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A NEW MINIMAL NON-σ-SCATTERED LINEAR ORDER

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Abstract. We will show it is consistent with GCH that there is a minimal non-σ-scattered linear order which does not contain any real or Aronszajn type. In particular the assumption PFA^+ in the main result of [5] is necessary, and there are other obstructions than real and Aronszajn types to the sharpness of Laver's theorem in [8].

§1. Introduction. Fraïssé in [4] conjectured that every descending sequence of countable order types is finite and every antichain of countable order types is finite. That is, the class of countable linear orders is *well quasi-ordered*. Laver confirmed the conjecture by proving a stronger statement.

Theorem 1.1 ([8]). *The class of σ-scattered linear orders is well quasi-ordered. In particular every descending chain of σ-scattered linear orders is finite.*

Here the class of linear orders is considered with the quasi-order of embeddability. Recall that a linear order L is said to be scattered if it does not contain a copy of the rationals. L is called σ-scattered if it is a countable union of scattered linear orders. At the end of his article, Laver asks about the behavior of non-σ-scattered linear orders under embeddability. For instance, *to what extent Laver's theorem is sharp*? If the answer to this question is independent of ZFC, *what are the obstructions to the sharpness of Laver's theorem*?

Not very long after Laver proved Theorem 1.1, various theorems in the direction of showing that Laver's theorem is consistently not sharp were proved. Baumgartner in [2], showed that it is consistent that all $\aleph_1$-dense sets of the reals are isomorphic. In the same article, he mentions that one can add all $\aleph_1$ sized subsets of the reals to the class of all σ-scattered linear orders in order to obtain a class $\mathcal{L}$ of linear orders such that $\mathcal{L}$ is strictly larger than the class of σ-scattered linear orders, $\mathcal{L}$ is closed under taking suborders and it is consistent that $\mathcal{L}$ is well quasi-ordered.

Another result in the direction of "Laver's theorem is consistently not sharp" is due to Abraham and Shelah. In [1], they showed that PFA, the proper forcing axiom, implies that every two nonstationary Countryman lines are either isomorphic or reverse isomorphic. An Aronszajn line A is said to be nonstationary if there is a continuous increasing sequence $\langle A_\xi : \xi \in \omega_1\rangle$ of countable subsets of A which covers A such that for each $\xi \in \omega_1$ no maximal interval of $A \setminus A_\xi$ has a least or greatest element. Since every Countryman line contains an uncountable

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nonstationary suborder, one can even consider a larger class of linear orders than what Baumgartner and Laver considered and still have a class of linear orders which is consistently well quasi-ordered and which is closed under taking suborders. Later Martinez–Ranero in [9] showed that under PFA the class of all Aronszajn lines is well quasi-ordered.

Baumgartner seems to be the first person who considered the other side of Laver's question, i.e., to what extent is Laver's theorem sharp? In [3], he introduces a class of non-σ-scattered linear orders and proves in ZFC that his examples are not minimal with respect to being non-σ-scattered. Baumgartner's example can be described as follows. Let $L = \{C_\alpha : \alpha \in S\}$ ordered lexicographically, where S is a stationary subset of ω_1 consisting of limit ordinals and C_α is a cofinal sequence in α that has order type ω. Baumgartner's example L has the property that a suborder $\{C_\xi : \xi \in A\}$ is σ-scattered if and only if A is not stationary. This together with pressing down lemma implies that if $f : L \longrightarrow L$ is an embedding then the set $\{\xi \in S : f(C_\xi) \neq C_\xi\}$ is not stationary. Therefore if $S_0 \subset S$ is such that $S \setminus S_0$ and S_0 are stationary then L does not embed into the linear order corresponding to S_0. In this article, *Baumgartber type* refers to Baumgartner's examples or the revers of them.

The behavior of Baumgartner types inspired Ishiu and Moore to generalize the situation above for a broader class of linear orders and prove the following theorem.

THEOREM 1.2 ([5]). PFA$^+$ *implies that every minimal non-σ-scattered linear order is either a real or a Countryman type.*

In other words under PFA$^+$, the only obstructions to the sharpness of Laver's theorem are real and Countryman types. This breakthrough should be considered with the following result.

THEOREM 1.3 ([10]). *It is consistent with* CH *that ω_1 and ω_1^* are the only linear orders that are minimal with respect to being uncountable.*

Later the methods in [5] and [10] were used to prove Laver's theorem is sharp, i.e., it is impossible to improve the theorem in ZFC.

THEOREM 1.4 ([7]). *If there is a supercompact cardinal, then there is a forcing extension which satisfies* CH *in which there are no minimal non-σ-scattered linear orders.*

Note that all of the results proving that Laver's theorem is consistently not sharp were based on the consistency of the minimality of real types or Aronszajn types. So it is natural to ask the following question.

QUESTION 1.5. *Does any minimal non-σ-scattered linear order have to be real or Aronszajn type?*

This question is also important from the point of view of the work involved in proving Theorems 1.2, 1.3, and 1.4. An affirmative answer to this question would assert that the assumption PFA$^+$ would not be needed in order to obtain the results in [5]. Consequently, the model Moore came up with in order to prove Theorem 1.3 would already satisfy "Laver's thorem is sharp." Therefore the work in [7] as well as the large cardinal assumption would not be needed to prove Theorem 1.4. In this article, we will provide a negative answer to this question. In particular real and

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Aronszajn types are not the only possible obstructions to the sharpness of Laver's theorem.

THEOREM 1.6. *It is consistent with* GCH *that there is a non-σ-scattered linear order L which contains no real or Aronszajn type and is minimal with respect to not being σ-scattered.*

Moreover, Theorem 1.6 is related the following question which is due to Galvin.

QUESTION 1.7 ([3], Problem 4). *Is there a linear order which is minimal with respect to not being σ-scattered and which has the property that all of its uncountable suborders contain a copy of ω_1?*

Note that a consistent negative answer is already given by Theorem 1.2. Theorem 1.6 does not answer Galvin's question because the linear order we introduce has a lot of copies of ω_1^*.

This article, is organized as follows. Section 2 reviews some notations, definitions and facts regarding linear orders. Section 3 is devoted to constructing a specific Kurepa tree that is a suitable candidate for having suborders that witness Theorem 1.6. We also show that this tree contains a lot of non-σ-scattered linear orders which become σ-scattered in order to witness the main result. In Section 4 we introduce the posets that add isomorphisms we need. Section 5 finishes the proof of Theorem 1.6.

§2. Preliminaries.

This section is devoted to some background, notation and definitions regarding trees, linearly ordered sets, forcings and their iterations. More discussion can be found in [5–7] and [11].

To avoid ambiguity we fix some terminology and notations. An ω_1-*tree* is a tree which is of height ω_1, has countable levels and does not branch at limit heights, i.e., if s, t are of the same limit height and have the same predecessors then they are equal. A *branch* of a tree T is a chain in T which intersects all levels. An ω_1-tree T is called *Aronszajn* if it has no branches. It is called *Kurepa* if it has at least ω_2 many branches.

For a tree T and $t \in T$, $T(t)$ is the collection of all elements of T that are comparable with t. If T is a tree and A is a set of ordinals, by $T \restriction A$ we mean $\{t \in T : \mathrm{ht}(t) \in A\}$, with the order inherited from T. If S, T are trees of height κ and $C \subset \kappa$ is a club and $f : T \restriction C \longrightarrow S \restriction C$ is one to one, level and order preserving then f is called a *club embedding* from T to S. $\mathcal{B}(T)$ refers to the collection of all branches of T. If L is a linearly ordered set, $\hat{L}$ denotes the completion of L. Formally $\hat{L}$ consists of all Dedekind cuts of L.

The following few definitions and facts give a characterization of σ-scatteredness which we use in the proof of Theorem 1.6. They also generalize the behavior of Baumgartner types that causes them to be nonminimal. We will use this to show that the generic tree that we build in Section 3 has suborders that are obstructions to minimality.

DEFINITION 2.1 ([5]). Assume L is a linear order and Z is a countable set. We say Z *captures* $x \in L$ if there is a $z \in Z \cap \hat{L}$ such that there is no element of $Z \cap L$ strictly in between x and z.

FACT 2.2 ([5]). *Suppose L is a linear order and κ is a regular large enough cardinal. If M is a countable elementary submodel of H_κ such that $L \in M$ and $x \in L \setminus M$, then M captures $x \in L$ iff it there is a unique $z \in \hat{L} \cap M$ such that there is no element of $M \cap L$ strictly in between x and z. In this case we say M captures x via z.*

DEFINITION 2.3 ([5]). Assume L is a linear order. $\Omega(L)$ is the set of all countable $Z \subset \hat{L}$ which capture all elements of L. $\Gamma(L) = [\hat{L}]^\omega \setminus \Omega(L)$.

PROPOSITION 2.4 ([5]). *A linear order L is σ-scattered iff $\Gamma(L)$ is not stationary in $[\hat{L}]^\omega$.*

DEFINITION 2.5 ([5]). Assume L is a linear order, $x \in L$, and M is a countable elementary submodel of H_θ where $\theta > 2^{|L|}$ is a large enough regular cardinal. We say x is *internal* to M if there is a club $E \subset [L]^\omega$ in M such that whenever $Z \in M \cap E$, Z captures $x \in L$. We say L is *amenable* if for all large enough regular cardinals θ, for all countable elementary submodels M of H_θ that contain L as an element, and for all $x \in L$, x is internal to M.

The following proposition shows that amenability is what causes Baumgartner types and consistently more linear orders to be nonminimal, see [7], discussion after the proof of Theorem 3.1.

PROPOSITION 2.6 ([5]). *If L is an amenable non-σ-scattered linear order of size $\aleph_1$, then it is not minimal with respect to being non-σ-scattered.*

During this article, we consider the invariants Ω, and Γ for trees and linear orders with different definitions. The point is that all these definitions coincide modulo an equivalence relation that is defined here.

DEFINITION 2.7. Assume X, Y are two uncountable sets and A, B are two collections of countable subsets of X, Y such that $\bigcup A = X$ and $\bigcup B = Y$. We say A, B are equivalent if for all large enough regular cardinal θ there is a club E of countable elementary submodels M of H_θ such that $M \cap X \in A$ if and only if $M \cap Y \in B$.

The invariant Ω together with the equivalence relation mentioned above was used in [5]. By the work in [5], if $L_0 \subset L$ and L embedds into L_0 then $\Omega(L)$ is equivalent to $\Omega(L_0)$. In fact the strategy in that work was to find a suborder L_0 of a given non-σ-scattered linear order L such that $\Omega(L_0)$ is stationary and not equivalent to $\Omega(L)$. This seems to be the motivation of Problem 5.10 in [5]. The problem asks, assuming that S is stationary, is the class of all linear orders L with $\Omega(L) \equiv S$ well quasi-ordered? We will show that even with such a restriction on the Ω of non-σ-scattered linear orders it is impossible to obtain a well quasi ordered class. Here for linear orders A and B, the linear order consisting of the disjoint union of A, B in which every element of A is less than every element of B is denoted by $A \oplus B$.

PROPOSITION 2.8. *Assume $S \subset \omega_1$ is a stationary set consisting of limit ordinals, and $\{S_i : i \in \omega\}$ is a partition of S into infinitely many stationary pieces. Let $\langle C_\alpha \subset \alpha : \alpha \in S\rangle$ be a collection of cofinal sequences of order type ω. Let $L = \{C_\alpha : \alpha \in S\}$ and $L_i = \{C_\alpha : \alpha \in \bigcup_{j \geq i} S_j\}$ ordered with the lex order. Then the sequence $\langle L \oplus L_i : i \in \omega\rangle$ is a descending chain of linear orders and $\Omega(L \oplus L_i) \equiv \Omega(L \oplus L_j)$ for all i, j in ω.*

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PROOF. We start with an observation. Assume $L_X = \{x_\alpha : \alpha \in X\}$ and $L_Y = \{y_\alpha : \alpha \in Y\}$ are two arbitrary Baumgartner types where X, Y are stationary subsets of ω_1 consisting of limit ordinals and x_α, y_α are increasing cofinal ω-sequences in α. Suppose $f : L_X \longrightarrow L_Y$ is an embedding. Then the set of α with $f(x_\alpha) \neq y_\alpha$ is nonstationary. In order to see this, note that the set of all α with $f(x_\alpha) = y_\xi$, for some $\xi \in \alpha$, is nonstationary by pressing down lemma. Similarly, the set of all β with $y_\beta = f(x_\alpha)$, for some $\alpha \in \beta$ is nonstationary. Hence $\{x_\alpha : f(x_\alpha) = y_\beta(\exists \beta > \alpha)\}$ is σ-scattered. Therefore the set of all α with $f(x_\alpha) = y_\beta$ for some $\beta > \alpha$ is nonstationary as desired.

Now we will show that the sequence $\langle L \oplus L_i : i \in \omega\rangle$ is a descending chain. Assume for some $m \in \omega$, $L \oplus L_m$ embeds into $L \oplus L_{m+1}$. Let $A = \{a_\alpha : \alpha \in \bigcup_{i\geq m} S_i\}$ and $B = \{b_\alpha : \alpha \in \bigcup_{i\geq m+1} S_i\}$ be disjoint from L and be isomorphic to L_m and L_{m+1}, respectively, via the maps $a_\alpha \mapsto C_\alpha$ and $b_\alpha \mapsto C_\alpha$. Also let $f : L \oplus A \longrightarrow L \oplus B$ be an embedding. Then the sets $\{\alpha : f(C_\alpha) \neq C_\alpha$ and $f(C_\alpha) \neq b_\alpha\}$ and $\{\alpha : f(a_\alpha) \neq C_\alpha$ and $f(a_\alpha) \neq b_\alpha\}$ are nonstationary. Therefore, there is a nonstationary $N \subset \omega_1$ such that for all $\alpha \in S_m \setminus N$, $f(a_\alpha) = C_\alpha$ and for all $\alpha \in S_m \setminus N$, $f(C_\alpha) = C_\alpha$. This contradicts the injectivity of f.

If M is a countable elementary submodel of H_θ $(\theta > 2^{\omega_1})$, M captures all elements of $L \oplus L_i$ if and only if $M \cap \omega_1 \notin S$. This shows that $\Omega(L \oplus L_i) \equiv \Omega(L \oplus L_j)$. ⊣

Assume T is an ω_1-tree that is equipped with a lexicographic order such that for all $t \in T$, the set $\{s \in T : \mathrm{ht(s)} = \mathrm{ht}(t) + 1$ and $t <_T s\}$ is isomorphic to $\mathbb{Q}$, when it is considered with the lex order of the tree T. Let $L = (T, <_{\mathrm{lex}})$, then $\Omega(L)$ defined here is equivalent to the $\Omega(T)$ defined in [6].

DEFINITION 2.9 ([6]). Assume T is an ω_1-tree. $\Omega(T)$ is the set of all countable $Z \subset \mathcal{B}(T)$ with the property that for all $t \in T_{\alpha_Z}$ there is a $b \in Z$ with $t \in b$, where $\alpha_Z = \sup\{b \,\Delta\, b' : b, b' \in Z\}$.

Now we have two notions of capturing: for linear orders and ω_1-trees. The following trivial fact asserts that in the cases that we are interested in, these two notions coincide.

FACT 2.10. *Assume T is an ω_1-tree equipped with a lex order such that the set of all immediate successors of each element of T is isomorphic to $\mathbb{Q}$. Suppose $\mathcal{B}(T)$ is the collection of all cofinal branches in T, M is a countable elementary submodel of H_θ for some large enough regular θ, and $t \in T$. Then M captures t if and only if either $t \in M$ or there is a branch $b \in M \cap \mathcal{B}(T)$ such that $t \,\Delta\, b \geq M \cap \omega_1$. Here $t \,\Delta\, b$ is the height of the least element of b that is not a predecessor of t.*

The following fact is also easy to check.

FACT 2.11. *Assume $L' \subset L$ are linear orders, $x \in L'$ and M is a countable elementary submodel of H_θ that has L, L' as elements, where $\theta > 2^{|\hat{L}|}$ is regular. Then M captures x as an element of L iff M captures x as an element of L'.*

We will need the following fact in the final section. We include the proof for more clarity.

FACT 2.12. *Assume L is a linear order which has size $\aleph_2$, all elements of L have cofinality and coinitiality ω_1, and $L' \subset L$ is dense and has cardinality $\aleph_1$. Then L' is not σ-scattered.*

Proof. Assume L' is σ-scattered. Since all $x \in L$ have cofinality and coinitiality ω_1, there is a scattered suborder L_0 of L' whose closure in L has cardinality $\aleph_2$. For $x, y \in L_0$ let $x \sim y$ if there are at most $\aleph_1$ many elements of the closure of L_0 in between x and y. Note that there are exactly $\aleph_1$ many equivalence classes and between every two distinct equivalence classes there are $\aleph_1$ many, equivalence classes. Now let L_1 be a suborder of L_0 which intersect each equivalence class at exactly one point. L_1 is an infinite dense linear order which contradicts scatteredness of L_0. ⊣

We will be using forcings which are not proper. The rest of this section is devoted to the facts and lemmas which enable us to show that countable support iterations of these forcings are robust enough to preserve cardinals, under mild assumptions like CH. More discussion can be found in [6] and [11].

For a regular cardinal θ, H_θ is the collection of all sets of hereditary cardinality less than θ. We assume H_θ is equipped with a fixed well ordering without mentioning it. Assume $\mathcal{P}$ is an arbitrary set and θ is a regular cardinal such that $\mathcal{P}$ and the powerset of $\mathcal{P}$ are in H_θ. A countable elementary submodel N of H_θ is said to be *suitable* for $\mathcal{P}$ if $\mathcal{P} \in N$. If $\mathcal{P}$ is a forcing notion and $\langle p_n : n \in \omega \rangle$ is a decreasing sequence of conditions in $\mathcal{P} \cap N$, $\langle p_n : n \in \omega \rangle$ is said to be $(N, \mathcal{P})$-*generic* if for all dense subsets D of $\mathcal{P}$ that are in N, there is an $n \in \omega$ such that $p_n \in D$.

Definition 2.13. Assume X is uncountable and $S \subset [X]^\omega$ is stationary. A poset $\mathcal{P}$ is said to be S-*complete*, if every descending $(M, \mathcal{P})$-generic sequence, $\langle p_n : n \in \omega \rangle$ has a lower bound, for all M with $M \cap X \in S$ and M suitable for $X, \mathcal{P}$.

Fact 2.14 ([11]). *Assume X is uncountable and $S \subset [X]^\omega$ is stationary. If $\mathcal{P}$ is an S-complete forcing then it preserves ω_1 and adds no new countable sequences of ordinals.*

Corollary 2.15 ([11]). *Assume X is uncountable and $S \subset [X]^\omega$ is stationary. Then S-completeness is preserved under countable support iterations.*

We will use the following lemmas from [6]. Note that no forcing can add a new cofinal branch or Aronszajn subtree to T when T has no Aronszajn subtree and has only countably many cofinal branches.

Lemma 2.16 ([6]). *Assume T is an ω_1-tree which has uncountably many cofinal branches and which has no Aronszajn subtree in the ground model $\mathbf{V}$. Also assume $\Omega(T) \subset [\mathcal{B}(T)]^\omega$ is stationary and $\mathcal{P}$ is an $\Omega(T)$-complete forcing. Then T has no Aronszajn subtree in $\mathbf{V}^{\mathcal{P}}$.*

Lemma 2.17. *Assume T is an ω_1-tree, X is an uncountable set, $S \subset [X]^\omega$ is stationary, and $\mathcal{P}$ is an S-complete forcing. Then $\mathcal{P}$ does not add new cofinal branches to T.*

The following definition is a modification of the Shelah's notion for chain condition, κ-proper isomorphism condition. We will be using it for verifying certain chain conditions.

Definition 2.18. Assume S, X are as above and κ is an regular cardinal. We say that $\mathcal{P}$ satisfies the S-*closedness isomorphism condition for* κ, or $\mathcal{P}$ has the S-*cic for* κ, if whenever

- M, N are suitable models for $\mathcal{P}$,
- both $M \cap X, N \cap X$ are in S,
- $h : M \to N$ is an isomorphism such that $h \restriction (M \cap N) = id$,
- $\min(N \setminus M \cap \kappa) > \sup(M \cap \kappa)$, and
- $\langle p_n : n \in \omega \rangle$ is an $(M, \mathcal{P})$-generic sequence,

then there is a common lower bound $q \in \mathcal{P}$ for $\langle p_n : n \in \omega \rangle$ and $\langle h(p_n) : n \in \omega \rangle$.

LEMMA 2.19. *Assume $2^{\aleph_0} < \kappa$, κ is a regular cardinal and that S, X are as above. If $\mathcal{P}$ satisfies the S-cic for κ then it has the κ-c.c.*

The proof of the following fact, which is useful in verifying the chain condition properties of an iteration of posets, can be found in [6].

LEMMA 2.20. *Suppose $\langle \mathcal{P}_i, \dot{\mathcal{Q}}_j : i \le \delta, j < \delta \rangle$ is a countable support iteration of S-complete forcings, where $S \subset [X]^\omega$ is stationary and X is uncountable. Assume in addition that*

$$\Vdash_{\mathcal{P}_i} \text{“}\dot{\mathcal{Q}}_i \text{ has the } \check{S}\text{-cic for } \kappa\text{”},$$

for all $i \in \delta$. Then $\mathcal{P}_\delta$ has the S-cic for κ.

§3. The generic homogeneous Kurepa tree.

DEFINITION 3.1. Assume $\langle M_\xi : \xi \in \omega_2 \rangle$ is a continuous $\in$-chain of $\aleph_1$-sized elementary submodels of $H_{(2^{\omega_2})^+}$, such that $\xi \cup \omega_1 \subset M_\xi$ and $\langle M_\eta : \eta \le \xi \rangle$ is in $M_{\xi+1}$. Fix $C \subset \omega_2$ consisting of all $\sup(M_\xi \cap \omega_2)$ for $\xi \in \omega_2$. The poset $\mathcal{H}$ is the collection of all conditions $q = (T_q, b_q, \Pi_q)$ for which the following statements hold:

1. T_q is a countable tree of height $\alpha_q + 1$ which is equipped with a lexicographic order such that for all $t \in (T_q)_{<\alpha_q}$, the set t^+, consisting of all immediate successors of t, is isomorphic to the rationals when considered with the lexicographic order.
2. b_q is a bijective function from a countable subset of ω_2 to the last level of T_q.
3. The collection, $\Pi_q = \langle \pi^q_{t,s} : (t,s) \in \bigcup_{\xi \in \alpha_q} ((T_q)_\xi)^2 \rangle$ such that $\pi^q_{t,s}$ is a tree isomorphism from $T_q(t)$ to $T_q(s)$, which preserves the lexicographic order.
4. The collection Π_q is coherent, in the sense that if $t' > t$ and $\pi^q_{t,s}(t') = s'$ then $\pi^q_{t',s'} = \pi^q_{t,s} \restriction T_q(t')$.
5. The collection Π_q is symmetric in the sense that $\pi^q_{s,t} = (\pi^q_{t,s})^{-1}$.
6. The collection Π_q respects the club C in the following sense. If $\alpha \in C$, t, s are in T_q and have the same height, then $\xi < \alpha$ iff $b_q^{-1}(\pi^q_{t,s}(b_q(\xi))) < \alpha$.
7. The collection Π_q respects the composition operation, in the sense that if t, s, u are in $(T_q)_\xi$ and $\xi < \alpha_q$ then $\pi^q_{s,u} \circ \pi^q_{t,s} = \pi^q_{t,u}$.

For $p, q \in \mathcal{H}$ we let $q \le p$ if

1. $(T_q)_{\le \alpha_p} = T_p$ and the lex order on T_p is the same as the one on T_q,
2. $\mathrm{dom}(b_p) \subset \mathrm{dom}(b_q)$,
3. for all $\xi \in \mathrm{dom}(b_p)$, $b_p(\xi) \le b_q(\xi)$,
4. for all $(t,s) \in \bigcup_{\xi \in \alpha_q} (T_p)^2_\xi$, $\pi^p_{t,s}$ is equal to $\pi^q_{t,s} \restriction T_p$, and
5. for all $(t,s) \in \bigcup_{\xi \in \alpha_q} (T_p)^2_\xi$, and $\xi, \eta \in \mathrm{dom}(b_p)$, if $\pi^p_{t,s}(b_p(\xi)) = b_p(\eta)$ then $\pi^q_{t,s}(b_q(\xi)) = b_q(\eta)$.

NOTATION 3.2. *Assume G is a generic filter for $\mathcal{H}$. Define T_G to be $\bigcup_{q\in G} T_q$ and b_ξ to be the branch $\{b_q(\xi) : q \in G\}$. If t, s are in T_G and have the same height $\pi_{t,s}$ denotes $\bigcup_{q\in G} \pi^q_{t,s}$.*

LEMMA 3.3. *$\mathcal{H}$ is σ-closed.*

PROOF. Let $\langle p_n : n \in \omega\rangle$ be a decreasing sequence in $\mathcal{H}$ and $\sup(\alpha_{p_n})_{n\in\omega} = \alpha$. Let $T = \bigcup_{n\in\omega} T_{p_n}$. Note that $(b_{p_n}(\xi) : n \in \omega)$ is a cofinal chain in T for all $\xi \in \bigcup_{n\in\omega} \mathrm{dom}(b_{p_n})$. Let T_q be a countable tree of height $\alpha + 1$ such that

- $(T_q)_{<\alpha} = T$,
- for all $\xi \in \bigcup_{n\in\omega} \mathrm{dom}(b_{p_n})$, $(b_{p_n}(\xi) : n \in \omega)$ has an upper bound in T_q, and
- every element of height α is an upper bound for $(b_{p_n}(\xi) : n \in \omega)$, for some $\xi \in \bigcup_{n\in\omega} \mathrm{dom}(b_{p_n})$.

Now let q be the condition with $\alpha_q = \alpha$ and T_q as above. Let b_q be the function form $\bigcup_{n\in\omega} \mathrm{dom}(b_{p_n})$ to the last level of T_q such that for all ξ in the domain, $b_q(\xi)$ is the upper bound for the chain $(b_{p_n}(\xi) : n \in \omega)$. Similarly $\bigcup_{n\in\omega} \pi^{p_n}_{t,s}$, can be extended to the last level of T_q, for all t, s that are of the same height and are in T. It is easy to see that the condition q described above is a lower bound for the sequence $\langle p_n : n \in \omega\rangle$. ⊣

LEMMA 3.4. GCH *implies that $\mathcal{H}$ has the $\aleph_2$-cc.*

PROOF. Let $\langle q_\xi : \xi \in \omega_2\rangle$ be a collection of conditions in $\mathcal{H}$. Since there are $\aleph_1$-many possibilities for T_q and Π_q, we can thin down this collection to a subset of the same cardinality so that T_{q_ξ} and Π_{q_ξ} do not depend on ξ. Now define $f : C \longrightarrow \omega_2$ by $f(\xi) = \sup(\mathrm{dom}(b_{q_\xi}) \cap \xi)$, where C is the club that all elements of $\mathcal{H}$ respect. Note that for all $\xi \in C$ with $cf(\xi) > \omega$, $f(\xi) < \xi$. So there is a stationary $S \subset C$, and $\alpha \in \omega_2$ such that $f \restriction S$ is the constant α. We can thin down S to a stationary subset S' if necessary, so that in $\langle q_\xi : \xi \in S'\rangle$, $\mathrm{dom}(b_{q_\xi}) \cap \alpha$ and $b_{q_\xi} \restriction \alpha$ do not depend on ξ. Let $S'' \subset S' \setminus (\alpha + 1)$ be of size $\aleph_2$ and whenever $\xi < \eta$ are in S'', $\sup(\mathrm{dom}(b_{q_\xi})) < \eta$. Note that $\langle b_{q_\xi} : \xi \in S''\rangle$ forms a Δ-system with root r such that the $\mathrm{dom}(r) \subset \alpha$. Moreover for all $\xi \in S''$, $\min(\mathrm{dom}(b_{q_\xi}) \setminus r) \geq \xi$. Since $S'' \subset C$, every two conditions in $\langle q_\xi : \xi \in S''\rangle$ are compatible. ⊣

The following can routinely be verified.

FACT 3.5. *The following sets are dense in $\mathcal{H}$.*

- *$H_\alpha := \{q \in \mathcal{H} | \alpha_q > \alpha\}$.*
- *For $\xi \in \omega_2$, $I_\xi := \{q \in \mathcal{H} : \xi \in \mathrm{dom}(b_q)\}$.*

The proof of the following lemma is the same as Lemma 3.3.

LEMMA 3.6. *If M is suitable for $\mathcal{H}$ and $\langle p_n : n \in \omega\rangle$ is a decreasing $(M, \mathcal{H})$-generic sequence, then there is a lower bound q for $\langle p_n : n \in \omega\rangle$ such that $\mathrm{dom}(b_q) = M \cap \omega_2$, and $\alpha_q = M \cap \omega_1$.*

FACT 3.7.

- *Assume G is a generic filter for $\mathcal{H}$. Then the generic tree $T := \bigcup_{q\in G} T_q$ is a Kurepa tree such that $\langle\{b_q(\xi) : q \in G\} : \xi \in \omega_2\rangle$ is an enumeration of the set of all branches.*
- *T has no Aronszajn subtree. Moreover, any uncountable downward closed subtree of T contains a branch b_ξ for some $\xi \in \omega_2$.*
- *Assume L is the linear order consisting of all branches of T, $\mathcal{B}(T)$, ordered by the lexicographic order of the tree T. Then $\Omega(L)$ is stationary.*

Proof. The first statement follows from the second one. The second statement follows from the third one and Proposition 2.9, or the stronger statement Theorem 4.1 of [5]. For the last statement, let M be suitable for $\mathcal{H}$ and $p \in M \cap \mathcal{H}$. Then the $(M, \mathcal{H})$-generic condition from the last lemma forces that $M[\dot{G}] \cap L \in \Omega(L)$. ⊣

From now on, T is the generic Kurepa tree generated by $\mathcal{H}$ unless otherwise mentioned. Also K is the linear order $\mathcal{B}(T)$ ordered by the lexicographic order of the tree T. We fix an enumeration of $\mathcal{B}(T) = \langle b_\xi : \xi \in \omega_2 \rangle$.

The rest of this section is devoted to showing that K has a lot of non-σ-scattered suborders that are amenable. These facts are not used in the proof of the results in the next sections but show some possible obstruction for the minimality of suborders of K. In the next sections, these non-σ-scattered suborders are forced to be σ-scattered by an improper forcing. Here we say a countable sequence of conditions in $\mathcal{H}$ forces a statement if every lower bound of that sequence forces that statement, equivalently all generic filter that contain the sequence extends the model to the one in which the statement holds.

Definition 3.8. Let T be the $\mathcal{H}$-generic tree and $t \in T$. The element t is said to be *simple* if whenever $\theta > 2^{\omega_1}$ is a regular cardinal and M is a countable elementary submodel of H_θ containing T, then M captures $t \in T$. Otherwise t is said to be *complex*.

Lemma 3.9. *Assume* GCH *holds in* **V**, *M is suitable for $\mathcal{H}$, $\langle p_n : n \in \omega \rangle$ is an $(M, \mathcal{H})$-generic sequence, $t \in T_0 := \bigcup_{n \in \omega} T_{p_n}$, $\langle p_n \rangle_{n\in\omega} \Vdash$ "t is simple", b is a branch in T_0 and $ht(t) < \alpha < \delta := M \cap \omega_1$. Then there exists $s \in T_0$ such that $ht(s) = \alpha$, $t < s$, $s \notin b$ and $\langle p_n \rangle_{n\in\omega} \Vdash$ "s is simple".*

Proof. First note that if G is $\mathcal{H}$-generic over $\mathbf{V}$ then $H_{\omega_3}[G] = H_{\omega_3}^{\mathbf{V}[G]}$ has a well ordering $\lhd$. Let $\dot{\lhd}$ be an $\mathcal{H}$-name for $\lhd$. Since $\langle p_n \rangle_{n\in\omega}$ is M-generic it decides $\dot{\lhd} \cap (M[\dot{G}])^2$, in the sense that, if τ and π are two $\mathcal{H}$-names that are in M then there is an $n \in \omega$ such that $p_n \Vdash$ "$\tau \dot{\lhd} \pi$" or $p_n \Vdash$ "$\pi \dot{\lhd} \tau$".

Also note that if t is simple then so is every $t' \in t^+$. Now let $\sigma \in M$ be an $\mathcal{H}$-name for a branch of the $\mathcal{H}$-generic tree such that $\langle p_n \rangle_{n\in\omega}$ forces that

- $t \in \sigma$,
- $\sigma(\mathrm{ht}(t) + 1) \neq b(\mathrm{ht}(t) + 1)$, and
- σ is the $\dot{\lhd}$-minimum branch of $\dot{T}$ with the properties above.

Let $s \in T_0$ such that $\langle p_n \rangle_{n\in\omega}$ forces that $s = \sigma(\alpha)$. We will show that $\langle p_n \rangle_{n\in\omega} \Vdash$ "s is simple". Let G be an $\mathcal{H}$-generic filter containing $\langle p_n \rangle_{n\in\omega}$ and in $\mathbf{V}[G]$, N be a countable elementary submodel of H_{ω_3}. If $N \cap \omega_1 \leq \mathrm{ht}(t)$, by simplicity of t, N captures s. if $\mathrm{ht}(t) < N \cap \omega_1$ then $t^+ \subset N$ so $\sigma_G = \min_{\lhd}\{c \in \mathcal{B}(T) : c(\mathrm{ht}(t)+1) = s(\mathrm{ht}(t) + 1)\}$. So by elementarity $\sigma_G \in N$ and N captures s. ⊣

Proposition 3.10. *Assume* GCH *holds in* **V** *and G is* **V***-generic for $\mathcal{H}$. Then K has an amenable non-σ-scattered suborder in* **V**$[G]$.

Proof. Let $L = \{t \in T : t \text{ is minimal complex}\}$ ordered by the lexicographic order of the $\mathcal{H}$-generic tree T. To see L is amenable, let M be a countable elementary submodel of H_θ with $T, L \in M$, where θ is a regular large enough cardinal and $t \in L$. We need to show that t is internal to M. If $\mathrm{ht}(t) < M \cap \omega_1$ then $t \in M$ and there is nothing to prove. If $\mathrm{ht}(t) > M \cap \omega_1$, note that $t \restriction (M \cap \omega_1)$ is simple

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and M captures it. If $\mathrm{ht}(t) = M \cap \omega_1$, let $E = \{N \cap \mathcal{B}(T) : N$ is a countable elementary submodel of H_{ω_3} with $T, L \in N\}$. For every $Z \in E \cap M$ there is a countable elementary submodel N of H_{ω_3} such that $N \in M$ and $N \cap \mathcal{B}(T) = Z$. In particular $N \cap \omega_1 < \mathrm{ht}(t)$, and since t is a minimal complex element of T, Z captures t. So t is internal to M and L is amenable.

In order to see L is not σ-scattered we will show that $\Gamma(L)$ is stationary in $[\hat{L}]^\omega$. Assume $\dot{E}$ is an $\mathcal{H}$-name for a club in $[\hat{L}]^\omega$ and $q \in \mathcal{H}$. In **V**, let M be suitable for $\mathcal{H}$ with $q, \dot{E}$ in M and $\langle p_n : n \in \omega\rangle$ be an M-generic sequence such that $p_0 = q$. Also let $\langle b_n : n \in \omega\rangle$ be an enumeration of all branches of $T_0 = \bigcup_{n\in\omega} T_{p_n}$ which are downward closure of $\{b_{p_n}(\xi) : n \in \omega\}$ for some $\xi \in M \cap \omega_2$. By the previous lemma there is a sequence $\langle t_k : k \in \omega\rangle$ of elements in T_0 such that for all $k \in \omega$, $\langle p_n\rangle_{n\in\omega}$ forces that t_k is simple, $t_k < t_{k+1}$, $t_k \notin b_k$ and $\sup\{\mathrm{ht}(t_k) : k \in \omega\} = \delta := M \cap \omega_1$.

Now let $T_p = T_0 \cup (T_p)_\delta$ where $(T_p)_\delta$ is a minimal set such that

- for each $\xi \in M \cap \omega_2$, $\{b_{p_n}(\xi) : n \in \omega\}$ has a unique upper bound in $(T_p)_\delta$,
- the sequence $\langle t_k : k \in \omega\rangle$ has a unique upper bound for in $(T_p)_\delta$, and
- for each $u, v \in T_0$ and $t \in (T_p)_\delta$ with $u < t$, $(\pi^{p_n}_{u,v})[\{s \in T_0 : s < t\}]$ has a unique upper bound in $(T_p)_\delta$.

It is easy to see that there is a b_p which is a function from a countable subset of ω_2 to $(T_p)_\delta$ and Π_p consisting of natural extensions of the maps $\pi^{p_n}_{u,v}$ where u, v are in T_0, such that $p = (T_p, b_p, \Pi_p)$ is a lower bound for p.

On the other hand p forces the following statement.

- There are minimal complex elements at the δth level of the $\mathcal{H}$-generic tree T.
- $M[\dot{G}] \cap \tau \in \dot{E}$, where τ is an $\mathcal{H}$-name for $\hat{L}$ in M.
- $M[\dot{G}] \cap \tau$ does not capture all elements of $\dot{L}$.

Therefore $\mathbb{1}_{\mathcal{H}} \Vdash$ "$\dot{L}$ is not σ-scattered." Note that the elements of L form an antichain in T. Let $L' \subset K$ such that for every $t \in L$ there is a unique branch $b \in L'$ with $t \in b$. Then L' is isomorphic to L, hence K has an amenable non-σ-scattered suborder. ⊣

§4. Adding embeddings. In the previous section we introduced a forcing which generates a Kurepa tree T equipped with a lexicographic order which also has some homogeneity properties. In this section we use the homogeneity of T to prove the countable support iteration of some forcings that add embeddings among the $\aleph_1$-sized dense subsets of the linear order $K = (\mathcal{B}(T), <_{\mathrm{lex}})$ do not collapse cardinals. We fix an enumeration $\langle b_\xi : \xi \in \omega_2\rangle$ of the branches of the tree T for the rest of the article, and recall that for each $t \in T$, the set t^+, consisting of all immediate successors of t with respect to $<_T$, is isomorphic to the rationals when considered with $<_{\mathrm{lex}}$. Here homogeneity of T means that there is a collection $\Pi = \langle \pi_{t,s} : t, s \in T \wedge \mathrm{ht}(t) = \mathrm{ht}(s)\rangle$ with the following properties.

(1) for all t, s in T which have the same height, $\pi_{t,s}$ is a tree and lex order isomorphism from $T(t)$ to $T(s)$.
(2) Π is symmetric, in the sense that $\pi_{t,s} = (\pi_{s,t})^{-1}$.
(3) Π is coherent in the sense that if t, s, t', s' are in T, $\mathrm{ht}(t) = \mathrm{ht}(s)$, $t < t'$, $s < s'$ and $\pi_{t,s}(t') = s'$, then $\pi_{t,s} \restriction (T(t')) = \pi_{t',s'}$, where $t' \uparrow = \{u \in T : t'$ is compatible with $u\}$.

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Definition 4.1. Assume T is as above and X, Y are two subsets of ω_2 such that $|X| = |Y| = \aleph_1$ and both $\{b_\xi : \xi \in X\}, \{b_\xi : \xi \in Y\}$ are dense in K. $\mathcal{F}_{XY}(= \mathcal{F})$ is the poset consisting of all conditions $p = (f_p, \phi_p)$ for which the following holds:

(1) $f_p : T \restriction A_p \longrightarrow T \restriction A_p$ is a lex order and level preserving tree isomorphism where $A_p \subset \omega_1$ is countable and closed with $\max A_p = \alpha_p$.

(2) ϕ_p is a countable partial injection from ω_2 to ω_2 such that:

  (a) for all $\xi \in \mathrm{dom}(\phi_p)$, if $\xi \in X$ then $\phi_p(\xi) \in Y$,

  (b) for all $\xi \in \mathrm{dom}(\phi_p) \setminus X$, $b_{\phi_p(\xi)} = \pi_{t,s}[b_\xi]$, where $t = b_\xi(\alpha_p + 1)$ and s is an immediate successor of $f_p(b_\xi(\alpha_p))$, and

  (c) the map $b_\xi \mapsto b_{\phi_p(\xi)}$ is lexicographic order preserving.

(3) For all $t \in T_{\alpha_p}$ there are at most finitely many $\xi \in \mathrm{dom}(\phi_p)$ with $t \in b_\xi$.

(4) For all $\xi \in \mathrm{dom}(\phi_p)$, $f_p(b_\xi(\alpha_p)) = b_{\phi_p(\xi)}(\alpha_p)$.

We let $q \leq p$ if $f_p \subset f_q$, $A_q \cap \alpha_p = A_p$ and $\phi_p \subset \phi_q$.

It is obvious that the sets $\{q \in \mathcal{F} : \alpha_q > \beta\}$ and $\{q \in \mathcal{F} : \xi \in \mathrm{dom}(\phi_q)\}$ are dense for all $\beta \in \omega_1$ and $\xi \in \omega_2$. Therefore the forcing $\mathcal{F}$ adds a lexicographic order embedding from X to Y via the map $\Phi \restriction X$ where $\Phi = \bigcup_{p \in G} \phi_p$ and G is the generic filter for $\mathcal{F}$. We will show that countable support iterations of these forcings do not collapse cardinals.

Lemma 4.2. *Assume $\mathcal{P}$ is an S-complete forcing where $S = \Omega(T)$, and $\dot{X}, \dot{Y}$ are $\mathcal{P}$-names for the indexes of the elements of $\aleph_1$-sized dense subsets of K. Then*

(1) $\Vdash$ "$\dot{\mathcal{F}_{XY}}$ *is $\check{S}$-complete", and*

(2) $\Vdash$ "$\dot{\mathcal{F}_{XY}}$ *has the $\check{S}$-cic for $\check{\omega}_2$".*

Proof. Let $G \subset \mathcal{P}$ be generic. We work in $\mathbf{V}[G]$. To see (1), assume M is suitable for $\mathcal{F}$ and $M \cap K \in S$. Also let $\langle p_n = (f_n, \phi_n) : n \in \omega \rangle$ be a descending $(M, \mathcal{F})$-generic sequence and $\delta = M \cap \omega_1$. Note that $M \cap \omega_2 = \bigcup_{n \in \omega} \mathrm{dom}(\phi_n)$ and $\bigcup_{n \in \omega} A_{p_n}$ is cofinal in δ. Now let $\phi_p = \bigcup_{n \in \omega} \phi_n$, and $f_p = \bigcup_{n \in \omega} f_n \cup f$, where $f(b_\xi(\delta)) = b_{\phi_p(\xi)}(\delta)$ for all $\xi \in M \cap \omega_2$. This makes p a lower bound for $\langle p_n : n \in \omega \rangle$, since $M \cap K \in S$, and $\{b_\xi(\delta) : \xi \in M \cap \omega_2\} = T_\delta$.

For (2), still in $\mathbf{V}[G]$, let $M, N, \langle p_n = (f_n, \phi_n) : n \in \omega \rangle$, and h be as in Definition 2.18 with $M \cap \omega_1 = N \cap \omega_1 = \delta$. Since h fixes the intersection $h(f_n) = f_n$ and $b(\delta) = [h(b)](\delta)$, for all $b \in M \cap \mathcal{B}(T)$. Let $\phi_p = \bigcup_{n \in \omega} (\phi_n \cup h(\phi_n))$ and $f_p = \bigcup_{n \in \omega} f_n \cup f$, where $f(b_\xi(\delta)) = b_{\phi_p(\xi)}(\delta)$ for all $\xi \in M \cap \omega_2$.

We need to show that p is a condition and a common lower bound for $\langle p_n : n \in \omega \rangle$ and its image under h. We will prove the map $b_\xi \mapsto b_{\phi_p(\xi)}$ preserves the order $<_{\mathrm{lex}}$. The rest of the requirements are obvious. Let ξ, η be in $(M \cup N) \cap \omega_2$ and $b_\xi <_{\mathrm{lex}} b_\eta$. If one of ξ or η is in $M \cap N$, we are done. We are also done if $b_\xi(\delta) \neq b_\eta(\delta)$. So assume that $\xi \in M$, $\eta \in N$, and $b_\xi(\delta) = b_\eta(\delta)$. By elementarity $\eta = h(\xi)$. Fix $n \in \omega$ such that $\xi \in \mathrm{dom}(\phi_n)$. Since $|X| = \aleph_1$ and $X \in M \cap N$, $M \cap X = N \cap X$. In particular, ξ, η are not in X. Let $t = b_\xi(\alpha_{p_n} + 1)$ and s be the immediate successor of $f_{p_n}(b_\xi(\alpha_{p_n}))$ such that $b_{\phi_n(\xi)} = \pi_{t,s}[b_\xi]$. Then $b_{h\phi_n(\eta)} = \pi_{t,s}[b_\eta]$. But $\pi_{t,s}$ preserves $<_{\mathrm{lex}}$, so $b_{\phi_n(\xi)} <_{\mathrm{lex}} b_{h\phi_n(\eta)}$. Hence, ϕ_p preserves $<_{\mathrm{lex}}$. ⊣

§**5. Proof of the main theorem.** In this section we will finish the proof of Theorem 1.6. The strategy is to show that if two $\aleph_1$-sized $L, L' \subset K$ have closure of cardinality $\aleph_2$, then they are isomorphic. Note that by Lemma 3.10, K has non-σ-scattered

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suborders whose closure have cardinality $\aleph_1$. So in order to use the strategy mentioned above, we need to make these suborders σ-scattered by forcings for which the analogue of Lemma 4.2 holds. We finish this section with a proof of Theorem 1.6.

Definition 5.1. Assume $L \subset K$, $|\bar{L}| \leq \aleph_1$. $\mathcal{P}_L(= \mathcal{P})$ is the poset consisting of all conditions $p : \alpha_p + 1 \longrightarrow [\bar{L}]^\omega \cap \Omega(L)$ that are $\subset$- increasing and continuous.

Lemma 5.2. *Assume $S = \Omega(K)$, $\mathcal{Q}$ is an S-complete forcing, and $\dot{L}$ is a $\mathcal{Q}$-name for a suborder of K whose closure has size $\leq\aleph_1$. Then*

(1) $\Vdash$ "*$\dot{\mathcal{P}}_L$ is $\check{S}$-complete*", *and*
(2) $\Vdash$ "*$\dot{\mathcal{P}}_L$ has the $\check{S}$-cic for $\check{\omega}_2$*".

Proof. Let $G \subset \mathcal{Q}$ be generic. We work in $\mathbf{V}[G]$. To see (1), let M be suitable for $\mathcal{P}$ and $M \cap K \in S$. It is enough to show that $M \cap \bar{L} \in \Omega(L)$. First note that M does not capture any $x \in K \setminus M$ via cuts of countable cofinality or initiality. In order to see this, assume M captures $x \in K$ via a cut z where $z = \sup\{y_n : n \in \omega\}$ and $\langle y_n : n \in \omega\rangle$ is an increasing sequence in K. Let $\alpha = \sup\{x\Delta y_n : n \in \omega\}$. Obviously $\alpha \in M \cap \omega_1$ and we can find $x' \in K \cap M$ that is strictly in between z and x. This contradicts the assumption that M captures x via z. So if M captures an element that is not in M, it has to capture it via a cut $z \in \hat{K}$ of cofinality and coinitiality $\aleph_1$. But then z determines a branch in T which means that $z \in K$.

Now let M capture $x \in L \setminus M$ via $z \in K \cap M$. We will show that $z \in \bar{L}$. Note that $K \setminus \bar{L}$ is the union of a collection consisting of pairwise disjoint convex open subsets of K. So if $z \in (K \setminus \bar{L}) \cap M$ there is a convex open set I containing z which is in M. Since $I \in M$ the endpoints of I are in $M \cap \hat{K}$. But M captures x via a unique cut, so z is an endpoint of I which contradicts the fact that I is open.

For (2), note that if $h : M \longrightarrow N$ is an isomorphism that fixes $M \cap N$, then h fixes $\bar{L} \cap M$ because $|\bar{L}| = \aleph_1$. So any lower bound for an M-generic sequence is a lower bound for an N-generic sequence. ⊣

Now we are ready to prove Theorem 1.6. Assume GCH holds in $\mathbf{V}$ and T is the generic Kurepa tree from the forcing $\mathcal{H}$ in $\mathbf{V}^{\mathcal{H}}$. By Facts and Lemmas 2.16, 2.17, 3.7, 4.2, and 5.2, and the work in [6] there is countable support iteration of forcings of length ω_2 which is $\Omega(T)$-complete and extends $\mathbf{V}^{\mathcal{H}}$ to a model in which the following holds.

(1) T is club isomorphic to all of its everywhere Kurepa subtrees and has no Aronszajn subtree.
(2) If X, Y are two dense suborders of $K = (\mathcal{B}(T), <_{\text{lex}})$ and $|X| = |Y| = \aleph_1$ then X embeds into Y as a linear order.
(3) If $X \subset K$ and $|\bar{X}| \leq \aleph_1$ then X is σ-scattered.

Note that if $L \subset K, |L| = \aleph_1, |\bar{L}| = \aleph_2$, then there is $L_0 \subset L$ such that $\bar{L}_0$ is $\aleph_2$-dense. To see this, for $b, b' \in L$, let $b \sim b'$ if there are at most $\aleph_1$ many elements of $\bar{L}$ in between b, b'. It is obvious that there are at least two distinct equivalence classes. We consider the set of equivalence classes as a linear order. Here, the equivalence classes are ordered by the order of their elements. Since the equivalence classes are convex subsets of L, this order is well defied.

The set of equivalence classes is $\aleph_1$-dense. In order to see this, let b, b' be two non-equivalent elements of L such that there are only countably many equivalence classes in between them. Note that these equivalence classes are disjoint convex sets. Let $\alpha \in \omega_1$ be large enough such that for each $t \in T_\alpha$ with $b <_{\text{lex}} t <_{\text{lex}} b'$, the set of all branches containing t intersects at most one equivalence class. For each $c \in \bar{L} \cap (b, b')$ there exists $t \in T_\alpha$ with $b <_{\text{lex}} t <_{\text{lex}} b'$ such that $t \in c$. So, there are only $\aleph_1$ many elements of $\bar{L}$ in between b, b', which is a contradiction. Now, let L_0 be a suborder of L that intersects each equivalence class at exactly one point. $\bar{L}_0$ is $\aleph_2$-dense. In order to see this, let $b \in \bar{L} \setminus \bar{L}_0$. Fix $t \in b$ such that $\mathcal{B}(T_t)$ intersect at most one equivalence class. Since $|\bar{L} \cap \mathcal{B}(T_t)| \leq \aleph_1$ and there are at most $\aleph_1$ many such $t \in T$, $|\bar{L} \setminus \bar{L}_0| \leq \aleph_1$.

Note that for such an L_0, the tree $\bigcup \bar{L}_0$ is an everywhere Kurepa subtree of T. So L_0 is isomorphic to an $\aleph_1$-sized dense suborder of K. This finishes the proof because all $\aleph_1$-sized dense suborders of K are biembeddable.

We will finish the article, with some remarks about the iteration of the forcings we used. The most important features of the forcings we used are $\Omega(T)$-completeness and $\aleph_2$-chain conditions. These forcings preserve the stationarity of stationary subsets of $\Omega(T)$, but they do not need to preserve the stationarity of stationary subsets of $\Gamma(T)$. In fact, some of the iterands we considered shoot clubs into the complement of some stationary subsets of $\Gamma(T)$. On the other hand the set $\Gamma(T)$ itself remains stationary in the final model we obtain, by Proposition 2.4. The only way to see that $\Gamma(T)$ is stationary is that ω_2 is preserved and consequently K is not σ-scattered. The phenomenon that only preserving ω_2 without any control on countable structures which come from $\Gamma(T)$ guarantees that $\Gamma(T)$ remains stationary seems to be new and mysterious. For instance, assume $S \subset \Gamma(T)$ is stationary and is not in the form of Ω or Γ of any suborder of K. Is there any way to determine whether or not S remains stationary in the extension under counatble support iterations of these forcings?

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