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MINIMALITY OF STRONG STABLE AND UNSTABLE FOLIATIONS FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

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Abstract We give a topological criterion for the minimality of the strong unstable (or stable) foliation of robustly transitive partially hyperbolic diffeomorphisms.

As a consequence we prove that, for 3-manifolds, there is an open and dense subset of robustly transitive diffeomorphisms (far from homoclinic tangencies) such that either the strong stable or the strong unstable foliation is robustly minimal.

We also give a topological condition (existence of a central periodic compact leaf) guaranteeing (for an open and dense subset) the simultaneous minimality of the two strong foliations.

Keywords: minimal foliation; partial hyperbolicity; strong stable and unstable foliations; transitive

AMS 2000 *Mathematics subject classification:* Primary 37D25; 37C70; 37C20; 37C29

1. Introduction

Robustly transitive diffeomorphisms are one of the archetypes of chaotic dynamics: they are *transitive* (i.e. there are points whose forward orbits are dense in the whole ambient manifold) and the transitivity persists after C^1-perturbations. The first known examples are the Anosov (hyperbolic) ones. More recently non-hyperbolic robustly transitive diffeomorphisms were constructed using one of the following three methods: bifurcations from hyperbolic diffeomorphisms (see the *derived from Anosov* examples in [**4**, **10**, **20**]), skew products [**2**, **27**], and perturbations of the time-one of Anosov flows [**2**].

For volume preserving diffeomorphisms the prototype should be the *stably ergodic* diffeomorphisms, that is, volume preserving diffeomorphisms whose volume preserving perturbations are ergodic (with respect to the volume). These two theories (robust transitivity and stable ergodicity) have been systematically developed in recent years and run in parallel. As a general rule, examples can be translated from one theory to the other (see, for example, the example in [**28**]), and many problems are common to both theories (see the survey [**9**] on stable ergodicity).

Robustly transitive diffeomorphisms always satisfy some weak form of hyperbolicity: they are uniformly hyperbolic in dimension two [**21**], partially hyperbolic in dimension three [**13**], and they admit a dominated splitting in higher dimensions [**5**].

In this paper we deal with robustly transitive diffeomorphisms defined on 3-manifolds which are partially hyperbolic (but not Anosov). This means that the tangent bundle of the ambient manifold can be split into two or three invariant sub-bundles as follows.

- $TM = E^s \oplus E^c$, where E^s is one dimensional and uniformly hyperbolic and E^c is two dimensional, *undecomposable* (i.e. it does not admit any dominated splitting) and not uniformly hyperbolic.
- $TM = E^c \oplus E^u$, where E^u is one dimensional and uniformly hyperbolic (expanding) and E^c is two dimensional, undecomposable and not uniformly hyperbolic.
- $TM = E^s \oplus E^c \oplus E^u$, where the three bundles are one dimensional, E^s and E^u are uniformly hyperbolic (contracting and expanding, respectively) and E^c is not uniformly hyperbolic.

In the first two cases it is possible to perturb the initial diffeomorphism to get a new diffeomorphism exhibiting a *homoclinic tangency* (i.e. there is a hyperbolic periodic point whose invariant manifolds are non-transverse) (see [**13**, Corollary G]). In this paper we deal with the third situation, so-called *strongly partially hyperbolic* (see a more precise definition below), where the diffeomorphisms are *far from homoclinic tangencies.* For instance, the three-dimensional examples in [**2**, **20**, **28**] have this property.

In the strongly partially hyperbolic context, due to the uniform hyperbolicity of the bundles E^s and E^u, there exist one-dimensional foliations tangent to E^s and E^u, called strong stable $\mathcal{F}^s$ and strong unstable $\mathcal{F}^u$ foliations, respectively (see [**8**, **19**]). However, we do not know if in general the central bundle is (uniquely) integrable and so the central foliation is not *a priori* defined, in fact this is an important open problem in this area.

Some key problems in the study of (robustly) transitive diffeomorphisms are the following.

- *Characterization of robust transitivity or stable ergodicity.*
- *Which* 3*-manifolds can support robustly transitive diffeomorphisms*? Let us recall that Anosov diffeomorphisms in dimension 3 are always conjugate to a finite quotient of a linear Anosov map on the torus T^3 (see [**16**, **22**, **23**]). We conjecture that *there are no robustly transitive diffeomorphisms on the sphere* S^3. Theorem H in [**13**] verifies this conjecture in a more restrictive context.
- *State the ergodic properties of these systems, in particular, the existence and finiteness of SRB measures* (for results on this subject see [**1**, **4**, **14**]).

In this paper we will begin the topological description of the strong stable or unstable foliation, motivated by the two following ideas.

- For describing the global dynamics of strongly partially hyperbolic diffeomorphisms one often attempts to propagate topological or measurable local properties along the leaves of the strong stable or unstable foliations to the whole manifold.

- The support of SRB-measures of strongly partially hyperbolic diffeomorphisms is saturated for the strong unstable foliation. For instance, Bonatti and Viana [**4**] use the hypothesis of the density of the unstable leaves to prove the uniqueness of the SRB-measure.

A foliation is *minimal* if all its leaves are dense in the ambient manifold.

Problem 1.1. *Are the strong stable and strong unstable foliations minimal for an open and dense subset of the set of robustly transitive, strongly partially hyperbolic diffeomorphisms?*

Before stating our results, let us recall that this problem has a long history in the uniformly hyperbolic context.

- The stable and unstable foliations of transitive Anosov diffeomorphisms are always dense.

- For transitive Anosov flows on 3-manifolds, the strong stable and unstable foliations are both minimal, except for a very specific case: the suspension of an Anosov diffeomorphism of T^2 (see [**25**]). This study begun with the geodesic flows of surface having constant negative curvature, where the strong stable and unstable foliations correspond with the horocyclic flows and are uniquely ergodic (see [**17**]). Then the property of unique ergodicity was generalized to the strong stable or unstable foliation of Anosov flows (see [**7**]).

In this paper, assuming that the central bundle admits an invariant orientation, we prove that *the minimality of at least one of the two strong foliations is an open and dense property* (see Theorem 1.3). Moreover, if the central bundle is uniquely integrable, we also prove that *the minimality of both strong foliations is an open and dense property among the diffeomorphisms having a periodic compact central leaf* (see Theorem 1.6).

Let us now state our results precisely.

Statement of results

Let $f\colon M \to M$ be a diffeomorphism of a compact manifold. Let us recall that an f_*-invariant splitting $E \oplus F$ of TM is *dominated* if the fibres of the bundles have constant dimension and there are a Riemannian metric $\|\cdot\|$ and a constant $\lambda < 1$ such that $\|f_*(x)_{|E}\| \cdot \|f_*^{-1}(f(x))_{|F}\| < \lambda$, for all $x \in M$.

An f_*-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ with three bundles is called dominated if the splittings $E^{cs} \oplus E^u$ and $E^s \oplus E^{cu}$ are both dominated, where $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$.

Definition 1.2. A diffeomorphism f of a compact 3-manifold M is *strongly partially hyperbolic* if there is a f_*-invariant *dominated* splitting $TM = E^s \oplus E^c \oplus E^u$ with three one-dimensional bundles such that E^s and E^u are, respectively, uniformly contracting and expanding.

Given a compact closed manifold M of dimension 3 the set of robustly transitive diffeomorphisms of M is, by definition, open in $\mathrm{Diff}^1(M)$. As dominated splittings persist by C^1-perturbations, the set of strongly partially hyperbolic diffeomorphisms is open. The property of the central bundle not being hyperbolic is not open. However, Mañé's result [**21**] implies that every diffeomorphism which is robustly transitive and strongly partially hyperbolic but not Anosov can be C^1-approximated by one having hyperbolic periodic points of different *indices* (dimension of the stable manifold). This property is clearly open.

We now denote by $\mathcal{T}(M)$ the open set of robustly transitive and strongly partially hyperbolic diffeomorphisms having hyperbolic periodic points of different indices (of index 1 and 2). We endow $\mathcal{T}(M)$ with the C^1-topology. Define by $\mathcal{T}^+(M)$ the subset of $\mathcal{T}(M)$ of diffeomorphisms f such that the bundles E^s, E^u and E^c are orientable and f_* preserves these orientations. This set is open in $\mathcal{T}(M)$.

A property of diffeomorphisms is called *robust* if whenever it holds for a diffeomorphism it also holds for a C^1-neighbourhood of that diffeomorphism.

Denote by $\mathcal{O}^s(M)$ (respectively $\mathcal{O}^u(M)$) the subset of $\mathcal{T}(M)$ of diffeomorphisms f whose strong stable (respectively unstable) foliation is robustly minimal. In other words, $\mathcal{O}^s(M)$ (respectively $\mathcal{O}^u(M)$) is the interior of the subset of $\mathcal{T}(M)$ of diffeomorphisms f with a minimal strong stable (respectively unstable) foliation.

Similarly, let $\mathcal{O}^{i+}(M) = \mathcal{O}^i(M) \cap \mathcal{T}^+(M)$, $i = s, u$.

Theorem 1.3. *The open subset $\mathcal{O}^+(M) = \mathcal{O}^{s+}(M) \cup \mathcal{O}^{u+}(M)$ of diffeomorphisms with a robustly minimal strong stable or unstable foliation is dense in $\mathcal{T}^+(M)$.*

Let us first state a dynamical consequence of this result. Recall that a diffeomorphism f is *topologically mixing* if for every pair of open sets U and V there is a positive $n_0 = n(U, V)$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geqslant n_0$. One easily verifies that, if f has a strong foliation which is minimal, then it is topologically mixing. Then we have the following corollary.

Corollary 1.4. *There is an open and dense subset of $\mathcal{T}^+(M)$ consisting of topologically mixing diffeomorphisms.*

Now let us discuss the hypotheses of Theorem 1.3. The condition on the existence of an f_*-invariant orientation on the stable and unstable bundles is purely technical and can easily be dropped (see Proposition 7.1 in § 7). The hypothesis on the invariant orientation of the central bundle of f is much more delicate and subtle. Of course, this difficulty can be bypassed if either its lift to the orientation covering remains transitive or if the central bundle is orientable (f does not preserve this orientation) and f^2 is transitive. The robust transitivity of f^2 is guaranteed if f has some periodic point of odd period (see Lemma 7.3). The robust transitivity of the lift of f to the orientation covering is guaranteed if f has a hyperbolic periodic point p of period k such that $f^k_*(p)$ reverses the orientation of $E^c(p)$ (see Remark 7.5).

Another way to guarantee the transitivity of the lifts is to assume the *accessibility* hypothesis. More precisely, a point y is *accessible* from a point x if there is a path going from x to y consisting of segments contained in strong stable or strong unstable leaves. Accessibility is an equivalence relation. A diffeomorphism f has the *accessibility property* if there is only one accessibility class. Similarly, f has the *robust accessibility property* if every diffeomorphism g C^1-close to f has the accessibility property.

Theorem 1.5. *The open set of diffeomorphisms with a robustly minimal strong stable or unstable foliation is dense in the set $\mathcal{T}_a(M)$ of diffeomorphisms $f \in \mathcal{T}(M)$ having the robust accessibility property.*

The accessibility assumption does not seem to be too restrictive: Pugh and Shub conjecture that *the accessibility property is valid for an open and dense subset of $\mathcal{T}(M)$* [**26**], and recently Dolgopyat and Wilkinson [**15**]* have announced the proof of this conjecture. For a partial answer for this conjecture see [**24**]. Assuming this result our theorems can be reformulated as follows.

The set $\mathcal{O}(M) = \mathcal{O}^s(M) \cup \mathcal{O}^u(M)$ of diffeomorphisms with a robustly minimal strong stable or unstable foliation is dense in $\mathcal{T}(M)$.

The previous results only give the robust minimality of one of the strong foliations. We now introduce a subset of $\mathcal{T}(M)$ for which we have been able to prove the minimality of both foliations. Consider the subset $\mathcal{T}_0(M)$ of $\mathcal{T}(M)$ consisting of the diffeomorphisms f verifying the following two extra geometrical assumptions:

- *robust coherence*, that is, unique integrability of the bundles E^c, $E^c \oplus E^s$ and $E^c \oplus E^u$, for every diffeomorphism g in a C^1-neighbourhood of f; and
- existence of a compact and periodic central leaf.

A compact periodic central leaf is a normally hyperbolic invariant manifold; therefore, from [**19**], to have such a compact periodic central leaf is a C^1-open property in the set of partially hyperbolic systems, so that $\mathcal{T}_0(M)$ is a C^1-open subset of $\mathcal{T}(M)$.

In the partially hyperbolic setting the coherence assumption is quite common and natural, and it has often been used in the context of stably ergodic systems (see, for example, [**14**, **26**]). Unfortunately, it is not known whether dynamical coherence is a C^1-open property. If the central foliation is *plaque expansive* (see [**19**] for the definition), then dynamical coherence is verified in a C^1-neighbourhood of f. On the other hand, all the known examples of robustly transitive diffeomorphisms are plaque expansive.

As above we consider the subset $\mathcal{T}_0^+(M) = \mathcal{T}_0(M) \cap \mathcal{T}^+(M)$.

Theorem 1.6. *The open set $\mathcal{O}^s(M) \cap \mathcal{O}^u(M)$ of diffeomorphisms whose strong stable and unstable foliations are both robustly minimal is dense in $\mathcal{T}_0^+(M)$.*

* The definition of partial hyperbolicity in these papers is a bit more restrictive than the one we use here: in their definition, for any pair of points x and y and every unitary vectors $u \in E$ and $v \in F$ one has $|f_*(x)(u)|/|f_*(y)(v)| < k$ for some $k < 1$, and in our definition we only consider the case $x = y$.

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Again, we can substitute the hypothesis on the orientations by the robust accessibility property (see § 7.2).

Finally, Theorem 1.3 remains valid in any dimension for robustly transitive diffeomorphisms whose central bundle has dimension 1 and is uniquely integrable.

Theorem 1.7. *Let M be a compact manifold of dimension $n \geqslant 3$. Denote by $\mathcal{T}_1(M) \subset \mathrm{Diff}^1(M)$ the set of robustly transitive diffeomorphisms whose central bundle has dimension one, is uniquely integrable, and has an f-invariant orientation.*

Then $\mathcal{O}^u(M) \cup \mathcal{O}^s(M)$ contains a dense open subset of $\mathcal{T}_1(M)$, where $\mathcal{O}^u(M) \cup \mathcal{O}^s(M)$ is the set of diffeomorphisms $f \in \mathcal{T}_1(M)$ having a robustly minimal strong unstable or stable foliation.

The theorems above follow from a result on the existence of complete transverse sections adapted to the dynamics.

Definition 1.8. Let f be a strongly partially hyperbolic diffeomorphism of a compact 3-manifold M.

- A compact surface T with boundary, transverse to the strong unstable foliation of f, is a *u-section for f* if $f(T)$ is contained in the interior of T and $\omega(T) = \bigcap_{n \geqslant 0} f^n(T)$ is the union of finitely many segments and circles tangent to the central direction.
- A u-section T is *complete* if its interior intersects each strong unstable leaf $F^u(x)$ transversely, $x \in M$.

We define *s-section* and *complete s-section for f* by replacing f by f^{-1} in the definition above.

By the f-invariance, every complete u-section is tangent to the centre stable bundle $E^c \oplus E^s$.

The proof of our results follows from the following heuristic principle: generically a complete u-section contains finitely many periodic points; we will show that the closure of any leaf of $\mathcal{F}^u$ contains the leaf through one of these periodic points. So the minimality of the foliation will follows from the density of these finitely many unstable manifolds.

Denote by $\mathcal{U}(M)$ (respectively $\mathcal{S}(M)$) the subset of $\mathcal{T}(M)$ of diffeomorphisms admitting complete u-sections and s-sections, respectively.

Theorem 1.9.

(1) *The sets $\mathcal{S}(M)$ and $\mathcal{U}(M)$ are open in $\mathcal{T}(M)$.*

(2) *The union $\mathcal{S}(M) \cup \mathcal{U}(M)$ is dense in $\mathcal{T}^+(M)$.*

(3) *Both $\mathcal{S}(M)$ and $\mathcal{U}(M)$ are dense in $\mathcal{T}_0^+(M)$.*

(4) *There is an open and dense subset $\mathcal{M}^u(M)$ of $\mathcal{U}(M)$ such that the strong unstable foliation is minimal for all f in $\mathcal{M}^u(M)$. Similarly, there is an open and dense subset $\mathcal{M}^s(M)$ of $\mathcal{S}(M)$ such that the strong stable foliation is f-minimal for all f in $\mathcal{M}^s(M)$.*

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Theorems 1.3 and 1.6 are both direct consequences of Theorem 1.9, and the different items of this theorem correspond to the steps of the proofs of Theorems 1.3 and 1.6.

Let us return to the known examples of three-dimensional robustly transitive diffeomorphisms with a strongly partially hyperbolic structure [**2**, **20**, **28**]. The examples in [**2**, **28**] have closed periodic orbits and thus the two strong foliations are minimal. On the other hand, in the derived from Anosov examples in [**20**] our results only ensure the minimality of one of the strong foliations. In fact, our proof shows that if the initial Anosov diffeomorphism has a one-dimensional stable (respectively unstable) bundle then the strong stable (respectively unstable) foliation of the derived map is minimal. This leads us to the following question.

Problem 1.10. *Is having (simultaneously) minimal strong stable and unstable foliations a dense property in* $\mathcal{T}(M)$*?*

Concerning three-dimensional robustly transitive diffeomorphisms, we can now consider those admitting only a splitting with two bundles (see the constructions in [**4**]), for instance, having a splitting of the form $E^u \oplus E^c$, where E^u is one dimensional and uniformly hyperbolic (expanding) and E^c is two dimensional, undecomposable and non-hyperbolic. The hyperbolicity of E^u ensures the existence of the strong stable foliation.

Problem 1.11. *Consider the robustly transitive diffeomorphisms of a compact* 3*-manifold having a splitting* $E^c \oplus E^u$ *where* $\dim(E^c) = 2$*. Is the robust minimality of* $\mathcal{F}^u$ *a dense property for these diffeomorphisms?*

In higher dimensions the panorama of robustly transitive diffeomorphisms seems to be much more open. For instance, the existence of a hyperbolic direction (and therefore of an invariant foliation) is not guaranteed (see examples in [**4**]). So it should be important to understand which geometrical object would replace the invariant foliations of the partially hyperbolic theory.

This paper is organized as follows.

In § 2 we prove that there is an open and dense subset of $\mathcal{T}(M)$ of diffeomorphisms such that both stable and unstable manifolds of a periodic point are dense.

In § 3 we prove that the existence of a complete s- or u-section is an open property.

In § 4 we show that the existence of a complete u-section generically implies the robust minimality of the unstable foliation.

In § 5 we show that every Kupka–Smale diffeomorphism of $\mathcal{T}^+(M)$ has a complete s- or u-section. The first step of this section is to prove the existence of invariant manifolds tangent to the central bundle of the periodic points, this property will substitute the unique integrability of the central bundle. In the second step we get a complete section by considering the union of the strong stable or strong unstable leaves through the points in these central curves.

In § 6 (under the hypotheses of Theorem 1.6) we see that the local stable and unstable manifolds of any compact periodic leaf of the central foliation are complete u- and s-sections.

In § 7 we explain how to extend our results to the non-orientable case, assuming the accessibility property (Theorem 1.5).

In § 8 we extend our results to higher dimensions (Theorem 1.7).

2. Robust density of invariant manifolds of hyperbolic periodic points

Given a diffeomorphism f and a hyperbolic periodic orbit P_f of f we say that the stable manifold of P_f, $W^s(P_f)$, is *robustly dense* in M if there is a C^1-neighbourhood of f consisting of diffeomorphisms g such that $W^s(P_g)$ is dense in M, where P_g is the continuation of P_f for the diffeomorphism g. We define in the same way the robust density of the stable manifold of the orbit of P_f, and of the unstable manifold of P_f or its orbit.

In this section we show that the invariant manifolds of periodic orbits of generic diffeomorphisms of $\mathcal{T}(M)$ are robustly dense in M. This property is just a remark (see Proposition 2.1) when the invariant manifold is two dimensional. For the one-dimensional manifolds the key idea is to use the Hayashi's Connecting Lemma [**18**], to get *heterodimensional cycles*. After unfolding this cycle one gets that the closure of this one-dimensional manifold contains in a robust way the two-dimensional stable manifold of a saddle of different index (see [**12**] and § 2.1). Then, using Proposition 2.1, this one-dimensional invariant manifold is robustly dense in the whole M.

Proposition 2.1. *For every diffeomorphism $f \in \mathcal{T}(M)$ and every hyperbolic periodic point P of index 1 of f, the unstable manifold of the orbit of P is robustly dense in M. Similarly, the stable manifold of the orbit of any hyperbolic periodic point of f of index 2 is robustly dense in M.*

Proof. Consider $f \in \mathcal{T}(M)$ and any hyperbolic periodic point P of f of index 1. Consider any point $z \in M$ whose forward orbit is dense in M. Then there is some big i such that $w = f^i(z)$ is close enough to P so that the strong stable leaf $F^s(w)$ of w intersects transversely the interior of $W^u(P)$ at some point y. Now, using that f exponentially contracts the leaves of the strong stable foliation of f, one has that the ω-limit sets of y, w and z are equal. In particular, one has $\omega(y) = \omega(w) = \omega(z) = M$. So the orbit of $W^u(P)$ is dense in M, that is, the stable manifold of the orbit of P is dense in M. As the set $\mathcal{T}(M)$ is open and the density holds for any $f \in \mathcal{T}(M)$, the stable manifold of the orbit of P is robustly dense in M, ending the proof of the lemma. □

The arguments in the proof of the previous proposition give somewhat more.

Scholium 2.2.

- *Every f-invariant surface immersed in M transversely to $\mathcal{F}^s_f$ or $\mathcal{F}^u_f$ is dense in M.*
- *Proposition 2.1 and the item above hold in any dimension for robustly transitive diffeomorphisms having a partial hyperbolic structure $E^s \oplus E^c$ (E^s is a uniformly contracting bundle). In that case, every $\dim(E^c)$-dimensional unstable manifold of a periodic orbit is dense in M. A similar result holds for the stable manifolds.*

The main result of this section is the following proposition.

Proposition 2.3. *Let $\mathcal{V}$ be an open subset of $\mathcal{T}(M)$ such that, for every f in $\mathcal{V}$, there is a hyperbolic periodic point P_f of index 1 depending continuously on $f \in \mathcal{V}$.*

Then there is an open and dense subset $\mathcal{V}_1$ of $\mathcal{V}$ such that, for every $f \in \mathcal{V}_1$, the stable manifold of the orbit of P_f is dense in M.

Proof. By definition of $\mathcal{T}(M)$ any $f \in \mathcal{T}(M)$ has some periodic point Q_f of index 2. Up to cover $\mathcal{V}$ by open subsets, we can assume that Q_f depends continuously on $f \in \mathcal{V}$.

By the robust transitivity of $f \in \mathcal{V}$ we have that P_f and Q_f belong to the same transitive set (the ambient manifold M). Now [**3**, Proposition 1.1] implies that there is a dense subset $\mathcal{W} \subset \mathcal{V}$ such that, for every $f \in \mathcal{W}$, one has that $W^u(P_f) \cap W^s(Q_f) \neq \emptyset$ and $W^s(P_f) \cap W^u(Q_f) \neq \emptyset$ (in fact, this part of the proposition is a direct consequence of Hayashi's Connecting Lemma).

Observe that, by the strong partial hyperbolicity of f, every point in $W^u(P_f) \cap W^s(Q_f)$ corresponds to a transverse intersection of these two-dimensional invariant manifolds, and each point in $W^s(P_f) \cap W^u(Q_f)$ is a quasi-transverse intersection of these one-dimensional invariant manifolds. So P_f and Q_f belong to a generic heterodimensional cycle of codimension one as in [**12**]. Moreover, the strong partial hyperbolicity of f implies that this cycle is far from homoclinic tangencies.

Now the fact in the proof of [**6**, Lemma 7.2] (which is just a reformulation of [**12**, Proposition 3.6(b)]) implies that any $f \in \mathcal{W}$ belongs to the closure of an open set $\mathcal{V}_f$ consisting of diffeomorphisms g such that the stable manifold of P_g meets any 2-disk intersecting $W^s(Q_g)$ and transverse to the strong stable foliation. In particular, the closure of $W^s(P_g)$ contains $W^s(Q_g)$, which is dense in M (recall Proposition 2.1). Thus $W^s(P_g)$ is dense in M for every $g \in \mathcal{V}_f$. In the next subsection we will state more precisely the result on heterodimensional cycles which is the key of this argument (see Proposition 2.6).

Finally, by construction, the set $\mathcal{V}_1 = \bigcup_{f \in \mathcal{W}} \mathcal{V}_f$ is the announced dense open subset of $\mathcal{V}$. The proof of the proposition is now complete. □

Remark 2.4. Let P be a hyperbolic periodic point (of any index) of $f \in \mathcal{T}(M)$ such that the stable manifold $W^s(P)$ of its orbit is dense in M. Then given any surface Σ transverse to $\mathcal{F}^s_f$ one has that $W^s(P) \cap \Sigma$ is dense in Σ. This follows from the fact that $W^s(P)$ is saturated by the leaves of the strong stable foliation.

Corollary 2.5. *Under the hypotheses of Proposition 2.3, the homoclinic class of P_f is equal to M for all $f \in \mathcal{V}_1$.*

As a consequence, there is an open and dense subset of $\mathcal{T}(M)$ of diffeomorphisms whose hyperbolic periodic points of index 1 (or index 2) are dense in M.

2.1. Heterodimensional cycles: Proposition 2.6

We say that a diffeomorphism f has a *codimension-one heterodimensional cycle* if there are hyperbolic periodic points P_f and Q_f of indices $k+1$ and k, respectively, such that $W^s(P_f)$ and $W^u(Q_f)$ have non-empty transverse intersections and $W^u(P_f)$ and $W^s(Q_f)$ have a quasi-transverse intersection (i.e. there is a point x of $W^u(P_f) \cap W^s(Q_f)$

such that $T_xW^u(P_f) + T_xW^s(Q_f)$ has dimension $\dim(M) - 1$, where M is the ambient manifold). This heterodimensional cycle is *far from homoclinic tangencies* if every g in a C^1-neighbourhood of f has no homoclinic tangencies associated neither to P_g nor Q_g.

Proposition 2.6. *Let f be a C^1-diffeomorphism having a codimension-one heterodimensional cycle associated to the hyperbolic periodic points P_f and Q_f, where the index of P_f is bigger than the index of Q_f. Suppose that the cycle is far from homoclinic tangencies. Then there is a C^1-open set $\mathcal{U}_f$ whose closure contains f such that*

$$W^s(P_g) \subset \text{closure}(W^s(Q_g)) \quad \textit{and} \quad W^u(Q_g) \subset \text{closure}(W^u(P_g))$$

for every diffeomorphism $g \in \mathcal{U}_f$.

This kind of phenomenon, an invariant manifold of dimension $k+1$ being contained in the closure of the corresponding invariant manifold of dimension k, was first exhibited in [**11**] for the unfolding of some heterodimensional cycles (in the context of parametrized families of diffeomorphisms). In [**2**] using the *blenders* (a kind of skew horseshoe) was shown that the unfolding of some heterodimensional cycles is a key mechanism for obtaining robust transitivity. In fact, the blenders give a mechanism guaranteeing that (for instance) the closure of a stable manifold of dimension k contains a stable manifold of dimension strictly bigger than k in a robust way.

Later, in [**12**] was introduced the *generalized blenders* (hyperbolic sets with skew Markov partitions) and proved that the previous result always holds in the context of bifurcations of heterodimensional cycles in parametrized families of diffeomorphisms (i.e. the closure of a stable manifold of the point of index k in the cycle contains the stable manifold of the point of index $k+1$ of the cycle in a robust way).

Proof. Proposition 2.6 is obtained as follows. The fact in the proof of [**6**, Lemma 7.2] (which is just a reformulation of [**12**, Proposition 3.6(b)]) ensures that f belongs to the closure of an open set $\mathcal{V}_f$ of diffeomorphisms g such that the stable manifold of Q_g transversely meets any $(\dim(M)-k)$-disk intersecting $W^s(P_g)$ and transverse to a strong stable foliation of $W^s(P_g)$. In particular, the closure of $W^s(Q_g)$ contains $W^s(P_g)$.

To prove the proposition observe that P_g has a transverse homoclinic point and that the stable manifold of P_g is contained in the closure of the stable manifold of Q_g for all $g \in \mathcal{V}_f$. The Connecting Lemma now implies that there is a dense subset of $\mathcal{V}_f$ consisting of diffeomorphisms g having a heterodimensional cycle associated to P_g and Q_g. By hypothesis, these cycles are far from tangencies. In this way one gets a sequence g_n, $g_n \in \mathcal{V}_f$, converging to f and such that every g_n has a heterodimensional cycle far from homoclinic tangencies.

Arguing as above, but now applying the previous construction to g_n^{-1}, for each n we get an open set $\mathcal{V}_n \subset \mathcal{V}_f$ whose closure contains g_n such that the closure of $W^u(P_g)$ contains $W^u(Q_g)$ for all $g \in \mathcal{V}_n$. Finally, it is enough to take $\mathcal{U}_f = \cup_n \mathcal{V}_n$. $\square$

3. Proof of Theorem 1.9 (1): the existence of a complete section is an open property

The aim of this section is to prove the following proposition (corresponding to Theorem 1.9 (1)).

Proposition 3.1. *The sets $\mathcal{S}(M)$ and $\mathcal{U}(M)$ of diffeomorphism admitting complete s and u-sections, respectively, are open in $\mathcal{T}(M)$.*

The first step of the proof is a general lemma about transverse sections of one-dimensional foliations. Consider a foliation $\mathcal{G}$ of dimension one, a submanifold (with boundary) T of M is a *complete section* if it cuts transversely every leaf of $\mathcal{G}$.

Lemma 3.2. *Consider a foliation $\mathcal{G}$ of dimension one having a complete section T. Then there is $K_T = K > 0$ such that every segment γ of length bigger than K contained in a leaf of $\mathcal{G}$ intersects transversely the interior of T. As a consequence, the interior of T contains a compact subset T_0 which also is a complete section.*

Proof. The proof is by contradiction. Suppose that the result is false. Then there is a sequence of segments γ_n of length n contained in leaves of $\mathcal{G}$ such that $\gamma_n \cap T = \emptyset$ for all n. Let x_n be the middle point of γ_n. Taking a subsequence, if necessary, we can assume that (x_n) converges to some point x. Since T is a complete section, there is a segment γ containing x and contained in a leaf of $\mathcal{G}$ that intersects transversely T in its interior. This implies that γ_n also intersects transversely T for all n big enough. This leads to a contradiction and ends the first part of the lemma.

To prove the second part of the lemma consider a finite compact covering C_i of M such that for each i there is a continuous function $\phi_i\colon C_i \to \operatorname{int}(T)$ which associates to each point $x \in C_i$ a point of intersection of $\operatorname{int}(T)$ and the segment of length K centred at x and contained in a leaf of $\mathcal{G}$. Then the compact set T_0 is the union of the compact sets $\phi_i(C_i)$. This completes the proof of the lemma. □

We are now ready to prove Proposition 3.1.

Proof. We prove, for instance, that the set $\mathcal{U}(M)$ is open. Let $f \in \mathcal{T}(M)$ and T_f be a complete u-section. By Lemma 3.2 above, there are $K > 0$ and a compact subset T'_f of the interior of T_f such that every unstable segment of length K intersects transversely T'_f.

Consider now a compact neighbourhood T^*_f of T'_f in the interior of T_f such that $f(T^*_f) \subset \operatorname{int}(T^*_f)$. One can choose T^*_f containing $f(T_f)$. By the normal hyperbolicity of the transverse section T^*_f, for every g sufficiently C^1-close to f, there is defined a continuation T^*_g of T^*_f which is a g-invariant compact submanifold with boundary, i.e. $g(T^*_g) \subset \operatorname{int}(T^*_g)$. For g close enough to f denote by T'_g the projection of T'_f into T^*_g along the leaves of $\mathcal{F}^u_f$. Thus every segment of length $2K$ contained in a leaf of $\mathcal{F}^u_f$ meets transversely T'_g.

Using now the continuous dependence of the foliations $\mathcal{F}^u_g$ on g, for every g sufficiently C^1-close to f, we have that every segment of length $3K$ of $\mathcal{F}^u_g$ intersects transversely T^*_g. So T^*_g is a complete section for g.

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It remains to see that T_g^* is a u-section. This follows from the normal hyperbolicity of $\omega(T_f) = \cap_{n\geqslant 0} f^n(T_f)$. More precisely, since T_f is a complete u-section, $\omega(T_f)$ consists of normally hyperbolic circles and central segments (not necessarily pairwise disjoint) whose extremes are periodic points of f (not necessarily hyperbolic). By the choice of T_f^*, $\omega(T_f) = \omega(T_f^*)$. Observe now that the circles of $\omega(T_f^*)$ admit a continuous continuation for every g close to f. On the other hand, the central segments of $\omega(T_f^*)$ only have (in general) a lower semicontinuous continuation. Such a semicontinuity is due to the possible disappearance of the non-hyperbolic extremities of the segments.

Take $\omega(T_g^*) = \cap_{n\geqslant 0} g^n(T_g)$. By the previous comment, for every g sufficiently close to f, the set $\omega(T_g^*)$ is the union of circles and central segments: the circles being continuations of the circle components of $\omega(T_f^*)$ and the segments being the lower semicontinuous continuation of the central segments of $\omega(T_f^*)$. This ends the proof of the proposition. □

Remark 3.3.

(1) The sections T_g^* in the proof of Proposition 3.1 are complete u-sections depending continuously on g in a neighbourhood of f.

(2) Observe that the constants $K_{T_g^*}$ in Lemma 3.2 can be taken independent of the diffeomorphism g if the neighbourhood of f is small enough.

4. Proof of Theorem 1.9 (4)

In this section we prove the following proposition (corresponding to Theorem 1.9 (4)).

Proposition 4.1. *There is a dense open subset $\mathcal{U}_m(M)$ (respectively $\mathcal{S}_m(M)$) of $\mathcal{U}(M)$ (respectively $\mathcal{S}(M)$) consisting of diffeomorphisms having a minimal strong unstable (respectively stable) foliation.*

We prove this proposition for the set $\mathcal{U}(M)$, the proof for $\mathcal{S}(M)$ is the same.

4.1. Perfect sections

A complete u-section Σ of $\mathcal{F}_f^u$ is called *perfect* if it is the (disjoint) union of finitely many local stable manifolds of hyperbolic periodic points of f of index 2. In particular, every perfect u-section is a complete u-section.

Let $\mathcal{U}_p(M)$ be the subset of $\mathcal{U}(M)$ of diffeomorphism having a perfect u-section.

Proposition 4.2. *The set $\mathcal{U}_p(M)$ is a dense open subset of $\mathcal{U}(M)$.*

Proof. We only need to prove that the set $\mathcal{U}_p(M)$ is dense in $\mathcal{U}(M)$: in fact, Proposition 3.1 and Remark 3.3 (1) ensure that if f admits a perfect u-section then every g close to f also has a perfect u-section (i.e. to have perfect sections is an open property).

We now prove the density of $\mathcal{U}_p(M)$ in $\mathcal{U}(M)$. Take any $f \in \mathcal{U}(M)$ and let T_f be a complete u-section for f. Observe first that, after a perturbation of f, we can assume that f is Kupka–Smale (i.e. density and hyperbolicity of the periodic points and general position of their invariant manifolds). Hence its dynamics is Morse-Smale in restriction

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to $\omega(T_f)$. Thus given any point z of T_f there are two possibilities: either it belongs to the stable manifold of a (hyperbolic) periodic point of index 2, or it belongs to the stable manifold of a point of index 1.

Consider now all the periodic points $Q_1, \dots, Q_m$ of index 1 of $\omega(T_f)$. Take a constant $K = K_{T_f} > 0$ as in Lemma 3.2 and for each $i \in \{1, \dots, m\}$ consider a segment γ_i of length K contained in the strong unstable leaf of Q_i and such that γ_i does not contain Q_i. By Lemma 3.2, every curve γ_i intersects T_f at some point z_i (which is, by construction, different from Q_i). Now, after an arbitrarily small perturbation of f preserving the set $\omega(T_f)$ and its Morse-Smale dynamics, we can assume that the points z_i do not belong to the (one-dimensional) stable manifolds of the points $Q_1, \dots, Q_m$.

The λ-lemma now implies that the orbit of any strong unstable leaf of the new diffeomorphism f intersects the stable manifold of some point of index 2 in $\omega(T_f)$.

To end the proof of the proposition we need the following lemma.

Lemma 4.3. *Let Σ be a u-section of $\mathcal{F}^u_f$ intersecting the forward orbit of any leaf of $\mathcal{F}^u_f$. Then Σ is a complete u-section.*

Proof. By hypothesis, given any $x \in M$ there is $n = n_x$ such that the unstable leaf through $f^{n-1}(x)$ intersects Σ, so the unstable leaf of $f^n(x)$ meets $f(\Sigma) \subset \mathrm{int}(\Sigma)$. By transversality, there is a finite covering of M by open sets U_i such that for each i there is n_i such that the unstable leaf through $f^{n_i}(x)$ intersects the interior of Σ for every $x \in U_i$.

Take $n = \sup(n_i)$. Then $f^{-n}(\Sigma)$ is a complete u-section. By Lemma 3.2 there is K such that any unstable segment of length K intersects the interior of $f^{-n}(\Sigma)$. Let $\lambda \geqslant \sup_{x \in M} \|f_*(x)\|$ and take any unstable segment γ of length bigger than $\lambda^n \cdot K$. Then $f^{-n}(\gamma)$ is a unstable segment of length larger than K, and so $f^{-n}(\gamma)$ intersects the interior of $f^{-n}(\Sigma)$. Thus γ intersects the interior of Σ, and so Σ is a u-complete section, ending the proof of the lemma. □

To end the proof of the proposition consider the periodic points $P_1, \dots, P_k$ of index 2 of $\omega(T_f)$ and the union Σ of the local stable manifolds of these points. Recall that (by the comment before the Lemma 4.3) the forward orbit of any strong unstable leaf of f intersects Σ. Lemma 4.3 now implies that Σ is a complete u-section. To see that this section is perfect just observe that, by construction, $\omega(\Sigma) = \{P_1, \dots, P_k\}$. The proof of Proposition 4.2 is now complete. □

4.2. Minimality of the strong unstable foliation

In this section we end the proof of Proposition 4.1 by proving that the strong unstable foliation of any diffeomorphism f in a dense open subset $\mathcal{U}_m(M)$ of $\mathcal{U}_p(M)$ is minimal. Since, by Proposition 4.2, $\mathcal{U}_p(M)$ is an open and dense subset of $\mathcal{U}(M)$, this implies Proposition 4.1.

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We begin with the following lemma.

Lemma 4.4. *Let $f \in \mathcal{U}_p(M)$ and T_f be a perfect u-section for f with $\omega(T_f) = \{P_1, \dots, P_m\}$. Then the closure of any leaf of the strong unstable foliation $\mathcal{F}_f^u$ contains the closure of the unstable manifold of some $P_i \in \omega(T_f)$.*

Proof. Observe that Lemma 4.3 implies that $f^n(T_f)$ is a perfect u-section for every $n \in \mathbb{Z}$. Moreover, for every $n > 0$ sufficiently large, the set $f^n(T_f)$ is contained in the union of arbitrarily small local stable manifolds of the points P_i. This means that each unstable leaf passes arbitrarily close to some of the P_i. The lemma now follows easily from this fact. □

Lemma 4.5. *Let f be a diffeomorphism of a compact manifold M and $\mathcal{G}$ an f-invariant foliation such that the orbit of any leaf of $\mathcal{G}$ is dense in M. Then the leaf of $\mathcal{G}$ of any periodic point p of f is dense in M.*

Proof. Consider a periodic point p of f (of period k) and let C_0 be the closure of the leaf of $\mathcal{G}$ containing p. We need to prove that $C_0 = M$. Let $C_i = f^i(C_0)$. By hypothesis, $M = C_0 \cup C_1 \cup \cdots \cup C_k$. As a finite union of compact sets with empty interiors has empty interior, there is i such that C_i has non-empty interior. Thus every C_j also has non-empty interior. Observe that, by construction, each set C_i is saturated for $\mathcal{G}$.

We now claim that given any $j \in \{1, \dots, k\}$ either the interiors of C_0 and C_j are disjoint or $C_0 = C_j$. To prove this claim it suffices to observe that if $\mathrm{int}(C_0) \cap C_j \neq \emptyset$ then the leaf through $f^j(p)$ intersects C_0 and so (since the set C_0 is saturated by the foliation $\mathcal{G}$) the set C_j is contained in C_0. This implies that $C_j \subset C_0$. As a consequence, $C_0 \cap \mathrm{int}(C_j) \neq \emptyset$. The inclusion $C_0 \subset C_j$ follows analogously, ending the proof of our claim.

If all the sets C_i are equal to C_0 we have $C_0 = M$ and we are done. Otherwise, the boundary ∂C_0 of C_0 is a non-empty set saturated for $\mathcal{G}$. Hence, by hypothesis, the f-orbit of ∂C_0 is dense in M. Thus there is j such that $f^j(\partial C_0) \cap \mathrm{int}(C_0) \neq \emptyset$. In other words, $\partial C_j \cap \mathrm{int}(C_0) \neq \emptyset$. By the claim $C_0 = C_j$ and so $\partial C_0 \cap \mathrm{int}(C_0) \neq \emptyset$, which is a contradiction. This ends the proof of the lemma. □

We can now finish the proof of Proposition 4.1.

Proof. Recall that, as $\mathcal{U}_p(M)$ is a dense open subset of $\mathcal{U}(M)$ (Proposition 4.2), it is enough to exhibit a dense open subset of $\mathcal{U}_p(M)$ with minimal strong unstable foliations.

Consider any $f \in \mathcal{U}_p(M)$ and let T_f be a perfect u-section for f. Let $P_{1,f}, \dots, P_{n,f}$ be the periodic points of index 2 whose union of local stable manifolds is T_f. Take a small open neighbourhood $\mathcal{U}_f$ of f such that, for any $g \in \mathcal{U}_f$, the union T_g of the local stable manifolds of the continuations $P_{1,g}, \dots, P_{n,g}$ is a perfect u-section for g.

Proposition 2.3 implies that there is an open and dense subset $\mathcal{V}_f$ of $\mathcal{U}_f$ of diffeomorphisms g such that the unstable manifold of the orbit of any point $P_{i,g}$ is dense in M.

Lemma 4.4 now implies that, for any $g \in \mathcal{V}_f$, the closure of any leaf of $\mathcal{F}_g^u$ contains a leaf through one of the $P_{i,g}$. In particular, the g-orbit of any leaf of $\mathcal{F}_g^u$ is dense in

M. By Lemma 4.5 the unstable leaf through each $P_{i,g}$ is dense. Finally, applying again Lemma 4.4, we get that every unstable leaf of $\mathcal{F}_g^u$ is dense in M, for all $g \in V_f$. This ends the proof of Proposition 4.1. □

5. Generic robust minimality of one of the two strong foliations: Proof of Theorem 1.9 (2)

The goal of this section is to prove Theorem 1.9 (2). Recall that the Kupka–Smale diffeomorphisms form a residual subset of $\mathrm{Diff}^1(M)$. By Corollary 2.5, the set of diffeomorphisms of $\mathcal{T}(M)$ whose hyperbolic periodic points are dense in M is open and dense in $\mathcal{T}(M)$. Let now $\mathcal{T}_K^+(M)$ be the residual subset of $\mathcal{T}^+(M)$ consisting of Kupka–Smale diffeomorphisms having periodic points dense in M. Recall that if $f \in \mathcal{T}^+(M)$ then the bundles E^s, E^u and E^c are orientable and f preserves these orientations.

Theorem 1.9 (2) claims the density in $\mathcal{T}^+(M)$ of the diffeomorphisms admitting complete u- or s-sections, this fact follows immediately from the next proposition.

Proposition 5.1. *Every diffeomorphism $f \in \mathcal{T}_K^+(M)$ has a complete u- or s-section.*

The steps of the proof of Proposition 5.1 are the following.

- Construction of invariant central curves of periodic points (§ 5.1 and Lemma 5.2).
- Analysis of the topology and the dynamics of the invariant central curves and classification of the periodic points according to this analysis (§ 5.2 and Proposition 5.6).
- Using the invariant central curves one constructs u-sections (or s-sections according to the case) by considering the union of the local stable leaves of the points in this central curve (§ 5.3 and Proposition 5.8).

5.1. Invariant central curves for periodic points

In this section we prove the following lemma.

Lemma 5.2. *Let $f \in \mathcal{T}_K^+(M)$ and p be a periodic point of f (of period k). Then there is a periodic curve (of period k) $\gamma_p\colon [-1, +\infty[\to M$ starting at p (i.e. $\gamma_p(0) = p$), tangent to E^c and positively oriented (according to the orientation of E^c) such that either γ_p contains a periodic circle or it is an injective immersion of infinite length.*

Observe that this lemma is trivial if the central bundle E^c is uniquely integrable. For a generalization of this lemma without the orientation assumption see Remark 5.5.

In what follows we call the curves tangent to the central bundle E^c *central curves*.

We start the proof of Lemma 5.2 with a local argument guaranteeing the existence of an f^k-invariant local central curves tangent to E^c.

Lemma 5.3. *Let p be a hyperbolic fixed point of index 2. Then there is an f-invariant curve γ tangent to E^c and embedded in $W^s_{\mathrm{loc}}(p)$, $\gamma\colon [-1, 1] \to M$, such that $\gamma(0) = p$ and $f(\gamma) \subset \mathrm{int}(\gamma)$.*

Observe that this result does not follow directly from the existence of weak stable manifolds: we need a curve tangent to the central bundle at any point.

Proof. Observe first that $W^s(p)$ is tangent at any point to the centre stable bundle $E^c \oplus E^s$, hence E^s and E^c define invariant line-fields in $W^s_{\mathrm{loc}}(p)$. In fact, E^s is tangent (in $W^s(p)$) to the local strong stable foliation of the restriction of f to $W^s_{\mathrm{loc}}(p)$.

Consider now a rectangle R contained in $W^s_{\mathrm{loc}}(p)$ and centred at p such that $f(R)$ is contained in the interior of R and whose boundary is the union of four segments ℓ, r, u and d, where ℓ and r are contained in leaves of the local strong stable foliation and u and d are transverse to the strong stable foliation. Moreover, oriented local stable leaves go from d to u and oriented local central leaves go from ℓ to r.

Choosing the rectangle R sufficiently thin in the central direction, we can assume that any curve in R tangent to E^c and containing p does not intersect $u \cup d$. Thus, by the Cauchy's theorem of existence of solutions of continuous differential equations, every local solution through p (i.e. a curve tangent to E^c containing p) can be extended to a maximal solution reaching both ℓ and r.

By transversality, every maximal solution in R intersects any local leaf of $\mathcal{F}^s_f$ in R in exactly one point. Since the stable leaves are oriented, for any family (γ_σ) of maximal solutions starting at p in R and any local stable leaf we can consider the supremum of the intersections of such a local leaf with the solutions γ_σ. We now define the supremum γ of the family of solutions (γ_σ) taking the set of the suprema in each leaf.

It is easy to verify that the supremum of two maximal solutions is also a maximal solution. From this one deduces that the supremum of an arbitrary family of maximal solutions (γ_σ) can be obtained as the limit of an increasing sequence of maximal solutions. Then, by the compacity of the set of maximal solutions (Ascoli–Arzelá), the supremum is a maximal solution.

Denote by γ the supremum of the family of all the maximal solutions in R. We claim that γ is f-invariant (this claim implies the lemma).

To prove the claim observe that, since f preserves the orientations of E^s and E^c and these bundles are f_*-invariant, the image of γ by f is the supremum of all maximal solutions in $f(R) \subset R$. Notice that $\gamma \cap f(R)$ is a maximal solution in $f(R)$, so it is below $f(\gamma)$. Let γ_1 be an extension of $f(\gamma)$ to a maximal solution in R. By definition of γ, the curve γ_1 is below γ. Hence the curves γ and γ_1 coincide over $f(R)$. We have shown that $f(\gamma) = \gamma \cap f(R)$, ending the proof of the claim (and thus of the lemma). □

Let us observe that there are similar results for periodic points of index 2 and for periodic points of index 1.

Lemma 5.4. *Let p be a hyperbolic periodic point of period k and index 2 of f and $\gamma\colon [0,1] \to M$ be an f^k-invariant central curve starting at p (i.e. $\gamma(0) = p$) and contained in $W^s(p)$. Let $\Upsilon_p = \cup_{n\in\mathbb{N}} f^{-nk}(\gamma)$. Then there are two possibilities:*

- *either the length of Υ_p is infinite; or*
- *the closure of Υ_p is a compact central segment whose extremes are p and q, where q is a hyperbolic periodic point of index 1 whose period is a divisor of k.*

There is a similar result for points p of index 1, in this case if the length of Υ_p is finite then the extreme q has index 2.

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Proof. Assume that the length of Υ_p is finite (otherwise we are done). By construction, the extreme q of Υ_p different from p is a periodic point whose period divide k. Analysing the dynamics of f^k in this segment, since it is tangent to E^c, we have that (except the point p) this curve is contained in the unstable manifold of q. Thus $E^c \oplus E^u$ is the unstable bundle of q. Thus q has index 1, ending the proof of the lemma. □

We are now ready to end the proof of Lemma 5.2. We repeat the procedure in Lemma 5.4 starting at the point $p = p_0$ and argue inductively. Observe that the existence of the curves γ satisfying Lemma 5.4 follows from Lemma 5.2. At each inductive step we obtain an f^k-invariant curve having either infinite length (and in this case we are done) or a point of different index and period bounded by k (the period of the initial periodic point).

In this way we get a sequence of points p_0, q_0, p_1, $q_1, \dots, p_k, q_k$, where the p_i have index 2 and the q_i index 1, q_i is a extreme of Υ_{p_i}, and p_{i+1} is a extreme of Υ_{q_i}. Since the number of periodic points of f of period less than k is bounded, at some stage of the construction we get a curve of infinite length (and we are done) or a first j such that $p_i = p_j$ or $q_i = q_j$ for some $i < j$. In this last case and assuming (for instance) that $p_i = p_j$, we have that the curve $\Upsilon = \cup_{t=0}^{j-1}(\Upsilon_{p_t} \cup \Upsilon_{q_t})$ contains a circle. More precisely, observing that, by construction, $\Upsilon_{q_{j-1}}$ is tangent to p_j and that the curves Υ_r follows the orientation of E^c we have that the curve $\cup_{t=i}^{j-1}(\Upsilon_{p_t} \cup \Upsilon_{q_i})$ is a circle. The proof of Lemma 5.2 is now complete. □

Let us make two remarks about the proof of Lemma 5.2 that we will use in the non-orientable case.

Remark 5.5 (extensions of Lemma 5.2).

Non-transitive case: the conclusions in Lemma 5.2 also hold (and the proof is exactly the same) for Kupka–Smale diffeomorphisms f with a partially hyperbolic splitting $TM = E^s \oplus E^c \oplus E^u$ defined in the whole manifold such that the bundles E^s, E^u and E^c are orientable and f preserve these orientations.

Non-transitive and non-orientable case: suppose now that E^c is orientable and f preserves this orientation, but either some of the bundles E^s or E^u is non-orientable or the bundles E^s and E^u are orientable and f does not preserve some of these orientations. In this case Lemma 5.2 can be stated in the following way.

> Given any hyperbolic periodic point p of f of period k there is a periodic curve (of period less than $4k$) $\gamma_p\colon [-1, +\infty[\to M$ starting at p (i.e. $\gamma_p(0) = p$), tangent to E^c and positively oriented (according to the orientation of E^c) such that either γ_p contains a periodic circle or it is an injective immersion of infinite length.

Proof: it is enough to consider a lift $\tilde{f}$ of f to a covering of M corresponding to the possible local orientations of E^s and E^u. This lift may be not transitive, but it is Kupka–Smale and strongly partially hyperbolic (for completeness see comments in § 7). Thus we can apply the first item of this remark.

5.2. Topology and the dynamics of central manifolds: classification of the periodic points

We now classify the periodic points p of $f \in \mathcal{T}_K^+(M)$ in three (*a priori* non-disjoint) classes. Observe that, since E^c is not necessarily uniquely integrable, to a periodic point p it is possible to associate (*a priori*) several central curves γ_p satisfying Lemma 5.2.

By construction, the period of any periodic point in γ_p divide the period of p. Thus, since the diffeomorphism f is Kupka–Smale, the curve γ_p only contains finitely many periodic points. Hence, if γ_p is an injective immersion, the last periodic point of γ_p (running γ_p in the positive orientation) is well defined. Denote such a point by $q(\gamma_p)$.

(1) Let Γ_1 be the set of periodic points p of f such that there is an f^k-invariant central curve γ_p containing a periodic closed curve (k is the period of p).

(2) Let Γ_2 be the set of periodic points p of f such that there is an f^k-invariant central curve γ_p being an injective immersion and whose last periodic point $q(\gamma_p)$ has index 2 (so $q(\gamma_p)$ is an attractor for the restriction of f^k to γ_p).

(3) Let Γ_3 be the set of periodic points p of f such that there is an f^k-invariant central curve γ_p which is an injective immersion and whose last periodic point $q(\gamma_p)$ has index 1 (so $q(\gamma_p)$ is a repellor for the restriction of f^k to γ_p).

As f preserves the orientation of E^c each set Γ_i is f-invariant. Moreover, the union of the sets Γ_i is the set of periodic points of f which (by hypothesis) is dense in M. Hence the closure of some Γ_i has non-empty interior. Thus, by the transitivity of f, such a Γ_i is dense in M. Proposition 5.1 now follows directly from Proposition 5.6 below.

Proposition 5.6. *Let $f \in \mathcal{T}_K^+(M)$. The following hold.*

- *If Γ_2 is dense in M, then there is a complete u-section for f.*
- *If Γ_3 is dense in M, then there is a complete s-section for f.*
- *If Γ_1 is dense in M, then there are (simultaneously) complete u- and s-sections for f.*

We will prove that if the union Γ of Γ_1 and Γ_2 is dense in M, then there exists a complete u-section for f. This will be done in the next section.

5.3. Construction of u-sections

We say that a compact central curve J is *c-contracting* if it is a circle or a simple segment such that $f^k(J)$ is contained in the interior of J for some $k > 0$. The *local stable manifold of a central curve* J, denoted by $W^s_{\mathrm{loc}}(J)$, is the union of the local stable leaves of the points in J.

Lemma 5.7. *Let J be a c-contracting central curve. Then the local stable manifold of the orbit of J is a u-section.*

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Proof. Remark that the only thing to verify is that this local stable manifold is a surface tangent to $E^s \oplus E^c$. This fact follows as in the proof of the local stable manifold theorem. □

Proposition 5.8. *Suppose that $\Gamma = \Gamma_1 \cup \Gamma_2$ is dense in M. Then given any point z there are a c-contracting central curve J and a neighbourhood U_z of z such that the local unstable leaf of y intersects $W^s_{\rm loc}(J)$ for every $y \in U_z$.*

This proposition immediately implies Proposition 5.6: by compacity, there are finitely many points $z_1, \dots, z_n$ such that the neighbourhoods $U_{z_1}, \dots, U_{z_n}$ cover M, then the union of their corresponding u-sections is a complete u-section.

We will use the following remark which follows immediately from the continuity of the central bundle E^c.

Remark 5.9. There is $\delta > 0$ such that every central curve of length less than δ has no auto-intersections. In particular, the length of any circle contained in a central curve is at least δ.

Lemma 5.10. *Suppose that the set Γ is dense in M. Then for every $z \in M$ there are a neighbourhood U_z of z, a periodic point $p \in \Gamma$, and a segment I of length δ (δ as Remark 5.9) contained in a curve γ_p given by Lemma 5.2 such that the union of the local strong stable manifolds of the points in the interior of I intersects the local unstable leaf of every point in U_z.*

Proof. Consider the local unstable leaf $F^u_{\rm loc}(z)$ of z and the topological surface Δ obtained considering the union of the local stable leaves of the points in $F^u_{\rm loc}(z)$. The set $(U_z \setminus \Delta)$ has two connected components, say U^ℓ_z and U^r_z, where the points in U^ℓ_z are at the left of the points U^r_z (following the orientation of the central leaves). By the density of Γ in M, there is a hyperbolic periodic point $p \in \Gamma \cap U^\ell_z$. If U_z is sufficiently small (diameter less than $\delta/2$), then the segment I of length δ in γ_p starting at p meets Δ, ending the proof of the lemma. □

Proof of Proposition 5.8. To prove the proposition we now need to modify (by extending or shrinking) the segment I given by Lemma 5.10 in order to get the announced c-contracting curve J.

Recall that I is contained in the curve γ_p, $\gamma_p \colon]-1, +\infty[\to M$, parametrized following the positive orientation of E^c. Let $I = \gamma_p([0, s])$ and k the period of p.

To construct J we need to distinguish four cases, according to the indices of the first and the last periodic points of I.

The point p has index 2 and the last periodic point $\gamma_p(t)$ in I has index 2. Then $s \geqslant t$ and, for every $\varepsilon > 0$ small enough, the curve $J = \gamma_p([-\varepsilon, s + \varepsilon])$ contains the same periodic points as $\gamma_p([0, t])$ and $f^k(J)$ is contained in the interior of J. In this case J is a c-contracting curve.

The point p has index 1 and the last periodic point $\gamma_p(t)$ in I has index 2. Then $s \geqslant t$ and, for every $\varepsilon > 0$ small enough, the curve $J = \gamma_p([\varepsilon, s + \varepsilon])$ contains every

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periodic point of $\gamma_p([0,s])$ except the point p, and $f^k(J)$ is contained in the interior of J. In this case J is a c-contracting curve.

Moreover, the union of the local strong stable manifolds of the points in J intersects transversely the local strong unstable manifold of every point close to z.

The point p has index 2 and the last periodic point $\gamma_p(t)$ in I has index 1. As $p \in \Gamma$, there is a periodic point $q = \gamma_p(r)$, $r > s$, of index 2. Then, for every $\varepsilon > 0$ small enough, the curve $J' = \gamma_p([-\varepsilon, r+\varepsilon])$ is such that $f^k(J')$ is contained in the interior of J'.

Observe that J' may exhibit auto-intersections and there are the following possibilities: J' is either a circle or a c-contracting curve (in these cases we take $J = J'$) or the union of a circle C and a segment H having one extremity in the circle (where $C \cap H$ is a periodic point). In this last case we need to shrink J'.

The point p has index 1 and the last periodic point $\gamma_p(t)$ in I has index 1. Since $p \in \Gamma$, there is a periodic point $q = \gamma_p(r)$, $r > s$, of index 2. Then, for every $\varepsilon > 0$ small enough, the curve $J' = \gamma_p([\varepsilon, r+\varepsilon])$ is such that $f^k(J')$ is contained in the interior of J'.

Moreover, if ε is small enough, the union of the local strong stable manifolds of the points in J' meets transversely the local strong unstable manifold of every point close to z.

As above, J' is either a circle or a c-contracting curve (in this case, $J = J'$) or the union of a circle C and a segment H. In this last case we will shrink J'.

It remains to consider the case where $J' = C \cup H$ (C is a circle, H is a segment, and $H \cap C$ is a periodic point), corresponding to items (3) and (4) above. In such a case we will replace J' by a smaller curve J being a circle or a c-contracting curve.

Let w be the point of J' whose local strong stable leaf intersects the local unstable leaf of z. There are the following possibilities for the position of w in J'.

- $w \in C$: in this case we let $C = J$ (a circle).
- $w \notin C$ (i.e. w is in the interior of H): by construction of J', there are two possibilities.
  - Either the point w belongs to the stable manifold of some periodic point q of index 2: in this case let J be the intersection of the local stable manifold of q and J'. Observe that J is a c-contracting curve.
  - Or w is a periodic point of index 1: In this case, by construction of J', there are two periodic points q_1 and q_2 of index 2 in the curve J', $q_1 \in H$ and q_2 at the right of w, such that the sub-interval $[q_1, q_2]$ of J' joining q_1 and q_2 contains w. Then we take J as the intersection of J' and the local stable manifold of $[q_1, q_2]$. The resulting interval is a c-contracting curve.

The comments above end the proof of Proposition 5.8. □

The proof of Proposition 5.6 is now complete. □

6. Proof of Theorem 1.6 (Theorem 1.9 (3))

First let us recall that, for $f \in \mathcal{T}_0(M)$ the bundle E^c is uniquely integrable, so there is a central foliation $\mathcal{F}^c_f$ tangent to E^c. Moreover, for $f \in \mathcal{T}_0^+(M)$ the foliations $\mathcal{F}^s_f$, $\mathcal{F}^c_f$ and $\mathcal{F}^u_f$ are all orientable and the diffeomorphism f preserves these orientations. Finally, by Propositions 3.1 and 4.1, to prove Theorem 1.6 it is enough to construct a pair of complete (u- and s-) sections for a dense subset of $\mathcal{T}_0^+(M)$.

Let γ be a periodic compact leaf of the central foliation. For simplicity, we will assume that γ is fixed, the periodic case follows similarly. The curve γ is a normally hyperbolic compact manifold, so that one can speak of the stable and unstable manifolds, $W^s(\gamma)$ and $W^u(\gamma)$, of γ. These manifolds are the union of the stable and unstable leaves (respectively) through the points in γ. As the bundles E^s and E^u are one dimensional and oriented, the manifolds $W^s(\gamma)$ and $W^u(\gamma)$ are two-dimensional cylinders. Then the set $(W^s(\gamma) \setminus \gamma)$ has two connected components, denoted by $W^s_-(\gamma)$ and $W^s_+(\gamma)$, following the convention that the oriented leaves of $\mathcal{F}^s_f$ go from $W^s_-(\gamma)$ to $W^s_+(\gamma)$. Both $W^s_-(\gamma)$ and $W^s_+(\gamma)$ are f-invariant so, by Scholium 2.2, they are both dense in M.

Similarly, given any point x we let $F^c_-(x)$ and $F^c_+(x)$ be the two connected components of $(F^c(x) \setminus x)$, where the points in $F^c_-(x)$ are at the left of the points of $F^c_+(x)$ following the orientation of the central manifold.

We now construct a complete u-section for f. The construction of the complete s-section is obtained analogously.

Proposition 6.1. *Every local stable manifold of γ is a complete u-section for f.*

Proof. One of the difficulties of the proof of the proposition arises from the fact that $W^s(\gamma)$ is not *a priori* saturated by the central foliation (recall that it is saturated for $\mathcal{F}^s_f$). The next lemma allows us to bypass this difficulty.

Lemma 6.2. *Consider the component $W^s_+(\gamma)$ of $W^s(\gamma)$. There is $i \in \{-,+\}$ such that, for all $x \in W^s_+(\gamma)$, the component $F^c_i(x)$ is contained in $W^s_+(\gamma)$.*

There is a similar result for the component $W^s_-(\gamma)$.

Proof. The fact that the lemma holds for x in a small neighbourhood U of γ in W^s_+ is a general fact for foliations on surfaces in the neighbourhood of a compact leaf. Now, observing that $W^s_+(\gamma) = \bigcup_{i\in\mathbb{Z}} f^i(U)$, we extend this property to the whole $W^s_+(\gamma)$. □

End of the proof of Proposition 6.1. Given $f \in \mathcal{T}_0^+(M)$ consider any leaf F^u of the strong unstable foliation and an (arbitrarily small) segment L^u of it. For each point $x \in L^u$ take a small segment of the central foliation centred at x, say $L^c(x) \subset F^c(x)$. As, by hypothesis, f is dynamically coherent the union of these central segments is a disk Σ tangent to $E^c \oplus E^u$, thus transverse to the strong stable foliation. Denote by Σ^- and Σ^+ the two connected components of $(\Sigma \setminus L^u)$, where the central leaves go from Σ^- to Σ^+.

Notice that $W^s_+(\gamma)$ remains dense after removing from it the local stable manifold of γ. Let $W^s_{+,0}(\gamma)$ be this reduced separatrix. By construction, there is $\delta > 0$ such that, for

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each $x \in W^s_{+,0}(\gamma)$, the stable segment of length δ centred at x is contained in $W^s_+(\gamma)$. Now, using the density of $W^s_{+,0}(\gamma)$, we get that $W^s_+(\gamma) \cap \Sigma$ is dense in Σ.

We fix a pair of points $x^- \in L^c_-(x) \subset \Sigma^-$ and $x^+ \in L^c_+(x) \subset \Sigma^+$. Since $W^s_+(\gamma) \cap \Sigma$ is dense in Σ there are intersections $z^\pm$ of $W^s_+(\gamma) \cap \Sigma^\pm$ arbitrarily close to the points $x^\pm$.

Observe that by definition (orientation criterion) one has that

- $F^c_+(z^-)$ intersects L^u,
- $F^c_-(z^+)$ intersects L^u.

Let $i \in \{+,-\}$ be such that the component $F^c_i(w)$ of $(F^c(w) \setminus w)$ is contained in $W^s_+(\gamma)$ for all $w \in W^s_+(\gamma)$, recall Lemma 6.2. By the previous comment it follows that if $i = +$ then $F^c_+(z^-)$ intersects L^u. Otherwise, $i = -$, $F^c_-(z^+)$ intersects L^u. In both cases we have that $W^s_+(\gamma)$ intersects (transversely) L^u, ending the proof of the proposition. □

7. Extensions of the theorems for non-orientable bundles

The proof of Proposition 5.1 in the general case allows many possibilities: each bundle E^s, E^u and E^c may be orientable or not, and the diffeomorphism f may preserve or not these orientations. We split $\mathcal{T}(M)$ into the following subsets.

- $\mathcal{T}^{c+}(M)$ is the subset of $\mathcal{T}(M)$ of diffeomorphisms f such that E^c is orientable and f preserves this orientation. This set is open and closed in $\mathcal{T}(M)$ and contains $\mathcal{T}^+(M)$.
- $\mathcal{T}^{\mathrm{odd}}(M)$ is the subset of $\mathcal{T}(M)$ of diffeomorphisms f such that E^c is orientable, f reverses this orientation, and f has a hyperbolic periodic of odd period. This set is open in $\mathcal{T}(M)$.
- $\mathcal{T}^-(M)$ is the set of diffeomorphisms f in $\mathcal{T}(M)$ having a hyperbolic periodic orbit p (of period k) such that $f^k_*(p)$ reverses the orientations of $E^c(p)$. Observe that this set is open.
- $\mathcal{T}^\pm(M)$ is the interior of $(\mathcal{T}(M) \setminus (\mathcal{T}^{c+}(M) \cup \mathcal{T}^-(M)))$, that is, the set of diffeomorphisms $f \in \mathcal{T}(M)$ such that E^c does not admit any f-invariant orientation, but for every periodic point q of f and every g close enough to f, $g^r_*(q)$ preserves the orientations of $E^c(q)$ (where r is the period of q).

As in the previous sections we denote by $\mathcal{T}^{c+}_K(M)$, $\mathcal{T}^-_K(M)$ and $\mathcal{T}^\pm_K(M)$ the subsets of $\mathcal{T}^{c+}(M)$, $\mathcal{T}^-(M)$ and $\mathcal{T}^\pm(M)$, respectively, consisting of Kupka–Smale diffeomorphisms whose periodic points are dense in M.

By definition, the union $\mathcal{T}^\pm(M) \cup \mathcal{T}^{c+}(M) \cup \mathcal{T}^-(M)$ is a dense open subset of $\mathcal{T}(M)$. Recall that $\mathcal{T}_a(M)$ is the subset of $\mathcal{T}(M)$ of diffeomorphisms having the robust accessibility property.

The next result generalizes Proposition 5.1 and implies Theorem 1.5.

Proposition 7.1. *The set $\mathcal{S}(M) \cup \mathcal{U}(M)$ contains a dense open subset of $\mathcal{T}^{c+}(M) \cup \mathcal{T}^{\mathrm{odd}}(M) \cup \mathcal{T}^-(M) \cup \mathcal{T}_a(M)$.*

In the next subsections we will prove this result.

7.1. Existence of central invariant curves in the non-orientable case. Proof of Proposition 5.1 for $f \in \mathcal{T}^{c+}(M)$

We begin this section by doing some general comments about coverings of M, the lift of transitive diffeomorphisms to these coverings, and their u- and s-sections.

Lifts of transitive diffeomorphisms

Given $f \in \mathcal{T}(M)$ let $\pi\colon \tilde{M} \to M$ be the covering of 8 leaves corresponding to the different orientations of the stable, unstable and central bundles of f. The diffeomorphism f acts in a natural way on these orientations, so that f has a lift $\tilde{f}$ defined on $\tilde{M}$. Observe that diffeomorphism $\tilde{f}$ is strongly partially hyperbolic, its stable, unstable and central bundles are orientable, and $\tilde{f}$ preserves these orientations. However, $\tilde{f}$ may fail to be transitive. In fact, this is the main difficulty that appears in the non-orientable case. For diffeomorphisms in $\mathcal{T}^{c+}(M) \cup \mathcal{T}^{-}(M) \cup \mathcal{T}^{\mathrm{odd}}(M)$ this difficulty may be bypassed. For diffeomorphism in $\mathcal{T}^{\pm}(M)$ we will use the accessibility property (in fact, this is the only point in the paper where this condition is used).

Consider also the covering $\pi_c\colon M_c \to M$ of two leaves corresponding to the two orientations of the central bundle. Denote by f_c the corresponding lift of f to M_c. As above, f_c is strongly partially hyperbolic and it is not necessarily transitive.

Finally, observe that f is Kupka–Smale if and only if f_c is Kupka–Smale. Moreover, the periodic points of f are dense in M if and only the periodic points of f_c are dense in M_c. Similar results hold replacing f_c and M_c by $\tilde{f}$ and $\tilde{M}$.

Projections and lifts of sections

First, observe that $\tilde{M}$ is a covering of M_c and $\tilde{f}$ is a lift of f_c.

Observe that the projection by π_c of any complete s- or u-section for f_c (in M_c) onto M is a complete s- or u-section for f. Conversely, the lift of any complete u-section for f (in M) to M_c is a complete u-section for f_c.

By the comments above and Remark 5.5, Lemma 5.2 can be now applied to the lift $\tilde{f}$ of f, obtaining the following.

Remark 7.2. Let p be a periodic point of f (of period k), p_c a lift of p in M_c, and $\tilde{p}$ a lift of p_c in $\tilde{M}$. Denote by $\tilde{\gamma}$ the $\tilde{f}^{\tilde{k}}$-invariant curve starting at $\tilde{p}$ ($\tilde{k}$ is the period of $\tilde{p}$) given by Remark 5.9. Let γ_c be the projection of $\tilde{\gamma}$ onto M_c. Then γ_c is $f_c^{\tilde{k}}$-invariant curve positively oriented. Observe that the period of p_c is $\tilde{k}$, $\tilde{k}/2$ or $\tilde{k}/4$.

Moreover, if the bundle E^c is orientable and f preserves this orientation, the projection γ of $\tilde{\gamma}$ is a $f^{\tilde{k}}$-invariant curve positively oriented.

7.1.1. *Proof of Proposition 7.1 for diffeomorphisms in $\mathcal{T}^{c+}(M)$*

Consider any $f \in \mathcal{T}_K^{c+}(M)$, that is the bundle E^c is orientable and f preserves this orientation. Then the sets Γ_1, Γ_2 and Γ_3 defined exactly as in § 5 are f-invariant and their union is dense in M. As in the case $f \in \mathcal{T}_K^{+}(M)$, the closure of some Γ_i has non-empty interior. Thus, by the transitivity of f, such a closure is the whole M. From this fact we now deduce that any $f \in \mathcal{T}_K^{c+}(M)$ has a complete u- or s-section, ending the proof of Proposition 7.1 for diffeomorphisms in $\mathcal{T}^{c+}(M)$.

7.1.2. *Proof of Proposition 7.1 for diffeomorphisms in* $\mathcal{T}^{\mathrm{odd}}(M)$

As in the previous case, for $f \in \mathcal{T}_K^{\mathrm{odd}}(M)$ the orientability of E^c allows us to define the sets Γ_i, whose union is dense in M. These sets are not necessarily invariant, but they are f^2-invariant. Thus if f^2 is transitive the proof of the previous case works in this one. The next lemma ensures that f^2 is transitive for generic diffeomorphisms f of $\mathcal{T}_K^{\mathrm{odd}}(M)$, implying Proposition 7.1 for $f \in \mathcal{T}_K^{\mathrm{odd}}(M)$.

Lemma 7.3. *Let $f \in \mathcal{T}_K^{\mathrm{odd}}(M)$. Suppose that f has a periodic point p of odd period such that its homoclinic class is the whole M. Then the set of periodic points of odd period of f is dense in M and f^2 is transitive.*

Proof. Just observe that the homoclinic class of p is the closure of a union of hyperbolic sets containing p and having a Markov partition. Now given any point q there are periodic orbits passing arbitrarily close to p and q. If all these orbits have even periods then we can consider new orbits whose itineraries are obtained by adding the itinerary of p (i.e. an odd number of iterates). This completes the proof of the first part of the lemma.

Consider now a point x whose forward orbit is dense in M and a point q of odd period k. Then there is a sequence n_i such that $f^{n_i}(x)$ converges to q. Observe that $f^{n_i+k}(x)$ also converges to q. Then either n_i or $n_i + k$ contains a subsequence of even numbers. Thus the closure of the orbit of x by f^2 contains all the periodic points of odd period and so it is dense in M. This completes the proof of the lemma. □

7.1.3. *Lift of homoclinic classes. Proof of Proposition 7.1 for diffeomorphisms in* $\mathcal{T}^{-}(M)$

Let us state a general result about the lift of homoclinic classes. Given a hyperbolic periodic point denote by $H(p, f)$ its homoclinic class (the closure of the transverse intersections of its invariant manifolds).

Lemma 7.4. *Let p be a hyperbolic periodic point of f and p_1 and p_2 the lifts of p to M_c. Then*

$$\pi_c(H(p_1, f_c)) = \pi_c(H(p_2, f_c)) = H(p, f),$$

the homoclinic classes of p_1 and p_2 are interchanged by the automorphism of the covering π_c, and

$$H(p_1, f_c) \cup H(p_2, f_c) = \pi_c^{-1}(H(p, f)).$$

Moreover, if the points p_1 and p_2 are in the same f_c-orbit or they are homoclinically related, then $H(p_1, f_c) = H(p_2, f_c) = \pi_c^{-1}(H(p, f))$.

Proof. Observe that the invariant manifolds of p_i projects onto the (corresponding) invariant manifold of p and that $W^i(p_1) \cup W^i(p_2) = \pi_c^{-1}(W^i(p))$, $i = s$ or u.

Consider a homoclinic point x of p, $x \in W^s(p) \cap W^u(p)$, and let x_1 and x_2 be the two lifts of x to M_c. Up to changing the indices of these points, we can assume that

$x_1 \in W^s(p_1)$. Then, using the automorphism of the covering, we get that $x_2 \in W^s(p_2)$. Similarly, if $x_1 \in W^u(p_2)$ then $x_2 \in W^s(p_1)$, and if $x_2 \in W^u(p_2)$ then $x_1 \in W^s(p_1)$.

Suppose first that $x_1 \in W^u(p_2)$ and $x_2 \in W^u(p_1)$. Therefore, $x_1 \in W^s(p_1) \cap W^u(p_2)$ and $x_2 \in W^u(p_1) \cap W^s(p_2)$. By the partial hyperbolicity of $\tilde{f}$, these intersections are transverse. Thus p_1 and p_2 are homoclinically related and their homoclinic classes are equal. Now it is easy to check that $H(p_i, f_c) = \pi_c^{-1}(H(p, f))$.

For the remaining case, i.e. for every homoclinic point $x \in W^s(p) \cap W^u(p)$ one has that the lifts x_1 and x_2 are such that (up to change of indices) $x_i \in W^s(p_i) \cap W^u(p_i)$, $i = 1, 2$. In this case, to get $H(p_i, f_c) = \pi_c^{-1}(H(p, f))$, it is enough to consider the closure of these transverse intersections of $W^s(p_i)$ and $W^u(p_i)$. This ends the proof of the lemma. □

Remark 7.5. Let p be a hyperbolic periodic point of f of period k.

- If f^k reverses the orientations of $E^c(p)$, then the lifts p_1 of p_2 of p in M_c are in the same f_c-orbit. Thus, by Lemma 7.4, the lift of the homoclinic class of p to M_c is a homoclinic class of f_c.

- Otherwise, if f^k preserves the orientations of $E^c(p)$, Lemma 7.4 ensures that the lift of the homoclinic class of p is the union of two homoclinic classes (corresponding to the homoclinic classes of the two lifts p_1 and p_2 of p), which (*a priori*) may be different.

We are now ready to prove Proposition 7.1 for diffeomorphisms f in $\mathcal{T}_K^-(M)$. Following [**3**, Théorème B], there is a residual subset $\mathcal{R}_K^-(M)$ of $\mathcal{T}_K^-(M)$ of diffeomorphisms such that the homoclinic class of any periodic point of f is the whole M. By definition of $\mathcal{T}_K^-(M)$, there is a periodic point p of f such that $f_*^k(p)$ (k the period of p) reverses the orientation of $E^c(p)$. The first part of Remark 7.5 implies that the homoclinic class of p_c is the whole M_c, where p_c is any lift of p in M_c. In particular, f_c is transitive.

Observing that f_c preserves the orientation of the central bundle E^c, the proof follows (in M_c) as for $f \in \mathcal{T}_K^+(M)$. This ends the proof of Proposition 7.1 for diffeomorphisms in $\mathcal{T}^-(M)$.

7.1.4. *End of the proof of Proposition 7.1: the case* $f \in \mathcal{T}_K^{\pm}(M)$

In view of the two previous subsections to end the proof of Proposition 7.1 it remains to consider the diffeomorphisms f in $\mathcal{T}_K^{\pm}(M)$. As in previous cases, there is a residual subset $\mathcal{R}_K^{\pm}(M)$ of $\mathcal{T}_K^{\pm}(M)$ of diffeomorphisms such that every homoclinic class is equal to M. By Lemma 7.4, if there is some periodic point p of f having a homoclinic class whose lift to M_c is a unique homoclinic class then such a class is M_c. In this case we can construct a complete section as in the previous section. Thus it remains to consider the case where *the lift to* M_c *of any homoclinic class of* f *is the union of two different homoclinic classes*. To deal with this situation we need the following proposition.

Proposition 7.6. *Let $f \in \mathcal{T}(M)$ be a diffeomorphism such that*

- *there is a hyperbolic periodic point p of f whose homoclinic class is equal to M, and*
- *f satisfies the accessibility property.*

Then the lift of the homoclinic class of p to M_c is the whole M. In particular, the lift f_c of f is transitive.

Arguing as in the previous cases we have that this proposition implies Proposition 7.1 for $f \in \mathcal{T}_K^{\pm}(M)$.

Proof. The proof is by contradiction: let p_1 and p_2 be the two lifts of p to M_c. Assume that $H(p_i, f_c) \neq M_c$, $i = 1, 2$. By Lemma 7.4, the orbits of p_1 and p_2 are disjoint and these two points are not homoclinically related. Moreover, again by Lemma 7.4, $H(p_1, f_c) \cup H(p_2, f_c) = M_c$ and these homoclinic classes are interchanged by the automorphism of the covering π_c. Thus these two classes have both non-empty interior and the union of their interiors is dense in M_c.

We claim that if the intersection A of these two interiors is non-empty, then A is dense in M_c. In fact, we have the following:

- the projection of A by π_c in M is f-invariant, hence it is dense in M (due to transitivity of f),
- the set A is invariant by the automorphism of the covering.

These two facts imply that the homoclinic classes of p_1 and p_2 are both M_c, ending the proof of the proposition when A is non-empty.

It remains to consider the case in which these two interiors are disjoint ($A = \emptyset$). In this case the boundaries of the homoclinic classes of p_1 and p_2 are equal and f_c-invariant. Denote by ∂ such a boundary.

Lemma 7.7. *The set ∂ is saturated by $\mathcal{F}^s_f$ and $\mathcal{F}^u_f$.*

Before proving this lemma, let us observe that it contradicts the accessibility hypothesis, ending the proof of Proposition 7.6.

Proof. By continuity of the foliations $\mathcal{F}^s_f$ and $\mathcal{F}^u_f$, it is enough to see that the interiors of the homoclinic classes of p_1 and p_2 are both saturated by $\mathcal{F}^s_f$ and $\mathcal{F}^u_f$.

As the periodic orbits homoclinically related to p_1 are dense in $H(p_1, f_c)$, there is a periodic orbit q homoclinically related to p_1 in the interior of $H(p_1, f_c)$. Observe that $H(p_1, f_c)$ is f_c-invariant, thus the invariant manifolds of q are contained in the interior of $H(p_1, f_c)$. Thus, since $H(p_1, f_c)$ is closed,

$$\text{closure}(W^s(q)) \cup \text{closure}(W^u(q)) \subset H(p_1, f_c).$$

To see the converse inclusion, observe that, since q is homoclinically related to p_1, the closure of each invariant manifold of q contains the corresponding invariant manifold of

p_1, which, on its turn, contains a dense subset of $H(p_1, f_c)$. Thus the invariant manifolds of q are both dense in $H(p_1, f_c)$. Hence

$$H(p_1, f_c) \subset \text{closure}(W^s(q)), \text{closure}(W^u(q)).$$

Thus

$$\text{closure}(W^s(q)) = \text{closure}(W^u(q)) = H(p_1, f_c).$$

As the closure of saturated set is saturated, and $W^s(q)$ and $W^u(q)$ are saturated sets, this ends the proof of the lemma. □

Now the proof of Proposition 7.6 is complete. □

7.2. Theorem 1.6 in the non-orientable case

We have the following generalization of Theorem 1.6.

Theorem 7.8. *Consider the subset $\mathcal{T}_0^a(M)$ of $\mathcal{T}_0(M)$ of diffeomorphisms satisfying the robust accessibility property. Then the open set $\mathcal{O}^s(M) \cap \mathcal{O}^u(M)$ of diffeomorphisms whose strong stable and unstable foliations are both robustly minimal is dense in $\mathcal{T}_0^a(M)$.*

The proof follows arguing exactly as in the previous subsections, where the orientation hypothesis was replaced by the robust accessibility property.

As in § 7.1.4, if $f \in \mathcal{T}_0(M)$ has a unique accessibility class, then it admits a transitive lift $\tilde{f}$ on some orientations covering.

8. Generalization to higher dimensions. Proof of Theorem 1.7

Let M be a compact manifold of any dimension. Let us recall that $\mathcal{T}_1(M)$ denotes the set of robustly transitive diffeomorphisms f of M whose central bundle has dimension 1, is uniquely integrable and admits an f-invariant orientation.

Proposition 8.1. *Every Kupka–Smale diffeomorphism of $\mathcal{T}_1(M)$ whose periodic points are dense in M has a complete s- or u-section.*

The proof of this proposition is identical to the proof of Proposition 5.1, observing that the unique integrability of the central bundle substitutes trivially Lemma 5.2. As the Kupka–Smale diffeomorphisms whose periodic points are dense in M is dense in $\mathcal{T}_1(M)$, this completes the proof of Theorem 1.7. □

As in the previous cases, we can state Theorem 1.7 in the non-orientable case for diffeomorphisms with the robust accessibility property.

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References

1. J. Alves, Ch. Bonatti and M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly expanding, *Invent. Math.* **140** (2000), 351–398.
2. Ch. Bonatti and L. J. Díaz, Persistence of transitive diffeomorphims, *Ann. Math.* **143** (1995), 367–396.
3. Ch. Bonatti and L. J. Díaz, Connexions hétéroclines et généricité d'une infinité de puits ou de sources, *Annls Sci. Ec. Norm. Super.* **32** (1999), 135–150.
4. Ch. Bonatti and M. Viana, SRB measures for partially hyperbolic attractors: the mostly contracting case, *Israel J. Math.* **115** (2000), 157–193.
5. Ch. Bonatti, L. J. Díaz and E. R. Pujals, A $\mathcal{C}^1$-generic dichotomy for diffeomorphisms: weak forms of hyperbolicicity or infinitely many sinks or sources, preprint PUC-Rio (1999), *Ann. Math.*, in press.
6. Ch. Bonatti, L. J. Díaz, E. R. Pujals and J. Rocha, Robust transitive sets and heterodimensional cycles, *Asterisque*, in press.
7. R. Bowen and B. Marcus, Unique ergodicity for horocycle foliations, *Israel J. Math.* **26** (1977), 43–67.
8. M. Brin and Ya. Pesin, Partially hyperbolic dynamical systems, *Math. USSR Izv.* **8** (1974), 177–218.
9. K. Burns, C. Pugh, M. Shub and A. Wilkinson, *Recent results about stable ergodicity*, Proceedings of Symposia in Pure Mathematics, vol. 69 (2001).
10. M. Carvalho, Sinai–Ruelle–Bowen measures for N-dimensional derived from Anosov diffeomorphisms, *Ergod. Theory Dynam. Syst.* **13** (1993), 21–44.
11. L. J. Díaz, Robust nonhyperbolic dynamics and heterodimensional cycles, *Ergod. Theory Dynam. Syst.* **15** (1995), 291–315.
12. L. J. Díaz and J. Rocha, Partially hyperbolic and transitive dynamics generated by heteroclinic cycles, *Ergod. Theory Dynam. Syst.* **21** (2001), 25–76.
13. L. J. Díaz, E. R. Pujals and R. Ures, Partial hyperbolicity and robust transitivity, *Acta Math.* **183** (1999), 1–43.
14. D. Dolgopyat, On the dynamics of mostly contracting diffeomorphisms, *Commun. Math. Phys.* **213** (2000), 181–201.
15. D. Dolgopyat and A. Wilkinson, Stable accesibility is C^1-dense, *Asterisque*, in press.
16. J. Franks, Anosov diffeomorphisms on tori, *Trans. Am. Math. Soc.* **145** (1969), 117–124.
17. H. Furstenberg, The unique ergodicity of the horocycle flow, In *Recent Advances in Topological Dynamics, Proc. Conf. Yale University, New Haven, CT, 1972, In honor of Gustav Arnold Hedlund*, pp. 95–115, Lecture Notes in Mathematics, vol. 318 (Springer, 1973).
18. S. Hayashi, Connecting invariant manifolds and the solution of the C^1-stability and Ω-stability conjectures for flows, *Ann. Math.* **145** (1997), 81–137.
19. M. Hirsch, C. Pugh and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, vol. 583 (Springer, 1977).
20. R. Mañé, Contributions to the stability conjecture, *Topology* **17** (1978), 386–396.
21. R. Mañé, An ergodic closing lemma, *Ann. Math.* **116** (1982), 503–540.
22. A. Manning, There are no new Anosov on tori, *Am. J. Math.* **96** (1974), 422–429.
23. S. Newhouse, On codimension one Anosov diffeomorphisms, *Am. J. Math.* **92** (1970), 761–770.
24. V. Niţikă and A. Török, An open dense set of stably ergodic diffeomorphisms in a neighbourhood of a non-ergodic one, *Topology* **40** (2001), 259–278.
25. J. F. Plante, Anosov flows, *Am. J. Math.* **94** (1972), 729–754.

26. C. Pugh and M. Shub, Stably ergodic dynamical systems and partial hyperbolicity, *J. Complexity* **13** (1997), 125–179.
27. M. Shub, *Topological transitive diffeomorphism on T^4*, Lecture Notes in Mathematics, vol. 206, p. 39 (Springer, 1971).
28. M. Shub and A. Wilkinson, Stably ergodic approximation: two examples, *Ergod. Theory Dynam. Syst.* **20** (2000), 875–894.