

Combining all this together,

$$\begin{aligned} \text{ospt}'_r(n) &\sim \frac{1}{4} p(n) - \sum_{j=1}^{r-1} \frac{j}{8\sqrt{2\pi}} \left(\frac{\pi}{\sqrt{6n}} \right)^{7/2} I_{-7/2} \left(\pi \sqrt{\frac{2n}{3}} \right) \\ &\sim \left(\frac{1}{4} - \frac{\pi^2}{96n} r(r-1) \right) p(n), \end{aligned}$$

where we have used $I_s(z) \sim e^z / \sqrt{2\pi z}$ and $p(n) \sim (4n\sqrt{3})^{-1} \exp(\pi\sqrt{2n/3})$.

3. Concluding remarks

For a fixed positive integer n , we can expect that

$$N_{1,\ell}^+(n) > M_{1,\ell}^+(n) \tag{3.1}$$

if ℓ is small enough, as the tail distribution of cranks is thicker than that of ranks. However, by the result of Andrews *et al.* [1], we know that the inequality will be reversed if ℓ is close enough to n as $N_{1,n}^+(n) < M_{1,n}^+(n)$. Let $f(n)$ be the largest ℓ satisfying (3.1). Then computations suggest the following conjecture:

$$f(n) \sim \sqrt{\frac{2n}{3}}.$$

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