Notes on the theory of series (III): *On the summability of the Fourier series of a nearly continuous function.* By Mr G. H. HARDY and Mr J. E. LITTLEWOOD.

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1. The theorem which we prove here seems obvious enough when stated, but it appears to have been overlooked by the numerous writers who have discussed the subject, and the proof is less immediate than might be expected.

We say that $\phi(t)$ is continuous (C, α), where $\alpha > 0$, if

$$\frac{1}{\Gamma(\alpha)t^{\alpha}}\int_0^t \phi(u)(t-u)^{\alpha-1}\,du$$

tends to a finite limit l when $t \to 0$. We shall also say that $\phi(t) \to l\ (C, \alpha)$. To assert continuity (C, α), for a small α, is to assert a little less than ordinary continuity.

If $f(t)$ is periodic and integrable, and

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2s\} \to 0,$$

then the Fourier series of $f(t)$, for $t = x$, is summable (C, δ) to sum s for every positive δ.* If $\phi(t) \to 0\ (C, 1)$, then the series is summable $(C, 1+\delta)$.† It is therefore natural to expect the truth of the following theorem.

THEOREM. *If* $\phi(t) \to 0\ (C, \alpha)$, *then the Fourier series of* $f(t)$, *for* $t = x$, *is summable* $(C, \alpha+\delta)$ *to sum* s *for every positive* δ.

We prove the theorem in the most interesting case when $0 < \alpha < 1$. The proof for $\alpha > 1$ will differ in complication but not in principle.

2. We suppose β and δ positive and $\beta + \delta < 1$. The $(\beta+\delta)$-th Rieszian mean of the Fourier series, for terms whose rank does not exceed ω, differs from s by

$$\frac{2\Gamma(1+\beta+\delta)}{\pi} J_{\beta+\delta}(\phi, \omega) = \frac{2\Gamma(1+\beta+\delta)}{\pi}\int_0^\infty t^{-1-\beta-\delta}\, C_{1+\beta+\delta}(t)\,\phi\left(\frac{t}{\omega}\right)dt,$$

where $C_q(t)$ is Young's function‡

$$C_q(t) = \frac{t^q}{\Gamma(1+q)}\left\{1 - \frac{t^2}{(1+q)(2+q)} + \frac{t^4}{(1+q)\ldots(4+q)} - \ldots\right\}.$$

Since $\quad C_{1+\beta+\delta}(\omega u) = \dfrac{\omega^\beta}{\Gamma(\beta)}\displaystyle\int_0^u C_{1+\delta}(\omega v)(u-v)^{\beta-1}\,dv,$

* M. Riesz, 4, 5; Chapman, 1. See also Hobson, 3.
† Young, 6, 8. ‡ Young, 7.

we have

$$J_{\beta+\delta}(\phi,\omega)=\omega^{-\beta-\delta}\int_0^\infty u^{-1-\beta-\delta}C_{1+\beta+\delta}(\omega u)\,\phi(u)\,du$$

$$=\frac{\omega^{-\delta}}{\Gamma(\beta)}\int_0^\infty u^{-1-\beta-\delta}\phi(u)\,du\int_0^u C_{1+\delta}(\omega v)(u-v)^{\beta-1}dv$$

$$=\frac{\omega^{-\delta}}{\Gamma(\beta)}\int_0^\infty C_{1+\delta}(\omega v)\,dv\int_v^\infty u^{-1-\beta-\delta}(u-v)^{\beta-1}\phi(u)\,du$$

$$=\frac{\omega^{-\delta}}{\Gamma(\beta)}\int_0^\infty v^{-1-\delta}C_{1+\delta}(\omega v)\,\psi(v)\,dv=\frac{1}{\Gamma(\beta)}J_\delta(\psi,\omega),$$

where $$\psi(v)=v^{1+\delta}\int_v^\infty u^{-1-\beta-\delta}(u-v)^{\beta-1}\phi(u)\,du.$$

There is no difficulty in the inversion of the order of integration, since $C_{1+\delta}(t)$ is bounded, and $O(t^{1+\delta})$ for small t, and the double integral is absolutely convergent.

Assume for the moment that it has been proved that $\psi(v)$ tends to zero with v. Then $J_\delta(\psi,\omega)\to 0$ when $\omega\to\infty$, this being indeed the kernel of the proof that the Fourier series of a continuous function is summable (C,δ).* Hence $J_{\beta+\delta}(\phi,\omega)\to 0$, and the Fourier series of $f(t)$ is summable $(C,\beta+\delta)$ for $t=x$. The theorem will accordingly be proved if we can shew that $\psi(v)\to 0$ with v whenever $\beta>\alpha$ and $\delta>0$.

3. We may suppose without loss of generality that $x=0$, that $s=0$, and that the Fourier series of $f(t)$ has no constant term, so that the mean value of $\phi(t)$ is zero. It is in fact easily verified that $\psi(v)\to 0$ with v when $f(t)$ is $f_0(t)=a_0-(a_0-s)\cos t$, and we may consider $f-f_0$ instead of f. Our hypothesis is that

$$\phi_\alpha(t)=A\int_0^t\phi(u)(t-u)^{\alpha-1}du=o(t^\alpha),\quad\ldots\ldots(3{\cdot}1)$$

A being, here and in the sequel, a constant (a different constant in different places). From (3·1) it follows that

$$\phi(t)=A\frac{d}{dt}\int_0^t\frac{\phi_\alpha(u)}{(t-u)^\alpha}du=\phi_1'(t),\quad\ldots\ldots\ldots(3{\cdot}2)$$

* See Young, 6, 7; Hardy, 2. Here ψ will play the part of the ϕ of the ordinary proof. The genesis of ψ is really irrelevant, and it is not important that ψ is not periodic; but we can, if we please, replace ψ by the periodic function ψ^* equal to ψ in $(0, 2\pi)$, it being easily proved that the difference between $J_\delta(\psi,\omega)$ and $J_\delta(\psi^*,\omega)$, for any ψ integrable in $(0, 2\pi)$, is $O(\omega^{-\delta})$ and therefore trivial.

say, for almost all t.* Also

$$\phi_1(t) = \int_0^t \frac{o(u^a)}{(t-u)^a}\, du = o(t) \quad \text{..............(3·3)}$$

for small t, while, being the integral of $\phi(t)$, it is $O(1)$ for large t.

We write

$$\psi(v) = v^{1+\delta} \int_v^\infty = v^{1+\delta} \int_v^{2v} + v^{1+\delta} \int_{2v}^\infty = \psi_1 + \psi_2,$$

and we dispose first of ψ_2. This is

$$-v^{1+\delta}(2v)^{-1-\beta-\delta} v^{\beta-1} \phi_1(2v) - v^{1+\delta} \int_{2v}^\infty \phi_1(u) \frac{d}{du}\{u^{-1-\beta-\delta}(u-v)^{\beta-1}\}\, du.$$

The first term is $o(1)$, by (3·3). The second is

$$v^{1+\delta} \int_{2v}^1 o(u)\, O\{u^{-2-\beta-\delta}(u-v)^{\beta-1}\}\, du$$

$$+ v^{1+\delta} \int_{2v}^1 o(u)\, O\{u^{-1-\beta-\delta}(u-v)^{\beta-2}\}\, du$$

$$+ v^{1+\delta} \int_1^\infty O(1)\, O(u^{-3-\delta})\, du$$

$$= o(1) + o(1) + O(v^{1+\delta}) = o(1).$$

Hence $$\psi_2 \to 0.$$

4. It remains to prove that $\psi_1 \to 0$. We observe first that if $0 < \xi \leqq \eta \leqq 2\xi$, then

$$\int_\xi^\eta \phi(u)\, du = o\{\xi^a (\eta - \xi)^{1-a}\}.$$

We have in fact

$$\phi_1(\eta) - \phi_1(\xi) = A \int_\xi^\eta \frac{\phi_a(t)}{(\eta-t)^a}\, dt - A \int_0^\xi \phi_a(t) \left\{\frac{1}{(\xi-t)^a} - \frac{1}{(\eta-t)^a}\right\} dt.$$

The first term here is

$$o\left\{\xi^a \int_\xi^\eta \frac{dt}{(\eta-t)^a}\right\} = o\{\xi^a(\eta-\xi)^{1-a}\};$$

and the second is

$$o\left[\xi^a \int_0^\xi \left\{\frac{1}{(\xi-t)^a} - \frac{1}{(\eta-t)^a}\right\} dt\right] = o[\xi^a\{(\eta-\xi)^{1-a} - \eta^{1-a} + \xi^{1-a}\}],$$

which is of the same form; and this establishes our assertion.

* This is simply the solution of 'Abel's integral equation'.

We have now

$$\int_v^{2v} u^{-1-\beta-\delta}(u-v)^{\beta-1}\phi(u)\,du = \left[u^{-1-\beta-\delta}(u-v)^{\beta-1}\left(\int_v^u \phi(t)\,dt\right)\right]_v^{2v}$$

$$-\int_v^{2v}\left[\frac{d}{du}\{u^{-1-\beta-\delta}(u-v)^{\beta-1}\}\int_v^u \phi(t)\,dt\right]du$$

$$= o\,(v^{a}\,.\,v^{1-a}\,.\,v^{-1-\beta-\delta}\,.\,v^{\beta-1})$$

$$+\int_v^{2v} O\,\{u^{-2-\beta-\delta}(u-v)^{\beta-1}+u^{-1-\beta-\delta}(u-v)^{\beta-2}\}\,o\,\{v^{a}(u-v)^{1-a}\}\,du$$

$$=\int_v^{2v} o\,\{v^{-2-\beta-\delta+a}(u-v)^{\beta-a}+v^{-1-\beta-\delta+a}(u-v)^{\beta-a-1}\}\,du$$

$$= o\,(v^{-2-\beta-\delta+a}\,.\,v^{\beta-a+1}+v^{-1-\beta-\delta+a}\,.\,v^{\beta-a}) = o\,(v^{-1-\delta}),$$

or $\psi_2 = o(1)$; which completes the proof.

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