Nontame Morse–Smale flows and odd Chern–Weil theory

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Abstract. Using a certain well-posed ODE problem introduced by Shilnikov in the sixties, Minervini proved the currential "fundamental Morse equation" of Harvey–Lawson but without the restrictive tameness condition for Morse gradient flows. Here, we construct local resolutions for the flow of a section of a fiber bundle endowed with a vertical vector field which is of Morse gradient type in every fiber in order to remove the tameness hypothesis from the currential homotopy formula proved by the first author. We apply this to produce currential deformations of *odd degree closed forms* naturally associated to any hermitian vector bundle endowed with a unitary endomorphism and metric compatible connection. A transgression formula involving smooth forms on a classifying space for odd K-theory is also given.

1 Introduction

The well-known Morse Lemma gives the canonical form of a Morse function f on a compact, Riemannian manifold (M, g) around a critical point but does not provide information about the gradient flow. On the other hand, the Hartman–Grobman theorem gives the *topological* conjugacy class of the gradient flow around a critical point. However, there are important situations where both of these classical results are insufficient as in the next context. Suppose one is interested in taking a smooth submanifold S and "flow it through" a critical point. More precisely, suppose S lies within a regular level $c - \varepsilon$ of the Morse function "before" a critical level and we look at its "trace" at a regular level $c + \varepsilon$, meaning the intersection of the (forward) flow lines determined by S with level $c + \varepsilon$. Obviously, this "trace" can be empty if S is contained in the stable manifold of the critical point. So a transversality condition with the stable manifold is naturally imposed. The natural question is can one say anything about the structure of the *closure* of the "trace" at level $c + \varepsilon$? One expects to get at least a rectifiable set because of transversality. In order to show this, one aims in general to prove something slightly stronger, namely the existence of a manifold with corners of the same dimension as the submanifold and a proper "projection" which maps to the closure of the trace and is one-to-one almost everywhere. This is the resolution. In order to make this rigorous, a "tameness" condition on the triple (M, g, f) is very helpful. We recall the definition from [12]. A Morse function f is called *tame* if around each critical point one can find coordinates that fulfill two requirements, the metric is

flat in these coordinates and the Morse function has the canonical form of the Morse Lemma. An immediate consequence of tameness is that the eigenvalues of the Hessian of f are ± 1. This gives an idea of how restrictive tameness is. On the positive side, the flow has the simplest form possible and one can prove, by performing a blow-up of the intersection of the submanifold S with the stable manifold of the critical point that a resolution of the closure of the "trace" is available. In fact, one can prove that such a manifold with corners resolution is available also for the closure of the entire "flow-out" of the submanifold between levels $c - \varepsilon$ and $c + \varepsilon$. More general situations are contemplated in [5, 15].

The existence of such resolutions has important consequences. The "fundamental Morse equation" proved by Harvey–Lawson in [12] says that for the gradient flow φ : $\mathbb{R} \times M \to M$ induced by a tame f, which additionally satisfies Smale's transversality condition there exists a rectifiable current T on $M \times M$ such that

$$dT = \Delta - \sum_{p \in \mathrm{Crit}(f)} U_p \times S_p,$$

where Δ is the diagonal and U_p and S_p are the unstable, resp. the stable manifold of the critical point p. To be a bit more precise, the submanifold in this case is the diagonal and the flow is on $M \times M$ via φ in the first component of the product and keeping fixed the second component. In his PhD thesis, Latschev [15] also used the resolution idea and extended these results to Morse–Bott–Smale flows. The first author developed this point of view further in [5] in order prove a generalization of the "fundamental Morse equation" this time for sections of fiber bundles, satisfying adequate transversality conditions. Even with the tameness condition in place, the rigorous details of the construction of the resolution are quite involved. This is related to the fact that flowing the image of the section of a fiber bundle through the critical locus raises new challenges, bearing in mind that the transversality conditions naturally imposed do not give information about how the section intersects the critical locus, but just how it intersects the *stable manifolds*.

Completely new ideas are necessary in order to deal with the *nontame* case. In his PhD thesis, Minervini [18] used a combination of results of Shilnikov [22] on a certain type of ODE problems together with objects he introduced, called "horned stratified spaces" in order to prove the Harvey–Lawson theorem without the tameness condition. Applications to Morse–Novikov theory were given by Harvey and Minervini in [14]. In this article, we take the next natural step and remove the tameness condition from the currential homotopy formula of [5]. One slight simplification is added in, the model flows in each fiber are assumed here Morse as opposed to Morse–Bott in [5]. We plan to return to the Morse–Bott case somewhere else.

The novelty here is that we implement a combination of ideas of Latschev [15] and Minervini [18] in our present approach and hence we do not rely on the "horned stratified spaces" of the latter. Roughly, we use Shilnikov–Minervini local analysis of the closure of the graph of the flow which will give a local resolution (see Theorem 2.6) of the flow-out of the section, but use induction on the critical levels ala Latschev for the proof of the next homotopy formula.

Theorem 1.1   *Let $\pi : P \longrightarrow B$ be a fiber bundle with compact fiber. Let X be a horizontally constant Morse–Smale vertical vector field and denote by $\Phi : \mathbb{R} \times P \to P$ the flow*

induced by X. Let $s : B \longrightarrow P$ be a section transverse to all the stable manifolds $S(F)$ associated to the critical manifolds F of X and let $\xi_t(b) := (\Phi_t(s(b)), s(b))$, $b \in B$. Then

$$T = \xi([0, +\infty) \times B)$$

defines a $(n+1)$-dimensional rectifiable current of locally finite mass and if B is compact then T is of finite mass.

  Moreover, the following equality of currents holds in $P \times_B P$:

(1.1)
$$dT = \sum_F U(F) \times_F s(s^{-1}(S(F))) - (\xi_0)_*(B),$$

where $U(F)$ are the unstable manifolds of X.

Recall that a vertical vector field X on the total space of a fiber bundle $P \to B$ is called horizontally constant if there exist local trivializations of the fiber bundle such that X has a zero horizontal component in this trivialization. As a consequence, the flow induce by X is, up to diffeomorphism, the same in every fiber. It also means that if the flow in the fiber is Morse–Bott, then the critical sets F of X are manifolds and so are the sets $S(F)$ and $U(F)$.

An immediate consequence (see Corollary 3.3) of Theorem 1.1 is the explicit computation of the following limits in the weak sense

$$\lim_{t \to \infty} s_t^* \omega,$$

where $s_t := \Phi_t \circ s$ and $\omega \in \Omega^*(P)$. This immediately justifies a transgression formula for closed forms ω

(1.2)
$$\lim_{t \to \infty} s_t^* \omega - s^* \omega = dT(\omega)$$

by using the standard yoga of kernels versus operators.

This Poincaré duality type of result is a source of many applications (see [6]) even in the tame case. In an early preprint of [5] posted on arXiv an application to (1.2) concerning certain odd degree forms on the unitary group was included. The flow used, however, did not satisfy the tameness hypothesis and the application was removed from the published version. We present it here in a more general context, but not before revisiting a classical topic and introducing some new objects which seem of independent interest.

Chern–Weil theory is an important source of closed forms arising from geometric data. To any complex vector bundle $E \to B$ of rank n endowed with a connection ∇ and a $GL(n)$ invariant polynomial P in the entries of an $n \times n$ matrix one has an associated closed form $P(F(\nabla))$. For homogeneous P one gets that $P(F(\nabla))$ is of *even* degree, more precisely twice the degree of P. The deRham cohomology class of $P(F(\nabla))$ does not depend on ∇.

In order to get odd degree forms we endow $E \to B$ with an automorphism $A : E \to E$. Then, we associate to the quadruple (E, A, ∇, P) a closed form $TP(E, A, \nabla)$ which satisfies the following properties: it is natural with respect to pull-back, the cohomology class determined by $TP(E, A, \nabla)$ does not depend on the connection ∇, the same cohomology class does not change under deformations of A in the same homotopy class. We prove all these properties in Section 5 for hermitian vector

bundles but the interested reader can adapt the results without difficulty to other structure groups. While these objects are instances of the Chern–Simons difference forms, they should not be confused with what are widely called Chern–Simons forms [2], forms that live on the *total space* of a principal bundle and which are not closed. Related formulas for the transgression of the Pfaffian form have appeared in [1, 8].

Let $P = c_k$ be the invariant polynomial induced by the kth elementary symmetric polynomial. The following statement, which generalizes a result of Nicolaescu ([19], Proposition 57) also gives a description of the Poincaré duals to $\mathrm{Tc}_k(E, g, \nabla)$.

Theorem 1.2 *Let $E \to B$ be a trivializable hermitian vector bundle of rank n over an oriented manifold with corners B endowed with a compatible connection. Let $g : E \to E$ be a smooth gauge transform. Suppose that a complete flag $E = W_0 \supset W_1 \supset \cdots \supset W_n = \{0\}$ (equivalently a trivialization of E) has been fixed such that g as a section of $\mathcal{U}(E)$ is completely transverse to certain (see (6.1)) submanifolds $S(U_I)$ determined by the flag. Then, for each $1 \le k \le n$ there exists a flat current T_k such that the following equality of currents of degree $2k - 1$ holds:*

$$(1.3) \qquad \mathrm{Tc}_k(E, g, \nabla) - g^{-1}(S(U_{\{k\}})) = dT_k,$$

where

$$g^{-1}(S(U_{\{k\}})) = \{ b \in B \mid \dim \mathrm{Ker}\,(1 + g_b) \cap (W_{k-1})_b = 1,$$
$$\dim \mathrm{Ker}\,(1 + g_b) \cap (W_k)_b = 0 \}.$$

In particular, when B is compact without boundary, then $\mathrm{Tc}_k(E, g, \nabla)$ and $g^{-1}(S(U_{\{k\}}))$ are Poincaré duals to each other and (1.3) is a spark equation [2, 13].

The condition that $E \to B$ be trivializable is related to the nontame flow used in the proof which requires the existence of a complete flag $E = W_0 \supset \cdots \supset W_n = \{0\}$ of vector sub-bundles. It is an interesting question of how one can describe the Poincaré duals to $\mathrm{Tc}_k(E, g, \nabla)$ for a general E.

The next application is to families of self-adjoint Fredholm operators. Fix H a Hilbert space. The space of unitary operators $U \in \mathcal{U}(H)$ such that $1 + U$ is Fredholm is a classifying space for odd K-theory. This space is a Banach manifold but is "too big" to build smooth differential forms. Restricting the attention to the Palais classifying spaces $\mathcal{U}^p$ which are unitary operators of type $1 + S$ where S belongs to some Schatten ideal, e.g., trace class or Hilbert–Schmidt operators then Quillen [21] was able to construct several families of smooth forms all representing the components of the odd Chern character. When one has a smooth family of Dirac operators $\mathcal{D}_{b \in B}$ parametrized by a smooth and finite dimensional manifold B then by taking the Cayley transforms one gets a smooth map $\varphi : B \to \mathcal{U}^p$. The pull-backs of the Quillen forms compute the cohomological analytic index determined of the family.

Let us remark that in the finite dimensional case, the Quillen forms on $U(n)$ have explicit formulas in terms of the odd Chern–Weil forms arising from the trivial vector bundle $\mathbb{C}^n$ over $U(n)$ endowed with the tautological unitary endomorphism and trivial connection, i.e., in terms of the standard deRham generators of the cohomology ring of $U(n)$.

On the other hand, in [4], we produced explicit representatives for the Poincaré duals of these classes using the infinite dimensional analogues of the stable manifolds

$S(U_{\{k\}})$ which appear in Theorem 1.2. We used sheaf theory in [4] in order to be able to define cohomology classes arising from certain stratified spaces, called quasi-manifolds on an infinite dimensional Banach manifold. Here, we exchange the sheaf theoretical approach from [4] with the currential approach and show that under the expected transversality hypothesis one can produce a transgression formula, strengthening thus the results from [4]. Transgression formulas in an infinite dimensional context have appeared also in [24].

Theorem 1.3   *Let $\varphi : B \to \mathcal{U}^p$ be a smooth map from a compact, oriented manifold B, possibly with corners such that $\varphi \pitchfork Z_I^p$ for every I. Let Ω_k be a Quillen form of degree $2k - 1$ that makes sense on $\mathcal{U}^p$. Then for every such Ω_k, there exists a flat current T_k such that:*

(1.4) $$\varphi^{-1} Z_{\{k\}} - (-1)^{k-1}(k-1)! \varphi^* \Omega_k = dT_k.$$

In particular, when B has no boundary, $\frac{(-1)^{k-1}}{(k-1)!} \varphi^{-1} Z_{\{k\}}^p$ represents the Poincaré dual of $\mathrm{ch}_{2k-1}([\varphi])$, where $[\varphi] \in K^{-1}(B)$ is the natural odd K theory class determined by φ.

The proof of this result reduces to Theorem 1.2 via symplectic reduction.

A few more comments about the structure of the article are in order. Section 2 which revisits Shilnikov theory, also adds some details to Minervini's presentation in [18]. In particular, Theorem 2.4 introduces some flow-convex neighborhoods that are fundamental later on. The main technical part of the proof of the main Theorem 1.1 is contained in the rather long Section 4. We felt it necessary to present many complete arguments. The proof is by induction and the amount of data one has to carry from one step to another is quite substantial. That is why we paid special care in proving properties like properness or injectivity of the flow-resolution map. To get a feel for the level of technicality the reader can take a quick glance at Proposition 4.9 which is the key step in the induction. In essence, the main idea of the proof of the main Theorem is to follow the same steps as the induction proof presented in the Appendix of [5], but to substitute the oriented blow-up technique which takes care of the local picture in [5] with Minervini's Theorem 1.3.21 which appears here as Theorem 2.6. The advantage of the presentation in [5] via blow-ups is that several maps are explicit and several properties come for free (e.g., a blow-down map is proper). This, of course, is a consequence of tameness. On the negative side, one works hard in [5] to show that the relevant maps have regularity C^1 while here the regularity is C^∞ and it is a consequence of the Minervini–Shilnikov theory.

The models in the fiber are classical Morse–Smale flows associated to gradients of Morse functions. The results ought to hold also for the Morse–Smale quasi-gradients as defined in [17]. One point that made us cautious is contained in Remark 2.5 and is related to the properties of the flow-convex neighborhoods of Theorem 2.4.

A proof of the main Theorem appeared in the PhD thesis [20] of the second author. Some arguments have been simplified in this presentation.

2 Minervini-Shilnikov theory

We review some results about certain well-posed ODE problems studied by Shilnikov in the 1960s. We borrowed the terminology that gives the title of this section from the

main reference [18]. Where the complete proofs were skipped, the reader will find the details in chapter 1 of [18].

Let (x, y) be coordinates in $\mathbb{R}^s \times \mathbb{R}^u$. With respect to this decomposition, let $L = \begin{bmatrix} L^- & 0 \\ 0 & L^+ \end{bmatrix}$ be a constant, real coefficients matrix, in which the real parts of the eigenvalues of L^- are strictly negative, say $-\lambda_s \leq \cdots \leq -\lambda_1 < 0$, and those of L^+ are strictly positive, say $0 < \mu_1 \leq \cdots \leq \mu_u$. For the situation, we are interested in, L is symmetric.

Consider the ODE system in $\mathbb{R}^s \times \mathbb{R}^u$:

$$(2.1) \qquad \begin{cases} \dot{x} = L^- x + f(x, y) \\ \dot{y} = L^+ y + g(x, y), \end{cases}$$

where $F = (f, g) : \mathbb{R}^s \times \mathbb{R}^u \longrightarrow \mathbb{R}^s \times \mathbb{R}^u$ is a differentiable function satisfying

$$F(0, 0) = (0, 0) \quad \text{and} \quad dF(0, 0) = (0, 0).$$

Given a triple $(x_0, y_1, \tau) \in \mathbb{R}^s \times \mathbb{R}^u \times [0, +\infty)$ a boundary value problem (BVP) for the ODE (2.1) has the following form

$$(2.2) \qquad \begin{cases} \dot{x} = L^- x + f(x, y) \\ \dot{y} = L^+ y + g(x, y) \\ x^*(0) = x_0 \\ y^*(\tau) = y_1, \end{cases}$$

where the solution $(x^*(t), y^*(t))$ is defined in the interval $[0, \tau]$. The solution at time t to the BVP (2.2) with data (x_0, y_1, τ) is denoted

$$(x^*(t, x_0, y_1, \tau), y^*(t, x_0, y_1, \tau)).$$

The "end point" (x_1^*, y_0^*) for the BVP solution is

$$(2.3) \qquad \begin{array}{l} x_1^*(x_0, y_1, \tau) = x^*(\tau, x_0, y_1, \tau) \\ y_0^*(x_0, y_1, \tau) = y^*(0, x_0, y_1, \tau). \end{array}$$

We compare this with the solution at time t of the initial value problem with data $(x_0; y_0, t = 0)$ for which the following notation is used

$$(x(t, x_0, y_0), y(t, x_0, y_0)).$$

Notice that

$$(2.4) \qquad \begin{array}{l} x(t, x_0, y_0) = x^*(t, x_0, y(\tau, x_0, y_0), \tau) \\ y(t, x_0, y_0) = y^*(t, x_0, y(\tau, x_0, y_0), \tau) \\ x^*(t, x_0, y_1, \tau) = x(t, x_0, y^*(x_0, y_1, \tau)) \\ y^*(t, x_0, y_1, \tau) = y(t, x_0, y_0^*(x_0, y_1, \tau)). \end{array}$$

Let

$$(2.5) \qquad \delta_\varepsilon^k := \sup_{|x, y| \leq \varepsilon} \sum_{|m| \leq k} \left| \frac{\partial^{|m|} F}{\partial(x, y)^m} \right| < +\infty,$$

where $|x, y| := \max\{|x|, |y|\}$ and $| \cdot |$ denotes the euclidian norm.
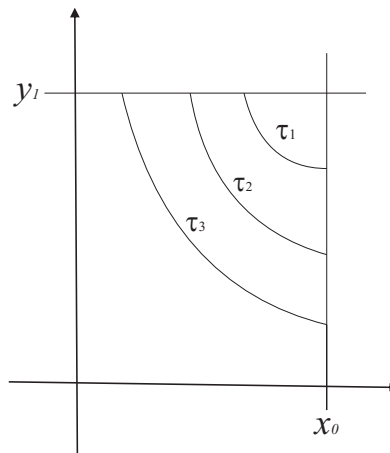
Figure 1: Boundary Value Problem

Quite similarly to the Cauchy problem for ODE and proceeding in the standard way, i.e., writing the BVP as a system of integral equations and using Banach Fixed Point theorem, the following general result holds:

Theorem 2.1 *Suppose $\varepsilon > 0$ is such that the the estimate $\delta^1_{2\varepsilon} < \min\{\lambda_1, \mu_1\}$ holds. Then the BVP for the system* (2.2) *is solvable for any data (x_0, y_1, τ) in the "ball" $|x_0, y_1| < \varepsilon$. The solution is unique, it depends smoothly on all its arguments and satisfies:*

$$|x^*(t), y^*(t)| \le 2|x_0, y_1|, \quad \forall t \in [0, \tau].$$

It turns out that the integral equations equivalent to the BVP (2.2) make sense also for $\tau = \infty$ in which the only given spatial coordinate is $x^*(0) := x_0$. The correspondence $x_0 \to y^*(0)$ is smooth and its graph is an invariant manifold of the flow, tangent to $y = 0$ at the origin. This is in fact the stable manifold of the origin and using the obvious change in coordinates that takes the graph diffeomorphically to the domain of definition one notices that the original vector field

$$X := (X_1, X_2) = (L^- x + f(x, y), L^+ y + g(x, y))$$

gets conjugated to one for which the stable and unstable manifolds coincide with the x and the y axes at least locally.

Theorem 2.2 *There are smooth coordinates centered at the origin such that* (2.1) *can be written as*

(2.6)
$$\begin{cases} \dot{x} = L^- x + \tilde{f}(x, y)x \\ \dot{y} = L^+ y + \tilde{g}(x, y)y, \end{cases}$$

where $\tilde{f} : \mathbb{R}^{s+u} \longrightarrow \mathrm{End}(\mathbb{R}^s)$ and $\tilde{g} : \mathbb{R}^{s+u} \longrightarrow \mathrm{End}(\mathbb{R}^u)$ are square matrices of functions that vanish at the origin. Moreover, in a neighborhood of the origin in the new coordinates, the stable and unstable manifolds are given by $S_0 = \{y = 0\}$ and $U_0 = \{x = 0\}$.

Gronwall Lemma is used to prove some useful estimates for solutions of BVP in straighten coordinates.

Theorem 2.3   *Let $\varepsilon, \alpha > 0$ be such that $\delta := \delta_{2\varepsilon}^1 < \alpha < max\{\lambda_1, \mu_1\}$. Then the solution of the BVP defined by system (2.6) with spatial data $|x_0, y_1| \leq \varepsilon$ satisfies for any $\tau \in [0, \infty)$ and $t \leq \tau$ the following inequality*

(2.7)
$$\begin{cases} |x^*(t, x_0, y_1, \tau)| \leq |x_0| e^{-(\alpha-\delta)t} \\ |y^*(t, x_0, y_1, \tau)| \leq |y_1| e^{(\alpha-\delta)(t-\tau)}. \end{cases}$$

In particular,

(2.8)
$$\begin{cases} |x_1^*(x_0, y_1, \tau)| \leq |x_0| e^{-(\alpha-\delta)\tau} \\ |y_0^*(x_0, y_1, \tau)| \leq |y_1| e^{-(\alpha-\delta)\tau}. \end{cases}$$

Estimates are available also for any partial derivative $\frac{\partial^k x_1^*}{\partial(x_0, y_1, \tau)}$ or $\frac{\partial^k y_0^*}{\partial(x_0, y_1, \tau)}$ of order k in the form:

(2.9)
$$\left| \frac{\partial^k x_1^*}{\partial(x_0, y_1, \tau)}\Big|_{(x_0, y_1, \tau)} \right| \leq C_k e^{-(\alpha-k\delta)\tau}, \qquad \left| \frac{\partial^k y_0^*}{\partial(x_0, y_1, \tau)}\Big|_{(x_0, y_1, \tau)} \right| \leq C_k e^{-(\alpha-k\delta)\tau}$$

for some constant C_k which does not depend on (x_0, y_1, τ).

Let Ω be a neighborhood around the origin and suppose the vector field X has the form (2.6).

We will choose a "cube" $C_\varepsilon = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u; |x, y| \leq \varepsilon\} \subset \Omega$ of radius $\varepsilon > 0$ which satisfies the hypothesis of Theorem 2.1. The cube has the following boundary pieces:

$$\partial^+ C_\varepsilon = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u; |x| = \varepsilon, |y| \leq \varepsilon\}$$

and

$$\partial^- C_\varepsilon = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u; |x| \leq \varepsilon, |y| = \varepsilon\}.$$

Denote

$$V_0^\varepsilon = \{(x, y) \in C_\varepsilon \mid |x| \cdot |y| = 0\} = S_0 \cup U_0.$$

In order to state the next result the partial order notation for the flow determined by X is useful, i.e.,

$$p_1 \prec p_2$$

will say that there exists a *forward* flow line from p_1 to p_2.

Theorem 2.4   *Suppose L^- and L^+ are symmetric.*
Then for ε small enough X is transverse to $\partial^+ C_\varepsilon$ and to $\partial^- C_\varepsilon$. In addition, the following properties hold for C_ε with respect to X:

(1) **Flow-convexity**: *for every pair $q_1 \prec q_2$ with $q_1, q_2 \in C_\varepsilon$ and every $q_1 \prec p \prec q_2$ one has $p \in C_\varepsilon$ and*
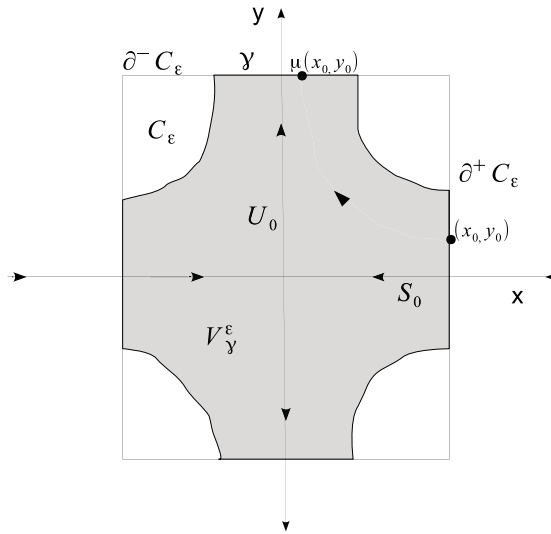
Figure 2: Dulac map and the neighborhoods V_γ^ε

(2) **Dulac map**: *there exists a "first encounter" diffeomorphism*

$$\mu = (\mu_1, \mu_2) : \partial^+ C_\varepsilon \backslash S_0 \longrightarrow \partial^- C_\varepsilon \backslash U_0$$

induced by the flow that satisfies

$$\mu(x, y) = (x, y), \qquad \forall (x, y) \in \partial^+ C_\varepsilon \cap \partial^- C_\varepsilon.$$

(3) **Continuity of μ_1 close to S_0**: *for every $0 < \gamma \le \varepsilon$, there exists $0 < \gamma_0 \le \varepsilon$ such that*

$$\forall (x, y) \in \partial^+ C_\varepsilon \text{ with } |y| \le \gamma_0 \text{ one has } |\mu_1(x, y)| \le \gamma;$$

(4) **Fundamental neighborhoods**: *let $0 < \gamma \le \varepsilon$ and*

$$V_\gamma^\varepsilon := \{ p \in C_\varepsilon \mid \exists \, q = (x_1, y_1) \in \partial^- C_\varepsilon, \ |x_1| < \gamma, \ p < q \} \cup V_0^\varepsilon$$

Then V_γ^ε is a flow-convex neighborhood of V_0^ε in C_ε such that

$$V_\gamma^\varepsilon \to V_0^\varepsilon,$$

i.e., for every neighborhood U of V_0^ε there exists $\gamma_0 > 0$ such that $V_{\gamma_0}^\varepsilon \subset U$.

The following figures illustrate the properties encountered in Theorem 2.4.

Remark 2.5 Without the symmetry property of $L = L^- \oplus L^+$ the claim about transversality of X with $\partial^+ C_\varepsilon$, $\partial^- C_\varepsilon$ fails even in the linear case. This seems to have been overlooked in [18]. Take $s = 0$, $u = 2$, $L^+ := \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\tilde{g} = 0$. Then

$$\langle L^+(y_1, y_2), (y_1, y_2) \rangle = (y_1 + y_2)^2.$$

Hence, $X = L^+$ on $\mathbb{R}^2$ is hyperbolic but has points of tangency along the antidiagonal with any coordinate sphere.

Proof The tangent spaces to the cylinders $\partial^+ C_\varepsilon$ and $\partial^- C_\varepsilon$ at points (x, y) are described via:

$$T_{(x,y)}\partial^+ C_\varepsilon = \{(v_1, v_2) \in \mathbb{R}^n \mid \langle v_1, x \rangle = 0\},$$

$$T_{(x,y)}\partial^- C_\varepsilon = \{(v_1, v_2) \in \mathbb{R}^n \mid \langle v_2, y \rangle = 0\}.$$

Hence, we need to look at

$$\langle X_1(x, y), x \rangle = \langle L^- x, x \rangle + \langle \tilde{f}(x, y)x, x \rangle$$

and

$$\langle X_2(x, y), y \rangle = \langle L^+ y, y \rangle + \langle \tilde{g}(x, y)y, y \rangle.$$

We have that $\langle \tilde{f}(x, y)x, x \rangle = o(|x|^2)$ uniformly in y and $\langle \tilde{g}(x, y)y, y \rangle = o(|y|^2)$ uniformly in y since $\tilde{f}$ and $\tilde{g}$ are continuous and vanish at the origin.

The symmetry and definiteness of L^- and L^+ imply now that there exists ε_1 and ε_2 such that

$$\langle L^- x, x \rangle + \langle \tilde{f}(x, y)x, x \rangle < 0 \qquad \forall\, 0 < |x| \le \varepsilon_1, \forall\, |y| \le \varepsilon_1$$

and

$$\langle L^+ y, y \rangle + \langle \tilde{g}(x, y)y, y \rangle > 0 \qquad \forall\, 0 < |y| \le \varepsilon_2, \forall\, |x| \le \varepsilon_2.$$

For $\varepsilon \le \min\{\varepsilon_1, \varepsilon_2\}$ we get the transversality of X with both $\partial^\pm C_\varepsilon$.

Proof of (1). Let $q_1 = (x_1, y_1) \prec q_2 = (x_2, y_2), q_1, q_2 \in C_\varepsilon$. Let $\tau > 0$ be the time needed to "travel" from q_1 to q_2, i.e., $(x_2, y_2) = (x(\tau, x_1, y_1), y(\tau, x_1, y_1))$. Now, consider the BVP defined by X with data (x_1, y_2, τ). Since $|x_1, y_2| \le \varepsilon$, it follows from the Theorem 2.3 that for all $0 \le t \le \tau$, the following holds:

$$|x^*(t, x_1, y_2, \tau)| \le |x_1| e^{-(\alpha - \delta)t} < |x_1| < \varepsilon,$$
$$|y^*(t, x_1, y_2, \tau)| \le |y_2| e^{(\alpha - \delta)(t - \tau)} < |y_2| < \varepsilon.$$

Therefore, the portion of the trajectory comprised between q_1 and q_2 is contained in C_ε.

Proof of (2). Consider $(x_0, y_0) \in \partial^+ C_\varepsilon$ with $|y_0| < \varepsilon$. The normal vectors v at (x_0, y_0) to $\partial^+ C_\varepsilon$ that point to the interior of C_ε are described by the inequality $\langle v_1, x_0 \rangle < 0$ and we already chose ε so that the vector field X satisfies such an inequality. Hence, either the trajectory $(x(t, x_0, y_0), y(t, x_0, y_0))$ will belong to the interior of C_ε for small $t > 0$ or $(x_0, y_0) \in \partial^+ C_\varepsilon \cap \partial^- C_\varepsilon$ i.e. $|y_0| = \varepsilon$, which is not the case.

Suppose $t > 0$. For $y_0 \ne 0$ we have $\lim_{t \to +\infty} (x(t), y(t)) \ne (0, 0)$ and the trajectory cuts again the boundary of C_ε. This happens because in the cube C_ε the function $t \to |x(t)|$ is decreasing while $t \to |y(t)|$ is increasing. Indeed the derivatives of the square of these functions are equal to

$$\langle X_1(x, y), x \rangle \text{ and } \langle X_2(x, y), y \rangle, \text{ respectively}$$

and by our choice of ε above the first one is negative while the second one is positive as long as they are not 0 which would happen for a stable or unstable flow line.

Let $(x_1, y_1) \in \partial^+ C_\varepsilon \cup \partial^- C_\varepsilon$ be the first point where this trajectory hits the boundary again and $\tau > 0$ the time needed to get from (x_0, y_0) to (x_1, y_1).

Since $\langle X_1(x, y), x \rangle$ is a continuous function we see that $(x_1, y_1) \notin \partial^+ C_\varepsilon \backslash S_0$ and moreover $|x_1| < \varepsilon$, because

$$(2.10) \qquad |x_1| = |x^*(\tau, x_0, y_1, \tau)| \le |x_0| e^{-(\alpha-\delta)\tau} < |x_0| \le \varepsilon.$$

Define μ, for the time being, as the map that associates to every $(x_0, y_0) \in \partial^+ C_\varepsilon \backslash (S_0 \cup \partial^- C_\varepsilon)$ the point of "first encounter" $(x_1, y_1) \in \partial^- C_\varepsilon \backslash (U_0 \cup \partial^+ C_\varepsilon)$ which lies on the same trajectory.

Suppose now that $|x_0| = |y_0| = \varepsilon$. If there exist $\tau_0 > 0$ such that $(x(\tau_0, x_0, y_0), y(\tau_0, x_0, y_0)) = (x_0', y_0') \in C_\varepsilon$, define the BVP with data (x_0, y_0', τ_0). Then, we get a contradiction from the estimates

$$\varepsilon = |y_0| = |y^*(0, x_0, y_0', \tau_0)| \le |y_0'| e^{-(\alpha-\delta)\tau_0} < |y_0'| < \varepsilon.$$

Hence, for $|x_0| = |y_0| = \varepsilon$, the trajectory determined by (x_0, y_0) only intersects C_ε in (x_0, y_0). Therefore, the natural extension of μ to $\partial^+ C_\varepsilon \cap \partial^- C_\varepsilon$ is equal to the identity on this set.

The map μ is clearly bijective. The differentiability of μ is standard and proved along the following lines. The essential part is to prove the differentiability of the time-function that associates to each (x_0, y_0) the time $t(x_0, y_0)$ it takes to get to $\mu(x_0, y_0)$. Fix one such (x_0, y_0) and use the diffeomorphism of the flow that corresponds to time $t(x_0, y_0)$ to flow a small open neighborhood of (x_0, y_0) inside $\partial^+ C_\varepsilon$ which does not contain points in the stable manifold to an $n-1$ dimensional manifold H which contains $\mu(x_0, y_0)$ and is still transverse to X. Since $\mu(x_0, y_0)$ is not a critical point put coordinates in order to turn X into the generator of the first coordinate-translation by unit-time. One arrives thus at the problem of having to prove differentiability of the time-function obtained by going from one hypersurface to another hypersurface through the origin and both transverse to the first coordinate. That is simply the difference in height functions where the height is the first coordinate, hence a differentiable function.

Proof of (3). Suppose that there exists $0 < \gamma \le \varepsilon$ and a sequence (x_{0n}, y_{0n}) such that $|x_{0n}| = \varepsilon, |y_{0n}| \longrightarrow 0 \, (y_{0n} \neq 0)$ when $n \to \infty$ and $|x_{1n}| \ge \gamma$ for all n where $\mu(x_{0n}, y_{0n}) = (x_{1n}, y_{1n}) \in \partial^- C_\varepsilon$. Let $\tau_n > 0$ be the sequence of moments such that

$$(x(\tau_n, x_{0n}, y_{0n}), y(\tau_n, x_{0n}, y_{0n})) = (x_{1n}, y_{1n}).$$

It follows from the following estimates (see also (2.10))

$$(2.11) \qquad \gamma \le |x_{1n}| \le |x_{0n}| e^{-(\alpha-\delta)\tau_n} \le \varepsilon e^{-(\alpha-\delta)\tau_n}$$

that $\tau_n \le \ln\left(\dfrac{\varepsilon}{\gamma}\right) \dfrac{1}{\alpha - \delta}$.

We can therefore select a convergent subsequence $(x_{0n_k}, \tau_{n_k}) \longrightarrow (x_0', \tau')$. Note that $\tau' > 0$, because if $\tau' = 0$ then $|y_{0n_k}| \longrightarrow \varepsilon$. Indeed, the only points for which it takes 0-time to go from $\partial^+ C_\varepsilon$ to $\partial^- C_\varepsilon$ are the points on $\partial^+ C_\varepsilon \cap \partial^- C_\varepsilon$ and it is not

hard to see that, due to the continuity of the time-function, the shorter the time it takes to get to $\partial^- C_\varepsilon$ the closer to $\partial^+ C_\varepsilon \cap \partial^- C_\varepsilon$ the starting point has to be.

Since $|y(\tau_{n_k}, x_{0n_k}, y_{0n_k})| = |y_{1n_k}| = \varepsilon$ we get that a contradiction with

$$y(\tau_{n_k}, x_{0n_k}, y_{0n_k}) \to y(\tau', x', 0) = 0.$$

The later holds because the trajectory determined by $(x', 0)$ is stable.

Proof of (4). The flow-convex property is immediate from the analogous property of C_ε.

Suppose the V_γ^ε is not a neighborhood of V_0. Hence, there exists a sequence $(x_n, y_n) \in C^\varepsilon \backslash V_\gamma^\varepsilon$ with $(x_n, y_n) \longrightarrow (v_1, v_2) \in V_0^\varepsilon$. Thus either $x_n \longrightarrow 0$ or $y_n \longrightarrow 0$.

As none of the points (x_n, y_n) is on $S_0 \cup U_0$ there exist points of "first encounter" $(x_{0n}, y_{0n}) \in \partial^+ C_\varepsilon$ and $(x_{1n}, y_{1n}) \in \partial^- C_\varepsilon$ such that

$$(x_{0n}, y_{0n}) \prec (x_n, y_n) \prec (x_{1n}, y_{1n}).$$

Since $(x_n, y_n) \notin V_\gamma^\varepsilon$ it follows that $|x_{1n}| \geq \gamma$.

The case $x_n \to 0$ is disposed immediately by considering the BVP (x_n, y_{1n}, τ_n) where τ_n is the time it takes to go from (x_n, y_n) to (x_{1n}, y_{1n}). Then, we get a contradiction in

$$\gamma \leq |x_{1n}| \leq |x_n| e^{-(\alpha-\delta)\tau_n} \leq |x_n| \to 0.$$

For the case $y_n \to 0$ consider the BVP with data $(x_{0n}, y_{1n}, \tau_n')$ where τ_n' now is the time it takes to get from (x_{0n}, y_{0n}) to (x_{1n}, y_{1n}). Recall that $|x_{0n}| = |y_{1n}| = \varepsilon$. We obtain

$$\gamma \leq |x_{1n}| \leq |x_{0n}| e^{-(\alpha-\delta)\tau_n'} \implies \tau_n \leq \ln\left(\frac{\varepsilon}{\delta}\right) \frac{1}{\alpha - \delta}.$$

$$|y_{0n}| \leq |y_n| e^{-(\alpha-\delta)s_n} \leq |y_n|,$$

where $0 < s_n \leq \tau_n'$. Since $|y_n| \longrightarrow 0$, we have $y_{0n} \longrightarrow 0$.

Now, since $|x_{0n}| = \varepsilon$ and τ_n' are bounded and $y_{0n} \to 0$, we are in the same scenario that led to a contradiction in the proof of item (3).

In order to prove that $\lim_{\gamma \to 0} V_\gamma^\varepsilon = V_0^\varepsilon$ as defined in the statement it is enough to prove that

$$(2.12) \qquad\qquad\qquad\qquad \bigcap V_{\gamma_n}^\varepsilon = V_0^\varepsilon$$

for any decreasing sequence $\gamma_n \to 0$. Indeed if (2.12) holds then fix one such sequence γ_n. Suppose $\exists u_n \in V_{\gamma_n}^\varepsilon \backslash U$ for all n. Since $\overline{V_{\gamma_n}^\varepsilon} \subset V_{\gamma_{n+1}}^\varepsilon$ and $\overline{V_{\gamma_n}^\varepsilon}$ (the closure in C_ε) is compact we get that we can extract a convergent subsequence u_{n_k} whose limit necessarily belongs to $\bigcap V_{\gamma_n}^\varepsilon = V_0^\varepsilon$. But this is a contradiction with U being a neighborhood of V_0^ε.

In order to prove (2.12), take a point $a = (x, y) \in \bigcap V_{\gamma_n}^\varepsilon$ such that $x \neq 0 \neq y$. Since $y \neq 0$ it follows that $a \notin S_0$ and therefore a lies on a trajectory which hits $\partial^- C_\varepsilon$ in a point $b = (x', y') \notin U_0$, i.e., $x' \neq 0$. But $a \in \bigcap V_{\gamma_n}^\varepsilon$ implies that $|x'| < \gamma_n$ for all n, hence $x' = 0$. Contradiction. ∎

The properties of these neighborhoods will enable us in Section 3 to control the deformation of subsets by the flow of a vector field.

The fundamental local tool we will use in the next sections is Theorem 1.3.21 in [18] which we now state.

Theorem 2.6 *Let $C_\varepsilon \subset \Omega$ be a cube as in Theorem 2.4 and let $\mathring{C}_\varepsilon$ be its interior. If ψ_t denotes the flow of the vector field $X = (L^- + f, L^+ + g)$, then the closure of the submanifold*

(2.13)
$$W = \left\{ (t, \psi_{\frac{t}{1-t}}(x,y), x, y);\ (x,y) \in \mathbb{R}^s \times \mathbb{R}^u,\ 0 < t < 1 \right\} \cap (\mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon)$$

inside $(\mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times \mathbb{R})$ is a smooth submanifold with boundary

(2.14)
$$\partial \overline{W} = \{1\} \times (\mathrm{U}_0 \cap \mathring{C}_\varepsilon) \times (\mathrm{S}_0 \cap \mathring{C}_\varepsilon) \bigcup \{0\} \times \triangle_{\mathring{C}_\varepsilon},$$

where $\triangle_{\mathring{C}_\varepsilon} = \{ (p,p) \mid p \in \mathring{C}_\varepsilon \}$ denotes the diagonal in $\mathring{C}_\varepsilon \times \mathring{C}_\varepsilon$.

Proof (Sketch) The idea is to turn to BVP coordinates via (2.4) in

$$W = \left\{ \left(\tau, x_1^* \left(x_0, y_1, \frac{\tau}{1-\tau} \right), y_1, x_0, y_0^* \left(x_0, y_1, \frac{\tau}{1-\tau} \right) \right) \,\middle|\, |x_0, y_1| < \varepsilon, \tau \in (0,1) \right\}.$$

Use now (2.9) to conclude that x_1^* and y_0^* converge uniformly to 0 when $\tau \to 1$ together with their derivatives, obtaining thus that W is the graph of a smooth function over $\mathring{C}_\varepsilon \times [0,1]$. ∎

Remark 2.7 It is easy to see that the projection of $\overline{W}$ onto the $\mathring{C}_\varepsilon \times \mathring{C}_\varepsilon$ components coincides with the intersection $\overline{\bigcup_{t \geq 0} \tilde{\psi}_t(\Delta_{\mathring{C}_\varepsilon})} \cap (\mathring{C}_\varepsilon \times \mathring{C}_\varepsilon)$ where $\tilde{\psi}$ is the flow on $\mathbb{R}^{s+u} \times \mathbb{R}^{s+u}$ that is equal to ψ in the first component and leaves the points fixed in the second component. Take $(a,b) \in \overline{\bigcup_{t \geq 0} \tilde{\psi}_t(\Delta_{\mathring{C}_\varepsilon})} \cap (\mathring{C}_\varepsilon \times \mathring{C}_\varepsilon)$. Then, there exist $t_n \geq 0$ and $u_n \in \mathbb{R}^{s+u}$ such that $(\psi_{t_n}(u_n), u_n) \to (a,b)$. Since $a, b \in \mathring{C}_\varepsilon$ one has that $u_n, \psi_{t_n}(u_n) \in \mathring{C}_\varepsilon$ for n big enough. In fact, by passing to a subsequence either t_n converges or $t_n \to \infty$. Let $s_n \in [0,1)$ be such that $t_n = \frac{s_n}{1-s_n}$ we get that $(a,b) = \lim z_n$ where $(z_n, s_n) \in W$ converges in $[0,1] \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon$. The other inclusion is also obvious.

Remark 2.8 We will need a slight extension of this results in the simplest situation when there exists a central manifold. If the vector field Y on $\mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$ is of type $(X, 0)$ with X as before, not depending on $z \in \mathbb{R}^c$ then rather than taking the graph of the flow ψ^Y of Y in $\mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$, it makes sense to forget about one stationary variable $\mathbb{R}^c$ and consider the corresponding W^Y of Theorem 2.6 to be

$$W^Y := \{ (t, \psi^X_{\frac{t}{1-t}}(x,y), x, y, z) \mid t \in (0,1) \} \cap [0,1] \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times \mathbb{R}^c.$$

The closure will be a manifold with boundary

$$\partial \overline{W^Y} = \{1\} \times (\mathrm{U}_0 \cap \mathring{C}_\varepsilon) \times (\mathrm{S}_0 \cap \mathring{C}_\varepsilon) \times \mathbb{R}^c \bigcup \{0\} \times \triangle_{\mathring{C}_\varepsilon} \times \mathbb{R}^c.$$

We do this rather than considering the graph of ψ^Y in the full ambient space with an eye to keep the bookkeeping simpler and avoid using fiber products later on.

We have a useful Corollary of Theorem 2.4.

Corollary 2.9 *Let* $(q, 0), (0, r) \in C_\varepsilon$ *be two points in the stable, respectively the unstable manifold of the origin. Let* $B^u(q, \varepsilon') := \{(q, y) \mid |y| \le \varepsilon'\}$ *be a small transverse slice to* S_0 *that passes through* $(q, 0)$*. Then there exists two sequences of points* $(q, y'_n), (x_n, y_n) \in C_\varepsilon$ *such that*

(i) $(q, y'_n) \in B^u(q, \varepsilon')$ *and* $y_n \to 0$;
(ii) $(x_n, y_n) \to (0, r)$; *and*
(iii) $(q, y'_n) \prec (x_n, y_n)$.

Finally, we will need the following consequence of the Flowout theorem [16] which is proved with the same ideas as the differentiability of the Dulac map μ from Theorem 2.4. Let M be a compact smooth Riemannian manifold and let ψ denote the gradient flow of a function f.

Proposition 2.10 *Let* $N \subset M$ *be a submanifold which does not contain any critical points of* f *and is transverse to the gradient vector field* ∇f*. Let* $f^{-1}(\theta)$ *be a regular level set of* f *such that* $N \cap f^{-1}(\theta) = \varnothing$ *and suppose that for every* $n \in N$ *there exists a time* $t_n \ge 0$ *such that* $\psi_{t_n}(n) \in f^{-1}(\theta)$*. Then the function* $n \to t_n$ *is smooth and moreover the set* $\{\psi_{t_n}(n) \mid n \in N\}$ *is a smooth submanifold of* $f^{-1}(\theta)$ *diffeomorphic to* N*.*

3 A general homotopy formula

Consider $\pi : P \longrightarrow B$ a locally trivial fiber bundle with compact fiber M over a n-dimensional, oriented manifold B. Let $X : P \longrightarrow VP$ be a vertical vector field on P where the vertical tangent space $VP := \operatorname{Ker} d\pi$ represents the collection of all the tangent spaces to the fibers. Suppose X a horizontally constant Morse–Smale vector field. Recall that this means that for every $b \in B$ there exists an open set $b \in B_0 \subset B$ and a local trivialization $\alpha : P|_{B_0} \longrightarrow M \times B_0$ such that

(3.1)
$$\alpha_*(X_p) = (\operatorname{grad} f_{\alpha_1(p)}, 0), \ \forall p \in P|_{B_0},$$

where $f : M \longrightarrow \mathbb{R}$ is a Morse–Smale function for same Riemannian metric on M. This in particular implies that the critical set of X is a fiber bundle over B (with various components) and the same thing stays true about the stable and unstable manifolds.

In fact the critical manifolds of X in the local trivialization α are $F = \{p\} \times B_0$ with p satisfying $\nabla_p f = 0$ and the stable and unstable manifolds are

$$S(F) = S(p) \times B_0, \qquad U(F) = U(p) \times B_0.$$

Let $s : B \longrightarrow P$ be a transversal section to all the stable bundles $S(F)$ relative to the critical manifold $F \subset P$.

We follow the ideas in [11]. Consider the fiber bundle $P \times_B P \longrightarrow B$ and the vertical vector field $\tilde{X} = (X, 0)$ on $P \times_B P$. It is not difficult to verify that the vector field $\tilde{X}$ is horizontally constant Morse–Bott–Smale and that its flow is $\Theta_t(v_1, v_2) = (\Phi_t(v_1), v_2)$, where $\Phi_t : P \longrightarrow P$ denotes the flow of the vector field X. Thus, the stable and unstable manifolds relative to a critical manifold $\tilde{F}$ of $\tilde{X}$ are

(3.2)
$$S(\tilde{F}) := S(F) \times_B P \text{ and } U(\tilde{F}) := U(F) \times_B P.$$

Now define the family of sections $\xi : [0, \infty) \times B \to P \times_B P$:

$$\xi_t(b) := \Theta_t(s(b), s(b)) = (\Phi_t(s(b)), s(b))$$

transverse to all stable manifolds of $\tilde{X}$.

Remark 3.1 The transversality of s and $S(F)$ translates into the transversality of ξ_0 and $S(\tilde{F})$

The question we will be concerned with in this section is whether the following family of currents in $P \times_B P$ has a limit:

$$\lim_{t \to +\infty} (\xi_t)_*(B)?$$

For each $t > 0$, Stokes Theorem and the commutativity of d with push-forward implies:

$$(3.3) \qquad (\xi_t)_*(B) - (\xi_0)_*(B) = d[\xi_*([0, t] \times B)].$$

From (3.3) and of the continuity of the (exterior) differential operator of currents, we have reduced our analysis to the existence of the limit $\lim_{t \to +\infty} \xi_*([0, t] \times B)$. The following result presents a positive response to the existence of this limit without the tameness condition of [5].

Theorem 3.2 *Let $\pi : P \longrightarrow B$ be a fiber bundle with compact fiber and let X be a horizontally constant Morse–Smale vertical vector field. If $s : B \longrightarrow P$ is a section transversal to the stable manifolds $S(F)$, then*

$$T = \xi([0, +\infty) \times B)$$

defines a $(n+1)$-dimensional current of locally finite mass and if B is compact then T is of finite mass. Moreover, the following equality of kernels holds:

$$(3.4) \qquad \mathrm{d}T = \sum_F \mathrm{U}(F) \times_F s(s^{-1}(S(F))) - \xi_*(B).$$

We deduce from Theorem 3.2 that

$$(3.5) \qquad \xi_\infty(B) := \lim_{t \to +\infty} (\xi_t)_*(B) = \sum_F \mathrm{U}(F) \times_F s(s^{-1}(S(F))).$$

Another consequence of the Theorem 3.2 is expressed in terms of operators:

Corollary 3.3 *Let $\pi : P \longrightarrow B$ be an fiber bundle with compact oriented fiber on a n-dimensional smooth, oriented B. Let $X : P \longrightarrow VP$ be a horizontally constant Morse–Smale vertical vector field and let $s : B \longrightarrow P$ be a section transversal to all stable manifolds $S(F)$ of X. Assume that for each critical manifold $F \subset P$ the bundle $\mathrm{U}(F) \longrightarrow F$ is oriented. Let $s_t = \Phi_t \circ s : B \longrightarrow P$ be the induced family of sections. Then for each closed form ω on P of degree $k \leq n$, the following identity of flat currents in B is true:*

$$(3.6) \qquad \lim_{t \to +\infty} s_t^* \omega = \sum_{\mathrm{codim} S(F) \leq k} \mathrm{Res}_F^u(\omega)[s^{-1}(S(F))],$$

where $\mathrm{Res}_F^u(\omega) = \tau_F^* \left(\int_{\mathrm{U}(F)/F} \omega \right)$ and $\tau_F : s^{-1}(\mathrm{S}(F)) \to F$ is the composition of $\pi_F^s :$ $\mathrm{S}(F) \to F$ with $s : s^{-1}(\mathrm{S}(F)) \longrightarrow \mathrm{S}(F)$. Moreover, there is a flat current $\mathcal{T}_\infty(\omega)$ such that

$$(3.7) \qquad \sum_{\mathrm{codim}\mathrm{S}(F) \le k} \mathrm{Res}_F^u(\omega)[s^{-1}(\mathrm{S}(F))] - s^*\omega = d[\mathcal{T}_\infty(\omega)].$$

Remark 3.4 One can extend Theorem 3.2 and Corollary 3.3 without extra effort to the case when B is a manifold with corners and $s : B \to P$ is a smooth section completely transverse to all $S(F)$. This happens because any smooth map on a manifold with corners can, by definition, be extended locally in a neighborhood of the corner a smooth map defined on an open set in $\mathbb{R}^n$. Since the same applies to the trivialization maps, one can extend smoothly the whole fiber bundle structure and the section to be defined on an open set in $\mathbb{R}^n$. The transversality condition being open, it will hold in an open subset. Then one uses the corresponding results of Theorem 3.2 and Corollary 3.3 in this open set, only to restrict it afterwards.

In order to prove Theorem 3.2, we notice that due to the sheaf property of currents it is enough to prove it for a convenient open covering of B. Namely, if there exists an open covering $B = \cup B_i$ such that $T_i := T\big|_{P_i}$ exists, where $P_i := P\big|_{B_i} \times_{B_i} P\big|_{B_i}$, it is of locally finite mass and (3.4) holds then it is of the same thing is true over B. This happens first because on the overlap $P_i \cap P_j$ the restriction of the limits $\xi_*([0,t] \times B_{i/j})$ are the same. This allows the patching of $(T_i)_{i \in I}$ to a single current. A similar argument works for the right hand side of (3.4).

Remark 3.5 One has to worry always if an embedded oriented submanifold really determines a current especially when it is not *properly* embedded, i.e., the inclusion is not a proper map. This is the case of each $\mathrm{U}(F) \times_F s(s^{-1}(\mathrm{S}(F)))$ and is not clear apriori that they exist globally. But if (3.4) holds on an open covering P_i then the patching of T_i to a single current implies the patching of $\mathrm{U}(F) \times_F s(s^{-1}(\mathrm{S}(F)))\big|_{P_i}$ to a single current globally and (3.4) stays true everywhere.

Remark 3.6 Notice that if $F_{\max}$ represents a "maximal" critical manifold, in other words, the critical manifold for which $S(F_{\max})$ is an open subset of P the transversality condition of $s \pitchfork S(F_{\max})$ is automatically satisfied for all points $b \in s^{-1}S(F_{\max})$. In the open set $s^{-1}(S(F_{\max})) \subset B$ relation (3.4) is trivial to prove and the limit

$$\lim_{t \to \infty} \xi_t(s^{-1}(S(F_{\max}))) = F_{\max} \times_{F_{\max}} s(s^{-1}(S(F_{\max})))$$

holds even in the C^∞ sense. The open sets $s^{-1}(S(F_{\max}))$ will always be part of the open coverings of B and we will not mention them again.

We will therefore work with a convenient covering of B, each of its members being contained into a trivializing neighborhood for X, by which we mean a neighborhood U where (3.1) holds.

The strategy is now *roughly* the following. We show that around each point $b_0 \in B$ there exists a trivializing neighborhood B_0 and a finite open covering M_j of the fiber

$M := P_{b_0}$ such that Theorem 3.2 holds for the restrictions of all the currents to the open sets:

$$M_j \times M \times B_0 \simeq (M_j \times B_0) \times_{B_0} (M \times B_0),$$

where we have already used a trivializing diffeomorphism $P\big|_{B_0} \simeq M \times B_0$.

In order to achieve this on each of the open sets $M_j \times M \times B_0$ we construct a "resolution of the flow," namely a *proper* map

$$\Psi : N \to M_j \times M \times B_0$$

from a manifolds with corners N of dimension $n+1$ such that

(a) $\operatorname{Im} \Psi = \overline{\xi([0,\infty) \times B_0)} \cap (M_j \times M \times B_0)$.
(b) Ψ is a diffeomorphism from an open subset of full measure in N to an open subset of full $\mathcal{H}^{n+1}$-measure in $\xi([0,\infty) \times B_0) \cap (M_j \times M \times B_0)$.

Point (b) makes sense in view of the fact that $\xi([0,\infty) \times B_0 \backslash Z)$ is a smooth submanifold of $P \times_B P$ of dimension $n+1$ where $Z := \bigcup_F s^{-1}(F)$ are the fixed points of the section with respect to the flow.

Remark 3.7 We assume of course that B_0 is a small neighborhood of a point $b_0 \notin F_{\max}$ and therefore Z will have zero $\mathcal{H}^n$ measure.

We then use the following

Lemma 3.8 *Let N^{n+1} be a manifold with corners and let $\Psi : N \longrightarrow X$ be a smooth map to a smooth manifold X. If Ψ is a proper map, then $\Psi(M)$ has locally finite $n+1$-dimensional Hausdorff measure and*

$$d(\Psi_*(N)) = \Psi_*(\partial N).$$

This Lemma will allow not only to conclude that the restriction of $T\big|_{B_0 \times M_j \times M}$ is of locally finite mass, but also to compute its boundary as the push-forward of ∂N. In order to do that, we will need a full understanding of ∂N and of the map Ψ.

4 The local flow resolution

This section contains the heart of the proof of Theorem 3.2.

As we discussed in the previous section, it is enough to localize around each point b_0 in the base space B. We will therefore consider first an open neighborhood B_0 of b_0 where the fiber bundle $P\big|_{B_0}$ is trivial and the vector field X is horizontally constant and given by the gradient of a function $f : P_{b_0} \to \mathbb{R}$. With the notation $M := P_{b_0}$ already introduced, we have the following data

(a) a flow on M induced by the gradient of f and denoted ψ.
(b) a family of (local) sections $s_t : B_0 \to M \times B_0$

$$s_t(b) = (\psi_t(s_0(b)), b)$$

originating from $s\big|_{B_0} = (s_0, \operatorname{id}_{B_0}) : B_0 \to P\big|_{B_0} = M \times B_0$.

We are interested in the closure of the forward flow-out of the graph of s. In other words, let $\xi_t : B_0 \to M \times M \times B_0$ be the family of sections:

$$\xi_t(b) = (\psi_t(s_0(b)), s_0(b), b).$$

The function $\tilde{f} : M \times M \times B_0 \to \mathbb{R}$, $\tilde{f}(m_1, m_2, b) := f(m_1)$ will be used occasionally.

We will aim to construct a "manifold with corners resolution" of a piece of $\overline{\bigcup_{t\geq 0} \xi_t(B_0)}$ to be described momentarily.

Let

$$\mathbf{p}_1 := \lim_{t\to\infty} \psi_t(s(b_0)) \in M$$

be a critical point. We will assume that $\mathbf{p}_1$ is not a point of a local maximum for f, i.e., $\dim U_{\mathbf{p}_1} > 0$. The case $\dim U_{\mathbf{p}_1} = 0$ can be treated quite easily as, locally, everything flows when $t \to \infty$ to the critical manifold determined by $\{\mathbf{p}_1\}$ as already noticed in Remark 3.6.

For simplicity, we will also assume that $f(\mathbf{p}_1) = 0$.

We allow the situation $\mathbf{p}_1 = s_0(b_0)$. At the other extreme, $s_0(b_0)$ might be "far" from $\mathbf{p}_1$. If that is the case, we notice that nothing interesting happens with the flow-out of $\xi_0(B_0)$ before we get close to $\{\mathbf{p}_1\} \times M \times B_0$. So, we might assume without restriction of the generality that $s_0(b_0)$ is in a neighborhood D of $\mathbf{p}_1$ where the Straighten Coordinates Theorem is valid. Then, we will also assume that B_0 was first chosen so that $s_0(B_0) \in \mathring{C}_\varepsilon$ for some fixed ε, where $C_\varepsilon \subset \mathbb{R}^s \times \mathbb{R}^u$ satisfies the hypothesis of Theorem 2.4. Hence, $\xi_0(B_0) \subset \mathring{C}_\varepsilon \times M \times B_0$.

We will need to work with certain particular neighborhoods of $\{|x| \cdot |y| = 0\} \times M \times B_0$ of type $V_\gamma^\varepsilon \times M \times B_0$ where V_γ^ε is as in Theorem 2.4.

The next technical statement prepares the field for the next step of the induction, namely when we will go from the first critical level (that of $\mathbf{p}_1$) to the second critical level in the direction of the flow.

Proposition 4.1

(a) *There exist $\gamma \leq \varepsilon$ and $\theta > 0$ a regular value of f such that the trajectory determined by any $p \in V_\gamma^\varepsilon \cap f^{-1}(-\infty, \theta]$ that intersects $f^{-1}(\theta)$ does so before intersecting ∂C_ε^- when $t \to \infty$.*

(b) *If $\mathbf{p}_1$ is not a point of minimum then we can take θ and γ such that the trajectory determined by any $p \in V_\gamma^\varepsilon \cap f^{-1}[-\theta, \theta]$ that intersects $f^{-1}(-\theta)$ does so before intersecting ∂C_ε^+ when $t \to -\infty$.*

Moreover, after shrinking B_0 the following holds:

(i) *$s_0(B_0) \subset V_\gamma^\varepsilon \cap f^{-1}(-\infty, \theta')$ for some θ' with $0 < \theta' < \theta$ and*

(ii) *$(\bigcup_{t\geq 0} \xi_t(B_0)) \cap \tilde{f}^{-1}(\theta) \subset V_\gamma^\varepsilon \times M \times B_0$.*

Proof   In the cube C_ε of Theorem 2.4, there exists $0 < \gamma < \varepsilon$ such that

$$\inf_{u \in T_\gamma} f(u) > 0,$$

where $T_\gamma = \{(x, y) \in \partial^- C_\varepsilon; |x| < \gamma, |y| = \varepsilon\}$ since $f > 0$ on the compact $\partial^- C_\varepsilon \cap U_{\mathbf{p}_1}$. Fix such a γ.

Choose now $\theta > 0$ with

$$(4.1) \qquad\qquad 0 < \theta < \inf_{u \in T_\gamma} f(u).$$

Notice that each trajectory that starts inside V_γ^ε either ends up at the critical point or leaves V_γ^ε through T_γ. Let $p \in V_\gamma^\varepsilon \cap f^{-1}(-\infty, \theta]$. On one hand, f is increasing and continuous along the trajectories and on the other hand, $V_\gamma^\varepsilon \cap \gamma_p$ [1] is connected by the flow-convex property of V_γ^ε. It follows that γ_p meets $f^{-1}(\theta)$ before reaching T_γ by (4.1).

Part (b) is analogous.

Since $s_0(b_0) \in S_{\mathbf{p_1}}$ we have that $f(s(b_0)) \leq 0$ and therefore one can choose B_0 with $s_0(B_0) \subset V_\gamma^\varepsilon \cap f^{-1}(-\infty, \theta')$.

Part (ii) is an immediate consequence of (i) and the first part of the proof. $\blacksquare$

From now on the neighborhood B_0 of b_0 will satisfy the properties of Proposition 4.1 for a certain θ and γ.

We define now the first piece of the transverse intersection, we will use later.

Let $\tilde{C}_\varepsilon := C_\varepsilon \times C_\varepsilon \times B_0$ and $\mathring{\tilde{C}}_\varepsilon := \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times B_0$ and consider $W_1 \subset [0,1] \times \mathring{\tilde{C}}_\varepsilon$ be the analogue of W from Theorem 2.6 for this context:

$$W_1 = \left\{ \left(t, \psi_{\frac{t}{1-t}}(x, y), x, y, b \right); \ 0 < t < 1 \right\} \cap \mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times B_0.$$

In other words, modulo a permutation of the last two coordinates we have:

$$W_1 = W \times B_0 \subset \left([0,1] \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \right) \times B_0,$$

where W is as in Theorem 2.6.

It follows then from Theorem 2.6 that $\overline{W_1}$, the closure inside $[0,1] \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times B_0$ is a smooth $(m + 1 + n)$-dimensional manifold with boundary:

$$\partial \overline{W_1} = \underbrace{\{1\} \times (\mathrm{U}_{\mathbf{p_1}} \cap \mathring{C}_\varepsilon) \times (\mathrm{S}_{\mathbf{p_1}} \cap \mathring{C}_\varepsilon) \times B_0}_{\partial_1 \overline{W_1}} \bigcup \underbrace{\{0\} \times \Delta_{\mathring{C}_\varepsilon} \times B_0}_{\partial_0 \overline{W_1}}.$$

For future reference, let

$$(4.2) \qquad\qquad \overline{W}_{\mathbf{p_1}} := [0,1] \times \{\mathbf{p_1}\} \times \{\mathbf{p_1}\} \times B_0 \subset \overline{W_1}.$$

be the set of fixed points in $\overline{W_1}$.

We look now at the second piece of transverse intersection mentioned before. Let

$$V_\theta := \mathring{V}_\gamma^\varepsilon \cap f^{-1}((-\infty, \theta]) \subset \mathring{C}_\varepsilon,$$

where ε, γ and θ are as in Proposition 4.1.

Consider the following set:

$$Z_1 := \mathbb{R} \times V_\theta \times s(B_0) \subset \mathbb{R} \times \mathring{C}_\varepsilon \times (\mathring{C}_\varepsilon \times B_0).$$

Lemma 4.2 *The set Z_1 is a manifold of dimension $m + n + 1$ with boundary*

$$\partial Z_1 = \mathbb{R} \times (\mathring{V}_\gamma^\varepsilon \cap f^{-1}(\theta)) \times s(B_0).$$
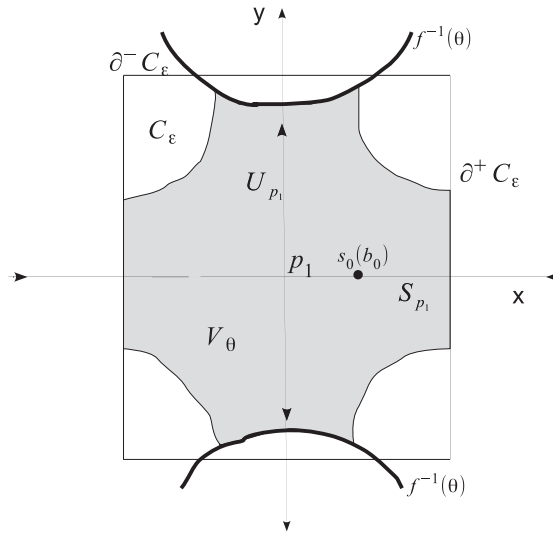
---

[1] the trajectory γ_p is determined by p.

Figure 3: The neighborhoods V_θ

Proof   For any regular value θ the intersection $U \cap f^{-1}(-\infty, \theta]$ is a manifold with boundary for any open $U \subset M$ such that $f^{-1}(\theta) \cap U \neq \varnothing$.   ∎

Lemma 4.3   *The manifolds with boundary Z_1 and $\overline{W_1}$ are completely transverse inside $\mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times B_0$, meaning that the different strata are all transverse. This implies that $Z_1 \cap \overline{W_1}$ is a manifold with corners of dimension $\dim B + 1$.*

Proof   Notice that $\overline{W_1} = W_1$.

We start with the intersection $W_1 \cap \mathring{Z}_\theta$. The transversality is immediate from the fact that W_1 is a graph over the first plus the last two variables, i.e., $\mathbb{R} \times \mathring{C}_\varepsilon \times B_0$ while the first component of $\mathring{Z}_\theta$ is an open subset of $\mathring{C}_\varepsilon$.

A similar reasoning applies for $q \in \partial_0 \overline{W_1} \cap \mathring{Z}_\theta$. In fact, due to the transversality of the flow to a regular level, set $f^{-1}(\theta)$ this also proves the transversality of W_1 with ∂Z_θ.

Notice that $\partial_0 \overline{W_1} \cap \partial Z_\theta = \varnothing$ since a point $q \in \xi_0(B_0)$ cannot satisfy $\tilde{f}(q) = \theta$ due to property (i) of Proposition 4.1.

The transversality of $\partial_1 \overline{W_1}$ with $\mathring{Z}_1$ follows from the transversality of $s(B_0)$ and $S_{\mathbf{p_1}} \times B_0$.

Finally, the transversality of $\partial_1 \overline{W_1}$ with ∂Z_1 follows from the transversality of $U_{\mathbf{p_1}}$ and $\tilde{f}^{-1}(\theta)$ together with the transversality of $s(B_0)$ and $S_{\mathbf{p_1}} \times B_0$.   ∎

Lemma 4.4   *Let N_1 and N_2 be submanifolds with corners of type k and l respectively, inside a manifold (with no corners) N. If they are completely transverse then $N_1 \cap N_2$ is a manifold with corners of type at most $k + l$.*

Proof   We use a standard trick. Clearly, $N_1 \times N_2$ is a manifold with corners of type $k + l$. Then the complete transversality of N_1 and N_2 is equivalent with the complete transversality of $N_1 \times N_2$ and the diagonal submanifold Δ in $N \times N$. The conclusion

then is that $(N_1 \times N_2) \cap \Delta$ is a manifold with corners (for this see Proposition A.3 in [7]). ∎

Following Lemma 4.2 denote

$$(4.3) \qquad \mathcal{A}_1 := Z_1 \cap \overline{W_1}.$$

The codimension 1 boundary has the following decomposition in components:

$$\partial^1 \mathcal{A}_1 = \underbrace{\{1\} \times (U_{\mathbf{p}_1} \cap f^{-1}(-\infty, \theta]) \times s(s^{-1}(S_{\mathbf{p}_1} \times B_0))}_{\partial_1^1 \mathcal{A}_1} \bigcup \underbrace{\{0\} \times \xi_0(B_0)}_{\partial_0^1 \mathcal{A}_1} \bigcup$$

$$(4.4) \qquad \bigcup \underbrace{\overline{W_1} \cap ([0,1] \times f^{-1}(\theta) \times s(B_0))}_{\partial_2^1 \mathcal{A}_1},$$

where

$$\partial_1^1 \mathcal{A}_1 := \partial_1 \overline{W_1} \cap Z_1, \quad \partial_0^1 \mathcal{A}_1 := \partial_0 \overline{W_1} \cap Z_1, \quad \partial_2^1 \mathcal{A}_1 := \overline{W_1} \cap \partial Z_1.$$

The codimension 2 stratum is given by

$$(4.5) \qquad \partial^2 \mathcal{A}_1 := \partial_1^1 \mathcal{A}_1 \cap \partial_2^1 \mathcal{A}_1 = \{1\} \times (U_{\mathbf{p}_1} \cap f^{-1}(\theta)) \times s(s^{-1}(S_{\mathbf{p}_1} \times B_0)).$$

We define the resolution map now. Let

$$\mathcal{R} : \mathcal{A}_1 \to V_\theta \times M \times B_0$$

be the restriction to $\mathcal{A}_1 \subset [0,1] \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times B_0$ of the projection onto the (last) three spacial coordinates. Notice that in fact the image of $\mathcal{R}$ is contained in $V_\theta \times V_\theta \times B_0$.

Proposition 4.5 *The map $\mathcal{R}$ is proper.*

Proof Recall that for locally compact metric spaces X, Y, a map $F : X \to Y$ is proper if and only if for any sequence $(x_n)_{n \in \mathbb{N}} \in X$ such that $\lim\limits_{n \to \infty} x_n = \infty$ one has $\lim\limits_{n \to \infty} F(x_n) = \infty$. By definition,

$$\lim_{n \to \infty} x_n = \infty$$

if for every $K \subset X$ compact there exists $n_0 \in \mathbb{N}$ such that $x_n \in X \backslash K$ for all $n \geq n_0$. Notice that such a sequence does not have any convergent subsequence in X and the converse is also true. It follows easily then that a map $F : X \to Y$ not proper if and only if there exists $(x_n)_{n \in \mathbb{N}} \in X, x_n \to \infty$ such that $F(x_n) \to y \in Y$.

Assume therefore that $(u_n)_{n \in \mathbb{N}} \in \mathcal{A}_1$ satisfies $u_n \to \infty$ while $\mathcal{R}(u_n) \to \tilde{u}$. Now u_n has 4 components:

$$(4.6) \qquad u_n = (t_n, a'_n, a_n, b_n) \in [0,1] \times V_\theta \times V_\theta \times B_0.$$

By passing to a subsequence of u_n, we can assume t_n converges. There are two possibilities.

Either $t_n \to t' < 1$ or $t_n \to 1$. We analyze them separately.

First, the triple $\mathcal{R}(u_n) = (a'_n, a_n, b_n)$ converges in $V_\theta \times V_\theta \times B_0$. From $u_n \in Z_1$ we get $(a_n, b_n) = (s_0(\beta_n), \beta_n)$ for some $\beta_n \in B_0$. Hence, β_n converges to $\beta \in B_0$ and $s_0(\beta_n) \to s_0(\beta)$.

If $t_n \to t' < 1$ then for n big enough we have that $u_n \in \overline{W_1} \backslash \{t = 1\}$ and therefore

$$a'_n = \psi_{\frac{t_n}{1-t_n}}(a_n).$$

Hence $a_n = s_0(\beta_n) \to s_0(\beta)$ and $a'_n \to \psi_{\frac{t'}{1-t'}}(s_0(\beta))$ and by hypothesis this is in V_θ. We conclude that

$$u_n \to \left(t', \psi_{\frac{t'}{1-t'}}(s_0(\beta)), s_0(\beta), \beta \right)$$

and this limit belongs to $Z_1 \cap \overline{W_1} = \mathcal{A}_1$. Contradiction with $u_n \to \infty$.

If $t_n \to 1$, we have that $u_n = (t_n, a'_n, s_0(\beta_n), \beta_n)$ and the convergence of $(a'_n, s_0(\beta_n), \beta_n)$ to a point in $V_\theta \times V_\theta \times B_0$ implies again that $\beta_n \to \beta \in B_0$. Since $s_0(B_0) \subset V_\theta$, we have that $s_0(\beta_n) \to s_0(\beta) \in V_\theta$. We get therefore that the sequence u_n converges in $[0,1] \times V_\theta \times V_\theta \times B_0$ since all its coordinates converge. In order to reach a contradiction, we need only check that it converges to some element u of $Z_1 \cap \overline{W_1}$. We have that

$$u_n \to u \in \mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times B_0.$$

On the other hand, since $u_n \in \overline{W_1}$ and the closure of W_1 is taken within $\mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times B_0$ we get that $u \in \overline{W_1}$. One sees easily that $u \in Z_1$ since $(a'_n, s_0(\beta_n), \beta_n) = \mathcal{R}(u_n)$ converges to a point $(a', s_0(\beta), \beta) \in V_\theta \times V_\theta \times B_0$. ∎

Let $\tilde{V}_\theta := V_\theta \times M \times B_0$ be the codomain of $\mathcal{R}$. We show two things:

(i) the currential formula (3.4) holds on $\mathring{\tilde{V}}_\theta$, the interior of $\tilde{V}_\theta$ and
(ii) there exists a map from a manifold with corners N to $\partial \tilde{V}_\theta = V_\gamma^\varepsilon \cap f^{-1}(\theta)$ that allows us to continue the process.

First we list some set-theoretic and differential properties of $\mathcal{R}$.

Proposition 4.6 *The map* $\mathcal{R} : \mathcal{A}_1 \longrightarrow \tilde{V}_\theta$ *satisfies:*

(1) $\mathcal{R}(\mathcal{A}_1) = \overline{\bigcup_{t \geq 0} \xi_t(B_0)} \cap \tilde{V}_\theta =: A_\theta$ *where the closure is taken within* $M \times M \times B_0$;
(2) $\mathcal{R}(\partial_1^1 \mathcal{A}_1) = (U_{\mathbf{p_1}} \cap f^{-1}(-\infty, \theta]) \times s(s^{-1}(S_{\mathbf{p_1}} \times B_0))$;
(3) $\mathcal{R}(\partial_0^1 \mathcal{A}_1) = \xi_0(B_0)$;
(4) $\mathcal{R}(\partial_2^1 \mathcal{A}_1) = A_\theta \cap \tilde{f}^{-1}(\theta)$;
(5) *the restriction of* $\mathcal{R}$ *is a bijection from* $\mathcal{A}_1 \backslash \overline{W}_{\mathbf{p_1}}$ *(see (4.2)) onto its image; and*
(6) *the restriction of* $\mathcal{R}$ *to* $\partial_2^1 \mathcal{A}_1$ *is a bijection onto the image.*

Proof At (1), let $q \in \overline{W_1} \cap Z_1$. On one hand

$$q = \lim_{n \to \infty} \left(t_n, \psi_{\frac{t_n}{1-t_n}}(x_n, y_n), x_n, y_n, b_n \right),$$

where $b_n \in B$, $(x_n, y_n) \in \mathring{C}_\varepsilon$, $t_n \in [0,1]$. From $q \in Z_1$ it follows that $(x_n, y_n, b_n) \to (s_0(\beta), \beta)$. Suppose $t_n \to t' < 1$. Then

$$\psi_{\frac{t_n}{1-t_n}}(x_n, y_n) \to \psi_{\frac{t'}{1-t}}(s_0(\beta)).$$

Hence, in this case, $\mathcal{R}(q) = \left(\psi_{\frac{t'}{1-t'}}(s_0(\beta)), s_0(\beta), \beta \right) \in \xi_{\frac{t'}{1-t'}}(B_0)$.

When $t_n \to 1$, $q \in \overline{W_1}$ implies that $(x_n, y_n, b_n) \to (s_0(\beta), \beta) \in S_{\mathbf{p}_1} \times B_0$, i.e., $\beta \in s^{-1}(S_{\mathbf{p}_1} \times B_0)$ and $\psi_{\frac{t_n}{1-t_n}}(x_n, y_n) \to q_1 \in U_{\mathbf{p}_1} \cap V_\theta$.

We argue that due to the transversality of s with $S_{\mathbf{p}_1} \times B_0$ all points in $U_{\mathbf{p}_1} \cap (V_\theta \times s(s^{-1}(S_{\mathbf{p}_1} \times B_0)))$ are limits of type $\xi_{t_n}(s_0(\beta_n), s_0(\beta_n), \beta_n)$, $t_n \geq 0$. Fix first $q_2 = (x_2, 0, b_2) \in s(s^{-1}(S_{\mathbf{p}_1} \times B_0))$. Since $s \pitchfork B_0$ we can provide a submanifold $B_0' \subset B_0$ of dimension equal to $\dim U_{\mathbf{p}_1}$ such that $s_0\big|_{B_0'} \pitchfork S_{\mathbf{p}_1}$ and $q_2 \in s(B_0')$. Take then a transverse small disk D in $q_2 + T_{q_2} s(B_0')$ centered at q_2. The trajectories originating in this disk will cut $s(B_0')$ exactly once due to the transversality of $s(B_0')$ to the flow. This stays true even if $q_2 = (0, 0, b_2)$ is critical. By Corollary 2.9 which can be applied also to "slanted" disks, given any point $(0, y_2) \in U_{\mathbf{p}_1}$ there exists a sequence of points $u_n \in D$ with $u_n \to q_2$ and a corresponding sequence of points on trajectories determined by u_n that converges to $(0, y_2)$.

This finishes the inclusion $\mathcal{R}(\mathcal{A}_1) \subset \overline{\bigcup_{t \geq 0} \xi_t(B_0)}$.

Conversely, let $(a, b, c) \in \overline{\bigcup_{t \geq 0}(\xi_t(B_0))} \cap \tilde{V}_\theta$. Then there exist $t_n \geq 0$ and $b_n \in B_0$ such that

$$\lim_{n \to \infty}(\psi_{t_n}(s_1(b_n)), s_1(b_n), b_n) = (a, b, c)$$

with $c \in B_0$. By passing to a subsequence one can assume that $t_n \to t_0$ or $t_n \to \infty$. Since $a \in V_\theta \subset \mathring{V}_\gamma^\varepsilon$ and the latter is open in M we can consider $\psi_{t_n}(s_1(b_n)) \in \mathring{V}_\gamma^\varepsilon$ for n big enough. However, $\psi_{t_n}(s_1(b_n))$ might not be in V_θ for infinitely many n, since it could happen that $f(\psi_{t_n}(s_1(b_n))) > \theta$ for a subsequence.

Let $r_n := \frac{t_n}{1+t_n}$. We have that

$$u_n := (r_n, \psi_{t_n}(s_1(b_n)), s_1(b_n), b_n) \in W_1.$$

Since r_n, b_n and $\psi_{t_n}(s_1(b_n))$ all converge we have that in fact u_n converges to a point $u \in [0, 1] \times V_\theta \times \mathring{C}_\varepsilon$ that necessarily lies in $\overline{W_1}$. The limit $u = (r, a, b, c)$ will also be a point in Z_1 since $(b, c) \in s(B_0)$ and $a \in V_\theta$. Hence, $(a, b, c) \in \mathcal{R}(\mathcal{A}_1)$.

Statements (2) and (3) are trivial from the description of $\partial_1^1 \mathcal{A}_1$ and $\partial_0^1 \mathcal{A}_1$.

For (4) the inclusion $\subset$ is straightforward. For the inclusion $\supset$ notice that if $(a, b, c) \in \mathcal{A}_1$ then either $f(a) < \theta$ in which case $a \in \mathring{V}_\theta$ and so $(a, b, c) \in \mathring{Z}_1 \cap \overline{W_1}$ or $f(a) = \theta$ in which case $(a, b, c) \in \partial Z_1 \cap \overline{W_1} =: \partial_2^1 \mathcal{A}$.

For (5) one notices that for $t \neq 1$ the (restriction of the) map $\mathcal{R}$ is injective away from the points corresponding to $s^{-1}(\{\mathbf{p}_1\} \times B_0)$. Moreover, $\mathcal{R}$ is injective when $t = 1$. For $p \neq q$ with $t_p \neq 1$ and $t_q = 1$, $\mathcal{R}(p) \neq \mathcal{R}(q)$ unless $p, q \in \overline{W}_{\mathbf{p}_1}$.

For (6) one notices that $\overline{W}_{\mathbf{p}_1} \cap \partial_2^1 \mathcal{A}_1 = \varnothing$. ∎

Corollary 4.7 *The equality of currents* (3.4) *holds on the open set* $\mathring{V}_\theta \times M \times B_0$.

Proof The intersection $\mathring{\mathcal{A}}_1 := \mathcal{A}_1 \cap (\mathring{V}_\theta \times M \times B_0)$ is a manifold with boundary since $\partial_2^1 \mathcal{A}_1$ gets eliminated in the intersection. Since $\mathcal{R}$ is proper we can push-forward any current. Use:

$$d(\mathcal{R}_*(\mathring{\mathcal{A}}_1)) = \mathcal{R}_*(d\mathring{\mathcal{A}}_1) = \mathcal{R}_*(\partial_1^1 \mathcal{A}_1) - \mathcal{R}_*(\partial_0^1 \mathcal{A}_1).$$

Now $\mathcal{R}\big|_{\mathring{A}_1}$ away from a set of zero measure[2] is a bijection onto

$$\xi([0, \infty) \times B_0) \cap \mathring{V}_\theta \times M \times B_0.$$

It follows from the area formula that

$$\mathcal{R}_*(\mathring{A}_1) = T\big|_{\mathring{V}_\theta \times M \times B_0},$$

where T is the current appearing in (3.4).   ∎

This is the first step. In order to proceed further we will use the restriction $\mathcal{R}$: $\partial_2^1 \mathcal{A}_1 \to f^{-1}(\theta) \times M \times B_0 = \tilde{f}^{-1}(\theta)$. We fix now another critical point $\mathbf{p}_2$ with $f(\mathbf{p}_2) > f(\mathbf{p}_1)$. In order to implement the program, we need the following.

Lemma 4.8   *The restriction of $\mathcal{R}$ denoted $\sigma : \partial_2^1 \mathcal{A}_1 \longrightarrow \tilde{f}^{-1}(\theta)$ is completely transverse to $S_{p_2} \times M \times B_0$ within $\tilde{f}^{-1}(\theta)$ for all critical points p_2.*

First, we need to clarify a point. The complete transversality for the map σ is meant here both within the ambient space $\tilde{f}^{-1}(\theta)$ (with $S_{p_2} \times M \times B_0 \cap \tilde{f}^{-1}(\theta)$) and within the ambient space $M \times M \times B_0$. The two statements are clearly equivalent due to the transversality of S_{p_2} to $f^{-1}(\theta)$ for every regular θ.

Proof   Since $\partial_2^1 \mathcal{A}_1$ is a manifold with boundary $\partial^2 \mathcal{A}_1$, as defined in (4.5), we need to show transversality at points

(1) $q \in \sigma(\partial^2 \mathcal{A}_1) \cap (S_{\mathbf{p}_2} \times M \times B_0)$ and
(2) $q \in \sigma(\partial_2^1 \mathcal{A}_1 \backslash \partial^2 \mathcal{A}_1) \cap (S_{\mathbf{p}_2} \times M \times B_0)$.

For both situations, take the unique (by (6) of Proposition 4.6) $q' \in \partial_2^1 \mathcal{A}_1$ such that $\sigma(q') = q$. The two situations are distinguished by $t_{q'} = 1$ (for (1)) or $t_{q'} \neq 1$.

From the explicit expression of $\partial^2 \mathcal{A}_1$ in (4.5), we see that transversality for (1) is implied by the Smale property of the flow.

For (2) since $t \neq 1$, we can give another description to $\overline{W_1} \backslash \{t = 1\} \cap (f^{-1}(\theta) \times M \times B_0)$ as the graph of the time map defined over $\xi_0(B_0 \backslash s^{-1}(S_{\mathbf{p}_1} \times B_0))$ that associates to a point p the time t_p it needs to reach $\tilde{f}^{-1}(\theta)$. We deduce that the intersection of this time map graph with the flow-invariant $S_{p_2} \times M \times B_0$ can be described as the flow-out of the intersection $\xi_0(B_0 \backslash s^{-1}(S_{\mathbf{p}_1} \times B_0)) \cap (S_{\mathbf{p}_2} \times M \times B_0)$ to the level set $\tilde{f}^{-1}(\theta)$. By the transversality of s with S_{p_2}, we get that this intersection is transverse. Moreover, since the flow preserves transversality, we get by Proposition 2.10 the transversality condition we are after within $\tilde{f}^{-1}(\theta)$.   ∎

We recall what we did so far. We started with the proper submanifold $\xi_0(B_0)$ of $V_\theta \times M \times B_0$ and we constructed a resolution of its flow-out in the open set $\mathring{V}_\theta \times M \times B_0$. The resolution took the form of a map $\mathcal{R}$ defined on a manifolds with corners $\mathcal{A}_1$ to $V_\theta \times M \times B_0$. Of course, this only solves the problem of the flow-out of $\xi_0(B_0)$ for the first critical point encountered. How do we go on from here?

The map $\sigma : \partial_2^1 \mathcal{A}_1 \to \tilde{f}^{-1}(\theta)$ will play now the role of $\xi_0 : B_0 \to M \times M \times B_0$ and we would like to apply the same ideas to σ.

So we would like to see what properties of σ can be preserved when going through the next critical level. Clearly by composing with the flow-diffeomorphisms we can

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[2]With respect to any measure induced by the volume form of a Riemannian metric on $\mathcal{A}_1$.

assume that the image of σ is contained in a regular level of $\tilde{f}$ close to the next critical level.

There is no harm in assuming that the next critical level lies within $\tilde{f}^{-1}(0)$ by changing f to $f + c$ for some constant c. Moreover we will be using certain neighborhoods of the critical sets. First, since the Shilnikov–Minervini results are local around the critical points we will use different neighborhoods around the critical sets of $\tilde{f}^{-1}(0)$.

- We will denote $\tilde{C}_\varepsilon := \bigcup_{p \in \mathrm{Crit}(f) \cap f^{-1}(0)} C_\varepsilon(p) \times M \times B_0$, where ε is chosen so that the results of Section 2 hold for each cube $C_\varepsilon(p)$ with p in the finite set $\mathrm{Crit}(f) \cap f^{-1}(0)$.
- For each $\mathbf{p} \in \mathrm{Crit}(f) \cap f^{-1}(0)$ we will choose a $\theta_{\mathbf{p}}$ and $\gamma_{\mathbf{p}}$ small enough so that Proposition 4.1 item (ii) is satisfied (observe that the points $\mathbf{p} \in \mathrm{Crit}(f) \cap f^{-1}(0)$ are not points of local minimum in our context). Then let

$$\theta_1 := \min \theta_{\mathbf{p}}, \qquad \gamma := \min \gamma_{\mathbf{p}}$$

$$V_{\theta_{\mathbf{p}}} := V_\gamma^\varepsilon(\mathbf{p}) \cap f^{-1}[-\theta_1, \theta_1], \qquad V_{\theta_1} := \cup_{\mathbf{p}} V_{\theta_{\mathbf{p}}}$$

and finally

$$\tilde{V}_{\theta_1} := V_{\theta_1} \times M \times B_0.$$

Here is the model result we are after.

Proposition 4.9 *Let N be an oriented manifold with corners of dimension n and type $k \geq 1$[3] and let $-\theta$ be a regular level for $\tilde{f}$ such that 0 is a critical level and θ is small enough. Let $\theta' < \theta_1$ and*

$$\sigma : N \to \tilde{f}^{-1}(-\theta') \cap \tilde{V}_{\theta_1}, \qquad \sigma = (\sigma_1, \sigma_2, \sigma_3)$$

be a smooth map such that

(a) *σ is proper.*
(b) *σ is completely transverse to all stable manifolds $S_p \times M \times B_0$ for $p \in \mathrm{Crit}(f)$; equivalently $\sigma_1 \pitchfork S_p$, for all $p \in \mathrm{Crit}(f)$.*
(c) *$\sigma|_{N \setminus \partial^2 N}$ is injective where $\partial^2 N$ is the collection of strata of codimension at least 2.*
(d) *$\sigma_1|_{N^0} = \alpha((\sigma_2, \sigma_3)|_{N^0})$ for some function α where N^0 is the top stratum of N.*

Then there exists a smooth map, called flow resolution $\mathcal{R}_\sigma : \mathcal{A}_\sigma \longrightarrow \tilde{V}_{\theta_1}$ defined over an oriented manifold with corners $\mathcal{A}_\sigma$ of dimension $n + 1$ and type $k + 2$ such that

(i) *$\mathcal{R}_\sigma$ is proper.*
(ii) *$\mathcal{R}_\sigma(\mathcal{A}_\sigma) = \overline{\left(\bigcup_{t \geq 0} \xi_t(\sigma(N)) \right)} \cap \tilde{V}_{\theta_1}$ where the closure is taken in $M \times M \times B_0$.*
(iii) *$\mathcal{R}_\sigma$ is a bijection from an open subset of $\mathcal{A}_\sigma$ of full measure to an open subset of full measure of $\left(\bigcup_{t \geq 0} \xi_t(\sigma(N)) \right) \cap \tilde{V}_{\theta_1}$.*
(iv) *$\mathcal{A}_\sigma$ has a distinguished boundary $N' := \partial_{\theta_1}^1 \mathcal{A}_\sigma$ of dimension n and type $k + 1$ such that $\sigma' := \mathcal{R}_\sigma|_{N'} : N' \longrightarrow \tilde{f}^{-1}(\theta_1) \cap \tilde{V}_{\theta_1}$ is completely transverse to all the stable manifolds $S_p \times M \times B_0$, is injective on $N' \setminus \partial^2 N'$ and the components of σ' satisfy property of item (d) when restricted to $(N')^0$.*

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[3]The type of a manifold with corners gives the codimension of the smallest strata in the boundary. Some authors prefer to call this "depth."

Remark 4.10 The word distinguished is related to the fact that $\partial^1_{\theta_1} \mathcal{A}_\sigma$ is not the full boundary of $\mathcal{A}_\sigma$ but one that has a collar neighborhood.

Remark 4.11 It is important to specify the codomain of σ in order to state the properness property. In the induction process, we are using, the original map $\mathcal{R}|_{\partial^1_2 \mathcal{A}_\theta}$ is proper when the codomain is $V_\theta \times M \times B_0 \cap \tilde{f}^{-1}(\theta)$. The later is an open set inside $\tilde{f}^{-1}(\theta)$. Clearly, the inclusion of an open set into the ambient space is not proper.

Remark 4.12 The injectivity property stated in item (c) appears because we want the current $(\mathcal{R}_\sigma)_*(\mathcal{A}_\sigma)$ to be determined by the image of $\mathcal{R}_\sigma$. Otherwise, one could have multiplicities or worse things happening. One cannot expect injectivity to hold everywhere.

Remark 4.13 The seemingly strange property (d) is to insure the "replication" of the injectivity property away from the codimension 2 stratum. Property (d) is fulfilled for the initial σ, i.e., the restriction of $\mathcal{R}$ to $\partial^1_2 \mathcal{A}$. In that case, N^0 is the graph of the time map $p \to t_p$ for $p \in \xi_0(B_0 \backslash \xi_0^{-1}(S_{\mathbf{p}_1} \times M \times B_0))$ as described in the proof of Lemma 4.8 while σ projects (t_p, p) to p. Since the first component of ξ_0 is dependent on the other two, we get the claim.

In order to prove Proposition 4.9, we also need to deal with the fact that $\sigma(N)$ is not necessarily a subspace of $V_{\theta_1} \times M \times B_0$. Hence rather than "flowing" $\sigma(N)$ we will consider the graph Γ_σ. It is convenient to consider first a proper embedding $N \hookrightarrow \mathbb{R}^j$ and look at the closure of the set

$$W_2 := \left\{ \left(t, \psi_{\frac{t}{1-t}}(m_1), m_1, m_2, b, n\right) \mid 0 < t < 1 \right\} \cap \mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times M \times B_0 \times \mathbb{R}^j$$

inside $\mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times M \times B_0 \times \mathbb{R}^j$.

By Remark 2.8, this closure is a manifold of dimension $2m + n + j + 1$ with boundary

$$\partial^1_1 \overline{W_2} := \bigcup_{\mathbf{p} \in f^{-1}(0) \cap \mathrm{Crit}(f)} \{1\} \times (U_{\mathbf{p}} \cap \mathring{C}_\varepsilon(\mathbf{p})) \times (S_{\mathbf{p}} \cap \mathring{C}_\varepsilon(\mathbf{p})) \times M \times B_0 \times \mathbb{R}^j \bigcup$$

$$\partial^1_0 \overline{W_2} := \{0\} \times \triangle_{\mathring{C}_\varepsilon} \times M \times B_0 \times \mathbb{R}^j.$$

Let

$$Z_\sigma := \mathbb{R} \times V_{\theta_1} \times \Gamma_\sigma \subset \mathbb{R} \times V_{\theta_1} \times V_{\theta_1} \times M \times B_0 \times N.$$

Since V_{θ_1} is a manifold with boundary we get that Z_σ is a manifold with corners of dimension $m + n + 1$ and type $k + 1$. The codimension 1 boundary components of Z_σ are

$$(4.7) \qquad \partial^1_0 Z_\sigma := \mathbb{R} \times (V_{\theta_1} \cap f^{-1}(-\theta_1)) \times \Gamma_\sigma$$

$$(4.8) \qquad \partial^1_1 Z_\sigma := \mathbb{R} \times (V_{\theta_1} \cap f^{-1}(\theta_1)) \times \Gamma_\sigma$$

$$(4.9) \qquad \partial^1_{j+1} Z_\sigma := \mathbb{R} \times V_{\theta_1} \times \Gamma_\sigma\big|_{\partial^1_j N}, \quad 1 \le j \le f_N,$$

where f_N is the number of codimension 1 boundary components of N, i.e., the number of connected components of $\partial^1 N \backslash \partial^2 N$, assumed finite.

Lemma 4.14 *The manifolds $\overline{W_2}$ and $\underline{Z_\sigma}$ are completely transverse inside $\mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times M \times B_0 \times \mathbb{R}^j$. This implies that $Z_\sigma \cap \overline{W_2}$ is a manifold with corners of dimension $n+1$ and type at most $k+2$.*

Moreover, the codimension 1 boundary of $Z_\sigma \cap \overline{W_2}$ has the following components which are themselves manifolds with corners

$$(4.10) \qquad \partial_1^1 \overline{W_2} \cap Z_\sigma \quad = \bigcup_{p \in \mathrm{Crit}(f) \cap f^{-1}(0)} \{1\} \times (U_p \cap f^{-1}([0,\theta_1])) \times \Gamma_{\sigma_{|\sigma^{-1}(S_p \times M \times B_0)}}$$

$$(4.11) \qquad \partial_0^1 \overline{W_2} \cap Z_\sigma \quad = \{0\} \times \Gamma_{\tilde{\sigma}}$$

$$(4.12) \qquad \overline{W_2} \cap \partial_1^1 Z_\sigma$$

$$(4.13) \qquad \overline{W_2} \cap \partial_{j+1}^1 Z_\sigma \, \mathrm{for} \, 1 \le j \le f_N,$$

where $\tilde{\sigma} = (\sigma_1, \sigma_1, \sigma_2, \sigma_3)$ given that $\sigma = (\sigma_1, \sigma_2, \sigma_3)$. Finally,

$$(4.14) \qquad \overline{W_2} \cap \partial_0^1 Z_\sigma \quad = \varnothing.$$

Proof Analogous to the proof of Lemma 4.3.

The reason for (4.14) is that the f-value of the second component of $\overline{W_2}$ is at least as big as the f-value of the third component while this is not the case for an element of $\partial_0^1 Z_\sigma$ due to $f \circ \sigma_1 = -\theta' > -\theta_1$. ∎

Remark 4.15 The reason for which we chose $\mathrm{Im}\,\sigma \subset \tilde{f}^{-1}(-\theta')$ is because we did not want $\mathrm{Im}\,\sigma \subset \partial \tilde{V}_{\theta_1}$. That would render Lemma 4.14 false. An alternative approach, if $\mathrm{Im}\,\sigma \subset \tilde{f}^{-1}(-\theta_1)$, would be to replace V_{θ_1} in the definition of Z_σ with $V_{\theta_1}^{\theta'} = f^{-1}([-\theta', \theta_1]) \cap V_\gamma^\varepsilon$ where $\theta' < \theta_1$. Then one has to be content with the construction of the resolution for the flow-out of $\sigma(N)$ in between levels $-\theta'$ and θ_1.

Let $\mathcal{A}_\sigma := \overline{W_2} \cap Z_\sigma$. This is an oriented manifolds with corners because $\overline{W_2}$ and Z_σ are both oriented. The convention here is that the components which correspond to graphs of smooth functions over a certain oriented base manifold B' inherit the orientation of the manifold B'. Hence, the direction of the flow gives the first vector of a positively oriented basis. Other than this, we respect the order of factors in a product.

Consider now

$$\mathcal{R}_\sigma : \mathcal{A}_\sigma \to V_{\theta_1} \times M \times B_0$$

to be the restriction of the projection onto the second, fourth, and fifth components of the product $\mathbb{R} \times \mathring{C}_\varepsilon \times \mathring{C}_\varepsilon \times M \times B_0 \times \mathbb{R}^j$.

Proposition 4.16 *The map $\mathcal{R}_\sigma$ is proper.*

Proof Suppose, just like in Proposition 4.5 that there exists a sequence $u_n \in \mathcal{A}_\sigma$ such that $u_n \to \infty$ and $\mathcal{R}_\sigma(u_n)$ converges. Now, u_n has six components

$$u_n = (t_n, a'_n, a_n, m_n, b_n, z_n)$$

with $t_n \in [0,1]$. By passing to a subsequence we can assume that $t_n \to t' \in [0,1]$. We will show that in both cases $t' = 1$ and $t' \neq 1$ one reaches a contradiction by showing that there exists a subsequence of u_n which converges in $\mathcal{A}_\sigma$.

For the case $t' = 1$, we will use the fact that $\{t = 1\} \cap \overline{W} \cap ([0,1] \times V_{\theta_1} \times V_{\theta_1})$ is compact and thus has a compact neighborhood. In fact, one notices that $\overline{W}$ is completely transverse to $V_{\theta_1} \times V_{\theta_1} \times \mathbb{R}$ and their intersection is a manifold with corners, with one of the codimension 1 boundaries being contained in $\{t = 1\}$

$$\bigcup_{\mathbf{p} \in \mathrm{Crit}(f) \cap f^{-1}(0)} \{1\} \times (U_\mathbf{p} \cap V_{\theta_1}) \times (S_\mathbf{p} \cap V_{\theta_1}).$$

This is compact. We conclude that from (a'_n, a_n) we can extract a subsequence denoted again (a'_n, a_n) that converges to a point in $V_{\theta_1} \times V_{\theta_1}$. On the other hand, we have by hypothesis that (a'_n, m_n, b_n) converges. Hence by passing to a subsequence we conclude that (a_n, m_n, b_n) converges to some point (a, m, b). But $(a_n, m_n, b_n) = (\sigma_1(z_n), \sigma_2(z_n), \sigma_3(z_n))$ and σ is proper. It follows that we can extract yet another subsequence this time from z_n that converges to $z \in N$ (just take $\sigma^{-1}(K)$ where K is a compact neighborhood of (a, m, b)). But then by the continuity of σ we get that (a_n, m_n, b_n, z_n) converges to $(a, m, b, z) \in \Gamma_\sigma$. Since a'_n converges to a point in V_{θ_1} we conclude that u_n converges to a point in $\mathcal{A}_\sigma$, contradiction with $u_n \to \infty$.

For $t' \neq 1$, we use that $a'_n = \psi_{\frac{t_n}{1-t_n}}(a_n)$ and since a'_n converges and $\frac{t_n}{1-t_n} \to \frac{t'}{1-t'}$, we conclude that a_n also converges. We claim that a_n converges to $a \in V_{\theta_1}$. First $a'_n \to a' \in V_{\theta_1}$ by hypothesis. Now V_{θ_1} is a flow-convex neighborhood and $a' = \psi_{\frac{t'}{1-t'}}(a)$. It follows that $a \in V_{\theta_1}$. Now the contradiction is obtained as before $(a_n, m_n, b_n) = (\sigma_1(z_n), \sigma_2(z_n), \sigma_3(z_n))$, and so on.   ∎

We now complete the proof of Proposition 4.9.

Proof   We need only be concerned with items (ii)–(iv). The distinguished boundary is defined as:

$$N' := \partial^1_{\theta_1} \mathcal{A}_\sigma := \overline{W_2} \cap \partial^1_1 Z_\sigma = \overline{W_2} \cap ([0,1] \times f^{-1}(\theta_1) \times \Gamma_\sigma).$$

Clearly $\mathcal{R}_\sigma(\partial^1_{\theta_1} \mathcal{A}_\sigma) \subset \tilde{f}^{-1}(\theta_1)$.

In order to prove (ii), it is useful to consider the projection $\tilde{\mathcal{R}}_\sigma$ from $\mathcal{A}_\sigma$ onto the second, fourth, fifth, and sixth components. Then the image of this map will give the closure of the flow-out of Γ_σ inside $V_{\theta_1} \times M \times B_0 \times \mathbb{R}^j$ and the proof of this fact follows the same lines as the proof of item (1) in Proposition 4.6. Then projecting the closure of the flow-out of Γ_σ onto $V_{\theta_1} \times M \times B_0$ equals $\overline{(\bigcup_{t \geq 0} \xi_t(\sigma(N)))} \cap \tilde{V}_{\theta_1}$ and this takes care of (ii).

The map $\tilde{R}_\sigma$ mentioned in the previous paragraph is injective on $\mathcal{A}_\sigma \backslash \{t = 1\}$ since on this set all points are of type

$$(t, \psi_{\frac{t}{1-t}}(\sigma_1(z)), \sigma_1(z), \sigma_2(z), \sigma_3(z), z), \qquad z \in N, t \in [0,1),$$

which get projected to $(\psi_{\frac{t}{1-t}}(\sigma_1(z)), \sigma_2(z), \sigma_3(z), z)$. Clearly all the points in the forward-flowout of Γ_σ that are inside $V_{\theta_1} \times M \times B_0 \times \mathbb{R}^j$ are also in the image of this projection. In order to obtain a set where $\mathcal{R}_\sigma$ is injective one needs to take out more points.

Since $\sigma\big|_{\partial^2 N}$ is completely transverse when restricted to all distinguished boundary pieces $\partial_i^2 N$ of $\partial^2 N$, it follows that, we can define a natural subspace $\mathcal{A}_\sigma\big|_{\partial^2 N}$ of $\mathcal{A}_\sigma$ by taking the union of the corresponding sets $\overline{W_2} \cap Z_\sigma\big|_{\partial_i^2 N}$. Since this $\mathcal{A}_\sigma\big|_{\partial^2 N}$ will be a union of manifolds with corners of lower dimension it will have measure zero inside $\mathcal{A}_\sigma$. Then $\mathcal{R}_\sigma$ will be injective on $\mathcal{A}_\sigma \setminus \left(\{t = 1\} \cup \mathcal{A}_\sigma\big|_{\partial^2 N} \right)$ and this takes care of (iii).

In order to prove transversality, we can use Lemma 4.17 in order to reduce to the proof of transversality of $\tilde{\mathcal{R}}_\sigma\big|_{\partial_{\theta_1}^1 \mathcal{A}_\sigma}$ with $S_p \times M \times B_0 \times \mathbb{R}^j$. This then follows the same scheme as Lemma 4.8.

For injectivity, we note that $\partial_0^1 \overline{W_2} \cap \partial_1^1 Z_\sigma = \varnothing$ and we separate $N' = \partial_{\theta_1}^1 \mathcal{A}_\sigma$ into two parts:

$$N_1 := \left(\overline{W_2} \setminus \{t = 1\} \right) \cap \partial_1^1 Z_\sigma \ \text{ and } \ N_2 := \partial_1^1 \overline{W_2} \cap \partial_1^1 Z_\sigma.$$

We observe that $\mathcal{R}_\sigma$ takes N_1 and N_2 to two disjoint sets in $\tilde{V}_\theta$ distinguished by the fact that the first component belongs to $\cup_{p \in \mathrm{Crit}(f) \cap f^{-1}(0)} U_p$ (in the case of points in $\mathcal{R}_\sigma(N_2)$) or does not belong to the same set (for $\mathcal{R}_\sigma(N_1)$). We have that N_2 is a distinguished boundary of N' while N_1 contains the top stratum of N'. Hence it is enough to prove the injectivity of $\mathcal{R}_\sigma$ separately on $N_1 \setminus \partial^2 N_1$ and on $N_2 \setminus \partial^1 N_2$.

The points in $N_1 \setminus \partial^2 N_1$ are of type $\left(t_0, \psi_{\frac{t_0}{1-t_0}}(\sigma_1(z)), \sigma_1(z), \sigma_2(z), \sigma_3(z), z \right)$ with $z \in N \setminus \partial^2 N$ and they get mapped to $(a, m, b) := \left(\psi_{\frac{t_0}{1-t_0}}(\sigma_1(z)), \sigma_2(z), \sigma_3(z) \right) \in f^{-1}(\theta_1) \times M \times B_0$. Then t_0 is the unique time it takes to flow backwards from point $a \in f^{-1}(\theta_1)$ to level $f^{-1}(-\theta')$, $\sigma_1(z)$ is the point of intersection with level $f^{-1}(-\theta')$ of the trajectory determined by a, and $z \in N \setminus \partial^2 N$ is uniquely determined by $\sigma(z)$ due to the hypothesis.

The description $N_2 = \bigcup_{p \in f^{-1}(0) \cap \mathrm{Crit}(f)} \{1\} \times (U_\mathbf{p} \cap f^{-1}(\theta_1)) \times \Gamma_\sigma\big|_{\sigma^{-1}(S_\mathbf{p} \times M \times B_0)}$ is useful. For the top stratum $(N_2)^0$ of N_2, one restricts Γ_σ to $\sigma^{-1}(S_\mathbf{p} \times M \times B_0) \cap N^0$. If $p = (u, \sigma_1(z), \sigma_2(z), \sigma_3(z), z) \in (N_2)^0$ then u does not determine $\sigma_1(z)$ anymore, but property (d) says that $\sigma_1(z)$ is determined by $(\sigma_2(z), \sigma_3(z))$. Since σ is also injective on N^0 we get that p determines z. Hence, the map on $(N_2)^0$ that takes p to $(u, \sigma_2(z), \sigma_3(z))$ is injective and this finishes the proof of this issue.

Finally, property (d) itself holds for $\sigma' : N' \to \tilde{V}_{\theta_1} \cap (f^{-1}(\theta_1) \times M \times B_0)$. This follows from the description of $(N')^0 = (N_1)^0$ above and the fact that σ is injective on N^0 and also satisfies property (d). ∎

Lemma 4.17 *Let $\tilde{\sigma} : \tilde{N} \to \tilde{M} \times \mathbb{R}^j$ be a smooth map. Then the (complete) transversality of $\tilde{\sigma}$ with $\tilde{S} \times \mathbb{R}^j$, for some submanifold $\tilde{S} \subset \tilde{M}$ implies the (complete) transversality of $\tilde{\pi} \circ \sigma$ with $\tilde{S}$ where $\tilde{\pi} : \tilde{M} \times \mathbb{R}^j \to \tilde{M}$ is the projection.*

Proof Straightforward. ∎

We now derive an important consequence of Proposition 4.9 in terms of currents. Let $T_\sigma = \mathcal{R}_\sigma(\mathcal{A}_\sigma)$ be the flow-out of $\sigma(N)$ between levels $-\theta'$ and θ_1. It is a rectifiable

current, once it is endowed, over the points where $\mathcal{R}_\sigma$ is a bijection, with the orientation induced by the direction of the flow and the orientation of N^0.

Let $T_\sigma\big|_{\partial_j^1 N} := \mathcal{R}_\sigma\big(\overline{W_2} \cap \partial_{j+1}^1 Z_\sigma\big)$ be the flow-out of $\sigma(\partial_j^1 N)$ for the jth codimension

1 boundary of N also between levels $-\theta'$ and θ_1. This is a rectifiable current of dimension n.

Corollary 4.18 *Let σ be a smooth map as in Proposition 4.9. The following currential equation holds in the open set $\tilde{V}_{\theta_1} \cap \tilde{f}^{-1}(-\theta', \theta_1)$.*

$$(4.15) \qquad dT_\sigma = \sum_{p \in \mathrm{Crit}(f) \cap f^{-1}(0)} \big(U_p \cap f^{-1}[0, \theta_1)\big) \times (\sigma_2, \sigma_3)\big(\sigma_1^{-1}(S_p)\big) + \sum_{j=1}^{f_N} T_\sigma\big|_{\partial_j^1 N} .$$

Proof One uses Lemma 4.14, Proposition 4.9 and Stokes on the manifold with corners $\mathcal{A}_\sigma$ pushed-forward via the proper map $\mathcal{R}_\sigma$. The injectivity property identifies $(\mathcal{R}_\sigma)_*(\mathcal{A}_\sigma)$ with T_σ and $(\mathcal{R}_\sigma)_*\big(\overline{W_2} \cap \partial_{j+1}^1 Z_\sigma\big)$ with $T_\sigma\big|_{\partial_j^1 N}$ for all j. ∎

Remark 4.19 It is important to understand why equation (4.15) was not stated as an identity directly in the open set $\tilde{f}^{-1}(-\theta', \theta_1)$. The conditions in the statement of Proposition 4.9 do not exclude the possibility that the image of σ oscillates wildly close to the topological boundary of $\tilde{V}_{\theta_1} \cap \tilde{f}^{-1}(-\theta')$ inside $\tilde{f}^{-1}(-\theta')$ so that $\sigma_*(N)$ might not be extendable as a current outside this neighborhood. This is of course not the case for the situation where we will apply Proposition 4.9. In the first step of the induction, σ is the restriction of $\mathcal{R}$ to $\partial_2^1 \mathcal{A}_\theta$ and the image of this map "away" from $U_{p_1} \times S_{p_1} \times B_0$ is simply the flow-out to level θ of $\xi_0(B_0 \backslash B_0')$ where $B_0' \subset B_0$ is a smaller neighborhood around the point of interest b_0. Hence, close to the topological boundary of $\tilde{V}_\theta$, or "away" from $U_{p_1} \times S_{p_1} \times B_0$ the image of the map is really an embedded submanifold, extendable beyond the topological boundary. For the other steps of the induction we make the following observation.

Remark 4.20 Proposition 4.9 is what allows us to cross critical levels at least if the image of the map lands close to the stable manifold(s) of the critical point(s) at level 0. But when flowing between two consecutive critical levels, nothing guarantees that the flow-out of the image of the resolution at the first critical level will end-up within a neighborhood of type $\tilde{V}_{\theta_1}$ so as to satisfy the hypothesis of Proposition 4.9.

One solution is to do the following. Suppose that in fact $\sigma : N \to \tilde{f}^{-1}(-\theta_1)$ and for each $p \in \mathrm{Crit}(f) \cap f^{-1}(0)$, there are neighborhoods D_p of $(f^{-1}(\theta_1) \cap S_p) \times M \times B_0$ such that $\sigma\big|_{\sigma^{-1}(D_p)}$ is proper. Then one gets a restriction map $\hat{\sigma}$ of σ to an open set of N, which is proper and whose image is contained in $\tilde{V}_{\theta_1}$ as in Propostion 4.9.

What about the rest of N? Take $D_{-\theta_1}$ to be *the complement* of $\bigcup_{p \in \mathrm{Crit}(f) \cap f^{-1}(0)} (f^{-1}(\theta_1) \cap S_p) \times M \times B_0$ in $\tilde{f}^{-1}(\theta_1)$ and let $D_N := \sigma^{-1}(D_{-\theta_1})$. Look at $\sigma_1 := \sigma\big|_{D_N}$. Use the flow diffeomorphism to get from σ_1 a map $\tilde{\sigma}_1 : D_N \to \tilde{f}^{-1}(\theta_1)$.

We claim that there exists an open subset of D_N and an open subset of $\partial_{\theta_1}^1 \mathcal{A}_\sigma$ which are diffeomorphic via a diffeomorphism α such that

$$\mathcal{R}_\sigma \circ \alpha = \tilde{\sigma}_1.$$

Take $\sigma^{-1}(U)$ where $U := D_{-\theta_1} \cap \tilde{V}_{\theta_1}$. This is obviously an open set diffeomorphic with a open subset of D_N. On the other hand, by taking $\overline{W_2} \cap \left(\mathbb{R} \times f^{-1}(\theta_1) \times \Gamma_\sigma \big|_{\sigma^{-1}(U)} \right)$, one obtains an open subset of $\partial^1_{\theta_1} \mathcal{A}_\sigma$ which is obviously diffeomorphic with $\sigma^{-1}(U)$.

We can then use the diffeomorphism α in order to "glue" $\mathcal{R}_\sigma$ and σ_1 to a smooth map going from a manifold with corners $\tilde{N}$ to $\tilde{f}^{-1}(\theta_1)$ and flow to the next critical level and apply again Proposition 4.9 and this Remark and so on.

Proof of Theorem 3.2 We have already sketched the proof strategy at the end of Section 3. We need to explain what are the open sets $M_j \times M \times B_0$ that cover $M \times M \times B_0$ where the currential identity is true. We will discuss only the situations where B_0 is a small neighborhood around a point b_0 such that the forward trajectory determined by $s(b_0)$ ends at a nonmaximal point. The remaining situation was already discussed in Remark 3.6.

Let $c_0 < \cdots < c_l$ be the consecutive critical levels of f excluding the maximum with c_0 the first encountered critical level by the forward trajectory of $s(b_0)$. It might even be a minimal level of f, i.e., a level that contains a local minimum if $s(b_0)$ is a local minimum. Let $c_0 < \delta_1 < \delta_2 < c_1 < \delta_3 < \delta_4 < \cdots < c_{l-1} < \delta_{2l-1} < \delta_{2l} < c_l < \delta_{2l+1}$ be regular level sets such that $\delta_1 = c_0 + \theta_0$ and for $k \geq 1$, $\delta_{2k} = c_k - \theta_k$ and $\delta_{2k+1} = c_k + \theta_k$ where θ_k is chosen small enough so that we can chose neighborhood V_{θ_k} around the critical points of level c_k satisfying the conditions of Proposition 4.1.

Corollary 4.7 takes care of the first step of induction and implies the formula for

$$f^{-1}(-\infty, \delta_1) \times M \times B_0.$$

By property (4) of Proposition 4.6 and Lemma 4.14 there exists a map for some $\varepsilon_1 > 0$:

$$\sigma : \partial^1_2 \mathcal{A}_1 \to f^{-1}(\delta_1 - \varepsilon_1) \times M \times B_0,$$

whose image contains the closure of the forward-flow of $\xi_0(B_0)$ intersected with level $\delta_1 - \varepsilon_1$ and is transverse to all $S_{\mathbf{p}} \times M \times B_0$ for all critical $\mathbf{p}$. Moreover, $\partial^1_2 \mathcal{A}_1$ is a manifold with boundary $\partial^2 \mathcal{A}_1 = U_{\mathbf{p}_1} \cap f^{-1}(\delta_1 - \varepsilon_1) \times s(s^{-1}(S_{\mathbf{p}_1}) \times B_0)$ and σ satisfies the conditions of Proposition 4.9.

Let T^1_σ to be the current determined by the flow-out of the image of σ in the open set $f^{-1}(\delta_1 - \varepsilon_1, \delta_2 + \varepsilon_2) \times M \times B_0$ for some small ε_2 where there are no critical points. Let $T^1_{\partial\sigma}$ be the flow-out of the image of $\sigma|_{\partial^2 \mathcal{A}_1}$. Both are rectifiable currents with the obvious orientation. The following identity of currents holds on $f^{-1}(\delta_1 - \varepsilon_1, \delta_2 + \varepsilon_2) \times M \times B_0$:

$$dT^1_\sigma = T^1_{\partial\sigma} = \left(U_{\mathbf{p}_1} \cap f^{-1}(\delta_1 - \varepsilon_1, \delta_2 + \varepsilon_2) \right) \times s(s^{-1}(S_{\mathbf{p}_1}) \times B_0)$$

proving thus the theorem on $f^{-1}(\delta_1 - \varepsilon_1, \delta_2 + \varepsilon_2) \times M \times B_0$.

From this point on we repeatedly apply Proposition 4.9 and Corollary 4.18. We notice that $(\sigma_2, \sigma_3)(\sigma_1^{-1}(S_p))$ can be substituted with $(\sigma_2, \sigma_3)(\sigma_1^{-1}(S_p) \cap N^0)$ and these points are easy to describe as $s(s^{-1}(S_p \times B_0))$ where p is any of the critical points

at ith step. It is not anymore difficult to see that $T_\sigma\big|_{\partial^1_j N}$ is a sum:

$$\sum_p \left(U_p \cap f^{-1}(c_i - \theta_i, c_i + \theta_i) \right) \times s(s^{-1}(S_p \times B_0)),$$

where the sum here runs over the critical points that have already been crossed. ∎

Remark 4.21 There is a subtlety in the proof of Theorem 3.2, that is easy to miss. In order to prove an equality of currents, these have to exist to begin with. In particular, T and $U(F) \times_F s(s^{-1}(S(F)))$ should be shown to have locally finite $n + 1$ and respectively n-Hausdorff measures. But this is a straightforward corollary of the existence of the flow resolutions we have constructed. As an immediate consequence we get via Fubini that $U(F)$ and $s(s^{-1}(S(F)))$ have finite Hausdorff measures if B is compact.

5 Odd Chern–Weil theory

We now start a new topic altogether. Let $E \to B$ be a hermitian vector bundle over a compact manifold B. Denote by $\mathscr{U}(E)$ the fiber bundle of unitary isomorphisms and let $U \in \Gamma(\mathscr{U}(E))$ be a section of this bundle. Let ∇ be a connection compatible with the metric.

Given any invariant polynomial P, we introduce odd degree forms $\mathrm{TP}(E, U, \nabla) \in \Omega^*(B)$, called odd Chern–Weil forms, which satisfy the following properties

(a) $d\,\mathrm{TP}(E, U, \nabla) = 0$;
(b) $\mathrm{TP}(E, U, \nabla) - \mathrm{TP}(E, U, \nabla')$ is exact for any two metric compatible connections ∇, ∇';
(c) if $U_0, U_1 \in \Gamma(\mathscr{U}(E))$ are homotopic then $\mathrm{TP}(E, U_0, \nabla) - \mathrm{TP}(E, U_1, \nabla)$ is exact; and
(d) if $\varphi : B_1 \to B$ is a smooth map then $\varphi^* \mathrm{TP}(E, U, \nabla) = \mathrm{TP}(\varphi^* E, \varphi^* U, \varphi^* \nabla)$.

Recall the fundamental Theorem of Chern–Weil theory. If P is an invariant polynomial and ∇_1 and ∇_2 are compatible connections then one can associate to ∇_0 and ∇_1 two forms $P(E, \nabla_1)$ and $P(E, \nabla_0)$ and there exists a nonunique form $\mathrm{TP}(\nabla_1, \nabla_0)$ such that

$$P(E, \nabla_1) - P(E, \nabla_0) = d\,\mathrm{TP}(\nabla_0, \nabla_1).$$

The construction of a particular such form $\mathrm{TP}(\nabla_1, \nabla_0)$ goes as follows. Let $\pi_2^* E \to [0,1] \times B$ be the pull-back of E with respect to the projection $[0,1] \times B \to B$. Consider the following connection on $\pi_2^* E$

$$\tilde{\nabla} := \frac{d}{dt} + (1 - t)\nabla_0 + t\nabla_1.$$

Then the standard homotopy formula implies that

$$d\left(\int_{[0,1]} P(\pi_2^* E, \tilde{\nabla}) \right) = P(E, \nabla_1) - P(E, \nabla_0),$$

where integration on the left is over the fibers of π_2. Define $\mathrm{TP}(\nabla_0, \nabla_1)$ to be $\int_{[0,1]} P(\pi_2^* E, \tilde{\nabla})$.

It is not hard to see that

$$\mathrm{TP}(\nabla_0, \nabla_1) = -\mathrm{TP}(\nabla_1, \nabla_0).$$

This is because the diffeomorphism on $[0,1] \times M \to [0,1] \times M$, $(t, m) \to (1 - t, m)$ reverses the orientation of the fiber and fiber integration is sensitive to this.

Remark 5.1 If one takes different paths between the connection ∇_0 and ∇_1, by a result of Simons and Sullivan [23] the transgression forms obtained for two different paths differ by an exact form.

In our context, consider the connections $\nabla_0 := \nabla$ and $\nabla_1 = U^{-1} \nabla U$ on E and write down the transgression formula from Chern–Weil theory:

$$P(F(U^{-1} \nabla U)) - P(F(\nabla)) = d \, \mathrm{TP}(\nabla, U^{-1} \nabla U).$$

Notice that $F(U^{-1} \nabla U) = U^{-1} F(\nabla) U$. It follows from the fact that P is invariant that $P(F(\nabla)) = P(F(U^{-1} \nabla U))$, hence the form $\mathrm{TP}(\nabla, U^{-1} \nabla U)$ satisfies property (a) above.

The next lemma helps prove property (b) for $\mathrm{TP}(\nabla, U^{-1} \nabla U)$.

Lemma 5.2 *If ∇_i, $i = 1, 4$ and are four metric compatible connections then*

$$\sum_i \mathrm{TP}(\nabla_i, \nabla_{i+1})$$

is exact where $\nabla_5 := \nabla_1$.

Proof If $H : C \times B \to M$ is a smooth map, C is an oriented, compact manifold with corners of dimension c and ω is a smooth form on M of degree $k \geq c - 1$ then:

$$\tag{5.1} \int_C H^* d\omega + (-1)^{c-1} d \int_C H^* \omega = \int_{\partial C} H^* \omega.$$

To see (5.1), apply first Stokes on $C \times B$ to

$$d(H^* \omega \wedge \pi_2^* \eta) = dH^* \omega \wedge \pi_2^* \eta + (-1)^k H^* \omega \wedge \pi_2^* d\eta,$$

where $\eta \in \Omega^{n-k+c-1}(B)$ is a smooth test form and $\pi_2 : C \times B \to B$ is the projection and then integrate over the fiber. On the closed, oriented B one has:

$$\tag{5.2} 0 = d \left(\int_C H^* \omega \right) \wedge \eta + (-1)^{k-c} \left(\int_C H^* \omega \right) \wedge d\eta.$$

Hence, using the orientation of the fiber first convention we get from (5.2)

$$(-1)^k \int_C H^* \omega \wedge \pi_2^* d\eta = (-1)^k \left(\int_C H^* \omega \right) \wedge d\eta = (-1)^{c-1} d \left(\int_C H^* \omega \right) \wedge \eta.$$

Take $C = \Delta^2$ be the standard simplex in $\mathbb{R}^2$ with coordinates (s, t) and on $\pi_2^* E \to C \times B$ consider the connection that "interpolates" between $\nabla_1, \nabla_2, \nabla_3$:

$$\tilde{\nabla} := \frac{d}{ds} + \frac{d}{dt} + \nabla_1 + s(\nabla_2 - \nabla_1) + t(\nabla_3 - \nabla_1),$$

where $\frac{d}{ds} + \frac{d}{dt}$ is the differential on C. The form $P(\pi_2^* E, \tilde{\nabla})$ on $C \times B$ is closed and

$$\int_{\partial C} P(\pi_2^* E, \tilde{\nabla}) = TP(\nabla_1, \nabla_2) + TP(\nabla_2, \nabla_3) + TP(\nabla_3, \nabla_1).$$

Use now (5.1) for $H = \mathrm{id}_{C \times B}$ to conclude that the Lemma works for three connections. Using that $\mathrm{TP}(\nabla, \nabla') = -\mathrm{TP}(\nabla', \nabla)$ one can extend by induction to any finite number of connections. ∎

One applies the lemma with $\nabla_1 := \nabla$, $\nabla_2 := U^{-1} \nabla U$, $\nabla_3 = U^{-1} \nabla' U$ and $\nabla_4 := \nabla'$ in order to conclude property (b). Indeed one has

$$\mathrm{TP}(\nabla_2, \nabla_3) = \mathrm{TP}(\nabla, \nabla') = -\mathrm{TP}(\nabla_4, \nabla_1).$$

Property (c) for $\mathrm{TP}(\nabla, U^{-1} \nabla U)$ is proved as follows. Consider the vector bundle $\pi^* E \to \mathscr{U}(E)$ where $\pi : \mathscr{U}(E) \to B$ is the natural projection. Then $\pi^* E$ has a connection $\pi^* \nabla$ and it also has a tautological unitary isomorphism U^τ : $\mathscr{U}(E) \to \mathscr{U}(\pi^* E)$. Therefore, there exists a natural transgression closed form $\mathrm{TP}(\pi^* \nabla, (U^\tau)^{-1}(\pi^* \nabla) U^\tau) \in \Omega^*(\mathscr{U}(E))$.

Let U_t be a smooth homotopy between U_0 and U_1. Then U_t is a section of

$$\mathscr{U}(\pi_2^* E) \to [0,1] \times B.$$

Then for any closed form ω on $\mathscr{U}(\pi_2^* E)$ the standard homotopy formula informs that $U_0^* \omega$ and $U_1^* \omega$ differ by an exact form on B. Take $\omega := p^* \mathrm{TP}(\pi^* \nabla, (U^\tau)^{-1} \pi^* \nabla U^\tau)$ where $p : \pi_2^* \mathscr{U}(E) \to \mathscr{U}(E)$ is the natural projection. It is not hard to check that

$$(5.3) \qquad U_0^* \omega = U_0^* \mathrm{TP}(\pi^* \nabla, (U^\tau)^{-1} \pi^* \nabla U^\tau) = \mathrm{TP}(\nabla, U_0^{-1} \nabla U_0) \quad \text{and}$$

$$(5.4) \qquad\qquad U_1^* \omega = \mathrm{TP}(\nabla, U_1^{-1} \nabla U_1)$$

and this finishes the proof of the third property.

One checks rather immediately the naturality of TP.

Hence, $\mathrm{TP}(\nabla, U^{-1} \nabla U)$ satisfies the four properties above. Define then the odd Chern–Weil forms associated to (P, E, U, ∇) by

$$\mathrm{TP}(E, U, \nabla) := \mathrm{TP}(\nabla, U^{-1} \nabla U).$$

We will also denote this by $\mathrm{TP}(U, \nabla)$ when the bundle is clear from the context.

Remark 5.3 There is an alternative way of thinking about $\mathrm{TP}(E, U, \nabla)$ that reminds one of the clutching construction. Consider the fiber bundle $\pi_2^* E \to \mathbb{R} \times B$ where now $\pi_2 : \mathbb{R} \times B \to B$.

Then $\mathbb{Z}$ acts on $\pi^* E \to \mathbb{R} \times B$ as follows:

$$k * (t, b, v) = (t - k, b, U_b^k v).$$

We get thus a vector bundle $\tilde{E} = T(E, U) = \pi_2^* E / \mathbb{Z}$. A smooth section of $\tilde{E}$ is a smooth family of sections $(s_t)_{t \in \mathbb{R}} \in \Gamma(E)$ satisfying:

$$s_{t-k}(b) = U_b^k s_t(b), \qquad \forall b \in B, \ k \in \mathbb{Z}, \ t \in \mathbb{R}.$$

Suppose ∇_t is a smooth family of connections on $E \to B$ satisfying:

$$(5.5) \qquad \nabla_{t+k} = U^{-k} \nabla_t U^k,$$

Then the connection $T(\nabla_t) = \frac{d}{dt} + \nabla_t$ on $\pi_2^* E$ "descends" to a connection on $\tilde{E}$ as the next computation shows:

$$(T(\nabla_t)s)_{t-k}(b) = \frac{\partial s}{\partial t}(t-k, b)\, dt + \nabla_{t-k} s_{t-k}(b) =$$

$$= U_b^k \frac{\partial s}{\partial t}(t, b)\, dt + U_b^k \nabla_t (U_b^{-k} U_b^k s_t)(b) = U_b^k (T(\nabla_t)s)_t)(b).$$

Now, the affine family $(1-t)\nabla + tU^{-1}\nabla U$ defined for $t \in [0,1]$ satisfies (5.5) for $k = 1$ and $t = 0$. Unfortunately, one cannot extend it to a *smooth* family satisfying (5.5) so one cannot say that $\tilde{\nabla}$ is a smooth connection on $\tilde{E}$.

However, one can show that $\mathrm{TP}(E, U, \nabla)$ represents the cohomology class

$$\int_{S^1} P(\tilde{E}),$$

where $P(\tilde{E})$ is the deRham cohomology class of the vector bundle $\tilde{E} \to S^1 \times B$. This is because the transgressions between ∇ and $U^{-1}\nabla U$ determined by the affine path and by a path satisfying (5.5) differ by an exact form. If one starts with a connection on $\tilde{E}$ of type $T(\nabla_t)$, the integral over S^1 of $P(F(T(\nabla_t)))$, is really the integral over $[0,1]$ of the same quantity and is just the transgression between ∇ and $U^{-1}\nabla U$ given by the path of connections ∇_t. This justifies the claim.

All vector bundles over $S^1 \times B$ when B is compact arise up to isomorphism via the clutching construction. This is because the pull-back of such a bundle to $[0,1] \times B$ is isomorphic with the pull-back of a vector bundle from B. The isomorphism of the fiber at 0 with the fiber at 1 gives the desired gauge transformation.

Example 5.4 Consider the trivial vector bundle $\underline{\mathbb{C}^n} \to U(n)$. It has a tautological gauge transformation $\widetilde{U}(U) := U$. Let d be the trivial connection. Then the difference between the $\widetilde{U}^{-1}d\widetilde{U}$ and d is just the Maurer-Cartan 1-form of $U(n)$ usually denoted $g^{-1}(dg)$. Then the transgression forms $\mathrm{Tc}_k(\tilde{U}, d)$ corresponding to the elementary symmetric polynomials c_k in the eigenvalues of a matrix, or if you want to the standard Chern classes are constant multiples of the forms

$$\mathrm{tr} \wedge^{2k-1} g^{-1}(dg).$$

We determine these constants now. Let $\omega := g^{-1}(dg)$. Consider the family of connections over $\underline{\mathbb{C}^n}$:

$$d + tg^{-1}(dg) = d + t\omega.$$

Then the induced connection on $\underline{\mathbb{C}^n} \to [0,1] \times B$ is $\tilde{\nabla} = d + t\pi_2^* \omega$. Using the Maurer–Cartan identity $d\omega + \omega \wedge \omega = 0$ we get that

$$F(\tilde{\nabla}) = dt \wedge \pi_2^* \omega + (t^2 - t)\pi_2^* \omega \wedge \pi_2^* \omega.$$

Then, by definition, the component of degree k of the Chern character is

$$\mathrm{ch}_k(\tilde{\nabla}) = \left(\frac{i}{2\pi}\right)^k \frac{1}{k!}\,\mathrm{tr}(F(\tilde{\nabla})^k).$$

But letting $A = dt \wedge \pi_2^*\omega$ and $B = (t^2 - t)\omega \wedge \omega$, we see that $A^2 = 0$ and $AB = BA$ and so $(A + B)^k = B^k + kB^{k-1}A$. We therefore get

$$\mathrm{ch}_k(\tilde{\nabla}) = \left(\frac{i}{2\pi}\right)^k \frac{1}{(k-1)!}(t^2 - t)^{k-1}dt \wedge \mathrm{tr}(\wedge^{2k-1}\pi_2^*\omega) + \left(\frac{i}{2\pi}\right)^k \frac{1}{k!}\,\mathrm{tr}(\wedge^{2k}\pi_2^*\omega).$$

The last term vanishes before integration because $\wedge^{2k}\pi_2^*\omega = \frac{1}{2}[\pi_2^*\omega, \pi_2^*\omega]$ and the trace vanishes on commutators. Hence,

$$(5.6)\qquad \mathrm{ch}_k(\tilde{\nabla}) = \left(\frac{i}{2\pi}\right)^k \frac{1}{(k-1)!}(t^2 - t)^{k-1}dt \wedge \mathrm{tr}(\wedge^{2k-1}\pi_2^*\omega).$$

On the other hand, $\int_{[0,1]}(t^2 - t)^{k-1}dt = (-1)^{k-1}B(k,k) = (-1)^{k-1}\frac{[(k-1)!]^2}{(2k-1)!}$.
We conclude that

$$(5.7)\qquad \mathrm{Tch}_k(\tilde{U}, d) = \int_{[0,1]}\mathrm{ch}_k(\tilde{\nabla}) = (-1)^{k-1}\left(\frac{i}{2\pi}\right)^k \frac{(k-1)!}{(2k-1)!}\,\mathrm{tr}(\wedge^{2k-1}g^{-1}(dg)).$$

We emphasize that this is the same form as $(-2\pi i)^{-k+1/2}\gamma_{2k-1}$ where γ_{2k-1} appears in Definition 5.1 of [21]. In fact, Quillen shows in his Proposition 5.23, by a rather long argument, that the form $(-2\pi i)^{-k+1/2}\gamma_{2k-1}$ represents the integral of $\mathrm{ch}_k(\tilde{E})$ over S^1, where $\tilde{E}$ is the vector bundle on $S^1 \times U(n)$ constructed from the trivial bundle via the clutching construction with respect to the tautological gauge transform. Remark 5.3 gives a more straightforward justification of this fact.

Now, the symmetric polynomials ch_k and c_k are related via the Newton identities. One has an identity of type:

$$c_k = (-1)^{k-1}(k-1)!\,\mathrm{ch}_k + R,$$

where R stands for a sum of products of ch_i with $i < k$. A quick glance at (5.6) convinces us that $R(\tilde{\nabla}) = 0$. Hence after integration over $[0,1]$ we get that

$$(5.8)\qquad \mathrm{Tc}_k(\tilde{U}, d) = \left(\frac{i}{2\pi}\right)^k \frac{[(k-1)!]^2}{(2k-1)!}\,\mathrm{tr}(\wedge^{2k-1}g^{-1}(dg)).$$

For $k = 1$ one gets $Tc_1(\tilde{U}, d) = -\frac{1}{2\pi i}\,\mathrm{tr}(g^{-1}(dg))$.

Let us end this example by giving a simple application.

Lemma 5.5 *Let $k \leq n$ and $\iota_{k-1,n} : U(k-1) \to U(n)$ be the natural inclusion. Then $\iota_{k-1,n}^*\,\mathrm{Tc}_l(\tilde{U}, d)$ is exact for $l \geq k$.*

Proof It is enough to prove the Lemma for $l = k$, and derive the general property from the inclusions $U(k-1) \hookrightarrow U(l-1) \hookrightarrow U(n)$. We look at the "clutching" bundle determined by $\mathbb{C}^n = \mathbb{C}^{k-1} \times \mathbb{C}^{n-k+1}$ over $S^1 \times U(k-1)$ where $U(k-1)$ acts trivially on $\mathbb{C}^{n-k+1}$. Denote this bundle by $\tilde{E}_{k,n}$. Clearly, $\tilde{E}_{k,n}$ will have $n - k + 1$ linearly independent sections, namely the vectors $e_k, \ldots, e_n$ of the canonical basis get glued to themselves and hence determine an trivial rank $n - k + 1$ subbundle of $\tilde{E}_{k,n}$. It

follows that $c_k(\tilde{E}_{k,n}) = 0$ in cohomology and hence also $\int_{S^1} c_k(\tilde{E}_{k,n}) = 0$. By Remark 5.3 and naturality of the odd Chern–Weil forms, this is the same as the class of $\iota_{k-1,n}^* \mathrm{Tc}_k(\tilde{U}, d)$. ∎

Remark 5.6 The odd Chern–Weil theory associates to a gauge transformation certain odd cohomology classes. For simplicity we presented it here for structure group $U(n)$, but it can be easily adapted to principal bundles $P \to B$ endowed with a gauge transform g and principal connection 1-form ω. Then one gets a new principal connection $g^{-1}\omega g$ and integrating over $[0,1]$ the invariant polynomial in the curvature entries of $(1-t)\omega + tg^{-1}\omega g$ gives a transgression closed form on the *base space* that satisfies analogous properties as $\mathrm{TP}(U, \nabla)$.

Recall that Chern–Simons theory constructs typically *nonclosed* odd forms living in the *total space* of the principal bundle as follows. Take a principal bundle $\pi : P \to B$ with structure group G, connection 1-form ω and an invariant polynomial Q. Then $\pi^* P = P \times_B P \to P$ is a trivial principal bundle with the trivializing section given by the diagonal embedding. It is endowed with two connections: $\pi^*\omega$ and the pullback of the Maurer–Cartan connection 1-form $\pi_2^* g^{-1}(dg)$ via the projection $\pi_2 : P \to G$ induced by the trivialization. Using the affine family of connections between $\pi_2^* g^{-1}(dg)$ and $\pi^*\omega$ one gets the Chern–Simons form $TQ(\omega)$ on P. Since the curvature of $g^{-1}(dg)$ is zero it follows that $dTQ(\omega) = \pi^* Q(\omega)$. The form $TQ(\omega)$ "descends" to a form on B only under very special circumstances.

The odd Chern–Weil theory on the other hand can be constructed in the same spirit, only that one is using the bundle of *gauge transformations* which is the bundle associated to $P \to B$ via the adjoint action of G onto itself.

6 The Chern classes of a gauge transform

In this section, we show how the main transgression identity from Corollary 3.3 can be applied to simplify the proof and give an extension to a result of Nicolaescu in [19]. This application was previously announced in a preprint of the first named author and posted on arxiv, using the *tame* version of the Corollary 3.3. However, the flow used did not satisfy the condition of tameness. We revisit this application under a renewed framework. The generalization has to do with the use of the odd Chern–Weil forms introduced in the previous section.

The starting question is the following. Let $E \to B$ be a hermitian vector bundle of rank n endowed with a gauge transform $U \in \Gamma(\mathcal{U}(E))$ and a compatible connection ∇.

Give a description of the Poincaré dual of $\mathrm{Tc}_k(U, \nabla)$, one in terms of pointwise spectral data of U.

In order to achieve this purpose, we will use a horizontally constant, vertical Morse–Smale vector field on fiber bundle over B with total space $\mathcal{U}(E)$.

Let $\pi : \mathcal{U}(E) \to B$ be the projection and $\pi^*\mathcal{U}(E) = \mathcal{U}(E) \times_B \mathcal{U}(E) \to \mathcal{U}(E)$ be the pull-back. It has a tautological section U^τ (the diagonal embedding of $\mathcal{U}(E)$ in $\pi^*\mathcal{U}(E)$) which is obviously a gauge transform of $\pi^* E$. With the help of the connection $\pi^* \nabla$ we can construct $\mathrm{Tc}_k(U^\tau, \pi^* \nabla) \in \Omega^*(\mathcal{U}(E))$ and the naturality of

Tc_k shows that

$$U^* \mathrm{Tc}_k(U^\tau, \pi^* \nabla) = \mathrm{Tc}_k(U, \nabla).$$

The form $\omega \in \Omega^*(\mathscr{U}(E))$ to be "flown" will be $\omega = \mathrm{Tc}_k(U^\tau, \pi^* \nabla)$. What is the flow then? In order to define it, we need to fix a complete flag of sub-bundles

$$E = W_0 \supset \cdots \supset W_n = \{0\}.$$

For $1 \le i \le n$ let $E_i := W_{i-1}/W_i$ (better said the orthogonal of W_i in W_{i-1}) and

$$A := \bigoplus_{1 \le k \le n} k \, \mathrm{id}_{E_i}$$

be a self-adjoint endomorphism of E with distinct eigenvalues. Then the function

$$f : \mathscr{U}(E) \to \mathbb{R}, \qquad f(U) = \mathrm{Re}\,\mathrm{Tr}(AU)$$

has a fiberwise or vertical gradient which can be described as $\mathrm{grad}^v f(U) = A - UAU$. The restriction to each fiber $\mathscr{U}(E_b)$ is a Morse–Smale function on the unitary group, thoroughly explored in [19][4] We recall the essential properties. The critical points are in one-to-one correspondence with the invariant subspaces of A. More precisely, let $\langle e_1, \ldots, e_n \rangle$ be a basis of E_b such that $(W_k)_b^\perp = \langle e_1 \ldots, e_k \rangle$. The critical points are reflections $U_I := -\mathrm{id}_V \oplus \mathrm{id}_{V^\perp}$ where $V = \langle e_{i_1}, \ldots, e_{i_k} \rangle$ for some ordered set $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$. There are 2^n such critical points with the absolute minimum (of $f\big|_{E_b}$) corresponding to $V = E_b$ and the absolute maximum to $V = \{0\}$.

The flow is given by the expression [9]

$$(t, U) \to (\sinh(tA) + \cosh(tA)U)(\cosh(tA) + \sinh(tA)U)^{-1}.$$

From this, we deduce that the stable and unstable manifolds can be described by the following incidence relations (compare with Corollary 16 in [19]):

$$S(U_I) = \{U \in \mathcal{U}(E_b) \mid \dim[\mathrm{Ker}\,(1 + U) \cap W_m] = k - p, \quad \forall 0 \le p \le k,$$
$$\forall\, i_p \le m < i_{p+1}\}$$
$$U(U_I) = \{U \in \mathcal{U}(E_b) \mid \dim[\mathrm{Ker}\,(1 - U) \cap W_m] = n - k - q, \quad \forall 0 \le q \le n - k,$$
$$\text{(6.1)} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, j_q \le m < j_{q+1}\},$$

where we set $i_0 := 0$, $i_{k+1} := \infty$ and $\{j_1 < \cdots < j_{n-k}\} = I^c$ is the complement of I. We note that $\dim U(U_I) = \mathrm{codim}\, S(U_I) = \sum_{i \in I} 2i - 1$.[5]

Remark 6.1   It is shown in [19] (Proposition 17 and Corollary 18) that the flow satisfies the Smale property. Moreover, their closures $\overline{U(U_I)}$ and $\overline{S(U_I)}$ are real algebraic sets. One can show that in fact $\overline{S(U_I)}$ has a stratification with no codimension 1 strata (see the comments after Corollary 5.1 in [4]) and by reversing the flow (or using the involution $U \to -U$) the same is true about $\overline{U(U_I)}$. In other words, $\overline{S(U_I)}$ and $\overline{U(U_I)}$ are pseudo-manifolds and $S(U_I)$ and $U(U_I)$ determine *closed* currents once

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[4]The analysis in [19] is on the Grassmannian of hermitian Lagrangians but, by a Theorem of Arnold a clever way of writing the Cayley transform makes this space diffeomorphic to the unitary group.

[5]We ignore the indication of the point b in the flag so as not to complicate notation. It should be clear from the context whether we refer to the fiber component or the entire fiber bundle.

an orientation is chosen. Their volume is finite by the results of [12] (see Remark 4.21). This also implies in particular that the Morse–Witten complex associated to the Morse–Smale flow induced by f is perfect, i.e., all the differentials are zero. This is not the case for the analogous flow on the real orthogonal group.

We will need the following two computational results interesting in their own right. Recall the forms $\mathrm{Tc}_k(\tilde U, d)$ from (5.8). We use the same notation for the analogous forms on $\mathcal U(E_b)$. In order to keep the notation simple, for this part, we will forget about the point b and use E and W for E_b and W_b, and so on.

Lemma 6.2 *Let $I \neq \{k\}$. Then*

$$\int_{U(U_I)} \mathrm{Tc}_k(\tilde U, d) = 0.$$

Proof Clearly this is true by definition if $\dim U(U_I) \neq \dim U(U_{\{k\}}) = 2k - 1$. If $\dim U(U_I) = 2k - 1$ but $U_I \neq U_{\{k\}}$ then we infer that $\iota := \max\{i \in I\} \leq k - 1$. We claim that this implies that

$$\mathrm{Ker}(1 - U) \supset W_{k-1}, \qquad \forall U \in U(U_I).$$

Indeed, let $m \geq \iota = \max I$. We first estimate p which satisfies $j_p \leq m < j_p + 1$, where $j_p \in I^c$. Let $l := |I|$. Then, for some $s \geq 0$, we have

(6.2) $$m = \iota + s < j_{\iota - l + s + 1}.$$

To see this more clearly consider first $s > 0$ and notice that in fact $\iota + s = j_{\iota - l + s}$ as there are exactly $\iota - l + s$ of j's in I^c which are smaller or equal than $\iota + s$, which itself lies in I^c. For the case $s = 0$, one still has $m = \iota < j_{\iota - l + 1}$ since there are exactly $\iota - l$ numbers smaller or equal ι which are not in I.

Then (6.2) implies that $p \leq \iota - l + s = m - l$. Therefore for $U \in U(U_I)$ one has:

$$\dim\left[\mathrm{Ker}\,(1 - U) \cap W_m\right] = n - l - p \geq n - m = \dim W_m.$$

Hence $U\big|_{W_m} = \mathrm{id}_{W_m}$ for all $m \geq \max I$ and this applies to $m = k - 1$. It follows then that the (proper) inclusion map:

$$\iota_{k-1,n} : \mathcal U(W_{k-1}^\perp) \to \mathcal U(E), \qquad U \to U \oplus \mathrm{id}_{W_{k-1}}$$

takes $U(U_I) \subset \mathcal U(W_{k-1}^\perp)$ diffeomorphically to $U(U_I) \subset \mathcal U(E)$.[6] Now $U(U_I)$ is a closed current in $\mathcal U(W_{k-1}^\perp)$ irrespective of the orientation. Then Lemma 5.5 and Remark 6.1 finish the proof. ∎

In order to compute $\mathrm{Tc}_k(\tilde U, d)$ over $U(U_{\{k\}})$, we need to fix an orientation. Notice first that $U(U_{\{k\}}) = \iota_{k,n}(U(U_{\{k\}}))$, where the latter lies within $\mathcal U(W_k^\perp)$ by the same type of argument that was used in Lemma 6.2 for $U(U_I)$. Moreover, by the naturality of $\mathrm{Tc}_k(\tilde U, d)$, one has $\iota_{k,n}^* \mathrm{Tc}_k(\tilde U, d) = \mathrm{Tc}_k(\tilde U, d)$.

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[6]Of course we abused notation by not making any difference between the unstable manifold corresponding to U_I in $\mathcal U(E)$ and in $\mathcal U(W_{k-1}^\perp)$, respectively.

It is only natural then to work on $\mathcal{U}(W_k^\perp)$. Notice that we have a natural flag on $W_k^\perp$ defined by $W_i' := W_i/W_k$, $0 \le i \le k$. Then

$$U(U_{\{k\}}) = \{U \in \mathcal{U}(W_k^\perp) \mid \dim[\mathrm{Ker}\,(1-U) \cap W_m] = k-1-m, \quad \forall 0 \le m \le k-1\}.$$

In fact $U(U_{\{k\}})$ is an open dense subset of the following manifold defined by a single incidence relation:

$$U(U_\kappa) := \{U \in \mathcal{U}(W_k^\perp) \mid \dim \mathrm{Ker}\,(1-U) = k-1\}.$$

This is because, generically, a hyperplane of $W_k^\perp$ (like $\mathrm{Ker}\,(1-U)$ for $U \in U(U_\kappa)$ will intersect W_m' in dimension $k-1-m$ for $m \ge 1$. Now

$$\overline{U(U_\kappa)} = \{U \in \mathcal{U}(W_k^\perp) \mid \dim \mathrm{Ker}\,(1-U) \ge k-1\} = U(U_\kappa) \cup \{\mathrm{id}_{W_k^\perp}\}.$$

We use the following map

$$\phi : S^1 \times \mathbb{P}(W_k^\perp) \to \mathcal{U}(W_k^\perp), \qquad (\lambda, L) \to \lambda\,\mathrm{id}_L \oplus \mathrm{id}_{L^\perp}$$

and note that for $\{\lambda \ne 1\}$, this map is a smooth bijection onto $U(U_\kappa)$ while on $\{\lambda = 1\}$ it "collapses" $\mathbb{P}(W_k^\perp)$ to $\mathrm{id}_{W_k^\perp}$.

Remark 6.3　Notice that the map ϕ induces an homeomorphism between $\Sigma\mathbb{CP}^{k-1}$ (the Thom space of the trivial real line bundle over $\mathbb{CP}^{k-1}$) and $\overline{U(U_\kappa)}$.

Now, $S^1 \times \mathbb{P}(W_k^\perp)$ has a canonical orientation. We put the orientation on $U(U_\kappa)$ (implicitly also on $U(U_{\{k\}})$) that makes ϕ *orientation reversing*. The reason for this choice is the next result.

Lemma 6.4

$$\int_{U(U_{\{k\}})} \mathrm{Tc}_k(\tilde{U}, d) = \int_{\overline{U(U_\kappa)}} \mathrm{Tc}_k(\tilde{U}, d) = 1.$$

Proof　Fix $L \in \mathbb{P}(W_k^\perp)$. We write ϕ in the "chart" $S^1 \times \mathrm{Hom}(L, L^\perp)$:

$$\phi(\lambda, A) = \begin{pmatrix} \frac{\lambda + A^*A}{1+A^*A} & (\lambda-1)(1+A^*A)^{-1}A^* \\ (\lambda-1)(1+AA^*)^{-1}A & \frac{1+\lambda AA*}{1+AA^*} \end{pmatrix},$$

where the decomposition of $\phi(\lambda, A)$ on the right is relative $L \oplus L^\perp$. The differential at the point $(\lambda, 0)$ in this chart is:

$$d\phi_{\lambda, L}(w, S) = \begin{pmatrix} w & (\lambda-1)S^* \\ (\lambda-1)S & 0 \end{pmatrix}$$

Hence,

$$\phi^{-1}(\lambda, L)d\phi_{\lambda, L}(w, S) = \begin{pmatrix} \bar\lambda w & (1-\bar\lambda)S^* \\ (\lambda-1)S & 0 \end{pmatrix}.$$

The right hand side is always a skew-symmetric matrix. This is of course the pull-back of $g^{-1}(dg)$ to $S^1 \times \mathbb{P}(W_k^\perp)$. It should be looked at as a 1-form with values in $\cong(k) = \cong(\tau \oplus \tau^\perp)$ where τ is the tautological bundle over $\mathbb{P}(W_k^\perp)$ pulled-back to $S^1 \times \mathbb{P}(W_k^\perp)$.

Then we see that

$$\phi^*(g^{-1}dg)(\lambda, L) = \begin{pmatrix} \lambda^{-1}d\lambda & -\overline{\alpha(\lambda)}dS_L^* \\ \alpha(\lambda)dS_L & 0 \end{pmatrix} \qquad \alpha(\lambda) = \lambda - 1,$$

where dS denotes (the pull-back of) the 1-form with values in the bundle $\mathrm{Hom}(\tau, \tau^\perp) \simeq T^{(1,0)}\mathbb{P}(W_k^\perp)$, obtained by differentiating $\mathrm{id}_{\mathbb{P}(W_k^\perp)}$ and dS^* is the conjugate of dS. The important point is that dS is globally defined not just in the chart centered at L.

We need to compute $\wedge^{2k-1}\phi^*(g^{-1}dg)$. Write then

$$\phi^*(g^{-1}dg) = C + B,$$

where

$$C := \begin{pmatrix} \lambda^{-1}d\lambda & 0 \\ 0 & 0 \end{pmatrix} \qquad B := \begin{pmatrix} 0 & -\overline{\alpha(\lambda)}dS^* \\ \alpha(\lambda)dS & 0 \end{pmatrix}.$$

We need also $C_1 = \begin{pmatrix} 0 & 0 \\ 0 & -\lambda^{-1}d\lambda \otimes \mathrm{id} \end{pmatrix}$.

The following relations are straightforward

$$C^2 = 0, \quad C_1^2 = 0, \quad B^2 C = CB^2, \quad B^2 C_1 = C_1 B^2,$$

$$CC_1 = C_1 C = 0, \quad BCB = B^2 C_1, \quad C_1 BC = 0, \quad CBC_1 = 0.$$

Let $\wedge^0 B := \mathrm{id}$. We prove by induction that for all $j \geq 1$ the following holds.

(6.3) $$\wedge^{2j-1}\phi^*(g^{-1}dg) = \wedge^{2j-2}B \wedge [jC + (j-1)C_1] + \wedge^{2j-1}B.$$

Indeed the equality is trivially true for $j = 1$. Let $\omega := \phi^*(g^{-1}dg)$ and write $\omega^j := \wedge^j \omega$. Then the following computation finishes the proof of (6.3):

$$\omega^{2j-1} = \omega^{2j-3} \wedge \omega^2 = [B^{2j-4}((j-1)C + (j-2)C_1) + B^{2j-3}](CB + BC + B^2) =$$

$$= B^{2j-4}((j-1)C + (j-2)C_1)B^2 + B^{2j-3}CB + B^{2j-2}C + B^{2j-1} =$$

$$B^{2j-2}((j-1)C + (j-2)C_1) + B^{2j-2}C_1 + B^{2j-2}C + B^{2j-1}$$
$$= B^{2j-2}(jC + (j-1)C_1) + B^{2j-1}.$$

Now B^{2j-1} is block anti-diagonal, hence $\mathrm{tr}\, B^{2k-1} = 0$ and we conclude that:

$$\mathrm{tr}\, \wedge^{2k-1}\omega = \mathrm{tr}\, B^{2k-2} \wedge (jC + (j-1)C_1) =$$

$$= [(-1)^{k-1}|\alpha(\lambda)|^{2k-2}\lambda^{-1}d\lambda] \cdot \text{w-str}(D^{k-1}),$$

where $\text{w-str}\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} = k\,\mathrm{tr}\, T_1 - (k-1)T_2$ and $D = \begin{pmatrix} dS^* \wedge dS & 0 \\ 0 & dS^* \wedge dS \end{pmatrix}$.

We have thus written $\operatorname{tr} \wedge^{2k-1}\phi^* g^{-1}(dg)$ as a product of pull-backs of forms from S^1 and $\mathbb{P}(W_k^\perp)$ respectively. Since ϕ is orientation reversing we have

$$(6.4) \qquad \int_{U(U_\kappa)} \operatorname{tr} \wedge^{2k-1} g^{-1}(dg) = \int_{S^1} (-1)^k |\alpha(\lambda)|^{2k-2}\lambda^{-1}d\lambda \cdot \int_{\mathbb{P}(W_k^\perp)} \text{w-str}(D^{k-1}).$$

We use the orientation preserving Cayley transform $t \to \frac{t-i}{t+i}$ to turn the integral:

$$\int_{S^1} (-1)^k |\alpha(\lambda)|^{2k-2}\lambda^{-1}d\lambda = (-1)^k 2^{k-1} \int_{S^1} (1 - \operatorname{Re}\lambda)^{k-1}\lambda^{-1}d\lambda$$

into

$$(6.5) \qquad 2^{2k-1}(-1)^k i \int_{\mathbb{R}} \frac{1}{(1+t^2)^k}\, dt = (-1)^k 2\pi i \binom{2k-2}{k-1}.$$

In order to compute the integral of w-str(D^{k-1}), we notice first that w-str(D^{k-1}) is a $\mathcal{U}(W_k^\perp)$ invariant form on $\mathbb{P}(W_k^\perp)$ and therefore has to equal a constant multiple times the Fubini–Study volume form of $\mathbb{P}(W_k^\perp)$. If we fix a point $L_0 \in \mathbb{P}(W_k^\perp)$ we can describe dS and dS^* in terms of a canonical basis of the chart centered at L_0 as

$$dS_{L_0} = \begin{pmatrix} dz_1 \\ dz_2 \\ \cdots \\ dz_{k-1} \end{pmatrix} \qquad dS^*_{L_0} = \begin{pmatrix} d\bar{z}_1 & d\bar{z}_2 & \cdots & d\bar{z}_{k-1} \end{pmatrix}$$

Therefore,

$$dS^*_{L_0} \wedge dS_{L_0} = \sum_{i=1}^{k-1} d\bar{z}_i \wedge dz_i, \text{ and } (dS_{L_0} \wedge dS^*_{L_0})_{ij} = dz_i \wedge d\bar{z}_j, \quad \forall\, 1 \le i, j \le k-1.$$

We get

$$(S^*_{L_0} \wedge dS_{L_0})^{k-1} = (k-1)!(-1)^{k-1} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{k-1} \wedge d\bar{z}_{k-1}.$$

One checks rather easily that $(dS_{L_0} \wedge dS^*_{L_0})^{k-1}$ is diagonal and each diagonal entry is up to a sign equal to $(k-2)! dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{k-1} \wedge d\bar{z}_{k-1}$. In fact

$$(dS_{L_0} \wedge dS^*_{L_0})^{k-1} = (-1)^{k-2}(k-2)! dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{k-1} \wedge d\bar{z}_{k-1} \otimes \text{id}$$

and thus

$$\text{w-str}\, D^{k-1}_{L_0} = (-1)^{k-1}(2k-1)(k-1)! dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_{k-1} \wedge d\bar{z}_{k-1}.$$

The Kähler form at the point L_0 on $\mathbb{P}(W_k^\perp)$ (see [10], p. 31) is

$$\eta_{L_0} := \frac{i}{2\pi} \sum_{j=1}^{k-1} dz_j \wedge d\bar{z}_j$$

and

$$\int_{\mathbb{P}(W_k^\perp)} \wedge^{k-1}\eta = 1.$$

We compare $\wedge^{k-1}\eta_{L_0}$ and w-str $(D_{L_0}^{k-1})$ and deduce, due to the fact that they are invariant forms that

$$\text{w-str}\left(D^{k-1}\right) = (-1)^{k-1}(2k-1)\left(\frac{2\pi}{i}\right)^{k-1}\wedge^{k-1}\eta$$

and therefore

(6.6) $$\int_{\mathbb{P}(W_k^\perp)} \text{w-str}\left(D^{k-1}\right) = (-1)^{k-1}(2k-1)\left(\frac{2\pi}{i}\right)^{k-1}.$$

Putting together (6.4)–(6.6) we conclude that

$$\int_{U(U_\kappa)} \text{tr}\wedge^{2k-1}g^{-1}(dg) = -2\pi i\binom{2k-2}{k-1}(2k-1)\left(\frac{2\pi}{i}\right)^{k-1} = \left(\frac{2\pi}{i}\right)^k\frac{(2k-1)!}{[(k-1)!]^2}. \quad\blacksquare$$

We will keep the notation U_I, $S(U_I)$ and $U(U_I)$ for the corresponding critical/stable/unstable manifolds in $\mathcal{U}(E)$.

Theorem 6.5 *Let $E \to B$ be a trivializable hermitian vector bundle of rank n over an oriented manifold with corners B endowed with a compatible connection. Let $g : E \to E$ be a smooth gauge transform. Suppose that a complete flag $E = W_0 \supset W_1 \supset \cdots \supset W_n = \{0\}$ (equivalently a trivialization of E) has been fixed such that g as a section of $\mathcal{U}(E)$ is completely transverse to all the manifolds $S(U_I)$ determined by the flag. Then, for each $1 \le k \le n$ there exists a flat current T_k such that the following equality of currents of degree $2k-1$ holds:*

(6.7) $$\text{Tc}_k(E, g, \nabla) - g^{-1}(S(U_{\{k\}})) = dT_k,$$

where

$$g^{-1}(S(U_{\{k\}})) = \{b \in B \mid \dim \text{Ker}\,(1+g_b) \cap (W_{k-1})_b = 1,$$
$$\dim \text{Ker}\,(1+g_b) \cap (W_k)_b = 0\}.$$

In particular, when B is compact without boundary, then $\text{Tc}_k(E, g, \nabla)$ and $g^{-1}(S(U_{\{k\}}))$ are Poincaré duals to each other.

Proof We use Corollary 3.3 for the fiber bundle $\mathcal{U}(E)$ with the flow described in this section and form $\omega = \text{Tc}_k(U^\tau, \pi^*\nabla)$ where $\pi : \mathcal{U}(E) \to B$. All the necessary residue computations have been performed. $\quad\blacksquare$

Remark 6.6 The current T_k is a spark in the terminology of Harvey and Lawson.

Remark 6.7 While the transversality condition of g with the stable manifolds $S(U_I)$ such that $S(U_I) \subset \overline{S(U_{\{k\}})}$ is a reasonable requirement for the existence of the current $g^{-1}(S(U_{\{k\}}))$, it seems unnatural that one needs to impose the transversality of g with *all* stable manifolds $S(U_I)$ in order to obtain (6.7) as one does in Theorem 7. We conjecture that (6.7) is true under the weaker hypothesis.

7 A Fredholm transgression formula

In [21], Quillen introduced various smooth differential forms that live on (infinite dimensional) Banach manifolds that are classifying for even and odd K-theory. Fix

H a complex, separable Hilbert space and let $\mathscr{L}$, $\mathscr{L}^+$, $\mathscr{K}$ be the space of bounded, bounded and self-adjoint, respectively compact operators on H. Inside $\mathscr{K}$ there exists a sequence of two-sided ideals, called Schatten spaces, denoted Sch^p.

The Palais spaces are the spaces of unitary operators $\mathcal{U}^p := \mathcal{U}(H) \cap (\mathrm{id}_H + \mathrm{Sch}^p)$ and it is well-known that they are smooth Banach manifolds (modeled on $\mathrm{Sch}^p \cap \mathscr{L}^+$) and also classifying for odd K-theory, i.e., they have the weak homotopy type of the topological direct limit of spaces $U(\infty) := \lim U(n)$. Quillen defined different families of smooth closed forms γ_{2k-1}^t, $\gamma_{2k-1,q}^t$, Φ_{2k-1}^u where $t \in \mathbb{C}$, $\mathrm{Re}\, t > 0$, $n \in \mathbb{N}$, $u > 0$ which are *well-defined* on the finite dimensional unitary groups $U(n)$ and on certain Palais spaces as follows (see Theorem 5 in op. cit.):

(1) γ_{2k-1}^t on $\mathcal{U}^p$ when $p \le 2k-1$;
(2) $\gamma_{2k-1,q}^t$ on $\mathcal{U}^p$ when $p \le 2k-1+2q$; and
(3) Φ_{2k-1}^u on $\mathcal{U}^p$ for all p.

The forms γ_{2k-1}^t, $\gamma_{2k-1,q}^t$, and Φ_{2k-1}^1 are all cohomologous and represent the degree $2k-1$-component of the odd Chern character ch_{2k-1} of the universal K^{-1}-class, i.e. the class induced by the identity map $\mathrm{id}_{U(\infty)}$. We will call each of them a Quillen form.

In [4] we gave alternative construction to the pull-backs $\varphi^* \mathrm{ch}_{2k-1}$ when $\varphi : B \to \mathcal{U}^p$ when B is a compact oriented manifold and φ is smooth. In fact, the theory works for maps $\varphi : B \to \mathcal{U}^-$, where $\mathcal{U}^-$ is the open subset of unitary operators U such that $1 + U$ is Fredholm. This is another manifold classifying space for K^{-1} that contains $\mathcal{U}^p$ for every p, however, it does not come with any easy to describe smooth differential forms on it. Under a certain finite set of transversality conditions, the classes $\varphi^* \mathrm{ch}_{2k-1}$ were described (up to multiplication by a rational number) via preimages $\varphi^{-1} \overline{Z_{\{k\}}}$ where $\overline{Z_{\{k\}}}$ are stratified subspaces of codimension $2k-1$ in $\mathcal{U}^-$. In fact, the Schubert cell $Z_{\{k\}}$ is defined by the same incidence relations as the stable manifold $S(U_{\{k\}})$ we saw in last section.

We used local (sheaf) cohomology in order to associate to a finite codimensional, cooriented stratified space a cohomology class, which behaves well under *transverse* pull-back. Here, we take a different path and show that in fact under a different but still finite set of transversality condition one can define the current $\varphi^* Z_{\{k\}}$ and this is Poincaré dual to $\frac{(-1)^{k-1}}{(k-1)!} \varphi^* \Omega_k$ where Ω_k is a Quillen form. In fact something stronger is true.

We will assume that a complete flag

$$H \supset W_0 \supset \cdots \supset W_k \supset$$

has been fixed with $\mathrm{codim}\, W_k = k$. For every $I = \{i_1 < \cdots < i_k\}$ a k-tuple of positive integers, let

$$Z_I^p := \{U \in \mathcal{U}^p \mid \dim \mathrm{Ker}(1+U) = k, \dim \mathrm{Ker}(1+U) \cap W_m = k - p, \ \forall 0 \le p \le k,$$
$$\forall i_p \le m < i_{p+1}\},$$

where as usual $i_0 = 0$, $i_{k+1} = \infty$.

We notice that for every smooth map $\varphi : B \to \mathcal{U}^p$ and every p from a compact manifold B there exists a subspace W_N of the flag such that $\mathrm{Ker}(1 + \varphi(b)) \cap W_N = \{0\}$

for all $b \in B$. This is because if $U \in \mathcal{U}^p$ then $1 + U$ is Fredholm and this is an open condition. It follows that the collection of transversality condition $\varphi \pitchfork Z_I^p$ is trivially satisfied if there exists $a > N$ such that $a \in I$ since then $\varphi^{-1}(Z_I^p) = \varnothing$. Hence, $\varphi \pitchfork Z_I^p$ for every I is a generic condition.

Theorem 7.1   *Let $\varphi : B \to \mathcal{U}^p$ be a smooth map from a compact, oriented manifold B, possibly with corners such that $\varphi \pitchfork Z_I^p$ for every I. Let Ω_k be a Quillen form of degree $2k - 1$ that makes sense on $\mathcal{U}^p$. Then for every such Ω_k, there exists a flat current T_k such that:*

$$(7.1) \qquad \varphi^{-1} Z_{\{k\}} - (-1)^{k-1} (k-1)! \varphi^* \Omega_k = d T_k.$$

In particular, when B has no boundary, $\frac{(-1)^{k-1}}{(k-1)!} \varphi^{-1} Z_{\{k\}}^p$ represents the Poincaré dual of $\mathrm{ch}_{2k-1}([\varphi])$, where $[\varphi] \in K^{-1}(B)$ is the natural odd K theory class determined by φ.

Proof   We use symplectic reduction. For each linear subspace $W \subset H$ of finite codimension there exists a smooth (even real analytic) map $\mathscr{R}^W : \mathcal{U}_W^p \to U(W^\perp)$, where

$$\mathcal{U}_W^p := \{U \in \mathcal{U}^p \mid \mathrm{Ker}(1 + U) \cap W = \{0\}\}$$

is an open subset of $\mathcal{U}^p$. The expression of $\mathscr{R}^W$ relative to the decomposition $U = \begin{pmatrix} X & Y \\ Z & T \end{pmatrix}$ vis-a-vis $H = W \oplus W^\perp$ is:

$$\mathscr{R}^W(U) = T - Z(1 + X)^{-1} Y.$$

The map $\mathscr{R}^{W\,7}$ has some nice properties. For example it can be shown that together with the "0-section":

$$\iota : U(W^\perp) \hookrightarrow \mathcal{U}_W^p, \qquad U \to -\mathrm{id}_W \oplus U$$

is diffeomorphic to a vector bundle over $U(W^\perp)$ (see Corollary 4.1 in [4]). Hence, by choosing $W = W_N$ a subspace of the flag, we get that $\mathrm{Im}\,\varphi \subset \mathcal{U}_W^p$ and there exists a smooth homotopy $h : [0, 1] \times B \to \mathcal{U}^p$ between $\psi := \mathscr{R}^W \circ \varphi$ and φ. The Quillen forms Ω_k have finite dimensional counterparts $\Omega_k^{W^\perp}$ such that $\iota^* \Omega_k = \Omega_k^{W^\perp}$. Hence there exists a smooth form $\beta(\Omega_k)$ on B such that

$$\varphi^* \Omega_k - \psi^* \Omega_k = d(\beta(\Omega_k)).$$

Another important, straightforward property is that $\mathscr{R}^W(Z_I^p) = S(U_I)$ and in fact the stronger $(\mathscr{R}^W)^{-1}(S(U_I)) = Z_I^p$ holds. It follows that $\varphi^{-1}(Z_{\{k\}}^p) = \psi^{-1}(S(U_{\{k\}}))$. Therefore we can use Theorem to conclude that (7.1) holds for $\Omega_k = \gamma_{2k-1}^1$ which coincides with Tch_k. Since the Quillen forms of degree $2k - 1$ are all cohomologous in the finite dimensional case we get the result for such forms.   ∎

Remark 7.2   The coorientation (implicitly the orientation) of $\varphi^{-1} Z_{\{k\}}$ is discussed in detail in [4].

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[7] It is called symplectic reduction because under Arnold's theorem which identifies the unitary group $\mathcal{U}^p$ with the (Hermitian) Lagrangian Grassmannian of Schatten class p it corresponds to the homonymous process well-known in symplectic topology.

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