

Order-Invariant Measures on Fixed Causal Sets

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A causal set is a countably infinite poset in which every element is above finitely many others; causal sets are exactly the posets that have a linear extension with the order-type of the natural numbers; we call such a linear extension a *natural extension*. We study probability measures on the set of natural extensions of a causal set, especially those measures having the property of *order-invariance*: if we condition on the set of the bottom k elements of the natural extension, each feasible ordering among these k elements is equally likely. We give sufficient conditions for the existence and uniqueness of an order-invariant measure on the set of natural extensions of a causal set.

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1. Introduction

For a finite partially ordered set (poset) $P = (X, <)$, a *linear extension* of P is a linear order on X extending the partial order $<$. The notion of a uniform random linear extension of P arises in a number of contexts (see for instance [6, 21]), enabling meaning to be given to the probability that x is below y , when x and y are incomparable.

We pick out one property possessed by the uniform measure in the finite case. A *down-set* in a poset $P = (X, <)$ is a subset D of X such that, if $x \in D$ and $y < x$, then $y \in D$. For A a down-set in P of size k , if we consider any linear extension of P in which the bottom k elements are the elements of A , then the order on these elements is a linear extension of the poset P_A induced by P on A . It is easy to see that, under the uniform probability measure, if we condition on the event that the bottom k elements are those in A , then each linear extension of P_A is equally likely.

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Our aim in this paper is to initiate study of the case where P is countably infinite, imposing the property above – which we shall call *order-invariance* – as an axiom. This condition, enabling a passage from the finite to the infinite, is hopefully reminiscent of the notion of a *Gibbs measure* from statistical physics.

As we shall see, depending on P , there may be one, many, or no order-invariant probability measures on the set of linear extensions of P (or ‘on P ’, for short). Our results include sufficient conditions for the existence of an order-invariant measure on P , and sufficient conditions for uniqueness. We also give a number of examples, including one class of posets – the downward-branching trees T – for which we give a surprisingly subtle answer to the question of when there is an order-invariant measure on T .

Our need to be able to discuss the ‘bottom k elements’ in a linear extension of P leads us to restrict the class of countable posets we deal with, and also the class of their linear extensions.

A *causal set* is a countably infinite partially ordered set $P = (Z, <)$ such that every element is above only finitely many others. A causal set is exactly a poset that has a linear extension with the order-type of \mathbb{N} , i.e., a bijection $\lambda : \mathbb{N} \rightarrow Z$ such that we never have $i < j$ and $\lambda(i) > \lambda(j)$. We call such a linear extension of a countable poset a *natural extension*.

A probability measure on the set of natural extensions of a causal set P is *order-invariant* if, for each $k \in \mathbb{N}$ and each k -element down-set A of P , conditioned on the event $\{\lambda(1), \dots, \lambda(k)\} = A$, each linear extension of P_A is equally likely to be the restriction of λ to $[k]$.

In this paper, all our measures will be probability measures, although we often omit explicit mention of this; for instance, we will write ‘order-invariant measure’ instead of ‘order-invariant probability measure’.

We give a simple example, to illustrate the definitions and to show that there are posets P with more than one order-invariant measure on P .

Example 1. Let P be the causal set made up of the disjoint union of two infinite chains $B : b_1 < b_2 < \dots$ and $C : c_1 < c_2 < \dots$. Not every linear extension of P is a natural extension: for instance $b_1 < b_2 < \dots < c_1 < c_2 < \dots$ is a linear extension that does not have the order-type of \mathbb{N} .

We shall consider natural extensions of P as constructed ‘from the bottom up’. At each stage, after we have selected the lowest k elements x_1, x_2, \dots, x_k of the linear extension, the next element x_{k+1} must be a minimal element among those not yet selected, and there will always be exactly two candidates, one in B and one in C . To prescribe how to generate a ‘random linear extension’ of P , we need to give a probabilistic rule stating how to choose between these two elements.

Given a parameter $q \in [0, 1]$, one such rule is ‘always choose the minimal remaining element of B with probability q , and the minimal remaining element of C with probability $1 - q$ ’. This rule gives us a probability measure μ_q on the set of natural extensions of P (equipped with a σ -field that we shall specify later).

To see that μ_q is order-invariant, consider any k -element down-set A of P , that is, $A = \{b_1, \dots, b_\ell, c_1, \dots, c_{k-\ell}\}$ for some ℓ . For any of the $\binom{k}{\ell}$ linear extensions $a_1 < \dots < a_k$ of P_A , the *a priori* probability that the random linear extension ‘starts’ $a_1 < \dots < a_k$ is equal to $q^\ell (1 - q)^{k-\ell}$. Thus, conditioned on the bottom k elements being the elements of A , each of the $\binom{k}{\ell}$ linear extensions of P_A is equally likely to be the order among the elements of A .

Thus we have an uncountable family of order-invariant measures on P .

An order-invariant measure on P is said to be *extremal* if it cannot be expressed as a convex combination of two other order-invariant measures on P . We shall return to this example later and show that the μ_q are the only extremal order-invariant measures on P . All other order-invariant measures can be constructed according to a two-stage rule: first choose q according to some probability distribution on $[0, 1]$, then choose the linear extension according to μ_q .

This work is part of a wider project, initiated in our companion paper [8]. In that paper, we consider probability measures where the causal set P is also random. More precisely, we consider processes that generate a causal set one element at a time, at each stage adding a maximal element, with a label drawn from a given set (which we take to be the interval $[0, 1]$), and putting the new element above some down-set in the current poset. Such processes are called *causal set processes*: formally they are Markov processes, whose states are pairs $(x_1 \cdots x_k, <^{[k]})$, where $x_1 \cdots x_k$ is a string of elements from $[0, 1]$, and $<^{[k]}$ is a partial order on the index set $[k]$ that is a suborder of the natural order on $[k]$. Each state corresponds to a partial order P_k on the set $X_k = \{x_1, \dots, x_k\}$ (given by $x_i < x_j$ if and only if $i <^{[k]} j$), together with a linear extension of P_k .

Let us indicate, fairly precisely, how a probability measure on a fixed causal set $P = (Z, <)$, with $Z \subset [0, 1]$, fits into this framework. Consider a causal set process where the only allowed transitions are to states $(x_1 \cdots x_k, <^{[k]})$, where $X_k = \{x_1, \dots, x_k\}$ is a finite down-set in P , and $i <^{[k]} j$ if and only if $x_i < x_j$. In other words, the derived poset P_k is the restriction of P to X_k . Effectively, a transition always adds a minimal element x_{k+1} of $P \setminus X_k$ to the end of the string $x_1 \cdots x_k$, and augments the poset $<^{[k]}$ according to which elements of X_k are below x_{k+1} in P . In such a process, the order $<^{[k]}$ can be derived from the string $x_1 \cdots x_k$ and the causal set P , and so it can be omitted from the notation. A sample path of the process gives rise to an infinite string $x_1 x_2 \cdots$ of elements of Z : if it happens that $X = \{x_1, x_2, \dots\} = Z$, then this will be a natural extension of P .

A consequence of the main result of [8] is that, to classify the extremal order-invariant measures in this broader setting, it is enough to classify the extremal order-invariant measures on fixed causal sets. However, that is likely to be a prohibitively difficult task: giving conditions for existence and/or uniqueness of order-invariant measures on a fixed P is a more realistic goal.

Besides the inherent interest, another motivation for studying order-invariant measures comes from physics, in the context of a proposal for a random causal set as a mathematical model of space–time. Rideout and Sorkin [15] gave various desirable conditions for such a model, including order-invariance. Although the proposed list of conditions turns out to be too narrow to include causal sets resembling the observed space–time universe (see [7]), we are led to ask whether order-invariance itself is an obstacle: we return to this in the open problems in Section 11 at the end of the paper.

We mention some other connections with earlier work.

Some years ago, the first author [4, 5] studied random linear extensions of locally finite posets. The main theorem of [4], interpreted in the present context, is as follows. If a causal set P has the property that, for some fixed k , every element is incomparable with at most k others, then there is a unique order-invariant measure on P . The interpretation is spelled out in Theorem 8.1 of the present paper.

The specific case where the causal set is the two-dimensional grid $G = (\mathbb{N} \times \mathbb{N}, <)$ has attracted attention from another direction, as it is connected with the representation theory of the

infinite symmetric group, and with harmonic functions on the Young lattice (which is the lattice of down-sets of G). A good account of this theory appears in Kerov [14], where a somewhat more general theory is also developed. Our concerns in this paper are rather different, but the two theories have various points of contact.

The case of order-invariant measures on a fixed causal set P can also be viewed as a (one-dimensional) spin system. There are (at least) two ways to do this: either we can treat the elements as particles, with the spin of an element z encoding its rank $\lambda^{-1}(z)$ in a natural extension λ , or we can treat the pairs of incomparable elements as particles, with the spin of a pair determining which is higher in the natural extension. Thus some of the general results discussed in, for instance, Bovier [3] or Georgii [10] apply. (Indeed, some of the results in [10] hold also for general order-invariant measures, as is explained in [8].)

The structure of the paper is as follows. Basic definitions and notation connected with causal sets and natural extensions are given in Section 2. In Section 3, we give a full specification of the probability spaces we work in, and of the notion of order-invariance. Section 4 is devoted to a simple example worked out in some detail. In Section 5, we state a consequence of a result from [8], giving different characterizations of extremal order-invariant measures.

Our formal definition of order-invariance includes processes that are not natural extensions of P , but instead are natural extensions of the restriction P_Y of P to some infinite down-set in P . An order-invariant measure that does a.s. give a natural extension of P is called *faithful*, and we investigate this concept in Section 6.

As we have mentioned, we are particularly interested in the following two questions. For which causal sets P is there an order-invariant measure on P ? For which causal sets P is there a unique order-invariant measure on P ? In Section 7, we show that any causal set P with no infinite antichain admits an order-invariant measure. In Section 8, we show that, for any causal set P where there is a uniform bound k on the number of elements incomparable with an element x , there is just one order-invariant measure on P . As mentioned above, this is a simple application of the main result of [4].

These conditions for existence and uniqueness are far from necessary, and in particular it seems that any description of which causal sets admit an order-invariant measure must be significantly more complicated. In Section 9, we show that a downward-branching tree T admits an order-invariant measure if and only if a certain series of numbers derived from T is convergent.

In Section 10, we briefly discuss the case of the two-dimensional grid poset studied by Kerov [14] and others.

One question that we have not answered is the one that originally motivated this research: is there an order-invariant process that gives rise to causal sets resembling discrete approximations to the space–time structure of the universe? This and other open problems are discussed in Section 11.

2. Causal sets and natural extensions

A (*labelled*) poset P is a pair $(Z, <)$, where Z is a set (for us, Z will always be countable), and $<$ is a partial order on Z . A *total order* or *linear order* on Z is a poset such that each pair of elements of Z is comparable.

A *down-set* in P is a subset $Y \subseteq Z$ such that, if $a \in Y$ and $b < a$, then $b \in Y$. A *stem* is a finite down-set (this term is less standard: it has been used in some physics papers). An *up-set* is the complement of a down-set.

If $P = (Z, <)$ is a poset, and $Y \subseteq Z$, then $<_Y$ denotes the restriction of the partial order to Y , and $P_Y = (Y, <_Y)$. For $W \subset Z$, we also write $P \setminus W$ to mean $P_{Z \setminus W}$.

A pair (x, y) of elements of Z is a *covering pair* if $x < y$, and there is no $z \in Z$ with $x < z < y$.

For a poset $P = (Z, <)$ and an element $x \in Z$, set $D(x) = \{y \in Z : y < x\}$, $U(x) = \{y \in Z : y > x\}$ and let $I(x)$ be the set of elements incomparable with x . We also define $D[x] = D(x) \cup \{x\}$ and $U[x] = U(x) \cup \{x\}$.

Let $P = (Z, <)$ be a poset on a countably infinite set Z . We say that P is a *causal set* (or *causet*) if $D(z)$ is finite for each $z \in Z$.

A *linear extension* of a poset $P = (Z, <)$ is a total order $<$ on Z such that, whenever $x < y$, we also have $x < y$.

The sets \mathbb{N} and $[k] = \{1, \dots, k\}$, for $k \in \mathbb{N}$, come equipped with a 'standard' linear order. In these cases, a *suborder* of \mathbb{N} or $[k]$ will be a partial order on that ground-set (typically denoted $<^{\mathbb{N}}$ or $<^{[k]}$) with the standard order as a linear extension, *i.e.*, if $<^{\mathbb{N}}$ is a suborder of \mathbb{N} and $i <^{\mathbb{N}} j$, then i is below j in the standard order on \mathbb{N} .

A *natural extension* of a causal set $P = (Z, <)$ is a bijection λ from \mathbb{N} to Z such that λ^{-1} is order-preserving: *i.e.*, if $\lambda(i) < \lambda(j)$, then $i < j$. We shall often write natural extensions as $x_1 x_2 \dots$, meaning that $\lambda(i) = x_i$. In this notation, an *initial segment* of λ is an initial substring $x_1 x_2 \dots x_k$, for some $k \in \mathbb{N}$.

A natural extension λ of $P = (Z, <)$ gives rise to a linear extension $<$ by setting $x < y$ whenever $\lambda^{-1}(x) < \lambda^{-1}(y)$. The linear extensions arising in this way are those with the order-type of \mathbb{N} .

Similarly, if $P = (Z, <)$ is a finite poset, with $|Z| = k$, we can think of a linear extension as a bijection $\lambda : [k] \rightarrow Z$ such that λ^{-1} is order-preserving, *i.e.*, if $\lambda(i) < \lambda(j)$, then $i < j$ in $[k]$. We shall sometimes write a linear extension of P as a string $x_1 \dots x_k$, meaning that $\lambda(i) = x_i$ for $i = 1, \dots, k$: in this sense, we can again talk of an initial segment of a linear extension. For finite partial orders, we shall use these various equivalent notions of linear extension interchangeably. For a finite poset P , let $e(P)$ denote the number of linear extensions of P .

An *ordered stem* of a causal set, or a finite poset, $P = (Z, <)$, is a finite string $x_1 \dots x_k$ such that $X = \{x_1, \dots, x_k\}$ is a down-set in P , and $x_1 \dots x_k$ is a linear extension of P_X . Ordered stems of a causal set (finite poset) P are exactly the strings that can arise as an initial segment of a natural (linear) extension of P .

For a causal set or finite poset P , and an ordered stem $x_1 \dots x_k$ of P , let $E^P(x_1 \dots x_k)$ denote the set of natural/linear extensions of P with initial segment $x_1 \dots x_k$. When there is only one poset P under consideration, we shall use the simpler notation $E(x_1 \dots x_k)$ instead.

For a causal set P , let $L(P)$ denote the set of natural extensions of P . Also, let $L'(P)$ denote the set of injections λ from \mathbb{N} to P such that, for each i , $D(\lambda(i)) \subseteq \{\lambda(1), \dots, \lambda(i-1)\}$. In general, elements of $L'(P)$ need not be bijections: those elements of $L'(P)$ that are bijections are exactly the natural extensions.

The following statements are all very straightforward to verify. A countable poset has a natural extension if and only if every element is above finitely many elements, *i.e.*, if and only if it is a

causal set. If a causal set P has no element x with $I(x)$ infinite, then all linear extensions of P are natural extensions, and $L(P) = L'(P)$. However, if there is an element x of P with $I(x)$ infinite, then there is (a) a linear extension of P that does not have the order-type of \mathbb{N} and (b) an element of $L'(P)$ whose image is the proper subset $I(x) \cup D(x)$ of P .

3. Order-invariant processes on fixed causal sets

Consider a fixed causal set $P = (Z, <)$, with Z a countable subset of $[0, 1]$. (The actual nature of the set Z is not crucial; we demand that the labels of our posets are taken from $[0, 1]$ only in order to incorporate the structures studied in this paper within the general framework of [8].)

For k a non-negative integer, let $\mathcal{E}_P^{[k]}$ denote the set of ordered stems of P with k elements. Let \mathcal{E}_P be the union of the $\mathcal{E}_P^{[k]}$, i.e., the set of all ordered stems of P .

A *causet process on P* is a discrete-time Markov chain with state space \mathcal{E}_P , such that the only allowed transitions from a state $x_1 \cdots x_k \in \mathcal{E}_P^{[k]}$ are those to a state $x_1 \cdots x_k x_{k+1} \in \mathcal{E}_P^{[k+1]}$, where x_{k+1} is a minimal element of $P \setminus X_k$, where $X_k = \{x_1, \dots, x_k\}$.

Sample paths of a causet process on P , starting from the empty ordered stem, correspond to natural extensions $x_1 x_2 \cdots$ of some restriction P_X to an infinite down-set $X = \{x_1, x_2, \dots\}$ of P . Indeed, given a natural extension $x_1 x_2 \cdots$, its finite initial segments form a possible sample path of a causet process on P . It is thus natural to work with a sample space whose elements are these natural extensions.

Accordingly, for a causal set P , we define Ω^P to be the set of infinite strings $\omega = x_1 x_2 \cdots$ that are natural extensions of P_X for some down-set $X = \{x_1, x_2, \dots\}$ in P . Equivalently, Ω^P is the set of strings $\omega = x_1 x_2 \cdots$ such that, for each $k \in \mathbb{N}$, x_k is a minimal element of $P \setminus X_{k-1}$, where $X_{k-1} = \{x_1, \dots, x_{k-1}\}$.

For $a_1 a_2 \cdots a_k$ an ordered stem of P , we define $E(a_1 \cdots a_k) = E^P(a_1 \cdots a_k)$ to be the set of elements of Ω^P with $a_1 \cdots a_k$ as an initial segment. In other words,

$$E(a_1 \cdots a_k) = \{\omega = x_1 x_2 \cdots \in \Omega^P : x_1 = a_1, \dots, x_k = a_k\}.$$

A set of this form is called a *basic event* (for P).

For fixed k , let \mathcal{F}_k^P be the σ -field generated by the events $E(a_1 \cdots a_k)$, for $a_1 \cdots a_k$ an ordered stem of length k . Also, let \mathcal{F}^P be the σ -field generated by the union of the \mathcal{F}_k^P .

A *causet measure* on P is a probability measure on $(\Omega^P, \mathcal{F}^P)$.

A *separating class* in $(\Omega^P, \mathcal{F}^P)$ is a subset \mathcal{H} of \mathcal{F}^P such that, if two probability measures agree on \mathcal{H} , then they are equal. For any causal set P , the collection of basic events $E(a_1 \cdots a_k)$, for $a_1 \cdots a_k$ an ordered stem of P , forms a separating class.

The sequence (\mathcal{F}_k^P) is the natural filtration for a causet process on P . The measure μ of a causet process on P is determined by the finite-dimensional distributions of the Markov process, i.e., by its values on the sets $E(a_1 \cdots a_k)$.

We can equip Ω^P with a metric in several natural ways, many of which lead to equivalent topologies. For instance we can define the metric by

$$d(x_1 x_2 \cdots, y_1 y_2 \cdots) = \sum_{i=1}^{\infty} 2^{-i} \mathbb{1}(x_i \neq y_i).$$

Theorem 3.1. *Let $P = (Z, <)$ be a causal set. The space Ω^P , with the metric above, is compact if and only if, for all stems A of P , $P \setminus A$ has finitely many minimal elements.*

If P has no infinite antichain, then the condition above is satisfied, since the set of minimal elements of $P \setminus A$, for any stem A , is an antichain. However, the condition in the theorem is weaker: consider a chain $a_1 < a_2 < \dots$, with incomparable infinite chains placed above each a_i . This poset has an infinite antichain, but deleting any stem leaves a causal set with finitely many minimal elements.

Proof. Suppose first that, for each stem A of P , $P \setminus A$ has finitely many minimal elements. Consider any sequence (ω^m) of elements of Ω^P . We show that there is a convergent subsequence (ω^{m_j}) of (ω^m) . The argument is very standard.

We construct an element $\omega^0 = a_1 a_2 \dots$ of Ω^P with the property that, for each $j \in \mathbb{N}$, the ordered stem $a_1 \dots a_j$ is an initial segment of infinitely many of the ω^m . Once we have done this, the result follows: for each j in turn, we choose $m_j > m_{j-1}$ so that ω^{m_j} has $a_1 \dots a_j$ as an initial segment – now the subsequence (ω^{m_j}) converges to ω^0 .

We construct ω^0 recursively. For $j \geq 0$, suppose that $a_1 \dots a_j$ is an ordered stem in P that is an initial segment of an infinite set $\{\omega^{m_1}, \omega^{m_2}, \dots\}$ of the elements ω^m . Now the set B of minimal elements of $P \setminus \{a_1, \dots, a_j\}$ is finite. Moreover, the next entry of each of the ω^{m_i} is an element of B , so some element a_{j+1} of B occurs infinitely often as the next element in ω^{m_i} , and hence the ordered stem $a_1 \dots a_j a_{j+1}$ occurs infinitely often as an initial segment. Proceeding in this way, we may construct a suitable ω^0 .

Conversely, suppose that there is a stem A of P such that the set M of minimal elements of $P \setminus A$ is infinite. We take some enumeration b_1, b_2, \dots of M , and any linear extension $a_1 \dots a_k$ of P_A , and define $\omega^i = a_1 \dots a_k b_i b_{i+1} b_{i+2} \dots$, for $i \in \mathbb{N}$. We see that each string ω^i is in Ω^P , and that $d(\omega^i, \omega^j) = \sum_{\ell=k+1}^\infty 2^{-\ell} = 2^{-k}$ whenever $i \neq j$. Therefore the sequence (ω^i) of elements of Ω^P does not have a convergent subsequence, and so the space Ω^P is not compact. \square

We need some notation for functions on Ω^P , i.e., random elements on our probability space. If $\omega = x_1 x_2 \dots$, then we set $\zeta_j(\omega) = x_j$, $\Xi_j(\omega) = \{x_1, \dots, x_j\}$, and $\Xi(\omega) = \{x_1, x_2, \dots\}$.

We say that a causet measure μ on P is *order-invariant* if, whenever $A = \{a_1, \dots, a_k\}$ is a stem of P , and s is a permutation of $[k]$ such that both $a_1 a_2 \dots a_k$ and $a_{s(1)} a_{s(2)} \dots a_{s(k)}$ are linear extensions of P_A , then

$$\mu(E(a_1 \dots a_k)) = \mu(E(a_{s(1)} \dots a_{s(k)})). \tag{3.1}$$

We say that a causet process on a causal set P is order-invariant if the corresponding causet measure is order-invariant.

We can rephrase the condition of order-invariance in several different ways.

For $A = \{a_1, \dots, a_k\}$ a stem of P , let v^A denote the uniform measure on linear extensions of the finite poset P_A . Then the causet measure μ on P is order-invariant if and only if, for every stem $A = \{a_1, \dots, a_k\}$ of P , and every linear extension $a_1 \dots a_k$ of P_A ,

$$\mu(E^P(a_1 \dots a_k) \mid \Xi_k = A) = v^A(\{a_1 \dots a_k\}) = \frac{1}{e(P_A)}. \tag{3.2}$$

More generally, if μ is an order-invariant measure on P , A is a stem of P of size k , ℓ is a natural number with $\ell \leq k$, and $a_1 \cdots a_\ell$ is an ordered stem whose elements are all in A , then

$$\mu(E^P(a_1 \cdots a_\ell) \mid \Xi_k = A) = v^A(E^{PA}(a_1 \cdots a_\ell)). \tag{3.3}$$

This identity is obtained by summing (3.2) over the elements of $E^{PA}(a_1 \cdots a_\ell)$.

There is a strong similarity between order-invariance and the *Gibbs measure* condition from statistical physics: if we take any finite patch of a space, and condition on the configuration outside that patch (here, that means conditioning on the event that the set Ξ_k of the first k elements – *i.e.*, those not accounted for outside the patch – is equal to a given set A), then all legal extensions of the configuration into the patch (here, all linear extensions of the order restricted to A) are equally likely (or, more generally, have some specified relative probabilities). See Georgii [10] or Bovier [3] for a very general treatment of Gibbs measures.

To check order-invariance, it is enough to verify condition (3.1) above when s is an adjacent transposition, and the two transposed elements are incomparable. This is an easy consequence of the fact that it is possible to step between any two linear extensions of a finite poset by exchanges of adjacent incomparable elements.

A causet process on P is *order-Markov* if the transition probabilities out of a state $x_1 \cdots x_k \in \mathcal{E}_P$ depend only on the set $X_k = \{x_1 \cdots x_k\}$, and not on the order of the elements. A causet measure μ on P is order-Markov if its associated process is order-Markov: this means that

$$\frac{\mu(E(a_1 \cdots a_k b))}{\mu(E(a_1 \cdots a_k))} = \frac{\mu(E(a_{s(1)} \cdots a_{s(k)} b))}{\mu(E(a_{s(1)} \cdots a_{s(k)})}$$

whenever $a_1 \cdots a_k$ and $a_{s(1)} \cdots a_{s(k)}$ are ordered stems of P , s is a permutation of $[k]$,

$$\mu(E(a_1 \cdots a_k)) > 0,$$

and b is a minimal element of $P \setminus \{a_1, \dots, a_k\}$.

If μ is an order-invariant measure on P , then it is also order-Markov, as the numerators and denominators above are equal. The converse is far from true: as an extreme example, consider a causet measure $\mu_{x_1 x_2 \dots}$ on a causal set P where the probability of one specified natural extension $x_1 x_2 \cdots$ of P is 1: this measure $\mu_{x_1 x_2 \dots}$ is trivially order-Markov, but not order-invariant unless $x_1 x_2 \cdots$ forms a chain.

However, if we know that a causet measure μ arises from an order-Markov process, then in order to check order-invariance, it is enough to verify that (3.1) holds when s is the permutation exchanging the *last* two incomparable elements: if this holds, then the order-Markov condition implies that (3.1) holds whenever s is an exchange of any pair of incomparable elements, and we have already remarked that this suffices for order-invariance. We shall make use of this later.

We next give an easy but useful lemma, telling us what conditions need to be checked to ensure that a given specification of values $\mu(E(a_1 \cdots a_k))$ defines a measure on $(\Omega^P, \mathcal{F}^P)$, for a given causal set P .

Lemma 3.2. *Let $P = (Z, <)$ be a causal set, and let f be a function from the set of ordered stems of P to $[0, 1]$. Setting $\mu(E(a_1 \cdots a_k)) = f(E(a_1 \cdots a_k))$ defines a measure on $(\Omega^P, \mathcal{F}^P)$ if and only if the following hold:*

- (i) $f(\phi) = 1$, where ϕ denotes the empty string,

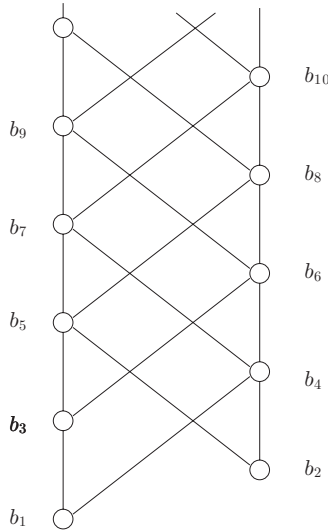


Figure 1. The causal set $P = (Z, <)$.

(ii) for each ordered stem $a_1 \cdots a_k$, we have

$$\sum_b f(a_1 \cdots a_k b) = f(a_1 \cdots a_k),$$

where the sum runs over all minimal elements b of $P \setminus \{a_1, \dots, a_k\}$.

The conditions of the lemma amount to Kolmogorov’s consistency conditions; see Chapter 8 in [13]. The proof is routine and omitted.

Thus, to check that μ is an order-invariant measure on a given causal set P , we need to check (i) (which is usually trivial) and (ii), and also the order-invariance condition.

4. An example

In this section, we study one specific example in detail, both to illustrate the definitions and themes of the paper and to provide an explicit (non-trivial) example of a causal set P such that there is exactly one order-invariant measure on P .

Example 2. Figure 1 shows the Hasse diagram of a labelled causal set $P = (Z, <)$, where $Z = \{b_1, b_2, \dots\}$, and $b_j > b_i$ if $j > i + 1$.

We will show, in some detail, that there is exactly one order-invariant measure on P . Some of the methods we use to study this example will be seen in more generality later.

For $n \in \mathbb{N}$, set $Z_n = \{b_1, \dots, b_n\}$, and $P_n = P_{Z_n}$, the restriction of P to Z_n . The linear extensions of P_n either have b_n as the top element, or have b_{n-1} top and b_n next top. The former set of linear extensions is in 1–1 correspondence with the set of linear extensions of P_{n-1} , and the latter set is in 1–1 correspondence with the set of linear extensions of P_{n-2} . Therefore the number $e(P_n)$ of linear extensions of P_n satisfies $e(P_n) = e(P_{n-1}) + e(P_{n-2})$, and so $e(P_n)$ is the n th Fibonacci

number F_n (with the convention that $F_0 = F_1 = 1$). Similarly, we see that the number of linear extensions of P_n with b_1 as the bottom element is equal to $e(P_{n-1}) = F_{n-1}$.

Let v^n denote the uniform measure on linear extensions of the finite poset P_n . The proportion $v^n(E^{P_n}(b_1))$ of linear extensions of P_n in which b_1 is the bottom element is equal to F_{n-1}/F_n , which tends to $\phi = \frac{1}{2}(\sqrt{5} - 1) = 0.618 \dots$ as $n \rightarrow \infty$. Similarly, for each fixed k ,

$$v^n(E^{P_n}(b_1 b_2 \dots b_k)) = \frac{F_{n-k}}{F_n} \rightarrow \phi^k, \quad \text{as } n \rightarrow \infty.$$

For any other ordered stem $b_{s(1)} b_{s(2)} \dots b_{s(k)}$, where s is a permutation of $[k]$ (so the set of elements in the stem is Z_k), and any $n \geq k$, the linear extensions of P_n with initial segment $b_{s(1)} \dots b_{s(k)}$ are in 1-1 correspondence with those with initial segment $b_1 \dots b_k$, so that $v^n(E^{P_n}(b_{s(1)} \dots b_{s(k)}))$ also tends to ϕ^k as $n \rightarrow \infty$.

The only other k -element down-set of P is $W_k = \{b_1, \dots, b_{k-1}, b_{k+1}\}$, and the same principle applies to initial segments that are orderings of this set, $v^n(E^{P_n}(b_1 \dots b_{k-1} b_{k+1})) = F_{n-k-1}/F_n \rightarrow \phi^{k+1}$, and the same is true for any other ordered stem whose elements are those of W_k .

It is now natural to define

$$\mu(E^P(a_1 \dots a_k)) = \lim_{n \rightarrow \infty} v^n(E^{P_n}(a_1 \dots a_k)),$$

for each ordered stem $a_1 \dots a_k$ of P : we have seen that all these limits exist, and we have found their values. We claim that μ is an order-invariant measure on P .

By Lemma 3.2, we need to verify identities of two types:

- (a) $\mu(E^P(a_1 \dots a_k)) = \sum_c \mu(E^P(a_1 \dots a_k c))$, for every ordered stem $a_1 \dots a_k$, where the sum is over minimal elements c of $P \setminus \{a_1, \dots, a_k\}$, of which there are at most two;
- (b) $\mu(E^P(a_1 \dots a_k)) = \mu(E^P(a_{s(1)} \dots a_{s(k)}))$, where s is a permutation of $[k]$, and $a_1 \dots a_k$ and $a_{s(1)} \dots a_{s(k)}$ are both ordered stems.

We could verify all these identities by direct calculation. However, it is just as easy to note that these identities all hold for each of the measures v^n with $n > k$, because the v^n are uniform measures on the set of linear extensions of finite posets, and therefore the identities hold in the limit. Here, it is crucial that the sums in (a) are all finite sums.

On the other hand, we claim that the measure μ defined above is the only order-invariant measure on P . To prove this, it is enough to show that $v(E(b_1 \dots b_k)) = \phi^k = \mu(E(b_1 \dots b_k))$ for each k , for any order-invariant measure v on P . Indeed, the values of v for all other basic events can be derived from the values of the $v(E(b_1 \dots b_k))$, assuming order-invariance, giving us that $v(E(a_1 \dots a_k)) = \mu(E(a_1 \dots a_k))$ for all basic events, and it follows that $v = \mu$, since the family of basic events forms a separating class.

Let v be an order-invariant measure on P , and take any $n > k$. The set Ξ_n , a down-set in P of size n , can take only the two values $Z_n = \{b_1, \dots, b_{n-1}, b_n\}$ and $W_n = \{b_1, \dots, b_{n-1}, b_{n+1}\}$. We now have

$$v(E^P(b_1 \dots b_k)) = v(E^P(b_1 \dots b_k) \mid \Xi_n = Z_n) v(\{\omega : \Xi_n(\omega) = Z_n\}) + v(E^P(b_1 \dots b_k) \mid \Xi_n = W_n) v(\{\omega : \Xi_n(\omega) = W_n\}).$$

Therefore $v(E^P(b_1 \cdots b_k))$ lies between the two values $v(E^P(b_1 \cdots b_k) \mid \Xi_n = Z_n)$ and $v(E^P(b_1 \cdots b_k) \mid \Xi_n = W_n)$. By (3.3), these two values are

$$\begin{aligned} v^{Z_n}(E^{P_n}(b_1 \cdots b_k)) &= v^n(E^{P_n}(b_1 \cdots b_k)), \\ v^{W_n}(E^{P_{W_n}}(b_1 \cdots b_k)) &= v^{n-1}(E^{P_{n-1}}(b_1 \cdots b_k)). \end{aligned}$$

As both $v^n(E^{P_n}(b_1 \cdots b_k))$ and $v^{n-1}(E^{P_{n-1}}(b_1 \cdots b_k))$ tend to ϕ^k as $n \rightarrow \infty$, we deduce that $v(E(b_1 \cdots b_k)) = \phi^k$, as required.

In summary, there is exactly one order-invariant measure on P .

This example is considered from a slightly different perspective in [8].

5. Extremal order-invariant measures

Recall that an order-invariant measure μ on P is *extremal* if it cannot be written as a convex combination of two different order-invariant measures on P .

Two elements $\omega = x_1x_2 \dots, \omega' = y_1y_2 \dots$ of Ω^P are said to be *finite rearrangements* if, for some $n \in \mathbb{N}$, $\{x_1, \dots, x_n\} = \{y_1, \dots, y_n\}$, and, for $m > n$, $x_m = y_m$. A *tail event* in Ω^P is a subset E of Ω^P such that, if $\omega \in E$ and ω' is a finite rearrangement of ω , then $\omega' \in E$. A measure μ is said to have *trivial tail* if $\mu(E) \in \{0, 1\}$ for every tail event E .

For $\omega = x_1x_2 \cdots \in \Omega^P$, and $k \in \mathbb{N}$, we can define a measure $v^k(\cdot)(\omega)$ on Ω^P as the uniform measure on the set of elements of Ω^P of the form $x_{s(1)} \cdots x_{s(k)}x_{k+1}x_{k+2} \cdots$, where s is a permutation of $[k]$. There are $e(P_{X_k})$ elements of this form, one corresponding to each linear extension $x_{s(1)} \cdots x_{s(k)}$ of P_{X_k} . We say that an order-invariant measure μ on P is *essential* if, for every event $E \in \mathcal{F}^P$, for μ -almost every ω , $v^k(E)(\omega) \rightarrow \mu(E)$ as $k \rightarrow \infty$.

We studied the property of extremality at length in [8], in the wider context mentioned earlier. In particular, we gave a number of equivalent conditions for an order-invariant measure to be extremal. These all transfer to our present setting: if an order-invariant measure on P is extremal in the space of all order-invariant measures, then it is certainly extremal in the space of order-invariant measures on P ; conversely, if an order-invariant measure μ on P is a convex combination of two other order-invariant measures μ_1 and μ_2 , then these must both be order-invariant measures on P – meaning that, for events A such that $\mu(A) = 0$, because μ is an order-invariant measure on the fixed causal set P , we also have $\mu_1(A) = \mu_2(A) = 0$ – so if μ is extremal among order-invariant measures on P , then it is extremal among all order-invariant measures.

Putting this observation together with Theorem 7.2 and Corollary 7.4 in [8] gives us the following result.

Theorem 5.1. *Let μ be an order-invariant measure on a causal set P , and let \mathcal{H} be a separating class in $(\Omega^P, \mathcal{F}^P)$. The following are equivalent:*

- μ is extremal,
- μ has trivial tails,
- μ is essential,
- for every event $E \in \mathcal{H}$, for μ -almost every ω , $v^k(E)(\omega) \rightarrow \mu(E)$ as $k \rightarrow \infty$.

We illustrate this result by returning to the example in the Introduction.

Example 1, revisited. As before, let P be the disjoint union of two infinite chains $B : b_1 < b_2 < \dots$ and $C : c_1 < c_2 < \dots$. For $q \in [0, 1]$, let μ_q be the order-invariant measure on P defined earlier.

The cases $q = 0$ and $q = 1$ are special. If $q = 0$, then elements from B are never chosen, and $\Xi = C$ a.s.; if $q = 1$, then $\Xi = B$ a.s. If $q \in (0, 1)$, then $\Xi = B \cup C$ a.s.

We claim that each measure μ_q is an extremal order-invariant measure. The easiest way to see this is to show that μ_q satisfies the final condition in Theorem 5.1. Consider the event $E(a_1 \dots a_k)$, where $a_1 \dots a_k$ is an ordered stem of P , and

$$\{a_1, \dots, a_k\} = \{b_1, \dots, b_\ell, c_1, \dots, c_{k-\ell}\}.$$

For μ_q -almost every ω , we have $|B \cap \Xi_n(\omega)|/n \rightarrow q$ as $n \rightarrow \infty$. Now suppose that $|B \cap \Xi_n(\omega)| = m_n(\omega) = m$; we have

$$v^n(E(a_1 \dots a_k))(\omega) = \frac{\binom{n-k}{m-\ell}}{\binom{n}{m}} = \left(\frac{m}{n}\right)^\ell \left(\frac{n-m}{n}\right)^{k-\ell} \left(1 - O\left(\frac{k^2}{\min(m, n-m)}\right)\right).$$

Therefore, for any ω such that $m_n(\omega)/n$ tends to q , we have

$$\lim_{n \rightarrow \infty} v^n(E(a_1 \dots a_k))(\omega) = q^\ell (1 - q)^{k-\ell} = \mu_q(E(a_1 \dots a_k)). \tag{5.1}$$

Therefore, μ_q satisfies the final condition given in Theorem 5.1, and hence is extremal.

Given any probability measure ρ on $[0, 1]$, define a probability measure μ_ρ by first choosing a random parameter $\chi \in [0, 1]$ according to ρ , then sampling according to μ_χ . In other words, μ_ρ is a convex combination of the order-invariant measures μ_q , so is also order-invariant. Suppose that ρ is not a.s. constant, so that there is some x such that $0 < p = \rho(\chi \leq x) < 1$; we claim that μ_ρ is not extremal. There are several easy arguments to show this, based on the various conditions in Theorem 5.1.

- (a) We can argue from the definition; for instance we can consider the conditional probability measures μ^1 and μ^2 obtained by conditioning μ_ρ on the events that $\chi \leq x$ and $\chi > x$ respectively, and write $\mu_\rho = p\mu^1 + (1 - p)\mu^2$.
- (b) We can consider the tail event $\limsup_{n \rightarrow \infty} |B \cap \Xi_n|/n \leq x$, which has probability p not equal to 0 or 1.
- (c) We can note that $v^n(E(b_1))(\omega)$ a.s. converges to the value χ chosen according to ρ , whereas $\mu_\rho(E(b_1)) = \mathbb{E}_\rho(\chi)$, so μ_ρ is not essential.

The description of μ_ρ includes several apparently different processes. For instance, consider the following process: having chosen the bottom n elements, m from B and $k = n - m$ from C , choose the next element to be from B with probability $(m + 1)/(n + 2)$. It is easy to check directly that this defines an order-invariant process on P . The theory of *Pólya's Urn* (see, for instance, Exercise E10.1 in Williams [20]) tells us that the proportion of elements taken from B in the first n steps converges to some limit χ as $n \rightarrow \infty$, and that this limit χ has the uniform distribution on $(0, 1)$. Moreover, it is possible to show that this process has the same finite-dimensional distributions as the one defined by choosing χ from the uniform distribution in advance, then choosing the natural extension according to μ_χ . See Ross [17], Section 3.6.3. Other urn processes correspond to other measures on $[0, 1]$.

We will now show that every extremal order-invariant measure μ on P is of the form μ_q , for some $q \in [0, 1]$. Given such a measure μ , we set $q = \mu(E(b_1))$, the probability that the

bottom element of the natural extension is in B . Our aim is to show that $\mu(E(a_1 \cdots a_k)) = \mu_q(E(a_1 \cdots a_k))$ for every ordered stem $a_1 \cdots a_k$ of P .

For any $n \in \mathbb{N}$, and any $\omega \in \Omega^P$ with $X_n = \Xi_n(\omega) = \{b_1, \dots, b_m, c_1, \dots, c_{n-m}\}$, the probability $v^n(E(b_1))(\omega)$ that the bottom element of a random linear extension of P_{X_n} is b_1 is equal to m/n , the proportion of elements of B in X_n . As μ is extremal, and therefore essential, we have that $v^n(E(b_1))(\omega) \rightarrow q$ a.s., and so the proportion of elements of B among the first n elements also a.s. tends to q .

Now, take any basic event $E(a_1 \cdots a_k)$, where the a_i include exactly ℓ elements of B , and any ω such that m of the first n elements are in B . As in (5.1), for any ω such that the ratio m/n of elements of B tends to q , we have

$$\lim_{n \rightarrow \infty} v^n(E(a_1 \cdots a_k))(\omega) = q^\ell (1 - q)^{k-\ell}.$$

We deduce that $\mu(E(a_1 \cdots a_k)) = q^\ell (1 - q)^{k-\ell} = \mu_q(E(a_1 \cdots a_k))$, since μ is essential. As μ agrees with μ_q on all basic events, μ and μ_q are equal.

Thus the μ_q are the only extremal order-invariant measures on P .

This example also appears in Section 2 of the paper of Kerov [14], and in [8].

It is not true that every extremal order-invariant measure is an extremal order-invariant measure on some fixed P . For instance, an extremal order-invariant measure is derived from the following process: at each step, take a label uniformly at random from $[0, 1]$, and take a new element incomparable with all existing elements. The causal set thus generated is a.s. an antichain.

As discussed at the end of Section 8 of [8], every order-invariant measure can be built from an order-invariant measure on some fixed P by a process of replacing some infinite chains of P by infinite antichains, with labels generated according to some probability distribution on $[0, 1]$. Thus the problem of classifying extremal order-invariant measures is reduced to the problem of classifying extremal order-invariant measures on a fixed P .

Another result of [8] is that every order-invariant measure μ has an expression, unique up to a.s., as a *mixture* of extremal order-invariant measures: there is a probability space (W, \mathcal{G}, ρ) , whose elements are extremal order-invariant measures μ_ω , and μ is given by sampling μ_ω from this space, and then sampling from μ_ω (more formally, $\mu(\cdot) = \int_W \mu_\omega(\cdot) d\rho(\mu_\omega)$). If μ is an order-invariant measure on some fixed causal set P , then the extremal order-invariant measures μ_ω are, ρ -a.s., measures on P , and so we can specify the mixture so that the μ_ω are all measures on P .

In Example 1, for instance, this implies that every order-invariant measure on P is a mixture of the μ_q , that is, of the form μ_ρ for some probability measure ρ on $[0, 1]$.

6. Faithful and non-faithful processes

A causet process on $P = (Z, <)$, and/or its associated measure, is said to be *faithful* if $\Xi(\omega) = Z$ a.s. If a causet process is faithful, then the associated probability measure μ is a measure on the space $L(P)$ of natural extensions of P .

For instance, in Example 1 above, the measure μ_ρ is faithful if and only if $\rho(\{0, 1\}) = 0$.

If, for all elements x of a causal set P , the set $I(x)$ of elements incomparable to x is finite, then P has no proper infinite down-sets, and therefore all causet processes are faithful. Conversely, if $I(x)$ is infinite for some x , then any causet process on the restriction $P_{I(x) \cup D(x)}$ is also a causet

process on P : if the restricted process is order-invariant, then it can be seen as an unfaithful order-invariant process on P . (In Section 9, we shall see a class of examples of causal sets P that admit a unique order-invariant measure, which is faithful, even though $I(x)$ is infinite for every element x : there is no order-invariant causet process on any restriction $P_{I(x) \cup D(x)}$.)

Let μ be an order-invariant measure on $P = (Z, <)$. An element $x \in Z$ is said to be *absent* in μ if $x \notin \Xi$ almost surely. Of course, if there is an absent element in μ , then μ is unfaithful. We shall prove that any maximal element x of P is absent in all order-invariant causet processes on P – more generally, any element x with no infinite chain above it is always absent.

Here and in future, when we are dealing with uniformly random linear extensions of a *finite* poset, we shall denote the linear extension $\zeta = \zeta_1 \cdots \zeta_n$.

Let $P = (Z, <)$ be a finite poset. For $x \in Z$ and $i \in [|Z|]$, we set $r_i(x) = v^Z(\{\zeta : \zeta_i = x\})$, the probability that, in a random linear extension of P , x is in position i .

Lemma 6.1. *If x is a maximal element in the finite poset $P = (Z, <)$, then the sequence $(r_i(x))$ is non-decreasing in i .*

Proof. Set $n = |Z|$ and, for each $i = 1, \dots, n$, let L_i denote the set of linear extensions $x_1 \cdots x_n$ of P in which $x_i = x$. For $i < n$, define a map $\phi_i : L_i \rightarrow L_{i+1}$ by

$$\phi_i(x_1 \cdots x_{i-1} x x_{i+1} \cdots x_n) = x_1 \cdots x_{i+1} x \cdots x_n.$$

This map ϕ_i is well-defined because, since x is maximal, $x_1 \cdots x_{i+1} x \cdots x_n$ is a linear extension of P whenever $x_1 \cdots x_{i-1} x x_{i+1} \cdots x_n$ is. For each i , the map ϕ_i is clearly an injection, and so $|L_i| \leq |L_{i+1}|$, and therefore $r_i(x) \leq r_{i+1}(x)$. □

Proposition 6.2. *Suppose μ is an order-invariant measure on a causal set $P = (Z, <)$. If $x \in Z$ is not absent in μ , then there is an infinite chain in P with bottom element x .*

In particular, if P has no infinite chain, then there is no order-invariant measure on P .

Proof. We start by proving that, if x is maximal in P , then x is absent in μ .

Suppose then that x is a maximal element that is not absent in μ . Now, for some $j, m \in \mathbb{N}$, we have $\mu(\{\omega : \xi_j(\omega) = x\}) > 1/m$. Set $n = m + j - 1$, so that $\mu(\{\omega : \xi_j(\omega) = x\}) > 1/(n - j + 1)$.

For any stem W of P , including x , with $|W| = n$, Lemma 6.1 tells us that $r_i^W(x) = v^W(\{\zeta : \zeta_i = x\})$ is non-decreasing in i . Therefore all of the $r_i^W(x)$, for $i = j, \dots, n$, are at least $r_j^W(x)$, and so $r_j^W(x) \leq 1/(n - j + 1)$.

Let \mathcal{W}_n denote the set of all n -element stems of P . For $W \in \mathcal{W}_n$, set $a_W = \mu(\{\omega : \Xi_n(\omega) = W\})$. Thus $\sum_{W \in \mathcal{W}_n} a_W = 1$.

By order-invariance, if $x \in W$,

$$\mu(\xi_j = x \mid \Xi_n = W) = r_j^W(x) \leq \frac{1}{n - j + 1},$$

and so

$$\mu(\{\omega : \xi_j(\omega) = x\}) = \sum_{W : x \in W} a_W \mu(\xi_j = x \mid \Xi_n = W) \leq \frac{1}{n - j + 1},$$

which is a contradiction. This proves that any maximal element x is absent in μ .

To prove the full result, suppose that μ is an order-invariant measure on $P = (Z, <)$, and let W be the set of non-absent elements. Now μ is also an order-invariant measure on $(W, <_W)$, so this causal set has no maximal elements. For any element $x \in W$, we can construct an infinite chain in $(W, <_W)$ with bottom element x recursively: having found $x = x_0 < x_1 < \dots < x_k$, let x_{k+1} be any element of W above x_k .

For the final statement, if there are no infinite chains in P , and μ is an order-invariant measure on P , then every element is absent in μ , which is not possible. \square

Example 3. Let $P = (Z, <)$ be a countably infinite antichain. As P contains no infinite chains, there is no order-invariant causet process on P , by Proposition 6.2.

In the more general context of [8], there is an order-invariant process giving rise to an antichain a.s., as discussed in that paper. However, such a process is not an order-invariant process on a particular labelled antichain: the labels on the elements of the generated antichain are random.

Example 4. Let P consist of one infinite chain $b_1 < b_2 < \dots$ together with a single incomparable element x . For any order-invariant measure μ on P , the maximal element x is absent in μ . Thus there is no faithful order-invariant process on P , and the only order-invariant process is the one whose measure is given by $\mu(b_1 b_2 \dots) = 1$; *i.e.*, at each stage i , the process a.s. selects the next element b_i of the infinite chain.

The causal sets in Examples 1, 3 and 4 are all upward-branching forests, *i.e.*, causal sets in which every element has at most one lower cover. Equivalently, P is an upward-branching forest if, for each element x of P , the set $D[x]$ is a finite chain. We can extend the arguments used in the analyses of these examples as follows.

Proposition 6.3. *Suppose the causal set $P = (Z, <)$ is an upward-branching forest. Then there is a faithful order-invariant process on P if and only if P has no maximal element.*

Proof. If there is a maximal element x , then Proposition 6.2 shows that x is absent, so there is no faithful order-invariant process.

If there is no maximal element, then we can define a faithful order-invariant process via a non-zero flow f through the forest, with value 1. To be precise, a *flow* in P is a function $f : Z \rightarrow \mathbb{R}^+$ satisfying $f(x) = \sum_{y>x} f(y)$ for all $x \in Z$, where the sum is over all elements y such that (x, y) is a covering pair. The *value* of the flow f is the sum over all minimal elements x of $f(x)$. (To obtain a flow g through the edges (covering pairs) of the forest, in the usual sense, we set $g(x, y) = f(y)$ for each covering pair (x, y) .)

A flow $f(x)$ can be constructed by working recursively up the forest, starting from the minimal elements. The set of minimal elements is non-empty and countable, so we can assign positive real numbers $f(x)$ to the minimal elements summing to 1. Once we have chosen $f(x)$, we note that there is at least one, but only countably many, upper covers of x , so we can choose positive numbers $f(y)$, for the upper covers y of x , so that $f(x) = \sum_{y>x} f(y)$.

Note that, given any stem A in P , the sum of the $f(x)$ over the minimal elements of $P \setminus A$ is 1.

Given a flow f , our rule defining an order-invariant causet process is: from any state $x_1 \dots x_k$, and for any minimal element x of $P \setminus \{x_1, \dots, x_k\}$, the probability of a transition to the state $x_1 \dots x_k x$ is equal to $f(x)$.

To see that a process defined in this way is order-invariant, observe that, if $a_1 \cdots a_k$ is an ordered stem of P , then $\mu(E(a_1 a_2 \cdots a_k)) = f(a_1) f(a_2) \cdots f(a_k)$, which depends only on the stem $\{a_1, \dots, a_k\}$, and not on the order of its elements. \square

One can show, using the same ideas as in Example 1, that faithful extremal order-invariant measures on an upward-branching forest are in 1–1 correspondence with flows through the forest.

A specific example is that where $P = (Z, <)$ is a countable union $\bigcup_{i=1}^\infty C_i$ of infinite chains. In this case, an extremal order-invariant measure is specified by a probability distribution on the index set \mathbb{N} : given non-negative numbers p_1, p_2, \dots summing to 1, an order-invariant process on P is defined by the rule that, at each step, the next element in chain C_i is chosen with probability p_i , independent of all other choices. This process is faithful if all the p_i are positive. This is an example of a faithful order-invariant measure on a causal set P containing an infinite antichain.

To conclude this section, we discuss the case where μ is an order-invariant measure on $P = (Z, <)$, and an element $b \in Z$ is in the random set Ξ with probability strictly between 0 and 1. In this situation, we can construct two new causet measures μ^+ and μ^- on P by conditioning on the events $b \in \Xi$ and $b \notin \Xi$ respectively:

$$\mu^+(E) = \mu(E \mid b \in \Xi) = \frac{\mu(E \cap \{\omega \in \Omega^P : b \in \Xi(\omega)\})}{\mu(\{\omega \in \Omega^P : b \in \Xi(\omega)\})},$$

for all $E \in \mathcal{F}^P$, and similarly for μ^- . Then $\mu(\cdot) = \mu^+(\cdot)\mu(b \in \Xi) + \mu^-(\cdot)\mu(b \notin \Xi)$, a convex combination of μ^+ and μ^- .

Proposition 6.4. *If μ is an order-invariant measure on $P = (Z, <)$, and b is an element of Z with $0 < \mu(\{\omega : b \in \Xi(\omega)\}) < 1$, then the measures μ^+ and μ^- defined above are order-invariant.*

Proof. We start by showing that μ^+ is order-invariant. Suppose that $a_1 \cdots a_k$ and $a_{s(1)} \cdots a_{s(k)}$ are two ordered stems of P , where s is a permutation of $[k]$: our task is to show that

$$\mu(E(a_1 \cdots a_k) \mid b \in \Xi) = \mu(E(a_{s(1)} \cdots a_{s(k)}) \mid b \in \Xi).$$

Since $\mu(\{\omega : b \in \Xi(\omega)\}) > 0$, this is equivalent to

$$\mu(E(a_1 \cdots a_k) \cap \{\omega : b \in \Xi(\omega)\}) = \mu(E(a_{s(1)} \cdots a_{s(k)}) \cap \{\omega : b \in \Xi(\omega)\}).$$

If b is one of the a_j , this holds directly by order-invariance. If not, then the set $E(a_1 \cdots a_k) \cap \{\omega : b \in \Xi(\omega)\}$ can be written as a countable disjoint union of events of the form $E(a_1 \cdots a_k c_1 \cdots c_t b)$. By order-invariance, each such event has the same probability as the corresponding event

$$E(a_{s(1)} \cdots a_{s(k)} c_1 \cdots c_t b).$$

Summing the probabilities now gives the required result.

We can write

$$\mu^-(E) = \frac{\mu(E) - \mu(b \in \Xi)\mu^+(E)}{\mu(b \notin \Xi)},$$

for every $E \in \mathcal{F}^P$. Using this identity, the fact that $\mu(b \notin \Xi) > 0$, and the order-invariance of μ and μ^+ , we see that μ^- is also order-invariant. \square

This result is analogous to Lemma 4.3.10 of Bovier [3].

One consequence of Proposition 6.4 is that, if μ is an order-invariant measure on $P = (Z, <)$, and b is an element of Z such that $P \setminus U[b]$ has no infinite chain, then $\mu(b \in \Xi) = 1$. Indeed, if not, then Proposition 6.4 says that μ^- is an order-invariant measure on $P \setminus U[b]$, in contradiction to Proposition 6.2.

7. Existence of order-invariant measures

We have seen examples where there are one, none, or many (faithful) order-invariant measures on a fixed labelled poset P . We now give a sufficient condition for the existence of an order-invariant measure on P .

Theorem 7.1. *Let $P = (Z, <)$ be a causal set. If $P \setminus A$ has finitely many minimal elements for each stem A of P , then there is an order-invariant measure on P . More generally, if P_Y has this property for some infinite down-set Y of P , then there is an order-invariant measure on P .*

Proof. Suppose that $P \setminus A$ has finitely many minimal elements for each stem A of P .

Let $Z_1 \subset Z_2 \subset \dots$ be an increasing sequence of stems of $P = (Z, <)$ whose union is Z . Note that, for each ordered stem $a_1 \dots a_k$, $v^{Z_n}(E(a_1 \dots a_k))$ is defined for all n large enough that all the a_j are in Z_n .

Since the set of all ordered stems of P is countable, a standard diagonalization argument shows that there is a subsequence (Z_{n_j}) of (Z_n) such that $\lim_{j \rightarrow \infty} v^{Z_{n_j}}(E(a_1 \dots a_k))$ exists for all ordered stems $a_1 \dots a_k$.

For each ordered stem $a_1 \dots a_k$, we now set

$$\mu(E(a_1 \dots a_k)) = \lim_{j \rightarrow \infty} v^{Z_{n_j}}(E(a_1 \dots a_k));$$

we claim that this defines an order-invariant measure on $(\Omega^P, \mathcal{F}^P)$.

For each ordered stem $a_1 \dots a_k$, the set $\{b_1, \dots, b_r\}$ of minimal elements of $P \setminus \{a_1, \dots, a_k\}$ is finite by assumption. Provided $|Z_{n_j}| > k$, we have

$$\sum_{i=1}^r v^{Z_{n_j}}(E(a_1 \dots a_k b_i)) = v^{Z_{n_j}}(E(a_1 \dots a_k)),$$

so this identity also holds for the limit μ . (Note that $v^{Z_n}(E(c_1 \dots c_t)) = 0$ unless all the c_i are in Z_n .) Thus, by Lemma 3.2, μ is a causet measure on P .

Checking that μ is order-invariant is also immediate: if $a_1 \dots a_k$ is an ordered stem of P , and s is a permutation of $[k]$ such that $a_{s(1)} \dots a_{s(k)}$ is also an ordered stem of P , then

$$v^{Z_{n_j}}(E(a_1 \dots a_k)) = v^{Z_{n_j}}(E(a_{s(1)} \dots a_{s(k)}))$$

for every n_j for which these are defined, so this identity holds in the limit too.

For the second statement in (1), we simply apply the first statement to P_Y . □

Corollary 7.2. *If $I(x)$ is finite for every element x of P , then there is a faithful order-invariant measure on P .*

Proof. If $I(x)$ is finite for all $x \in P$, then there is certainly no infinite antichain in P , and therefore the condition of Theorem 7.1 is satisfied, and there is an order-invariant measure on P .

Moreover, as we remarked at the beginning of Section 6, a causal set P in which $I(x)$ is finite for every x has no proper infinite down-sets – indeed, any causet process on P generates each element x no later than step $|I(x) + D[x]|$ – so all causet measures on P are faithful. \square

If we think of two elements of $P = (Z, <)$ as ‘interacting’ if they are incomparable, then the condition that $I(x)$ is finite for every $x \in Z$ is analogous to the condition that an interaction in a spin system be *regular* – see Section 4.2 of Bovier [3], which suffices for the existence of Gibbs measures in the context studied there (see Corollary 4.2.17 of [3]).

Example 4 illustrates these results: the poset P of that example has no infinite antichain, but there is one element x with $I(x)$ infinite; there is just one order-invariant measure on P , and it is not faithful.

The condition in Theorem 7.1 is certainly not necessary for the existence of an order-invariant measure on P . Indeed, we have already seen examples – see Proposition 6.3 and the remarks after it – where $P \setminus A$ has infinitely many minimal elements for every stem A , and yet there are infinitely many faithful extremal order-invariant measures on P .

However, we do have the following result.

Corollary 7.3. *Let $P = (Z, <)$ be a causal set. Then the following are equivalent.*

- (1) *For every infinite down-set Y of Z , there is an order-invariant measure on P_Y .*
- (2) *For every stem A of P , $P \setminus A$ has finitely many minimal elements.*

Proof. That (2) implies (1) follows from applying Theorem 7.1 to each P_Y , where Y is an infinite down-set of P .

If (2) fails, then there is a stem A such that the set M of minimal elements of $P \setminus A$ is infinite. Then $A \cup M$ is an infinite down-set of P with no infinite chains, so there is no order-invariant measure on $P_{A \cup M}$, by Proposition 6.2. \square

It is no accident that the condition of Theorem 7.1 for the existence of an order-invariant measure is the same as that in Theorem 3.1 for Ω^P to be compact. Indeed, we can use compactness to give an alternative proof of the first part of Theorem 7.1: we merely sketch this proof, which relies on the theory of weak compactness (see Billingsley [2]).

Since the space $(\Omega^P, \mathcal{F}^P)$ is compact, every family of measures in $\mathcal{P} = \mathcal{P}(\Omega^P, \mathcal{F}^P)$ is tight. Thus, by Prohorov’s theorem, every such family, and in particular the family $v^{Z_n}(\cdot)$ as defined in the proof, is relatively compact for weak convergence. Thus some sequence of measures $v^{Z_n}(\cdot)$ has a weak limit: we showed in [8] that a weak limit of such measures is order-invariant.

Some ‘compactness’ condition is required for either proof to work. For instance, suppose $P = (Z, <)$ is an antichain, with $Z = \{z_1, z_2, \dots\}$, and set $Z_n = \{z_1, \dots, z_n\}$ for each $n \in \mathbb{N}$. Now, for each fixed k , $v^{Z_n}(E(z_k)) = 1/n$ for $n \geq k$, so $v^{Z_n}(E(z_k)) \rightarrow 0$ as $n \rightarrow \infty$ for each $z_k \in Z$, although $\sum_{k=1}^{\infty} v^{Z_n}(E(z_k)) = 1$ for each n . A similar issue is explored in Example (4.16) in [10], where a sequence of measures tends weakly to a limit that is not a measure on the original space:

the limiting measure can be seen as a ‘point mass at infinity’ in the one-point compactification of the originally non-compact space.

8. Uniqueness of order-invariant measures

Our purpose in this section is to give a sufficient condition on a causal set P for P to admit a unique order-invariant measure.

The following result can be seen as an interpretation of a result from Brightwell [4].

Theorem 8.1. *Let $P = (Z, <)$ be a causal set, and suppose there is some k such that $|I(x)| \leq k$ for all $x \in P$. Then there is a unique order-invariant measure on P .*

Proof. For incomparable elements a and b of P , let $R(a, b)$ be the event that a appears below b in a natural extension of P . Formally, $R(a, b) = \{\omega \in \Omega^P : \exists i < j, \xi_i(\omega) = a, \xi_j(\omega) = b\}$.

Suppose P satisfies the condition of the theorem. It is proved in [4] that, for any increasing sequence (Z_1, Z_2, \dots) of stems in $P = (Z, <)$, whose union is Z , and any Boolean combination R of events of the form $R(a, b)$, the limit, as $n \rightarrow \infty$, of $v^{Z_n}(R)$ exists, and is independent of the choice of sequence (Z_n) .

Each basic event $E(a_1 \cdots a_k)$ can be written as an intersection of events $R(a, b)$. Also, for any $\omega = x_1 x_2 \cdots \in \Omega^P$, the union of the sequence (X_1, X_2, \dots) of stems is Z . Therefore, for each ordered stem $a_1 \cdots a_k$, and each $\omega \in \Omega^P$, the result of [4] tells us that $v^{X_n}(E(a_1 \cdots a_k))$ tends to a limit, which we denote $\mu(E(a_1 \cdots a_k))$, independent of the sequence (X_n) .

As in the proof of Theorem 7.1, this limit μ is an order-invariant causet measure on P .

Moreover, for every $\omega \in \Omega^P$, $v^n(E(a_1 \cdots a_k))(\omega)$ tends to $\mu(E(a_1 \cdots a_k))$. Every extremal order-invariant measure ν on P is essential, by Theorem 5.1, and so ν must agree with μ on the separating class consisting of the basic events $E(a_1 \cdots a_k)$, and therefore $\nu = \mu$.

Thus there is only one extremal order-invariant measure on P , namely μ . □

The condition that $I(x)$ be uniformly bounded in Theorem 8.1 is reminiscent of Dobrushin’s uniqueness criterion for interacting particle systems (see [3] or [10]), in that it bounds the strength of interactions.

Example 5. An example in Brightwell [4] shows that just having all the $I(x)$ finite is not sufficient to guarantee a unique order-invariant measure.

To construct this example, we start with P_1 the one-element poset on $Z_1 = \{a\}$ and P_2 the two-element antichain $Z_2 = \{a, b\}$. Each P_n , $n \geq 3$, is constructed from P_{n-1} by adding a chain of m_n elements above the elements of Z_{n-2} and incomparable with the chain $Z_{n-1} \setminus Z_{n-2}$, where m_n grows rapidly with n ($m_n = 2^{2^n}$ suffices). The infinite poset P is the union of the P_n . The point is that, as m_n is much larger than m_{n-1} , most linear extensions of the poset P_n have the elements of Z_{n-2} , in some order, as an initial segment, so $v^{Z_n}(E^{P_n}(a))$ can be made as close as is desired to $v^{Z_{n-2}}(E^{P_{n-2}}(a))$, for each $n \geq 3$. Thus $v^{Z_{2n}}(E^{P_{2n}}(a))$ and $v^{Z_{2n+1}}(E^{P_{2n+1}}(a))$ tend to different limits as $n \rightarrow \infty$. The proof of Theorem 7.1 then implies that there are at least two different order-invariant measures. These measures are necessarily faithful, as all the $I(x)$ are finite in this example.

For details, see [4].

On the other hand, the condition in Theorem 7.1 is not necessary for the uniqueness of an order-invariant measure on a causal set P . For instance, one can build a causal set by stacking finite posets on top of one another, with all elements of one poset in the stack being above all elements of all posets below it. It is easy to see that such a poset admits a unique order-invariant measure, constructed in an obvious way from the uniform measures on linear extensions of each poset in the stack. This class includes examples in which there is no uniform bound on $|I(x)|$.

9. Downward-branching trees

A *downward-branching forest* is a causal set in which every element has exactly one upper cover (equivalently, for each element x , $U[x]$ is a chain). A *downward-branching tree*, or simply *tree*, is a downward-branching forest with just one component, *i.e.*, such that every two elements have a common upper bound.

Our purpose in this section is to characterize the trees $T = (Z, <)$ that admit an order-invariant measure. Such a measure μ must be faithful: for any element $x \in Z$, there is no infinite chain in $Z \setminus U[x]$ (if the infinite chain $U[y]$ is disjoint from $U[x]$, then x and y have no common upper bound), and so, by the remark after Proposition 6.4, $\mu(x \in \Xi) = 1$.

Before giving this characterization, we state and prove two simple general lemmas that we shall need in the course of the proof, and later.

Lemma 9.1. *Let $P = (Z, <)$ be a causal set, and let $a_1 a_2 \cdots a_k$ be any ordered stem of P . If μ is an order-invariant measure on P such that, with positive probability, all the a_i appear, then $\mu(E(a_1 a_2 \cdots a_k)) > 0$.*

In particular, if a is a minimal element of P , then either $\mu(E(a)) > 0$, or a is absent.

Proof. The event that all the a_i appear is a countable union of events of the form $E(b_1 b_2 \cdots b_j)$, where all the a_i appear in the set $B = \{b_1, \dots, b_j\}$. Thus at least one such event has positive probability. Now, there is a linear extension $b_{s(1)} \cdots b_{s(j)}$ of P_B with initial segment $a_1 \cdots a_k$. We see that

$$\mu(E(a_1 a_2 \cdots a_k)) \geq \mu(E(b_{s(1)} \cdots b_{s(j)})) = \mu(E(b_1 b_2 \cdots b_j)) > 0,$$

as required. □

Lemma 9.2. *Let μ be a faithful order-invariant measure on $P = (Z, <)$ and let A be any stem of P . Take any linear extension $a_1 \dots a_m$ of P_A . For any ordered stem $b_1 \cdots b_k$ of $P \setminus A$, define*

$$\mu_A(E(b_1 \cdots b_k)) = \frac{\mu(E(a_1 \cdots a_m b_1 \cdots b_k))}{\mu(E(a_1 \cdots a_m))}.$$

Then μ_A is a faithful order-invariant measure on $P \setminus A$.

Proof. Note first that $\mu(E(a_1 \cdots a_m)) > 0$, by Lemma 9.1, so μ_A is well-defined. Also, by order-invariance, it is independent of the choice of the linear extension of P_A .

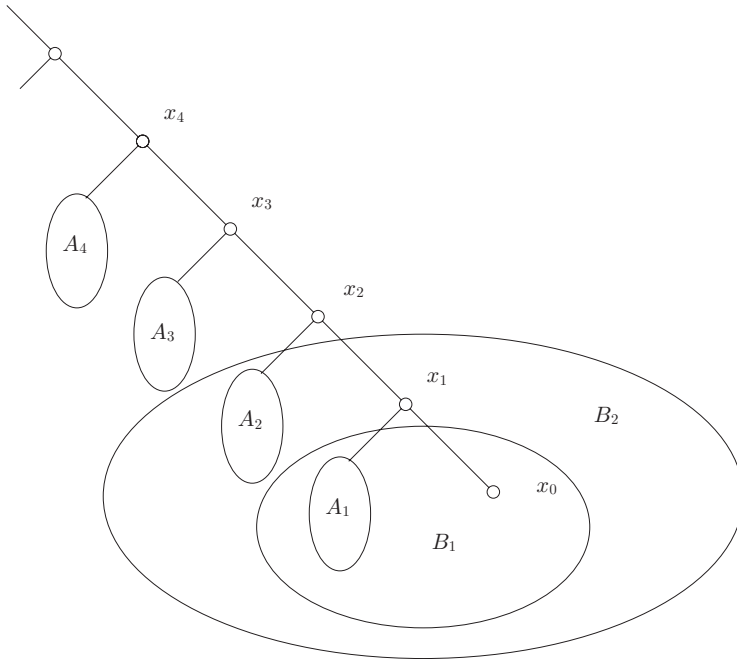


Figure 2. A downward-branching tree.

For any ordered stem $b_1 \cdots b_k$, we need to check that the sum, over all minimal elements b of $P \setminus (A \cup \{b_1, \dots, b_k\})$, of $\mu_A(E(b_1 \cdots b_k b))$ is equal to $\mu_A(E(b_1 \cdots b_k))$; this is immediate from the definition, since μ satisfies the analogous property.

Thus, by Lemma 3.2, μ_A is a causet measure on $P \setminus A$. Order-invariance and faithfulness are immediate from the definition. □

Let $T = (Z, <)$ be a downward-branching tree. Let $C : x_0 < x_1 < \cdots$ be an arbitrary maximal chain in T : the minimal element x_0 determines this chain C uniquely as the chain $U[x_0]$ of elements above x_0 .

For $i \geq 1$, set $B_i = D(x_i)$ and $A_i = D(x_i) \setminus D[x_{i-1}]$. Thus A_i is the finite forest of elements ‘hanging off’ C at x_i . The sets A_i partition $T \setminus C$. Also, for each i , $B_i = D(x_i) = A_i \cup D[x_{i-1}]$, and these two sets A_i and $D[x_{i-1}]$ have no comparabilities between them. See Figure 2.

Set $a_i = |A_i|$, $b_i = |B_i|$, and $t_i = a_i/b_i$, for each $i \geq 0$. So t_i is the proportion of elements below x_i that are in subtrees other than $D[x_{i-1}]$.

Proposition 9.3. *A tree $T = (Z, <)$ admits an order-invariant measure if and only if $\sum_{i=0}^{\infty} t_i$ converges. If the sum is convergent, there is just one order-invariant measure on T .*

This convergence condition is quite strong: a tree T such that $\sum t_i$ converges can be thought of as consisting of one chain C with elements hanging off it at widely spaced intervals. For instance, if each a_i is 1, so that there is one minimal element hanging off each element in the chain, then

$b_i = 2i$ for each i , so $t_i = 1/2i$, and $\sum t_i$ is divergent. This is therefore an example of a causal set with an infinite chain admitting no order-invariant measure.

Proof. We first note that the convergence condition is invariant under choice of the maximal chain: given any two chains, defined by their minimal elements, the elements have a least upper bound, which appears in both chains, and the sequence (t_i) is the same in both chains beyond this point.

We will now show that the convergence condition is invariant under the removal of a minimal element x . Unless T is a single chain – in which case the condition is satisfied both before and after removing the unique minimal element x – we can choose a reference chain C in which x is in one of the A_i . Removing x has the effect of reducing the one term t_i , and increasing all subsequent terms t_j by at most a factor of 2, so the convergence of $\sum t_i$ is not affected.

We deduce moreover that the convergence condition is invariant under the removal of any finite down-set of T .

Suppose that T admits an order-invariant measure μ , and consider the event $E^T(x_0)$ that x_0 is the bottom element in a random linear extension. By Lemma 9.1, $\mu(E^T(x_0)) > 0$.

Our basic intuition is that an order-invariant measure μ on T , if it exists at all, has to be the limit of the measures $\nu^{D[x_j]}$, as $j \rightarrow \infty$. (Indeed, if there is an order-invariant measure, then there is an extremal one, which is essential by Theorem 5.1, and therefore is certainly a limit of *some* sequence of measures ν^{D_k} , where (D_k) is an increasing sequence of down-sets of T .) Accordingly, our next step is to fix $j \geq 1$ and analyse the family of linear extensions of $T_{D[x_j]}$, which we call T_j for convenience. As x_j is the unique maximal element of T_j , a linear extension of T_j consists of a linear extension of $T_{D(x_j)}$ with x_j appended, so we may focus instead on the family of linear extensions of $T_{D(x_j)}$.

In $T_{D(x_j)}$, there are no comparabilities between the sets A_j and $D[x_{j-1}]$, so a linear extension of $T_{D(x_j)}$ is determined uniquely by: (i) a linear extension of T_{A_j} , (ii) a linear extension of T_{j-1} , and (iii) a set I of a_j elements of $[b_j]$. Given these three ingredients, the linear extension of $T_{D(x_j)}$ can be formed by mapping the elements of A_j to the elements of I , in the order given by the linear extension from (i), then mapping the elements of $D[x_{j-1}]$ to the elements of $[b_j] \setminus I$, in the order given by the linear extension from (ii).

The event that, in a uniformly random linear extension ζ of $T_{D(x_j)}$, the bottom element ζ_1 is in $D[x_{j-1}]$, depends only on the set I , and its probability is just the probability that $1 \notin I$, which is $(b_j - a_j)/b_j = 1 - t_j$.

Furthermore, the event that the lowest element of $D[x_{j-1}]$ is x_0 , in a uniformly random linear extension of T_j , depends only on the linear extension of T_{j-1} chosen in part (ii) of the process described above, so this event is independent of the event that the overall bottom element in the linear extension of T_j is in $D[x_{j-1}]$. Hence we have

$$\nu^{D[x_j]}(E^{T_j}(x_0)) = (1 - t_j)\nu^{D[x_{j-1}]}(E^{T_{j-1}}(x_0)),$$

and it follows by induction that

$$\nu^{D[x_j]}(E^{T_j}(x_0)) = \prod_{i=1}^j (1 - t_i).$$

Moreover, if W is any stem including x_j (and therefore all of $D[x_j]$), then $v^W(E^{T_W}(x_0)) \leq \prod_{i=1}^j (1 - t_i)$, as the product is the probability that x_0 is the lowest element of $D[x_j]$ in a uniformly random linear extension of T_W .

For $j, n \in \mathbb{N}$, let $A_{j,n} = \{\omega : x_j \in \Xi_n(\omega)\}$. We have that, for $\omega \in A_{j,n}$,

$$v^{\Xi_n(\omega)}(E^{T_{\Xi_n}}(x_0)) \leq \prod_{i=1}^j (1 - t_i).$$

For any $j \in \mathbb{N}$ and $\varepsilon > 0$, we may take n sufficiently large that $\mu(A_{j,n}) > 1 - \varepsilon$. Now, by (3.3), we have that

$$\mu(E^T(x_0)) = \sum_X \mu(E^T(x_0) \mid \Xi_n = X) \mu(\Xi_n = X) = \sum_X v^X(E^{T_X}(x_0)) \mu(\Xi_n = X),$$

where the sum is over all stems X of T of size n . Now we have

$$\mu(E^T(x_0)) \leq \sum_{X: x_j \notin X} \mu(\Xi_n = X) + \sum_{X: x_j \in X} v^X(E^{T_X}(x_0)) \mu(\Xi_n = X) \leq \varepsilon + \prod_{i=1}^j (1 - t_i).$$

As both ε and j are arbitrary, we conclude that $\mu(E^T(x_0)) \leq \prod_{i=1}^\infty (1 - t_i)$, which is positive if and only if $\sum t_i$ converges.

This proves that, if T admits an order-invariant process, then $\sum t_i$ converges.

Indeed, we can extract more information from the argument above. Suppose that $\sum t_i$ does converge. For any minimal element x , decompose the tree using the reference chain $C = U[x]$, calculate the constants $t_i = t_i(x)$ for this chain C , and set $p_T(x) = \prod_{i=1}^\infty (1 - t_i(x))$. We have seen that $\mu(E^T(x)) \leq p_T(x)$, for any order-invariant measure μ on T .

We claim that the sum of the $p_T(x)$ over all minimal x is equal to 1. This will imply that $\mu(E^T(x)) = p_T(x)$, for any order-invariant measure μ on T , and any minimal element x .

Note first that, for each fixed j , we have $\sum_{x \in M_j} \prod_{i=1}^j (1 - t_i(x)) = 1$, where the sum is over the set M_j of minimal elements of $D[x_j]$, as $\prod_{i=1}^j (1 - t_i(x))$ is the probability that x is the bottom element in a random linear extension of T_j .

Therefore $\sum_{x \in M_j} p_T(x) = \sum_{x \in M_j} \prod_{i=1}^\infty (1 - t_i(x)) \leq 1$, for each j . It follows that the sum of $p_T(x)$ over all minimal elements of T is at most 1.

To see the reverse inequality, we fix any $\varepsilon > 0$. As $\sum t_i$ converges, there is some n such that $\prod_{i=n+1}^\infty (1 - t_i) > 1 - \varepsilon$. Now, for all $x \in M_n$, $t_i(x) = t_i$ for $i \geq n$. Therefore

$$\sum_{x \in M_n} p_T(x) = \sum_{x \in M_n} \prod_{i=1}^n (1 - t_i(x)) \prod_{i=n+1}^\infty (1 - t_i) > (1 - \varepsilon) \sum_{x \in M_n} \prod_{i=1}^n (1 - t_i(x)) = 1 - \varepsilon.$$

What this shows is that, if there is an order-invariant measure μ on T , then $\mu(E^T(x))$ must be equal to $p_T(x)$ for every minimal element x of T .

Furthermore, from any state $a_1 \cdots a_k$, with $A = \{a_1, \dots, a_k\}$, all subsequent transitions must be those of an order-invariant process on $T \setminus A$, also a downward-branching tree, by Lemma 9.2. Therefore the probabilities for the next transition are necessarily obtained by selecting the next minimal element to be x with probability $p_{T \setminus A}(x)$.

This proves that, in the case where $\sum t_i$ converges, there is at most one order-invariant process on T , namely the one described above, with the rule that, if we have so far selected the elements

of the stem A , then the probability that a minimal element x of $T \setminus A$ is the next element selected is $p_{T \setminus A}(x)$.

It remains to show that this process is order-invariant.

The process is, by its definition, order-Markov. We need to check that, after the deletion of some stem A , $E^{T \setminus A}(yz)$ and $E^{T \setminus A}(zy)$ have the same probabilities, whenever y and z are minimal elements of $T \setminus A$. Without loss of generality, $A = \emptyset$ and $y = x_0$. We choose n so that $z < x_n$.

We see that

$$\mu(E^T(yz)) = p_T(y)p_{T \setminus \{y\}}(z) = \prod_{i=1}^{\infty} (1 - t_i(y)) \prod_{i=1}^{\infty} (1 - t'_i(z)),$$

where the t_i are calculated in T , and the t'_i in $T \setminus \{y\}$. Similarly

$$\mu(E^T(zy)) = \prod_{i=1}^{\infty} (1 - t_i(z)) \prod_{i=1}^{\infty} (1 - t''_i(y)),$$

where the t''_i are calculated in $T \setminus \{z\}$. In each product, all the terms beyond the n th are identical, so we need to prove that

$$\prod_{i=1}^n (1 - t_i(y)) \prod_{i=1}^n (1 - t'_i(z)) = \prod_{i=1}^n (1 - t_i(z)) \prod_{i=1}^n (1 - t''_i(y)).$$

But these products are exactly $\nu^{D[x_n]}(E^{T_n}(yz))$ and $\nu^{D[x_n]}(E^{T_n}(zy))$ respectively, so they are indeed equal. □

One explicit way of realizing the unique order-invariant measure in the case when $\sum t_i$ converges is as follows. Again, we need only describe how to generate the first element. Choose a reference chain C with minimal element x_0 , and define the A_i with respect to C as before. Mark each set A_i with probability t_i , independently of other marks. Note that no empty A_i is marked, and, by the Borel–Cantelli lemma, since $\sum t_i$ is finite, there are a.s. only a finite number of marked A_i . If there are any marked sets, let A_k be the last marked set, take a uniformly random linear extension of the finite poset A_k , and select the bottom element of this linear extension as our first element. If there are no marked sets, we choose x_0 as our first element. We omit the detailed analysis.

10. The two-dimensional grid poset

Let $G = (\mathbb{N} \times \mathbb{N}, <)$ be the infinite two-dimensional grid poset, with $(a, b) \leq (c, d)$ if $a \leq c$ and $b \leq d$. This is a causal set, with unique minimal element $(1, 1)$.

This example is studied in detail in papers of Gnedin and Kerov [11], Kerov [14] and Vershik and Tsilevich [19]. Our account will be a sketch only.

As G has no infinite antichain, Theorem 7.1 tells us that there is an order-invariant measure on G – however, this is actually trivial in this case, as the chain $H = \{(a, 1) : a \in \mathbb{N}\}$ forms an infinite down-set in G , and the process that always selects the next element of H is certainly order-invariant.

In fact, there is a faithful order-invariant measure on G . Although $I(x)$ is infinite for all elements x of G except the unique minimum, the method used in the proof of Theorem 7.1

can be used directly to construct such a measure. If we take $Z_n = [n] \times [n]$, a down-set in G , for each $n \in \mathbb{N}$, then the numbers of linear extensions of subposets of G_{Z_n} can be calculated using the hook formula of Frame, Robinson and Thrall [9], and so it is possible to write down an expression for $v^{Z_n}(E(a_1 \cdots a_k))$ for each n and any ordered stem $a_1 \cdots a_k$. It turns out that $v^{Z_n}(E(a_1 \cdots a_k))$ converges to a positive limit for each ordered stem $a_1 \cdots a_k$, and so the limit is a faithful order-invariant measure on G . This measure is the well-known *Plancherel measure* (see, for instance, [1, 18]).

However, this is far from the only faithful order-invariant measure on G . For example, for $\alpha \in (0, 1)$, we construct an order-invariant measure as follows. We decompose G as the union of the chain $H = \{(a, 1) : a \in \mathbb{N}\}$, and $G \setminus H$, which is isomorphic to G . On the poset formed as the disjoint union $H \cup (G \setminus H)$, where the relations between H and $G \setminus H$ are deleted, we construct a process which, at each step, takes the next element of the chain H with probability α , and otherwise takes an element from $G \setminus H$ according to the Plancherel measure. With positive probability, the sequence constructed is actually a natural extension of G : conditioning on this event gives an order-invariant measure on G . The order-invariant measure we obtain ‘favours the first row H ’, as elements of this row are chosen a positive proportion of the time in the process, unlike in the Plancherel measure.

It is easy to see that this can be extended, to obtain processes favouring more than one row, and/or favouring the low-numbered columns. Kerov [14] shows that the extremal order-invariant measures on G are in 1–1 correspondence with pairs of sequences $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq 0$, such that $\sum_{i=1}^{\infty} \alpha_i + \sum_{i=1}^{\infty} \beta_i \leq 1$. (The measure described above is the one corresponding to $\alpha_1 = \alpha$, with all other α_i and β_i equal to zero.)

11. Open problems

We finish by mentioning a number of open problems.

(1) Is there some reasonably simple description of the class of causal sets that admit a (faithful) order-invariant measure? We see from consideration of the class of downward-branching trees that there can be no *very* simple description. However, perhaps Theorem 9.3 may give some indication of the nature of a possible classification of causal sets admitting an order-invariant measure.

(2) Is there some reasonably simple description of the class of causal sets that admit a *unique* order-invariant measure? This seems likely to be harder than the previous problem.

In [8], we give a description of the general form of any extremal order-invariant measure on the space (Ω, \mathcal{F}) . In order to extend this to a classification of extremal order-invariant measures, it would suffice to be able to describe all extremal order-invariant measures on fixed causal sets. It is not clear what such a description might look like, but solving problems (1) and (2) would be progress towards this goal.

(3) One specific problem relates to a partial order obtained by taking a Poisson process X in \mathbb{R}_+^2 , and taking the partial order $<$ on X induced by the co-ordinate order. The poset $P = (X, <)$ is a.s. a causal set. Does such a poset (a.s.) admit an order-invariant measure?

If so, it seems that P will (a.s.) admit infinitely many order-invariant measures, because an order-invariant measure μ must have $\sum_{x \in M} \mu(E(x)) > 1 - \varepsilon$ for some finite set $M = M(\varepsilon)$ of minimal elements x , whereas the Poisson process itself has no distinguishable ‘minimum region’ of finite area. This is because the Lebesgue measure on \mathbb{R}_+^2 , and hence the Poisson process, are Lorentz invariant (*i.e.*, invariant under the measure-preserving transformations $(x, y) \rightarrow (ax, a^{-1}y)$ of \mathbb{R}_+^2).

The motivation behind this problem comes from physics. Any process generating a random causal set can be viewed as a potential discrete model for the space–time universe. Rideout and Sorkin [15, 16] proposed (essentially) order-invariance as a desirable feature of such a model. It would be good to know whether the (rich) class of order-invariant processes does include processes that produce outcomes resembling the observed space–time universe, *i.e.*, at least locally resembling a Poisson process in four-dimensional Minkowski space M^4 . If such a process exists, it will have an expression as a mixture of extremal order-invariant processes on fixed causal sets, where the causal sets ‘resemble’ those produced from a Poisson process.

It seems likely that either (i) causal sets arising from a Poisson process in M^4 (with an origin) a.s. admit an order-invariant measure, or (ii) there is some necessary structural condition for the existence of an order-invariant measure that is not satisfied by any causal set ‘faithfully embedded’ into M^4 . It would be very interesting to know which.

The two-dimensional version of this question, as proposed above, should be easier to settle.

(4) We give a more specific question, an answer to which is likely to lead to an answer to problem (3). Let $P = (X, <)$ denote the causal set defined from a Poisson process in the positive quadrant, as above. For each n , consider the restriction P_n of P to the set X_n of points in the square $[0, n]^2$.

Now let $x = (u, v)$ be the point in X with minimum sum of co-ordinates $u + v$. Consider the probability $q_n = \nu^{X_n}(E^{P_n}(x))$ that x is the bottom element of a uniform random linear extension of P_n . Does q_n a.s. tend to zero as $n \rightarrow \infty$?

If q_n does (a.s.) tend to zero, then it should be fairly easy to deduce that, in any order-invariant measure μ on P , $\mu(E^P(x)) = 0$, and thence that there is no order-invariant measure on P .

On the other hand, if q_n tends to some non-zero limit, and also $\nu^{X_n}(E^{P_n}(y))$ converges for every other minimal element y , with the sum of these limits being 1, then it seems very likely that the measures ν^{X_n} will have a limit that is an order-invariant measure on P .

The following version of the question seems likely to be equivalent, and may be slightly more appealing. If we generate P_n as above, and then take a random linear extension of P_n , does the probability that the bottom element lies in $[0, 1]^2$ tend to zero as $n \rightarrow \infty$?

(5) Can one say anything interesting about the causet properties ‘ P admits an order-invariant measure’ and ‘ P admits a unique order-invariant measure’. Could one or other be monotone (*i.e.*, preserved under adding relations)? The following example shows that the property ‘ P admits a faithful order-invariant measure’ is not monotone.

Example 6. Let $P = (Z, <)$ consist of two chains $B : b_1 < b_2 < \dots$ and $C : c_0 < c_1 < c_2 < \dots$, with also the ‘cross-relations’ $c_i > b_j$ if $j < 2^i$ – so each element c_i has $2^i - 1$ elements of

B below it. This causal set P is obtained from the one in Example 1, which does admit a faithful order-invariant measure, by adding relations. We shall show that, in any order-invariant measure μ on P , c_0 , and hence all the elements of C , are absent.

For $n \geq 1$, let X be any down-set of P of size 2^n containing c_0 . Thus $c_n \notin X$, and so X contains at most n elements of C . If $Q = (Y, <')$ is the poset with $Y = X$ consisting of the union of the two chains $C \cap X$ and $B \cap X$, without the cross-relations, then $v^Y(E^Q(c_0)) = |C \cap X|/|X| \leq n/2^n$. Now the theorem of Graham, Yao and Yao [12] implies that adding the cross-relations (which means conditioning on certain events that the c_j are higher than the b_i) cannot increase the probability that c_0 is below b_1 : thus

$$v^X(E^{P_X}(c_0)) \leq v^Y(E^Q(c_0)) \leq \frac{n}{2^n}.$$

As in the proof of Proposition 6.2, this implies that, in any order-invariant measure μ on P , $\mu(E^P(c_0)) \leq n/2^n$ for every n , so $\mu(E^P(c_0)) = 0$. Finally, by Lemma 9.1, we see that c_0 , and hence all the c_i , are absent in μ .

The property of admitting an order-invariant measure is preserved under the addition of *finitely many* relations to P : conditioning an order-invariant measure μ on the event that a linear extension of P respects those extra relations yields an order-invariant measure on the causal set with the relations added.

However, we do not know whether the property of admitting an order-invariant measure is preserved under the *removal* of finitely many relations.

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