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Comparing the Medvedev and Turing degrees of Π^0_1 classes

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Every co-c.e. closed set (Π^0_1 class) in Cantor space is represented by a co-c.e. tree. Our aim is to clarify the interaction between the Medvedev and Muchnik degrees of co-c.e. closed subsets of Cantor space and the Turing degrees of their co-c.e. representations. Among other results, we present the following theorems: if **v** and **w** are different c.e. degrees, then the collection of the Medvedev (Muchnik) degrees of all Π^0_1 classes represented by **v** and the collection represented by **w** are also different; the ideals generated from such collections are also different; the collections of the Medvedev and Muchnik degrees of all Π^0_1 classes represented by incomplete co-c.e. sets are upward dense; the collection of all Π^0_1 classes represented by K-trivial sets is Medvedev-bounded by a single Π^0_1 class represented by an incomplete co-c.e. set; and the Π^0_1 classes have neither nontrivial infinite suprema nor infima in the Medvedev lattice.

1. Introduction

1.1. *Summary*

The complexity Π^0_1 is known to be the first level in the arithmetical hierarchy, that can define a nonempty class $S \subseteq 2^\omega$ without computable elements. Therefore, the study of computability-theoretic complexities of Π^0_1-definable classes (Π^0_1 classes) in 2^ω is expected to be as interesting as that of the Turing degrees of Σ^0_1-definable sets (computably enumerable sets) in ω. Actually, in recent years, the degree-theoretic complexity of the Π^0_1 classes has become an important topic in the computability theory. It is to be noted that there are numerous nontrivial interactions between the global and local structures of Π^0_1 classes, which are described by basis and nonbasis theorems. These results motivate us to study the relationship between the global and local information contents of Π^0_1 classes. Formally, we deal with the following degree-theoretic notions.

1.1.1. *Turing degrees.* A closed set F in Cantor space is Π^0_1 (or *co-c.e.*) if the tree $T_F = \{\sigma \in 2^{<\omega} : [\sigma] \cap F \neq \varnothing\}$ is Π^0_1 (i.e. co-c.e.), where $[\sigma]$ is the clopen set in 2^ω consisting of all infinite binary strings extending σ. The Turing degree of T_F measures the *global* information content of every co-c.e. closed set F. A closed set F is *computable* if T_F is computable, i.e. the Turing degree of T_F is **0**.

1.1.2. *Medvedev and Muchnik degrees.* The notions of the Medvedev/Muchnik degree (Medvedev 1955; Muchnik 1963) offer alternatives to the Turing degree for analysing computability-theoretic complexities of type-two objects. Let us think of $F \subseteq \omega^\omega$ as the solution set of some mathematical problem. The Medvedev/Muchnik degree of F measures the degree of difficulty in constructing a solution to the problem. Roughly speaking, these degrees estimate locally easiest Turing degrees contained in F. In this sense, we may think of the Medvedev/Muchnik degree of F as a *local* information content of it. A set $F \subseteq \omega^\omega$ contains a computable element if and only if the Medvedev/Muchnik degree of F is $\mathbf{0}$. The Medvedev and Muchnik degree structures of Π^0_1 classes have also been widely studied by many authors. For more information on the degree structures of Π^0_1 classes, see also (Simpson 2005, 2011).

With regard to the relationship between the global and local information contents of Π^0_1 classes in Cantor space, the following results are known:

- Every computable closed subset of 2^ω contains a computable element. In other words, if the Turing degree of T_F is $\mathbf{0}$, then the Medvedev/Muchnik degree of F is also $\mathbf{0}$.
- If all the elements of a co-c.e. closed set $F \subseteq 2^\omega$ have PA degrees (or Martin-Löf random degrees), then the Turing degree of T_F is $\mathbf{0}'$, which is the Turing degree of the halting problem.
- Simpson (2005) and (Binns 2007) clarified that the thin Π^0_1 classes (i.e. Π^0_1 classes defined as the consistent complete extensions of Martin/Pour-El theories) show interesting behaviours for the Medvedev and Muchnik degree structures. These Π^0_1 classes have been known to be represented by co-c.e. trees of array noncomputable degrees (see Cholak *et al.* (2001)).
- Barmpalias *et al.* (2009) studied the structure of the Medvedev degrees of co-c.e. closed sets with K-trivial representations and showed that every such set has a K-trivial solution.

Our aim is to investigate techniques for controlling the Medvedev and Muchnik degrees of co-c.e. closed sets F and the Turing degrees of their representations T_F simultaneously and to clarify the relation between them. Among other results, we present the following theorems.

Theorem 1. If $\mathbf{v}$ and $\mathbf{w}$ are different c.e. degrees, then the collection of the Medvedev (Muchnik) degrees of all Π^0_1 classes represented by $\mathbf{v}$ and the collection represented by $\mathbf{w}$ are also different. Indeed, the ideals generated from such collections are different.

Theorem 2. The collections of the Medvedev and Muchnik degrees of all Π^0_1 classes P with $T_P <_T \varnothing'$ are upward dense.

Theorem 3. There is a Π^0_1 class P with $T_P <_T \varnothing'$ such that $Q \leqslant_M P$ for all Π^0_1 classes Q with K-trivial representations.

Theorem 4. The Π^0_1 classes have neither nontrivial infinite suprema nor infima in the Medvedev lattice. In particular, for every computable sequence $\{\mathbf{a}_i\}_{i\in\omega}$ of Medvedev degrees of Π^0_1 classes, we have the following:

1. If $\{\mathbf{a}_i\}_{i\in\omega}$ is strictly ascending, then there is no Π^0_1 class whose Medvedev degree is a least upper bound of $\{\mathbf{a}_i\}_{i\in\omega}$.

2. If $\{\mathbf{a}_i\}_{i\in\omega}$ is strictly descending, then there is no Π^0_1 class whose Medvedev degree is a least lower bound of $\{\mathbf{a}_i\}_{i\in\omega}$.

1.2. *Preliminaries*

We refer the reader to Soare (1987) and Nies (2009) for the basic notions of computability theory and algorithmic randomness, respectively.

Throughout this paper, we identify a set $A \subseteq \omega$ with its characteristic function, i.e. an element of 2^ω. We now prepare some of the formal basic notations and definitions. Fix an effective enumeration $\{\Phi_e\}_{e\in\omega}$ of all partial computable functionals. For $\sigma, \tau \in 2^{<\omega} \cup 2^\omega$, $|\sigma|$ denotes *the length* (*the height*) *of* σ, and $\sigma^\frown\tau$ denotes *the concatenation of* σ *and* τ, defined by $(\sigma^\frown\tau)(n) = \sigma(n)$ for $n < |\sigma|$ and $(\sigma^\frown\tau)(|\sigma| + n) = \tau(n)$ for $n < |\tau|$. *The restriction* σ *to* n (denoted by $\sigma \upharpoonright n$) is the unique initial segment of σ of height n. A predicate $\sigma \subseteq \tau$ expresses that $|\sigma| \leqslant |\tau|$ and $\tau \upharpoonright |\sigma| = \sigma$. *The join of* σ *and* τ is defined by $(\sigma \oplus \tau)(2n) = \sigma(n)$ and $(\sigma \oplus \tau)(2n + 1) = \tau(n)$ for each n. A *tree* T is a collection of strings closed under taking initial segments. For each tree $T \subseteq 2^{<\omega}$ let $[T]$ denote the set of all infinite paths through T.

A topology on the set 2^ω of all infinite binary sequences is induced by basic open sets $[\sigma] = \{f \supseteq \sigma\}$ for each string $\sigma \in 2^{<\omega}$. This topological space is called *Cantor space*. We let T_P denote *a tree representation* of a closed set P of Cantor space 2^ω, that is, $T_P = \{\sigma \in 2^{<\omega} : [\sigma] \cap P \neq \emptyset\}$. For a closed set $P \subseteq 2^\omega$, if T_P is co-c.e. (i.e. Π^0_1 definable) then we say that P is *a* Π^0_1 *class* or *a co-c.e. closed set*. (Whenever we say that P is a Π^0_1 class in this paper, it is assumed that P is a subset of Cantor space.) As T_P is co-c.e., we have a computable approximation $\{T^*_{P,s}\}_{s\in\omega}$ of T_P, i.e. $T^*_{P,0} = 2^{<\omega}$ and $T_P = \bigcap_s T^*_{P,s}$. In place of $\{T^*_{P,s}\}_{s\in\omega}$, we use $T_{P,s} = \{\tau \in 2^{<\omega} : (\forall \sigma \subseteq \tau)\ \sigma \notin T^*_{P,s}\}$. We note that computable approximations (of tree representations) enable us to perform priority constructions of Π^0_1 classes, in the same way as c.e. sets in ω do.

By $\leqslant_T$ and $\equiv_T$, we denote *Turing reducibility* and *Turing equivalence*, respectively. Additionally, $\deg_T(A)$ denotes *the Turing degree of* $A \subseteq \omega$. For each closed set $F \subseteq \omega^\omega$, $\deg_T(F)$ means $\deg_T(T_F)$. For sets $A, B \subseteq \omega^\omega$, *$A$ is Medvedev reducible to B* (written as $A \leqslant_M B$) if there exist a computable functional from B to A. Then *A is Muchnik reducible to B* (denoted by $A \leqslant_w B$) if every element of B computes an element of A. The *Medvedev* (resp. *Muchnik*) *degree of* A (denoted by $\deg_M(A)$ and $\deg_w(A)$) is the equivalent class of A by the *Medvedev* (resp. Muchnik) equivalence. The set $\mathcal{P}_M$ (resp. $\mathcal{P}_w$) of Medvedev (resp. Muchnik) degrees of all nonempty Π^0_1 classes forms a lattice with a supremum operator induced by $A \otimes B = \{f \oplus g : f \in A\ \&\ g \in B\}$ and an infimum operator induced by $A \oplus B = \{0^\frown f : f \in A\} \cup \{1^\frown g : g \in B\}$. It is known that there exists a Π^0_1 class of the greatest Medvedev (resp. Muchnik) degree among all nonempty Π^0_1 classes, and we call such a Π^0_1 class *Medvedev (resp. Muchnik) complete*. The set CPA of all consistent complete extensions of Peano arithmetic, and the set DNC_2 of all $\{0,1\}$-valued diagonally noncomputable functions, are major examples of Medvedev (also Muchnik) complete Π^0_1 classes. Here, a function f is *diagonally noncomputable* if $f(e) \neq \Phi_e(e)$ for every e. By $\mathbf{1}$ (resp. $\mathbf{0}$), we mean the top (resp. bottom) element of $\mathcal{P}_M$ and $\mathcal{P}_w$. For two given disjoint

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c.e. sets A and B, the set $S(A,B) = \{X \subseteq \omega : A \subseteq X \subseteq \omega \setminus B\}$ forms a Π^0_1 class in Cantor space. Such a Π^0_1 class is called *a separating class*.

2. Results

2.1. *Basic properties*

For any Medvedev (resp. Muchnik) degree $\mathbf{p}$ and c.e. degree $\mathbf{w}$, we say that $\mathbf{p}$ *is planted in* $\mathbf{w}$ if there is a nonempty Π^0_1 class P of Medvedev (resp. Muchnik) degree $\mathbf{p}$ such that the corresponding well-pruned co-c.e. tree $T_P = \{\sigma \in 2^{<\omega} : (\exists f \in P)\ \sigma \subseteq f\}$ is of Turing degree $\mathbf{w}$.

Proposition 5.

1. No nonzero Medvedev degree $\mathbf{p} \in \mathcal{P}_M$ is planted in the bottom c.e. degree $\mathbf{0}$.
2. For every Medvedev degree $\mathbf{p} \in \mathcal{P}_M$, if $\mathbf{p}$ is planted in a c.e. degree $\mathbf{v}$, then $\mathbf{p}$ is planted in any c.e. degree $\mathbf{w} \geqslant \mathbf{v}$. In particular, the bottom Medvedev degree $\mathbf{0}$ is planted in any c.e. degree.
3. There exists a Muchnik degree $\mathbf{d} \in \mathcal{P}_w$ such that $\mathbf{d} < \mathbf{1}$ and every $\mathbf{p} \geqslant \mathbf{d}$ cannot be planted in any c.e. degree $\mathbf{w} < \mathbf{0}'$. In particular, the top Muchnik (hence, Medvedev) degree $\mathbf{1}$ cannot be planted in any c.e. degree $\mathbf{w} < \mathbf{0}'$.

Proof. (2) For each c.e. set $W \subseteq \omega$, put $P_W = \{0^n 1^\omega : n \notin W\} \cup \{0^\omega\}$. Then P_W is clearly Π^0_1 class, $P_W \equiv_T W$, and every element of P_W is computable. Thus, for any Π^0_1 class P, we have $\deg_M(P \otimes P_W) = \deg_M(P)$ and $\deg_T(P \otimes P_W) = \deg_T(P) \vee \deg_T(P_W)$.

(3) By the Arslanov completeness criterion (see (Nies 2009, Theorem 4.1.11.)) if a c.e. set computes a diagonally noncomputable function then it is just c.e. complete. Simpson (2005) showed that there exists a Π^0_1 class $D \subseteq 2^\omega$ such that D is Muchnik equivalent to DNC, which is Muchnik incomplete. Here, DNC is the set of all diagonally noncomputable functions. Moreover, the leftmost path of any Π^0_1 class has a c.e. degree and is Turing reducible to its corresponding tree. Thus, any nonempty Π^0_1 class P with $T_P <_T \varnothing'$ has a c.e. path $f_P \leqslant_T T_P <_T \varnothing'$. Thus, we get $D \not\leqslant_w P$. □

For any c.e. degrees $\mathbf{v}$, by $\mathcal{P}_M(\mathbf{v})$ (resp. $\mathcal{P}_w(\mathbf{v})$), we denote the set of all Medvedev (resp. Muchnik) degrees of nonempty Π^0_1 classes planted in the c.e. degree $\mathbf{v}$. For any c.e. degrees $\mathbf{v}$, set $\mathcal{P}_r(<\mathbf{v}) = \bigcup_{\mathbf{w}<\mathbf{v}} \mathcal{P}_r(\mathbf{w})$ for each $r \in \{M, w\}$. For any collection of c.e. degrees, $\mathcal{C} \subseteq \mathcal{R}_T$, we also use the notation $\mathcal{P}_r(\mathcal{C}) = \bigcup_{\mathbf{w}\in\mathcal{C}} \mathcal{P}_r(\mathbf{w})$ for each $r \in \{M, w\}$. Then, the previous proposition states that (1) $\mathcal{P}_M(\mathbf{0}) = \{\mathbf{0}\}$; (2) If $\mathbf{v} \leqslant \mathbf{w}$ then $\mathcal{P}_M(\mathbf{v}) \subseteq \mathcal{P}_M(\mathbf{w})$; (3) $\mathcal{P}_w(<\mathbf{0}') \cap [\mathbf{d}, \mathbf{1}]_w = \varnothing$ for some Muchnik degree $\mathbf{d} < \mathbf{1}$. Here $[\mathbf{d}, \mathbf{1}]_w$ denotes the Muchnik interval $\{\mathbf{p} \in \mathcal{P}_w : \mathbf{d} \leqslant \mathbf{p}\}$.

Note that, if $\mathcal{I} \subseteq \mathcal{R}_T$ is an ideal of $\mathcal{R}_T$, then $\mathcal{P}_M(\mathcal{I})$ (resp. $\mathcal{P}_w(\mathcal{I})$) forms a sublattice of $\mathcal{P}_M$ (resp. $\mathcal{P}_w$). To see this, let $\mathcal{I} \subseteq \mathcal{R}_T$ be an ideal of $\mathcal{R}_T$. For any $\mathbf{p}, \mathbf{q} \in \mathcal{P}_M(\mathcal{I})$, there are Π^0_1 classes $P \in \mathbf{p}$ and $Q \in \mathbf{q}$ such that $\deg_T(P) = \mathbf{v}$ and $\deg_T(Q) = \mathbf{w}$ for some $\mathbf{v}, \mathbf{w} \in \mathcal{I}$. Then clearly $\deg_T(P \oplus Q) = \deg_T(P \otimes Q) = \mathbf{v} \vee \mathbf{w} \in \mathcal{I}$.

As a consequence, for every c.e. degree $\mathbf{v}$, we can see that $\mathcal{P}_r(\mathbf{v})$ forms a sublattice of $\mathcal{P}_r$ for each $r \in \{M, w\}$. This is because, for any c.e. degree $\mathbf{v}$, the interval $[\mathbf{0}, \mathbf{v}] = \{\mathbf{u} \in \mathcal{R}_T : \mathbf{u} \leqslant \mathbf{v}\}$

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forms an ideal of $\mathcal{R}_T$, and we have $\mathcal{P}_r(\mathbf{v}) = \mathcal{P}_r([\mathbf{0}, \mathbf{v}])$ for each $r \in \{M, w\}$ by Proposition 5 (2).

By using a technical tool from Section 3, we can show Theorem 1. In fact, we can see the following.

Theorem 6. For any c.e. degrees $\mathbf{v}$ and $\mathbf{w}$, the condition $\mathbf{v} \leqslant \mathbf{w}$ holds if and only if $\mathcal{P}_M(\mathbf{v}) \subseteq \mathcal{P}_M(\mathbf{w})$ if and only if $\mathcal{P}_w(\mathbf{v}) \subseteq \mathcal{P}_w(\mathbf{w})$.

Proof of Theorem 6 from Lemma 16 (see Section 3). We only show that $\mathcal{P}_w(\mathbf{v}) \subseteq \mathcal{P}_w(\mathbf{w})$ implies $\mathbf{v} \leqslant \mathbf{w}$. Assume that $\mathbf{v} \not\leqslant \mathbf{w}$. Then, by Lemma 16 (see Section 3), there is a nonempty Π^0_1 class $P \subseteq 2^\omega$ such that $\deg_w(P) \in \mathcal{P}_w(\mathbf{v})$, and P has no $\mathbf{w}$-computable path. Note that, for every $\mathbf{q} \in \mathcal{P}_w(\mathbf{w})$ and $Q \in \mathbf{q}$, $Q \leqslant_w R$ for some Π^0_1 class $R \subseteq 2^\omega$ whose corresponding tree T_R is $\mathbf{w}$-computable. Especially, the leftmost path f_R of the tree T_R must be $\mathbf{w}$-computable. Then Q contains an f_R-computable path since we have $Q \leqslant_w R$ and $f_R \in R$. Therefore, Q must have a $\mathbf{w}$-computable path. Hence, $\deg_w(P) \in \mathcal{P}_w(\mathbf{v}) \setminus \mathcal{P}_w(\mathbf{w})$. □

Remark 1. For a collection of Muchnik degrees, $\mathcal{Q} \subseteq \mathcal{P}_w$, let $\downarrow \mathcal{Q}$ denote the downward closure of $\mathcal{Q}$. Then, our proof of the previous theorem actually implies that, $\mathbf{v} \leqslant \mathbf{w}$ if and only if $\mathcal{P}_w(\mathbf{v}) \subseteq \downarrow \mathcal{P}_w(\mathbf{w})$, for all c.e. degrees $\mathbf{v}$ and $\mathbf{w}$. Moreover, by Remark 2 (Section 3), for each $r \in \{M, w\}$, if $\mathbf{v} \not\leqslant \mathbf{w}$, then there are $\mathbf{r}, \mathbf{s} \in \mathcal{P}_r(\mathbf{v}) \setminus \downarrow \mathcal{P}_r(\mathbf{w})$ such that $\mathbf{r}$ contains a positive measure Π^0_1 class and $\mathbf{s}$ contains a separating Π^0_1 class. Note that the order-preserving embedding from $(\mathcal{R}_T, \leqslant)$ into $(\mathfrak{P}(\mathcal{P}_\mathfrak{w}), \subseteq)$ given by $\mathbf{v} \mapsto \mathcal{P}_w(\mathbf{v})$ does not preserve the supremum, where $\mathfrak{P}(\mathcal{P}_\mathfrak{w})$ denotes the power set of $\mathcal{P}_w$. Indeed, for any c.e. degrees $\mathbf{v}, \mathbf{w} < \mathbf{0}'$, $\mathcal{P}_w(\mathbf{v}) \cup \mathcal{P}_w(\mathbf{w})$ does not contain the Muchnik interval $[\mathbf{d}, \mathbf{1}]_w$. However, if $\mathbf{v} \vee \mathbf{w} = \mathbf{0}'$ then $\mathcal{P}_w(\mathbf{v} \oplus \mathbf{w}) = \mathcal{P}_w \supsetneq \mathcal{P}_w(\mathbf{v}) \cup \mathcal{P}_w(\mathbf{w})$.

Question 7. Is $\mathbf{v} \mapsto \mathcal{P}_w(\mathbf{v})$ an embedding from $(\mathcal{R}_T, \leqslant)$ into $(\mathfrak{L}(\mathcal{P}_\mathfrak{w}), \subseteq)$ preserving the supremum? Here, $\mathfrak{L}(\mathcal{P}_\mathfrak{w})$ denotes the set of all sublattices of $\mathcal{P}_w$.

2.2. *Separating classes*

A set $A \subseteq \omega$ is *K-trivial* if there is a constant c such that $K(A \upharpoonright n) \leqslant K(0^n) + c$ for any n, and a closed set $P \subseteq 2^\omega$ is *K-trivial* if its tree representation T_P is K-trivial. Here, the symbol K refers to the prefix-free Kolmogorov complexity. For more information on K-trivial closed sets, see also Barmpalias *et al.* (2009) and Melnikov and Nies (2013). Note that, in Barmpalias *et al.* (2009), a closed set $P \subseteq 2^\omega$ is said to be K-trivial if its code x_P is K-trivial. However, x_P is Turing equivalent to T_P, and K-triviality is invariant under degree-preserving maps (see Nies (2009)). So our definition of K-triviality is equivalent to that of Barmpalias *et al.* (2009).

Let $\mathcal{K} \subseteq \mathcal{R}_T$ denote the set of all Turing degrees of K-trivial c.e. sets. In order to establish the dense splitting property in $\mathcal{P}_M(\mathcal{K})$, Barmpalias *et al.* (2009) asked whether every K-trivial Π^0_1 class should be bounded by some K-trivial Π^0_1 separating class, in the sense of Medvedev degrees. First, we need to show the following.

Proposition 8. For every Π^0_1 class Q, there exists a Π^0_1 separating class S such that $S \geqslant_M Q$ and $T_S \equiv_{tt} T_Q$ hold.

Proof. We may assume that Q is nonempty. For the eth binary string $\sigma \in 2^{<\omega}$ in the lexicographic order, we define $\mathtt{num}(\sigma) = e$. Clearly, num is computable. Put $S_Q^i = \{\mathtt{num}(\sigma) : (\exists s)\ \sigma \in T_{Q,s}\ \&\ \sigma^\frown i \notin T_{Q,s}\}$ for each $i < 2$. By our definition of the approximation $\{T_{Q,s}\}_{s\in\omega}$ of the tree T_Q, the sets S_Q^0 and S_Q^1 are disjoint. So we let S be the separating class generated by S_Q^0 and S_Q^1. We now show that $T_S \equiv_{tt} S_Q^0 \oplus S_Q^1 \equiv_{tt} T_Q$. First, in order to show $S_Q^0 \oplus S_Q^1 \leqslant_{tt} T_Q$, define a partial computable function m^{T_Q} which, for given input σ, computes the minimal stage s such that $\sigma \notin T_{Q,s}$. Then, it is easy to observe that $\mathtt{num}(\sigma) \notin S_Q^i$ if and only if either the condition $\sigma^\frown i \in T_Q$ holds or the conditions $\sigma \notin T_Q$, $\sigma^\frown i \notin T_Q$ and $m^{T_Q}(\sigma) \leqslant m^{T_Q}(\sigma^\frown i)$ hold. Next, to establish $T_Q \leqslant_{tt} S_Q^0 \oplus S_Q^1$, we show that $\sigma \in T_Q$ holds if and only if $\mathtt{num}(\sigma \restriction i) \notin S_Q^{\sigma(i)}$ for each $i < |\sigma|$. The 'only if' part follows easily by induction. The 'if' part also holds since, for minimal $i < |\sigma|$ such that $\sigma \restriction i+1 \notin T_Q$, we have $\sigma \restriction i \in T_{Q,s}$ for every stage s and $\sigma \restriction i+1 \notin T_{Q,t}$ for almost all stage t. Thus, the conditions $\sigma \restriction i \in T_{Q,s}$ and $\sigma \restriction i+1 \notin T_{Q,s}$ hold for some stage s simultaneously, and so $\mathtt{num}(\sigma \restriction i)$ is enumerated into $S_Q^{\sigma(i)}$.

Now, we construct a total computable procedure Γ such that, if X separates S_Q^0 and S_Q^1 (i.e. $S_Q^0 \subseteq X \subseteq (S_Q^1)^\complement$) then $\Gamma(X)$ must belong to Q. For $X \in 2^\omega$, put $\Gamma(X)(0) = X(\mathtt{num}(\emptyset))$, and $\Gamma(X)(n) = X(\mathtt{num}(\Gamma(X) \restriction n))$ for every n. For any $X \in S$, we inductively show that $\Gamma(X) \restriction n \in T_Q$ for all n. For $n = 0$, $\mathtt{num}(\emptyset) \in S_Q^i$ if and only if $\langle i\rangle \notin T_Q$, since $\emptyset \in T_Q$. Since $n \in S_Q^0$ implies $X(n) = 1$, and since $n \in S_Q^1$ implies $X(n) = 0$, if $\Gamma(X)(0) = X(\mathtt{num}(\emptyset)) = i$ then $\langle i\rangle \in T_Q$, that is, $\Gamma(X) \restriction 1 \in T_Q$. We assume $\Gamma(X) \restriction n \in T_Q$. We will show $\Gamma(X) \restriction n+1 \in T_Q$. By our assumption, we again observe that $\mathtt{num}(\Gamma(X) \restriction n) \in S_Q^i$ if and only if $(\Gamma(X) \restriction n)^\frown i \notin T_Q$. Hence, we also have that if $\Gamma(X)(n) = X(\mathtt{num}(\Gamma(X) \restriction n)) = i$ then $(\Gamma(X) \restriction n)^\frown i \in T_Q$, that is, $\Gamma(X) \restriction n+1 = (\Gamma(X) \restriction n)^\frown X(\mathtt{num}(\Gamma(X) \restriction n)) \in T_Q$. □

Recall that K-triviality is invariant under tt-preserving maps. By using an argument from Barmpalias *et al.* (2009), we now establish a dense splitting result for K-trivial Π_1^0 classes.

Corollary 9. For every K-trivial Π_1^0 class Q, there exists a K-trivial Π_1^0 separating class $S \geqslant_M Q$. Moreover, for any K-trivial Π_1^0 classes $Q <_M P$, there exist K-trivial Π_1^0 classes P^0 and P^1 such that $Q <_M P^0, P^1 <_M P$ and $P^0 \otimes P^1 \equiv_M P$.

Proof. By Proposition 8, we can straightforwardly show the first half of the statement. For the last half, let $S = S(A, B) \geqslant_M P$ be a K-trivial Π_1^0 separating class. By Binns' splitting theorem (Binns 2003), we get a partition $A^0 \sqcup A^1 = A$ such that $P \not\leqslant_M (\{A^i\} \otimes Q)$ for $i < 2$. Put $P^i = P \oplus (S(A^i, B) \otimes Q)$ for each $i < 2$. Then P^0 and P^1 are the desired Π_1^0 classes, by an observation of Barmpalias *et al.* (2009). □

Corollary 10. For every nonzero $\mathbf{a} \in \mathcal{R}_T$, the set $\mathcal{P}_M(\mathbf{a})$ contains $\mathbf{s}$ which is incomparable with $\mathbf{d}$ and $\mathbf{r}$, where $\mathbf{d}$ and $\mathbf{r}$ are the Muchnik degrees of the diagonally noncomputable functions and Martin-Löf random reals, respectively.

Proof. By Proposition 5 (3), if $\mathbf{a} < \mathbf{0}'$ then every $\mathbf{p} \in \mathcal{P}_w(\mathbf{a})$ satisfies $\mathbf{d} \not\leqslant \mathbf{p}$, and hence $\mathbf{r} \not\leqslant \mathbf{p}$ (see Simpson (2005)). Moreover, if $\mathbf{a}$ is nonzero, then, for any $\mathbf{p} \in \mathcal{P}_w(\mathbf{a})$, the degree $\mathbf{s}$ of the separating class S in Proposition 8 with $Q \in \mathbf{p}$ satisfies $\mathbf{p} \leqslant \mathbf{s} \in \mathcal{P}_w(\mathbf{a}) \setminus \{\mathbf{0}\}$. By Jockusch and Soare (1972), $\mathcal{U}(S)$, the upward closure in the Turing degrees of S, is of

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measure 0. Let MLR denote the set of all Martin-Löf random reals. Since $\mu(\text{MLR}) = 1$, almost all Martin-Löf random reals $\alpha \in \text{MLR}$ belong to the complement of $\mathcal{U}(S)$. This means $\mathbf{s} \not\leqslant \mathbf{r}$. In particular, $\mathbf{s} \not\leqslant \mathbf{d}$. □

Let $(\mathcal{L}, \leqslant, \cup, \cap, 0, 1)$ be a lattice. An element $a \in \mathcal{L}$ is *branching* if $a = b \cap c$ holds for some $b, c \in \mathcal{L}$. Alfeld (2007) showed that nonbranching degrees are downward dense in $\mathcal{P}_M \setminus \{\mathbf{0}\}$.

Corollary 11.

1. Nonbranching degrees are upward dense in $\mathcal{P}_M(\mathcal{K})$.
2. Nonbranching degrees are upward dense in $\mathcal{P}_M(< \mathbf{0}')$.

Proof. We will show that neither $\mathcal{P}_M(\mathcal{K})$ nor $\mathcal{P}_M(< \mathbf{0}')$ has a maximal element. Hence, for each $\mathbf{a} \in \mathcal{P}_M(\mathcal{K})$, by Proposition 8 we have the degree $\mathbf{s} > \mathbf{a}$ of a separating class, where $\mathbf{s}$ is contained in $\mathcal{P}_M(\mathcal{K})$. By an observation of Alfeld (2007), the Medvedev degree of a separating Π^0_1 class is nonbranching. □

2.3. *Upper bounds*

An ideal of $\mathcal{R}_T$ is *uniformly generated* if it is generated by a uniformly c.e. sequence of c.e. sets. Yates showed that each Σ^0_3 ideal of $\mathcal{R}_T$ is uniformly generated. In Barmpalias and Nies (2011), it is shown that $\mathcal{K}$ forms a Σ^0_3 ideal of $\mathcal{R}_T$, and $\mathcal{K}$ has a low$_2$ c.e. upper bound. A collection $\mathcal{A}$ of Π^0_1 classes is *T-incomplete* if there is a computable enumeration $\{Q_i\}_{i\in\omega}$ of $\mathcal{A}$, and the ideal generated by $\{\deg_T(Q_i)\}_{i\in\omega}$ is proper. Then, the set of all K-trivial Π^0_1 classes is T-incomplete. Moreover, there is a low$_2$ c.e. degree $\mathbf{v}$ such that $\mathcal{P}_M(\mathcal{K}) \subseteq \mathcal{P}_M(\mathbf{v})$. In particular, $\mathcal{P}_M(\mathcal{K}) \subseteq \mathcal{P}_M(< \mathbf{0}')$. However, $\mathcal{P}_M(\mathbf{v})$ may not be a principal ideal. Then it is natural to ask whether the Medvedev degrees of all nonempty K-trivial Π^0_1 classes, $\mathcal{P}_M(\mathcal{K})$, is bounded by a single Medvedev degree $\mathbf{p} \in \mathcal{P}_M(< \mathbf{0}')$. Theorem 3 provides an answer to this question. Indeed, by using results from Section 5, we can show the following theorem.

Theorem 12. Let $\mathcal{A}$ be a T-incomplete family of Π^0_1 classes. There exists a Π^0_1 class P with $\deg_T(P) <_T \varnothing'$ such that $Q \leqslant_M P$ for any $Q \in \mathcal{A}$.

Proof. By Lemma 23 (see Section 5). □

One may ask whether every T-incomplete family has the least upper bound which is planted in an incomplete c.e. degree. However, the answer is ‘no’. Our priority argument to show Theorem 12 also works for showing Theorem 4 which states that any nontrivial computable set $\{\mathbf{a}_i\}_{i\in\omega}$ of Medvedev degrees has no least upper bound. Here, a set of Medvedev degrees, A, is *nontrivial* if there is no finite subset $B \subset A$ such that every element of A is bounded by $\bigvee B$. For example, $\mathcal{P}_M(\mathbf{0}) = \{\mathbf{0}\}$ and $\mathcal{P}_M(\mathbf{0}') = \mathcal{P}_M$ is trivial, but $\mathcal{P}_M(\mathcal{K})$ is nontrivial by the result from Barmpalias *et al.* (2009). Then, the best possible answer to the previous question is the following:

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Theorem 13. Let $\{\mathbf{a}_i\}_{i\in\omega}$ be a nontrivial set of Medvedev degrees of a T-incomplete family of Π^0_1 classes. If $\mathbf{b} \in \mathcal{P}_M$ is an upper bound of $\{\mathbf{a}_i\}_{i\in\omega}$, then there is $\mathbf{a} \in \mathcal{P}_M(< \mathbf{0}')$ which bounds $\{\mathbf{a}_i\}_{i\in\omega}$ such that $\mathbf{b} \not\leqslant \mathbf{a}$.

Proof. See Section 5.2. □

Note that $\mathcal{P}_M(< \mathbf{0}')$ is not T-incomplete. It is natural to ask whether $\mathcal{P}_M(< \mathbf{0}')$ has a Medvedev incomplete upper bound. The first half of Theorem 2 provides an answer to this question. Indeed, we will show the following theorem in Section 4.

Theorem 14. For every Medvedev incomplete Π^0_1 class Q, there exists a Π^0_1 class P such that $\deg_T(P) < \mathbf{0}'$ and $P \not\leqslant_M Q$.

Proof. By Lemma 17 (see Section 4). □

As a corollary, we see that no upper bound of $\mathcal{P}_M(< \mathbf{0}')$ exists, except for $\max \mathcal{P}_M$. Cenzer and Hinman (2003) showed that $\mathcal{P}_M$ is (upward) dense. However, the similar problem for $\mathcal{P}_w$ is still open (Simpson 2005, Remark 5.6). In response, Barmpalias *et al.* (2009) showed that $\mathcal{P}_w(\mathcal{K})$ is upward dense by constructing a K-trivial Π^0_1 class without path computable in a fixed low c.e. degree. The remaining part of Theorem 2 provides the upward density of a large sublattice of $\mathcal{P}_w$.

Theorem 15. $\mathcal{P}_w(< \mathbf{0}')$ is upward dense.

Proof of Theorem 15 from Lemma 16 (Section 3). Let Q be a given Π^0_1 class with $Q <_T \varnothing'$. By the Sacks density theorem (see Soare (1987)), there exists a c.e. set V such that $Q <_T V <_T \varnothing'$. To prove our theorem, it is only necessary to construct a Π^0_1 class $P \leqslant_T V$ with no T_Q-computable paths. This is because the leftmost path f_Q of Q is computable in T_Q, so $P \not\leqslant_w Q$ is witnessed by f_Q, and then $Q <_w P \otimes Q \leqslant_T V \oplus T_Q \equiv_T V <_T \varnothing'$. Then, by setting W to be the complement of T_Q, Lemma 16 (Section 3) implies the existence of such P. □

As corollaries of Theorem 14 (resp. Theorem 15), we can see that there is no T-incomplete family of Π^0_1 classes whose Medvedev (resp. Muchnik) degrees includes all of $\mathcal{P}_M(< \mathbf{0}')$ (resp. $\mathcal{P}_w(< \mathbf{0}')$). To see this, by Theorem 12, a Π^0_1 class P such that $P <_T \varnothing'$ and $Q \leqslant_M P$ (hence $Q \leqslant_w P$) holds for every $Q \in \mathcal{A}$. In the case of the Muchnik degrees, by using Theorem 14, we get a Π^0_1 class $R <_T \varnothing'$ with $R \not\leqslant_M P$. If $S \equiv_M R$ then we also have $S \not\leqslant_M P$ and so S cannot belong to the family $\mathcal{A}$. In the case of the Muchnik degrees, by using Theorem 15, we get a Π^0_1 class $R <_T \varnothing'$ with $R >_w P$.

3. Main lemma for Theorem 1 and the last half of Theorem 2

This section deals mainly with Lemma 16, which is used to show Theorem 6 (hence, Theorem 1) and Theorem 15 (hence, the last half of Theorem 2).

3.1. *Permitting method*

Lemma 16. For c.e. sets $V, W \subseteq \omega$, if $V \not\leqslant_T W$, then there is a nonempty Π^0_1 class $P \subseteq 2^\omega$ such that $P \leqslant_T V$ and P has no W-computable path.

Proof. To prove the theorem, it suffices to meet the following requirements. □

Requirements. We need to construct a Π^0_1 class $P \subseteq 2^\omega$ such that, for each $e \in \omega$, the following conditions are met:

$$\mathcal{G} : (\exists \Gamma)\, \Gamma(V) = T_P;$$
$$\mathcal{P}_e : \Phi_e(W) \in P \;\rightarrow\; (\exists \Delta_e)\, \Delta_e(W) = V.$$

Here, Γ and Δ_e range over all partial computable functionals.

Construction. Fix $e \in \omega$. The *nth location* of the $\mathcal{P}_e$-strategy is defined as $\langle e, n\rangle$. The *length function* $l(e, s)$ is defined as follows:

$$l(e, s) = \max\{n \in \omega : \Phi_{e,s}(W_s) \upharpoonright n + 1 \in T_{P,s}\}.$$

For $n \in \omega$, If $n \in V_{s+1} \setminus V_s$, $\langle e, n\rangle \leqslant l(e, s)$, and V does not permit n at stage $t \leqslant s$, then *V permits n at stage $s + 1$*. At each stage $s + 1$, if V permits some n, then let $n(e, s + 1)$ denote the least such n. If V permits no n at stage $s+1$, then $n(e, s+1)$ is undefined. Then the tree-approximation of P at stage s is defined as follows:

$$T_{P,s+1} = 2^{<\omega} \setminus \{\Phi_{e,t}(W_t) \upharpoonright \langle e, n(e, t + 1)\rangle + 1 : e < t \leqslant s\}.$$

Finally, we set $P = [\bigcap_s T_{P,s+1}]$.

Claim. P is nonempty.

Proof. For each $e, n \in \omega$, at most one string of length $\langle e, n\rangle + 1$ is removed from T_P. □

For each σ of length $\langle e, n\rangle \leqslant s$, if $\Gamma(V ; \sigma)$ is undefined at the beginning of stage s, then set $\Gamma_s(V ; \sigma) = T_{P,s}(\sigma)$ with use n.

Claim. $\Gamma(V) = T_P$.

Proof. Fix $\sigma \in 2^{<\omega}$. Note that $\sigma \in T_{P,s} \setminus T_{P,s+1}$ holds only when there is a string $\tau \subseteq \sigma$ of length $\langle e, n\rangle + 1$ for some e, n such that $n \in V[s+1] \setminus V[s]$. Therefore, $T_{P,s+1}(\sigma) \neq T_{P,s}(\sigma)$, then $\Gamma(V ; \sigma)$ is undefined at the beginning of stage $s + 1$. Then $\Gamma_{s+1}(V ; \sigma)$ is defined to be $T_{P,s+1}(\sigma)$. □

For each stage s and n, if $\langle e, n\rangle \leqslant l(e, s)$ and $\Delta_e(W ; n)$ is undefined at the beginning of stage $s + 1$, then set $\Delta_{e,s+1}(W ; n) = V_{s+1}(n)$ with W-use $\varphi_e(\langle e, n\rangle)$, where $\varphi_e(\langle e, n\rangle)$ is the W-use of the computation of $\Phi_e(W) \upharpoonright \langle e, n\rangle + 1$.

Claim. For each e, n, s, if $\langle e, n\rangle \leqslant l(e, s)$, then $\Delta_{e,s+1}(W ; n) = V_{s+1}(n)$.

Proof. Assume that $\Delta_{e,s+1}(W ; n)$ is defined to be $V_{t+1}(n)$ because of $\langle e, n\rangle \leqslant l(e, t)$ for some $t < s - 1$. We also assume that, for some $u > t$, $\langle e, n\rangle \leqslant l(e, u)$ and $V_{u+1}(n) \neq V_u(n) = V_{t+1}(n)$ occur by enumerating n into V_{u+1}. Fix such u. Then $\Phi_e(W) \upharpoonright \langle e, n\rangle + 1$ is removed

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from T_P. If $\Delta_e(W;n)$ is not redefined between stages $u+1$ and $s+1$, i.e. $\Delta_e(W;n)$ is defined at the beginning of stage $v+1$ for $u \leqslant v \leqslant s$, then $W \upharpoonright \varphi_e(\langle e, \langle e, n\rangle\rangle) + 1$ does not change between stages $u+1$ and $s+1$. In other words, $\Phi_{e,s}(W_s) \upharpoonright \langle e, n\rangle + 1 = \Phi_{e,u}(W_u) \upharpoonright \langle e, n\rangle + 1$, and it is removed from $T_{P,u+1}$ since V permits n at stage $u+1$. Therefore, $l(e,v) < \langle e, n\rangle$ for $u < v \leqslant s$. Thus, if $l(e, s-1) < \langle e, n\rangle \leqslant l(e,s)$, then $\Delta_e(W;n)$ is undefined at the beginning of stage $s+1$, and we must redefine $\Delta_{e,s+1}(W;n)$ to be $V_{s+1}(n)$.

On the other hand, if, for some $u > t$, $\langle e, n\rangle > l(e,u)$ and $V_u(n) = V_{t+1}(n)$ (hence $l(e,u) < l(e,t)$), then there are two cases. The first case is that there is $m < n$ such that V permits m at some stage $v+1$ with $t < v \leqslant u$. In this case, for $s > u$, if $\langle e, n\rangle \leqslant l(e,s)$, then $\Delta_e(W;n)$ is undefined at the beginning of stage $s+1$ by a similar argument. The second case is that there is no $m < n$ such that V permits m at some stage $v+1$ with $t < v \leqslant u$. In this case, $l(e,t) > l(e,u)$ holds only when $\Phi_{e,u}(W_u;k) \neq \Phi_{e,t}(W_t;k)$ for some $k \leqslant l(e,u) < \langle e, n\rangle$. Thus, $p \in W_u \setminus W_t$ for some $p \leqslant \varphi_e(\langle e, n\rangle)$. Therefore, $\Delta_e(W;n)$ is undefined at the beginning of stage $u+1$. □

Claim. $\liminf_s l(e,s) < \infty$ for any e.

Proof. For each n, let t_n denote the least t such that $\langle e, n\rangle \leqslant l(e,u)$ for all stage $u \geqslant t$. If $\liminf_s l(e,s) = \infty$, then such t_n exists for each n. Therefore, by the previous lemma, $\Delta_e(W;n)$ is defined to be $V(n)$ for every n. This contradicts our assumption that $V \not\leqslant_T W$.

By these lemmata, eventually we construct a Π^0_1 class P which satisfies the $\mathcal{G}$ and $\mathcal{P}_e$-requirements. Then, the $\mathcal{G}$-requirement promises that T_P is computable in V, and $\mathcal{P}_e$-requirements ensures that P has no W-computable paths by our assumption of $V \not\leqslant_T W$. □

Remark 2.

— It is easy to see that we can construct P as a positive measure Π^0_1 set. Indeed, for any k, P is assured to have the measure $\geqslant 1 - 1/k$, by replacing the nth location of $\mathcal{P}_e$ by $\langle e, n\rangle + k$, since at most one string of height $\langle e, n\rangle + k + 1$ is removed from T_P.

— It is easy to see that we can construct P as a nonempty separating Π^0_1 set by replacing the definition of $T_{P,s+1}$ as follows:

$$T_{P,s+1} = 2^{<\omega} \setminus \{\sigma \in 2^{<\omega} : (\exists e,t)\ e < t \leqslant s \\ \& \ \sigma(\langle e, n(e,t+1)\rangle) = \Phi_{e,t}(W_t; \langle e, n(e,t+1)\rangle)\}.$$

4. Main lemma for the first half of Theorem 2

This section deals mainly with Lemma 17, which is used to prove Theorem 14 (hence, the first half of Theorem 2).

4.1. *Coding and preservation*

For a pair $\langle a, b\rangle \in (\omega \cup \{\uparrow\})^2$ and a string $\sigma \in \omega^{<\omega}$, we define $\sigma|^b_a \in \omega^{\leqslant b}$ as follows. If $a, b \in \omega$, then $\sigma|^b_a = \sigma(a+n)$ for each $n < \min\{b, |\sigma| - a\}$. If $a = \uparrow$, then $\sigma|^b_a = \langle\rangle$.

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If $a \in \omega$ and $b = \uparrow$, then $\sigma|_a^b = \sigma|_a^{|\sigma|}$. For any Π^0_1 class $P \subseteq 2^\omega$, let $P|_a^b$ denote the set $[\{\sigma \in 2^{<\omega} : \sigma|_a^b \in T_P\}]$.

For instance, $P|_0^\uparrow = P$, and $P|_\uparrow^\uparrow = 2^\omega$. Note that, if P is nonempty, then $P|_a^b$ is also nonempty. Moreover, if P is separating, then $P|_a^b$ is also separating.

Let Λ be a set of indices coded as a subset of ω. For a c.e. set $W_e \subseteq \Lambda \times \omega$, two partial computable functions $c_e, l_e : \Lambda \to \omega$ are defined as follows:

— $c_e(\alpha) = \min\{s \in \omega : 0 \in (W_e)_\alpha[s]\}$ if such s exists.
— $l_e(\alpha) = \#(W_e)_\alpha - 1$ if $0 < \#(W_e)_\alpha < \aleph_0$.

We also use the following notions.

— For strings $\sigma, \tau \in \omega^{<\omega}$, we say that σ *is left to* τ (written $\sigma <_{\rm left} \tau$) if $\gamma^\frown m \subseteq \sigma$, $\gamma^\frown n \subseteq \tau$, and $m < n$, for some string $\gamma \in \omega^{<\omega}$ and $m, n \in \omega$. Moreover, we write $\sigma \leqslant_{\rm left} \tau$ if $\sigma \subseteq \tau$ or $\sigma <_{\rm left} \tau$.
— An element $\delta \in \omega^\omega$ is *a* Σ^0_1*-path* (also called *a left-c.e. real*) if there is a uniformly computable $\leqslant_{\rm left}$-increasing sequence of computable reals, $\{\delta_s\}_{s\in\omega}$, such that $\delta(n) = \lim_s \delta_s(n)$ for any $n \in \omega$.
— Then *the c.e. active-stage set for a* Σ^0_1*-path* δ, denoted $\mathrm{Act}(\delta)$, is defined by

$$\mathrm{Act}(\delta)_\alpha = \{s \in \omega : \alpha \subset \delta_s, \delta_{s+1} \ \&\ \delta_s(|\alpha|) < \delta_{s+1}(|\alpha|)\},$$

and $\mathrm{Act}(\delta) = \{\langle \alpha, m\rangle \in \omega : m \in \mathrm{Act}(\delta)_\alpha\}$.

Note that, if $P \subseteq 2^\omega$ is a Π^0_1 class, and W_e is the c.e. active stage set for a Σ^0_1 path, then we can ensure the following properties.

1. $\bigotimes_{\alpha\in\omega^{<\omega}} P|_{c_e(\alpha)}^{l_e(\alpha)}$ is a Π^0_1 class.
2. $\bigotimes_{\alpha\in\omega^{<\omega}} P|_{c_e(\alpha)}^{l_e(\alpha)} \leqslant_M P$.

Now we start to show the main lemma in this section.

Lemma 17. Let A be a c.e. set, and let P and Q be nonempty Π^0_1 classes. If $A \not\leqslant_T \emptyset$ and $P \not\leqslant_M Q$, then there exists a nonempty Π^0_1 class $\widehat{P} \leqslant_M P$ such that $A \not\leqslant_T \widehat{P} \not\leqslant_M Q$.

Proof. We will construct $\widehat{P} = \bigotimes_{\alpha\in\omega^{<\omega}} P|_{C(\alpha)}^{L(\alpha)}$, where $L(\alpha) = l_e(\alpha)$ and $C(\alpha) = c_e(\alpha)$ for some c.e. active stage set W_e for a Σ^0_1 path. □

Requirements. We need to construct a Π^0_1 class $\widehat{P}$, for each index e, the following conditions are met:

$$\mathcal{N}_e : \Phi_e(T^{\rm ext}_{\widehat{P}}) = A \ \rightarrow\ (\exists \Gamma)\ \Gamma = A;$$
$$\mathcal{P}_e : (\forall g \in Q)\ \Phi_e(g) \in \widehat{P} \ \rightarrow\ (\exists \Delta)(\forall g \in Q)\ \Delta(g) \in P.$$

Here Γ and Δ range over all partial computable functionals.

Assume that a computable well-pruned tree T_s is given.

— The *length functions* $o(\alpha, s)$ *and* $l(\alpha, s)$ are defined as follows:

$$o(\alpha, s) = \max\{l < s : \Phi_{|\alpha|,s}(T_s) \upharpoonright l = A \upharpoonright l\},$$
$$l(\alpha, s) = \max\{l < s : (\forall \sigma \in T_{Q,s} \cap 2^s)\ \Phi_{|\alpha|,s}(\sigma) \upharpoonright l \in T_s\}.$$

— Let $\mathrm{use}_e(X, x, s)$ denote the X-use of computation $\Phi_{e,s}(X) \upharpoonright x$, where $\mathrm{use}_e(X, x, s) = 0$ if $\Phi_{e,s}(X; n)$ is undefined for some $n < x$. The *restraint function* $r(\alpha, s)$ is defined as follows:

$$r(\alpha^\frown o, s) = \mathrm{use}_{|\alpha|}(T_s, o+1, s) + 1.$$

— The *current true path δ_s at stage s* is inductively defined by $\delta_s(n) = o(\delta_s \upharpoonright n, s)$ for each $n < s$.

— Then, T_{s+1} is defined as the corresponding computable well-pruned tree for the Π^0_1 class $\bigotimes_{\alpha \in \omega^{<\omega}} P|^{L(\alpha,s+1)}_{C(\alpha,s+1)}$, where $L(\alpha, s+1)$ and $C(\alpha, s+1)$ are defined as follows:

$$L(\alpha, s+1) = \max\{l(\alpha, s) : (\exists t \leqslant s)\ \alpha \subseteq \delta_t\},$$

$$C(\alpha, s+1) = \begin{cases} \max(\{r(\beta, \mathrm{st}(\beta)) : \beta \subseteq \alpha\} \cup \{\mathrm{st}(\alpha)\}) & \text{if } (\exists t \leqslant s)\ \alpha \subseteq \delta_t, \\ \uparrow & \text{otherwise.} \end{cases}$$

Here $\mathrm{st}(\beta)$ denotes $\min\{u : \beta \subseteq \delta_u\}$.

Note that, for each stage s, we have $C(\alpha, s) =\uparrow$ for almost all α. Therefore, the Π^0_1 class $\bigotimes_\alpha P|^{L(\alpha,s+1)}_{C(\alpha,s+1)}$ is clopen, and an index of such a tree T_{s+1} can be calculated from an index of T_s, uniformly in s. Finally, put $L(\alpha) = \lim_s L(\alpha, s)$ and $C(\alpha) = \lim_s C(\alpha, s)$.

For each α and n, $\Gamma^\alpha(n)$ and $\Delta^\alpha(n)$ is defined as follows:

$$\Gamma^\alpha(n) = \Phi_{|\alpha|, \mathrm{st}(\alpha^\frown n)}(T_{\mathrm{st}(\alpha^\frown n)})(n),$$
$$\Delta^\alpha(g)(n) = g(\langle \alpha, n + C(\alpha, \mathrm{st}(\alpha)) \rangle).$$

Claim (N-Lemma). For every $\alpha \in \omega^{<\omega}$, $\lim_s o(\alpha, s)$ converges.

Proof. Inductively we assume that $\lim_s l(\beta, s)$ converges for each $\beta \subsetneq \alpha$. If $\alpha \not\subseteq \delta_s$ for almost all s, then $\lim_s l(\alpha, s)$ clearly converges. Otherwise, we have $\alpha \subseteq \delta_s$ for almost all s by our assumption. Fix t such that $\alpha \subseteq \delta_s$ for all $s \geqslant t$. Then, for each $\beta \subsetneq \alpha$, $l(\beta, s) = l(\beta, t)$ for all $s \geqslant t$. Note that $L(\beta, s) = L(\beta, t+1)$ and $C(\beta, s) = C(\beta, t+1)$ if $\beta \subsetneq \alpha$ or β is incomparable with α. Set $u_n = \mathrm{use}_{|\alpha|}(T_{\mathrm{st}(\alpha^\frown n)}, n+1, \mathrm{st}(\alpha^\frown n))$. Note that, if $\beta \supseteq \alpha^\frown m$ for some $m \geqslant n$, then and $C(\alpha, s) \geqslant u_n$ for every $s \geqslant \mathrm{st}(\alpha^\frown n)$, and, for any other β, we have $l(\beta, s) = l(\beta, \mathrm{st}(\alpha^\frown n) + 1)$ for every $s \geqslant \mathrm{st}(\alpha^\frown n)$. Therefore, $T_s \upharpoonright u_n + 1 = T_{\mathrm{st}(\alpha^\frown n)} \upharpoonright u_n + 1$. Hence, $\Gamma^\alpha(n) = \Phi_{|\alpha|,s}(T_s)(n)$ for every $s \geqslant \mathrm{st}(\alpha^\frown n)$. Thus, if $\lim_s o(\alpha, s) = \infty$, then $\Gamma^\alpha = A$. □

Claim (P-Lemma). For every $\alpha \in \omega^{<\omega}$, $\lim_s L(\alpha, s)$ converges.

Proof. Assume $C(\alpha)$ converges, and $L(\alpha) = \lim_s L(\alpha, s) =\uparrow$. Then $P|^{L(\alpha)}_{C(\alpha)}$ is Medvedev equivalent to P. Indeed, it is easy to see that $\Delta^\alpha(g) \in P$ for every $g \in Q$.

Set $\widehat{P} = \bigotimes_{\alpha \in \omega^{<\omega}} P|^{L(\alpha)}_{C(\alpha)}$. By N-Lemma and P-Lemma, for each e, there is l such that $\Phi_e(T^{\mathrm{ext}}_{\widehat{P}}) \upharpoonright l \neq A \upharpoonright l$, and $\Phi_e(g) \upharpoonright l \notin T^{\mathrm{ext}}_{\widehat{P}}$ for some $g \in Q$ by compactness. That is to say, we have $A \not\leqslant_T \widehat{P} \not\leqslant_M Q$ as desired. □

By modifying our construction, we can easily show the following.

Theorem 18. Let A be a c.e. set, and let $\{P_i\}_{i\in\omega}$ and $\{Q_i\}_{i\in\omega}$ be computable sequences of nonempty Π^0_1 classes. If $A \not\leqslant_T \varnothing$ and $P_i \not\leqslant_M Q_i$ for each $i \in \omega$, then there exists a nonempty Π^0_1 class $\widehat{P} \leqslant_M \bigotimes_i P_i$ such that $A \not\leqslant_T \widehat{P} \not\leqslant_M Q_i$ for each $i \in \omega$. □

4.2. *Applications*

By analysing the proof of Cenzer and Hinman (2003), we can easily check that, for every Π^0_1 classes $P, Q \subseteq 2^\omega$, if $P <_M Q$ then $P <_M P \otimes \bigotimes_\alpha \mathsf{CPA}|^{l_e(\alpha)}_{c_e(\alpha)} \oplus Q <_M Q$ for the c.e. active stage set W_e for a Σ^0_1 path.

Let $(\mathcal{L}, \leqslant, \cup, \cap, 0, 1)$ be a lattice. An element $a \in \mathcal{L}$ has *the anticapping property* if there exists $b \in (a, 1)$ such that $a < b \cap c$ for every $c > a$. It is not known whether every Π^0_1 Medvedev incomplete degree has the anticapping property. By combining the strategies for Theorem 18 and Cenzer and Hinman (2003), we can show the following anticapping result.

Theorem 19. Every element of $\mathcal{P}_M(< \mathbf{0}')$ has the anticapping property in $\mathcal{P}_M$.

Proof (sketch). Let Q be a Π^0_1 class with $T_Q \leqslant_T \varnothing'$. Note that $Q \cap C \leqslant_T \varnothing'$ for every clopen set C. Because, for any clopen set C, the corresponding tree T_C is clearly computable; hence, $T_{Q\cap C} = T_Q \cap T_C \equiv_T T_Q <_T \varnothing'$. In particular, $Q \cap C <_M \mathrm{CPA}$ for any clopen set C. Let $\{C_e\}_{e\in\omega}$ be an effective enumeration of all clopen sets. Set $Q_e = Q \cap C_e$.

Requirements. We need to construct a Π^0_1 class P such that for each e, i the following conditions are met:

$$\begin{aligned}
\mathcal{R}_{2\langle e,i\rangle} &: \big(Q_i \neq \varnothing \ \&\ (\forall f \in Q_i)\ \Phi_e(f) \in P\big) \\
&\qquad\qquad\qquad \rightarrow\ (\exists \Delta_{e,i})(\forall f \in Q_i)\ \Delta_{e,i}(f) \in \mathrm{CPA}; \\
\mathcal{R}_{2e+1} &: (\exists g \in Q)\ \big((\forall f \in P)\ \Phi_e(f \oplus g) \in \mathrm{CPA} \\
&\qquad\qquad\qquad \rightarrow\ (\exists \Gamma_e)(\forall f \in Q)\ \Gamma_e(f) \in \mathrm{CPA}\big).
\end{aligned}$$

Here, $\Delta_{e,i}$ and Γ_e range over all partial computable functionals, for each e and i.

Combine the strategies to show Theorem 18 and Cenzer and Hinman (2003). □

By analysing the previous proof, we can prove that for every Π^0_1 class Q, the Medvedev degree of Q has the anticapping property if $Q \cap C <_M \mathrm{CPA}$ holds for any clopen set C. Unfortunately, being $\mathcal{P}_M(< \mathbf{0}')$ is not characterized by the anticapping property, by following observation.

Proposition 20. There is a Medvedev degree $\mathbf{r} \in \mathcal{P}_M \setminus \mathcal{P}_M(< \mathbf{0}')$ which has the anticapping property.

Proof. Recall that MLR denotes the set of all Martin-Löf random reals. Then there exists a universal Martin-Löf test $\{U_i\}_{i\in\omega}$ such that $\mathrm{MLR} = \bigcup_i (2^\omega \setminus U_i)$. Then, the Π^0_1 class $R_1 = 2^\omega \setminus U_1 \subseteq \mathrm{MLR}$ has measure $\geqslant 1/2$. Hence, the set R_1 is nonempty. Since $\mathrm{DNC} \leqslant_w \mathrm{MLR}$, every $\alpha \in R_1 \subset \mathrm{MLR}$ computes a diagonally noncomputable function. The set T_{R_1} computes the leftmost path of R_1, so it computes a diagonally noncomputable function. By the Arslanov Completeness Criterion, $T_{R_1} \equiv_T \varnothing'$. Hence, $\mathbf{r} = \deg_M(R_1) \in \mathcal{P}_M \setminus \mathcal{P}_M(< \mathbf{0}')$. We claim that, for every clopen set C, if $R_1 \cap C \neq \varnothing$, then we have $\mu(R_1 \cap C) > 0$. Otherwise $R_1 \cap C$ is a null Π^0_1 class, so it contains no Kurtz random real. Hence, this contradicts that $R_1 \cap C$ contains a Martin-Löf random real. If $R_1 \cap C \neq \varnothing$ then $R_1 \cap C <_M \mathrm{CPA}$, since CPA is not Medvedev reducible to a

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positive measure Π^0_1 class (see Simpson (2005)). Thus, the Medvedev degree of R_1 has the anticapping property by our previous observation. □

Question 21. Does every degree $\mathbf{p} \in \mathcal{P}_M$ have the anticapping property?

Our finite injury priority construction also works for showing the absence of nontrivial computable infima in the co-c.e. closed Medvedev degrees $\mathcal{P}_M$.

Theorem 22 (The first half of Theorem 4). $\mathcal{P}_M$ has no infima of any nontrivial computable collection.

Proof (sketch). For any computable sequence $\{P_e\}_{e\in\omega}$ with $P_e >_M P_{e+1}$, and a Π^0_1 class $P \subseteq 2^\omega$, if $P \leqslant_M P_e$ for any $e \in \omega$, then we can construct a Π^0_1 class $Q \subseteq 2^\omega$ such that $P \not\geqslant_M Q$ and $Q \leqslant_M P_e$ by defining $Q = \mathsf{CPA}^\frown\{(\mathsf{CPA}|_0^{l_P(e)}) \otimes P_e\}_{e\in\omega}$ with some lower semi-computable function l_P. Here, we fix a computable set $\{\rho_e\}_{e\in\omega}$ of all leaves of the computable tree T with $[T] = \mathsf{CPA}$, and then $\mathsf{CPA}^\frown\{S_e\}_{e\in\omega}$ means $\mathsf{CPA} \cup \bigcup_{e\in\omega} \rho_e{}^\frown S_e$.

Requirements. We construct a Π^0_1 class Q meeting the following requirements:

$$\mathcal{G}_e : Q \leqslant_M P_e;$$
$$\mathcal{N}_e : \Phi_e(P) \subseteq Q \ \rightarrow\ (\exists \Delta_e)\ \Delta_e(P) \subseteq \bigotimes_{i<e} P_i.$$

Here, Δ_e ranges over all computable functionals.

The parameter l_P is defined to be the length of agreement of $\Phi_e(P) \subseteq Q$, i.e. $l_P(e)$ is the maximal value l such that there is $i \leqslant e$ with $\Phi_i(f) \upharpoonright l \in T_Q$ for any $f \in P$. If $\Phi_e(P) \subseteq Q$, then eventually Q is constructed to be a Π^0_1 class which is Medvedev equivalent to $\bigotimes_{i<e} P_i$, since $l_P(k) = \uparrow$ for any $k \geqslant e$. Therefore, the $\mathcal{N}_e$-requirements are satisfied. The definition of Q also ensures the success of the $\mathcal{G}_e$-requirements. □

5. Main lemma for Theorem 3

This section deals mainly with Lemma 23, which is used to prove Theorem 12 (hence, Theorem 3).

5.1. *Coding and preservation*

— An element $\delta \in \omega^{\leqslant\omega}$ is *a Σ^0_2-path* if there is a uniformly computable sequence of computable reals, $\{\delta_s\}_{s\in\omega}$, such that $\delta(n) = \liminf_s \delta_s(n)$ for any $n \in \omega$.

— For a given Σ^0_2-path $\delta \in \omega^{\leqslant\omega}$ and its computable approximation $\{\delta_s\}_{s\in\omega}$, we define *the number of times which α is initialized along δ by stage s* as follows:

$$\mathrm{in}_\delta(\alpha, s) = \#\{t \leqslant s : \delta_t <_{\mathrm{left}} \alpha\}.$$

— Then *the c.e. active-stage set for a Σ^0_2-path* δ, denoted $\mathrm{Act}(\delta)$, is defined by

$$\mathrm{Act}(\delta)_{\alpha,n} = \{s \in \omega : \mathrm{in}_\delta(\alpha, s) = n \ \&\ \alpha \subset \delta_s\},$$

and $\mathrm{Act}(\delta) = \{\langle \alpha, n, m\rangle \in \omega : m \in \mathrm{Act}(\delta)_{\alpha,n}\}$.

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Claim (coding). If a c.e. set W_e is an active-stage set for some Σ^0_2-path δ, then $Q_i \leqslant_M \bigotimes_{\alpha,n} Q_{|\alpha|}|^{l_e(\alpha,n)}_{c_e(\alpha,n)}$ for any $i \in \omega$.

Proof. If δ is a Σ^0_2-path then, for every $t \in \omega$, $\lim_s \mathrm{in}_\delta(\delta \upharpoonright t, s)$ converges to some $n(t) \in \omega$, since $\delta_s <_{\mathrm{left}} \delta \upharpoonright t$ occurs at most finitely many s. Then, $\mathrm{Act}(\delta)_{\delta\upharpoonright t,n(t)}$ is infinite, since $\delta(n) = \liminf_s \delta_s(n)$, hence $\delta \upharpoonright t \subset \delta_s$ for infinitely many s. Therefore, $c_e(\delta \upharpoonright t, n(t)) \downarrow$, and $l_e(\delta \upharpoonright t, n(t)) \uparrow$. Hence, $Q_t|^{l_e(\delta\upharpoonright t,n(t))}_{c_e(\delta\upharpoonright t,n(t))} \equiv_M Q_t$. □

Lemma 23. Let $\{Q_i\}_{i\in\omega}$ be a computable sequence of Π^0_1 classes, and assume that there is a c.e. set U with $U \not\leqslant_T \bigoplus_{i\leqslant n} Q_i$ for any $n \in \omega$. Then, there is a nonempty Π^0_1 class $P \subseteq 2^\omega$ such that $U \not\leqslant_T P$ and $P \geqslant_M Q_i$ for any $i \in \omega$.

Proof. We construct a Σ^0_2 path TP, and define $P = \bigotimes_{\alpha,n} Q_{|\alpha|}|^{l_e(\alpha,n)}_{c_e(\alpha,n)}$ for its active-stage set. Fix an effective enumeration $\{Q_e\}_{e\in\omega}$ of a given T-incomplete family. □

Requirements. We need to construct a Π^0_1 class P such that, for each index e, the following conditions are met:

$$\mathcal{G}_e : (\exists \Gamma_e)\, \Gamma_e(P) \subseteq Q_e;$$
$$\mathcal{N}_e : \Phi_e(T_P) = U \;\rightarrow\; (\exists \Delta_e)\, \Delta_e(\bigoplus_{i<e} T_{Q_i}) = U.$$

Here, Γ_e, Δ_e and Θ_e are Turing functionals which will be constructed by us, for each e.

Strategy. *The tree of strategies* $\mathcal{O}$ is $\omega^{<\omega}$. We assign a node of length $2e$ to a requirement $\mathcal{N}_e$, and $2e+1$ to a requirement $\mathcal{G}_e$. We say that α *is to the left of* β (or β *is to the right of* α) if $\gamma^\frown\langle m\rangle \subseteq \alpha$ and $\gamma^\frown\langle n\rangle \subseteq \beta$ hold for some $\gamma \in \mathcal{O}$ and $m < n$. We say that α *has higher priority than* β if α is to the left of β or $\alpha \subsetneq \beta$. For a fixed computable injection $\pi : \omega^{<\omega} \to \omega$, the symbol $\omega^{[\alpha]}$ denotes the $\pi(\alpha)$-section of ω, that is, $\omega^{[\alpha]} = \{\langle \pi(\alpha), n\rangle : n \in \omega\}$. First, we describe an outline of our strategy.

A. The $\mathcal{G}_e$-strategy α acts as follows at stage s. As a first step, for any n, pick *a coding location* $c^\alpha(n) \in \omega^{[\alpha]}$ which is the nth smallest *new* number greater than any restraints defined until now. If ρ has been removed from T_{Q_e} at this stage, then, we remove all $\tau \supseteq \sigma$ from T_P for any σ of height $c^\alpha(|\rho|-1)+1$ such that $\sigma(c^\alpha(n)) = \rho(n)$ for each $n < |\rho|$. Then, we put $\Gamma^\alpha_e(f)(n) = f(c^\alpha(n))$ for each $f \in 2^\omega$ and $n \in \omega$.

B. The $\mathcal{N}_e$-strategy α acts as follows at stage s.

B0. For the first step, put $l^\alpha = 0$.

B1. Wait until $U \upharpoonright l^\alpha + 1 = \Phi_e(T_P) \upharpoonright l^\alpha + 1$. Go to state (B2) when this occurs,

B2. Set $\Delta^\alpha_e(\bigoplus_{i<e} T_{Q_i}; l^\alpha) = U(l^\alpha)$ with $\bigoplus_{i<e} T_{Q_i}$-use $\delta(l^\alpha)$, where $\delta(l^\alpha)$ is the maximal k such that $c^\beta(k) \leqslant \varphi(l^\alpha)$ for some $\mathcal{G}$-strategy β of higher priority than α. Here, $\varphi(l^\alpha)$ denotes the T_P-use of computations $\Phi_e(T_P) \upharpoonright l^\alpha + 1$. Also protect the computation $\Phi_e(T_P) \upharpoonright l^\alpha + 1$ by restraining $T_P \upharpoonright \varphi(l^\alpha) + 1$. Now, go back to state (B1) with $l^\alpha + 1$ in place of l^α, and go to next state (B3).

B3. Wait until $(\bigoplus_{i<e} T_{Q_i}) \upharpoonright \delta(l^\alpha)$ changes, or until l^α is enumerated into U. Go to state (B4) when the former case occurs, and go to state (B5) when the latter case occurs.

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B4. Cancel all actions for $l' > l^\alpha$, remove any restraint for $l' \geqslant l^\alpha$, and go back to state (B2).

B5. Stop all actions for $l' > l^\alpha$, and wait for the former case in state (B3) to occur. When it does, go to state (B4).

Outcome. Intuitively, at stage s, the outcome of this strategy is the greatest l^α which is reached by some stage $t \leqslant s$ and uncanceled between $t+1$ and s. The *current true path* TP_s will be defined to be $TP_s(|\alpha|) = l^\alpha$.

Remark 3. In (B2), the use $\delta(l^\alpha)$ may be defined. Assume that the strategy α is never injured after some stage s, and $\delta(l^\alpha)$ is defined at some stage $t \geqslant s$. At some stage $u \geqslant t$, it can happen that $U_s(l^\alpha) \neq U_u(l^\alpha) = \Phi_e(T_P; l^\alpha)$, because of the enumeration of l^α into U and the change of $T_P \upharpoonright \varphi(l^\alpha)$. Pick the least such l^α. After stage s, until the computation $\Delta_e^\alpha(\bigoplus_{i \leqslant e} T_{Q_i}; l^\alpha)$ is destroyed, only strategies $\beta \subseteq \alpha^\frown l^\alpha$ or lower priority strategies than $\alpha^\frown l^\alpha$ are eligible to act, i.e. only such strategies can be an initial segment of current true paths. However, any lower priority strategy β than $\alpha^\frown l^\alpha$ can make changes of T_P only above $c^\beta(0)$. Moreover, in (A), the strategy β ensures $c^\beta(0) > \varphi(l^\alpha)$, since, if $c^\beta(0)$ has been already defined before (B1) happens, then $c^\beta(0)$ is initialized and redefined to be greater than $\varphi(l^\alpha)$. Thus, the change of $T_P \upharpoonright \varphi(l^\alpha)$ must be made by the actions (A) of strategies $\beta \subseteq \alpha^\frown l^\alpha$. In other words, $T_{Q_{|\beta|}} \upharpoonright c^\beta(k)$ changes for some $\beta \subseteq \alpha$, where $c^\beta(k) \leqslant \varphi(l^\alpha)$. Hence, $\bigoplus_{i \leqslant e} T_{Q_i} \upharpoonright \delta(l^\alpha)$ must change. Then, the strategy goes to (B4), and recovers $\Delta_e^\alpha(\bigoplus_{i \leqslant e} T_{Q_i}; l^\alpha) = U(l^\alpha)$. Therefore, this strategy ensures that $\Phi_e(T_P) \upharpoonright l^\alpha + 1 = U \upharpoonright l^\alpha + 1$ always implies $\Delta_e^\alpha(\bigoplus_{i \leqslant e} T_{Q_i}; l^\alpha) = U(l^\alpha)$.

Observation. Now, we assume that the infimum limit of outcomes of α is l^α. Then the $\mathcal{N}_e$-requirement is satisfied as follows:

1. If α waits at (B1) for l^α forever, then clearly $U \neq \Phi_e(T_P)$ is satisfied.
2. Suppose the former case of (B3) occurs infinitely many times for l^α. We notice that, when the former case of (B3) occurs for l^α, the computation $\Phi_e(T_P; l^\alpha)$ is also destroyed by the definition of δ in (B2). Thus, in this case, $\Phi_e(T_P; l^\alpha)$ is eventually undefined, and so $\Phi_e(T_P) \neq U$.
3. If α waits at (B5) for l^α forever, then we get $1 = U(l^\alpha) \neq \Phi_e(T_P; l^\alpha) = 0$.
4. If the infimum limit of l^α diverges, then we observe that $\Delta_e^\alpha(\bigoplus_{i<e} T_{Q_i}) = \Phi_e(T_P) = U$ holds, and this contradicts that $\bigoplus_{i<e} T_{Q_i} <_T U$.

Construction. Now, we describe the formal strategy of our construction. The construction proceeds in stages. A *stage* s consists of *substages* $t \leqslant s$. At each substage t of stage s, just one strategy $\alpha \subseteq TP_s$ of length t is eligible to act, where TP_s is inductively decided in our construction at substages of stage s. At the beginning of stage $s+1$, *initialize* all right strategies of TP_s by making all its parameters undefined, and at end of the stage $s+1$, for all α of the left of TP_{s+1}, put $r(\alpha, s+1) = r(\alpha, s)$. We now describe an action of $\alpha = TP_{s+1} \upharpoonright t$ at substage t of stage $s+1$.

Assume that a computable well-pruned tree T_s is given.

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— Let $\mathrm{in}(\alpha, t)$ denote the number of times α is initialized by stage t. Then the set of all *(α, n)-stages* is defined by $\mathrm{ST}_{\alpha,n}[s] = \{t \leqslant s : \mathrm{in}(\alpha, t) = n \ \&\ \alpha \subset TP_t\}$. If $\alpha \subset TP_s$, i.e. s is $(\alpha, \mathrm{in}(\alpha, s))$-stage, then we also say that α *is eligible to act at stage s*.

— The *length function* $l(\alpha, s)$ is defined as follows:

$$l(\alpha, s) = \max\{l < s : \Phi_{|\alpha|,s}(T_s) \upharpoonright l = U_s \upharpoonright l\}.$$

— The *outcome* $o(\alpha, s)$ is defined as follows. If s is the first (α, n)-stage, i.e. if $s = \min \mathrm{ST}_{\alpha,n}[s]$, then put $o(\alpha, s) = l(\alpha, s)$. Otherwise, let $s^- = \max\{t < s : t \in \mathrm{ST}_{\alpha,n}[s]\}$. In the case that $l(\alpha, s) < l(\alpha, s^-)$ because of enumerating $l(\alpha, s^-)$ into U at stage between s^- and s as in the latter case of (B3), we put $o(\alpha, s) = o(\alpha, s^-)$. Otherwise, we put $o(\alpha, s) = l(\alpha, s)$.

— The *restraint function* is defined as follows:

$$r(\alpha^\frown o, s) = \varphi(o + 1) + 1,$$

where $\varphi(k)$ denotes the T_s-use of computations $\Phi_{e,s}(T_s) \upharpoonright k$.

— The *current true path* $TP_s \in \omega^s$ at stage s is inductively defined by $TP_s(n) = o(TP_s \upharpoonright n, s)$ for each $n < s$.

— Then, T_{s+1} is defined as the corresponding computable well-pruned tree for the Π^0_1 class $\bigotimes_{(\alpha,n)\in\omega^{<\omega}} P|^{L(\alpha,n,s+1)}_{C(\alpha,n,s+1)}$, where $L(\alpha, n, s+1)$ and $C(\alpha, n, s+1)$ are defined as follows:

$$L(\alpha, n, s+1) = \max \mathrm{ST}_{\alpha,\mathrm{in}(\alpha,s)}[s],$$

$$C(\alpha, n, s+1) = \begin{cases} \max(\{r(\beta, \mathrm{st}(\beta)) : \beta \leqslant_{\mathrm{left}} \alpha\} \cup \{\mathrm{st}(\alpha)\}) & \text{if} \quad \mathrm{ST}_{\beta,n}[s] \neq \varnothing, \\ \uparrow & \text{otherwise.} \end{cases}$$

Here, $\mathrm{st}(\beta) = \min \mathrm{ST}_{\beta,\mathrm{in}(\alpha,s)}[s]$.

For our construction, by induction, *The true path* TP is defined to be $TP(n) = \liminf_{s\to\infty}\{o \in \mathcal{O} : (TP \upharpoonright n)^\frown\langle o\rangle \subseteq TP_s\}$. Finally, we define $P = \bigcap_s [T_s]$.

Claim. $P \neq \varnothing$.

Proof. Each $\mathcal{G}_e$-strategy α enumerates some strings into T_P only when some strings are enumerated into T_{Q_e}, and α's coding locations $\subseteq \omega^{[\alpha]}$ for distinct α's are disjoint. So, our assumption that $Q_e \neq \varnothing$ for each e ensures $P \neq \varnothing$. □

Claim. The $\mathcal{G}$-requirements are satisfied.

Proof. Fix e, and let $\alpha \in TP$ be a $\mathcal{G}_e$-strategy. Thus, α is not initialized after some stage s. Suppose $\Gamma^\alpha_e(f) = \langle f(c^\alpha(n))\rangle_{n\in\omega} \notin Q_e$. Then $\langle f(c^\alpha(n))\rangle_{n\leqslant k}$ for some k is enumerated into T_{Q_e} at some stage s', and so $f \upharpoonright c^\alpha(k) + 1$ must be removed from T_P since α acts infinitely many times for $s'' \geqslant \max\{s, s'\}$. This ensures that $\Gamma^\alpha_e(P) \subseteq Q_e$. □

Claim. The $\mathcal{N}$-requirements are satisfied.

Proof. Fix e and let $\alpha \in TP$ be a $\mathcal{N}_e$-strategy. We construct $\Delta^\alpha_e(\bigoplus_{e<i} T_{Q_i})$ as described in (B2). Let s be the stage after which α is not initialized, let $u + 1$ be the next stage $\geqslant s$ at which α is eligible to act, and let $v + 1$ also be the next stage $> u + 1$ with $\alpha \in TP_{v+1}$. We observe that, if $l(\alpha, v + 1)$ is differ from $l(\alpha, u + 1)$ because of changing $\Phi_e(T_P\,; k)$ for

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some k then such k must be less than $TP_{u+1}(|\alpha|)$, and this change forces us to destroy the computation of $\Delta_e^\alpha(\bigoplus_{i<e} T_{Q_i}; k)$. This is because any $\mathcal{G}$-strategies of lower priority than α cannot injure the computation $\Phi_e(T_P; k)$ since our construction makes a restraint $r(\alpha, u+1) = \varphi(TP_{u+1}(|\alpha|))$ on T_P, and the outcome $TP(|\alpha|)$ grows whenever $r(\alpha)$ grows. Moreover, $\mathcal{G}_j$-strategies for $j \geqslant e$ of higher priority than α never act after stage s since the actions of these strategies causes an initialization of the strategy α, and this contradicts the assumption of s. So only $\mathcal{G}_i$-strategies for $i < e$ can make a change of $T_P \upharpoonright \varphi(k)+1$ by changing $(\bigoplus_{i<e} T_{Q_i}) \upharpoonright \delta(k)$. Thus, $\Delta_{e,v}^\alpha(\bigoplus_{i<e} T_{Q_i,v}; x)$ correctly computes $\Phi_{e,v}(T_{P,v}; x)$ at every stage $v \geqslant s$ at which α is eligible to act, whenever $\Delta_{e,v}^\alpha(\bigoplus_{i<e} T_{Q_i,v}; x)$ is defined. So if $\liminf_s l(\alpha, s) = \infty$ then we get $\Delta_e^\alpha(\bigoplus_{i<e} T_{Q_i}) = \Phi_e(T_P) = U$ and this contradicts our assumption $\bigoplus_{i<e} T_{Q_i} <_T U$. Hence, $\liminf_s l(\alpha, s)$ must be some finite value $l(\alpha)$, and so $\Phi_e(T_P; l(\alpha)) \neq U(l(\alpha))$ as seen in previous observation. □

The $\mathcal{G}_e$-requirements assure that $Q_e \leqslant_M P$ via Γ_e, and $\mathcal{N}_e$-requirements assure that $U \not\leqslant_T T_P$ holds since the condition $\Delta_e(\bigoplus_{i<e} T_{Q_i}) = U$ cannot occur by our assumption. Hence, we obtain a desired Π_1^0 class P such that $U \not\leqslant_T T_P$ and $Q \leqslant_M P$ for every $Q \in \mathcal{A}$.

5.2. *Applications*

Recall that the c.e. Turing degrees have no nontrivial countable suprema, by Shoenfield's Thickness Lemma (see Soare (1987, Theorem VIII.2.3)). Our strategies construct the product of finite modifications of given Π_1^0 classes, as the Thickness Lemma constructs the sum of finite modifications of given c.e. sets. By a few modifications of our previous construction, we obtain the non-suprema result for Medvedev degrees of Π_1^0 classes.

Theorem 24 (the last half of Theorem 4). The Medvedev lattice $\mathcal{P}_M$ of the Π_1^0 classes has no nontrivial computable suprema.

Proof (sketch). Assume that a computable collection $\{Q_i\}_{i\in\omega}$ of Π_1^0 classes is given. Fix a Medvedev upper bound Q of $\{Q_i\}_{i\in\omega}$, i.e. $Q_i \leqslant_M Q$ for any $i \in \omega$. We will construct a Π_1^0 class P such that $Q_i \leqslant_M P$ for each $i \in \omega$, but $Q \not\leqslant_M P$.

Requirements. We need to construct a Π_1^0 class P such that, for each index e, the following conditions are met:

$$\mathcal{G}_e : (\exists \Gamma_e)\, \Gamma_e(P) \subseteq Q_e;$$
$$\mathcal{M}_e : \Phi_e(P) \subseteq Q \;\rightarrow\; (\exists \Theta_e)\, \Theta_e(\bigotimes_{i<e} Q_i) \subseteq Q.$$

Strategy (C). The $\mathcal{M}_e$-strategy α is quite similar to the $\mathcal{N}_e$-strategy. The strategy α acts as follows at stage s.

C0. For the first step, put $l^\alpha = 0$.

C1. Wait for the least s such that $|\Phi_e(\tau)| \geqslant l^\alpha + 1$ and $\Phi_e(\tau) \in T_Q$ hold, for any $\tau \in T_P$ of length s. Such s will be called *the use of the computation* $\Phi_e(P) \upharpoonright l^\alpha + 1$, and denoted by $\varphi(l^\alpha)$. Go to state (C2) when this happens.

C2. Choose the maximal k such that $c^\beta(k) \leqslant \varphi(l^\alpha)$ for some $\beta \subseteq \alpha$. For any choice $\sigma_i \in T_{Q_i}$ with $|\sigma_i| = k$, set $\Theta^\alpha_e(\bigoplus_{i\leqslant e}\sigma_i) = \Phi_e(\tau) \in T_Q$, where $\tau \in T_P$ satisfies the condition that $\tau^{[a\upharpoonright i,n]}|^{\uparrow}_{c^\alpha\upharpoonright i(0)}$ extends σ_i for each $i \leqslant e$.

C3. If we see that $\Phi_e(\tau)$ is removed from T_Q for some $\tau \in T_P$ of length $\varphi(l^\alpha)$, then go to state (C4).

C4. Stop all actions for $l' > l^\alpha$, and wait for the condition in (C1) to recover. When it does, reopen actions for $l' > l^\alpha$, and go back to state (C3).

Remark 4. Assume that the strategy α is never injured after some stage s, and the strategy reaches state (C2) at some stage $t \geqslant s$. After some stage, assume that α reaches state (C4) because of the decrease of T_Q, but the condition (C1) recovers because of the decrease of T_P. As discussed in the strategy (B), such recovering only happens due to some strategy $\beta \subseteq \alpha$. Therefore, if $\tau \in T_{P,t}$ below $\varphi(l^\alpha)+1$ is removed from $T_{P,u}$ for some stage $u \geqslant t$, then every corresponding $\bigoplus_{i\leqslant e}\sigma_i$ is also removed from $\bigotimes_{i\leqslant e} T_{Q_i,u}$, since some σ_i must be removed from $T_{Q_i,u}$. Hence, Θ^α_e recovers its role as a function from $\bigotimes Q_i$ to Q below $l^\alpha + 1$.

As the proof in Theorem 23, it is not hard to verify our construction. □

An ideal of the Medvedev lattice $\mathcal{P}_M$ of the Π^0_1 classes is *uniformly generated* if there is a computable sequence of Π^0_1 classes which generates the ideal. Theorem 24 implies that every uniformly generated proper ideal of the Medvedev lattice $\mathcal{P}_M$ of the Π^0_1 classes has a Medvedev incomplete upper bound. Furthermore, it is not hard to show the following theorem by combining our strategies (A), (B) and (C).

Theorem 25. Assume that $\{Q_i\}_{i\in\omega}$ is a computable sequence of Π^0_1 classes which satisfies $\bigotimes_{i<n} Q_i <_M \bigotimes_{i\in\omega} Q_i$. Let Q be a Π^0_1 class with $Q_i \leqslant_M Q$ for each $i \in \omega$, and U be a c.e. set which is not computable in $\bigoplus_{i<n} T_{Q_i}$ for any $n \in \omega$. Then, there is a nonempty Π^0_1 class $P \subseteq 2^\omega$ such that $U \not\leqslant_T T_P$ and $Q \not\leqslant_M P \geqslant_M Q_i$ for each $i \in \omega$. □

6. Conclusion

Our results indicate that if a Π^0_1-definable mass problem is difficult in the sense of the Medvedev or Muchnik degree, then its global information content must be complex in the sense of the Turing degree. For instance, if we know that a Π^0_1-definable mass problem is globally trivial in the sense of Kolmogorov complexity, then the problem must be easier than a predetermined Π^0_1 problem that has less information than the halting problem.

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