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DEGREES OF BRAUER CHARACTERS AND NORMAL SYLOW p-SUBGROUPS

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Abstract

Let p be a prime, G a solvable group and P a Sylow p-subgroup of G. We prove that P is normal in G if and only if $\varphi(1)_p^2$ divides $|G : \ker(\varphi)|_p$ for all monomial monolithic irreducible p-Brauer characters φ of G.

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All groups considered in this note are finite. We refer the reader for notation and terminology to [5] for character theory and [7] for Brauer character theory. For a group G and a prime p, we write $\mathrm{Irr}(G)$ and $\mathrm{IBr}(G)$ to denote the sets of irreducible (complex) characters and irreducible (p-)Brauer characters of G, respectively.

The authors with Cossey and Tong-Viet proved in [1] that if $\varphi(1)^2 \mid |G : \mathrm{Ker}(\varphi)|$ for all Brauer characters $\varphi \in \mathrm{IBr}(G)$, then $P \lhd G$ and G/P is nilpotent, where G is a p-solvable group and P is a Sylow p-subgroup of G. In [3], we showed that the full set of irreducible Brauer characters can be replaced by the monolithic irreducible Brauer characters. A group G is said to be a *monolith* if it contains a unique minimal normal subgroup, and the p-Brauer character $\varphi \in \mathrm{IBr}(G)$ is called *monolithic* if $G/\mathrm{Ker}(\varphi)$ is a monolith. In particular, when G is p-solvable and $P \in \mathrm{Syl}_p(G)$, we showed that $\varphi(1)^2 \mid |G : \mathrm{Ker}(\varphi)|$ for all monolithic Brauer characters $\varphi \in \mathrm{IBr}(G)$ if and only if $P \lhd G$ and G/P is nilpotent. In [2], we showed that $\mathrm{IBr}(G)$ can be replaced by the set of monomial irreducible Brauer characters; however, we had to strengthen the hypothesis that G is p-solvable to G being solvable. In particular, we proved in [2] when G is solvable that a Sylow p-subgroup P of G is normal and G/P is nilpotent if and only if $\varphi(1)^2 \mid |G : \mathrm{Ker}(\varphi)|$ for all monomial Brauer characters $\varphi \in \mathrm{IBr}(G)$.

Recently, Tong-Viet has shown that if we only consider when the Sylow p-subgroup is normal, then we may look at the p-parts of the degrees of the Brauer characters and of the indices of the kernels. Given an integer n and a prime p, define n_p to be the largest power of p dividing n. Tong-Viet proved in [8] that $\varphi(1)_p^2 \leq |G : \mathrm{Ker}(\varphi)|_p$

for all Brauer characters $\varphi \in \mathrm{IBr}(G)$ if and only if $P \trianglelefteq G$, where G is p-solvable and $P \in \mathrm{Syl}_p(G)$. Using Tong-Viet's proof, it is not difficult to see that $\mathrm{IBr}(G)$ can be replaced with the set of irreducible monolithic Brauer characters.

We show that if we strengthen the hypothesis of p-solvable to solvable, then we only need to look at the irreducible monomial Brauer characters. In fact, we only need to look at the irreducible, monolithic, monomial Brauer characters.

Theorem 1. *Let G be a solvable group, p a prime and $P \in \mathrm{Syl}_p(G)$. Then $\varphi(1)_p^2 \le |G : \mathrm{Ker}(\varphi)|_p$ for all monomial, monolithic Brauer characters $\varphi \in \mathrm{IBr}(G)$ if and only if $P \lhd G$.*

Notice that we cannot drop the hypothesis of G being solvable. For example, let S_5 be the symmetric group of degree five and $p = 2$. Then the principal 2-Brauer character is the unique monomial 2-Brauer character which satisfies the condition; however, S_5 contains no normal Sylow 2-subgroup.

Since the following proof is very similar to the proof of Theorem 1 in [2] and the solvable part of the proof of Theorem 1.3 in [3], we will suppress many of the details of this proof.

Proof of Theorem 1. If P is normal in G, then $P = \mathbf{O}_p(G) \subseteq \ker\varphi$ for every irreducible p-Brauer character φ of G. Thus, $\mathrm{IBr}(G) = \mathrm{IBr}(G/P) = \mathrm{Irr}(G/P)$ since G/P is a p'-group. It follows for every Brauer character $\varphi \in \mathrm{IBr}(G)$ that $p \nmid \varphi(1)$. So, $\varphi(1)_p = 1$ for every $\varphi \in \mathrm{IBr}(G)$. We have $\varphi(1)_p^2 \mid |G : \ker(\varphi)|_p$ for all monomial, monolithic $\varphi \in \mathrm{IBr}(G)$.

Conversely, assume that $\varphi(1)_p^2 \le |G : \mathrm{Ker}(\varphi)|_p$ for every monomial, monolithic Brauer character $\varphi \in \mathrm{IBr}(G)$. We work by induction on $|G|$. Suppose that $\mathbf{O}_p(G) > 1$. Since $\mathbf{O}_p(G)$ is contained in the kernel of every irreducible p-Brauer character of G, it follows that $G/\mathbf{O}_p(G)$ satisfies the induction hypothesis. This implies that $P = \mathbf{O}_p(G) \lhd G$.

Thus, we can assume that $\mathbf{O}_p(G) = 1$. Let M be a minimal normal subgroup of G; so M is a p'-subgroup of G. Notice that G/M satisfies the induction hypothesis; so PM will be a normal subgroup of G. Using the Frattini argument, we have $G = M\mathbf{N}_G(P)$.

It is not difficult to see that if G has more than one minimal normal subgroup, then we use the inductive hypothesis to show that G will have a normal Sylow p-subgroup. Therefore, we may assume that G is a monolith and M is its unique minimal normal subgroup. Notice that M is an elementary abelian q-group for some prime $q \neq p$. By the usual argument, $M \cap \mathbf{N}_G(P) = 1$. Also, using the usual arguments, P acts faithfully on M and thus it acts faithfully on $\mathrm{IBr}(M) = \mathrm{Irr}(M)$.

Now, it follows by Isaacs' large-orbit result [6, Theorem B] that there exists a Brauer character $\lambda \in \mathrm{IBr}(M)$ such that $|P : \mathbf{C}_P(\lambda)|^2 > |P|$. Denote by T the inertia group of λ in G. As usual, we can show that $|G : T|_p = |P : \mathbf{C}_P(\lambda)| > \sqrt{|P|}$.

Since M is complemented in T, it follows from a result of Gallagher (see [4, Lemma 1]) that there exists a nonprincipal Brauer character $\mu \in \mathrm{IBr}(T)$ such that $\mu_M = \lambda$. Thus, we have by the Clifford correspondence for Brauer characters

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[7, Theorem 8.9] that $\varphi = \mu^G \in \mathrm{IBr}(G)$. Since μ is linear, this implies that φ is monomial and $\varphi(1) = |G : T|$. Also, as M is the unique minimal normal subgroup of G, it follows that $\mathrm{Ker}(\varphi) = 1$. We have seen that the fact $|G : T|_p^2 \geq |G|_p$ implies that $\varphi(1)_p^2 \geq |G : \mathrm{Ker}(\varphi)|_p$, which contradicts the hypothesis. □

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