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Multidimensional inverse source problems of underwater acoustics

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We consider linear and nonlinear inverse source problems of sound radiation in an unbounded domain which models an oceanic waveguide. The method for analysing their solvability is based on analytical properties of generalized acoustic potentials and the theory of extremal problems.

1 Introduction

In this paper we are concerned with the theoretical study of acoustic inverse source problems for an unbounded domain $D \subset \mathbb{R}^n$ with a reflecting boundary S which models an oceanic waveguide. We shall refer to these problems as 'inverse source problems of underwater acoustics'. Four inverse source problems arising in underwater acoustics will be considered below. We refer to the first two as inverse source problems for the acoustic potential; they are acoustic analogues of linear and nonlinear inverse source problems for the gravitational potential which have been extensively studied by many authors (see Isakov [1] and Cherednichenko [2] and the references therein). The remaining two problems are linear and nonlinear inverse sound radiation extremum problems. They are infinite-dimensional analogues of the inverse extremal problems relevant to the active minimization of sound fields in regular acoustic waveguides. These problems have great practical importance (for a review of results and applications, see Elliott & Nelson [3] and Alekseev & Martynenko [4]).

Although inverse source problems for the acoustic potential are similar to inverse source problems for the gravitational potential, there exist two important differences between the former and the latter. The first is that the uniqueness of the corresponding direct boundary value problem is violated for domains of waveguide type when the sound frequency coincides with an eigenfrequency of the waveguide. The second is that the oceanic waveguide geometry is such that the Helmholtz equation should be considered not in the whole space $\mathbb{R}^n$ (as in the gravitational case), but in an unbounded region D with a reflecting boundary at which a homogeneous boundary condition is imposed. This complicates the analysis of inverse source problems of underwater acoustics because even the direct sound radiation problems for the above-mentioned domains are not well enough understood.

Due to these difficulties, the theory of acoustic inverse source problems started to be developed only recently. The first theoretical studies of linear problems were made for the case when $D = \mathbb{R}^3$ by Bleistein & Cohen [5], Porter & Devaney [6] and Devaney & Sherman [7], and for $D = \mathbb{R}^2$ by Porter & Devaney [8]. In these papers the non-uniqueness of inverse source problems was established by demonstrating the existence of so-called 'non-radiating sources' and sufficient conditions for an unknown density f, namely the minimum of its energy in the L^2-, were established which ensure the uniqueness of the recovered source. Later these results were generalized to inhomogeneous or dissipative media by Devaney & Porter [9] and Tsang *et al.* [10] and, for the case of unbounded domains in $\mathbb{R}^3$ with a reflecting boundary, by Alekseev & Chebotarev [11] and Alekseev [12]. We also mention recent papers by Alekseev & Chebotarev [13] and Alekseev [14], where the first results for acoustic nonlinear inverse source problems were presented for some model cases.

The purpose of the paper is to analyse the solvability, and partially the uniqueness, of inverse source problems of underwater acoustics outlined above. Exact formulations of these problems are given in § 2. In § 3 we present the solvability analysis of linear inverse source problems in the space $L^2(\Omega)$. We also suggest a procedure for extracting a unique stable solution by introducing a notion of a 'normal solution'. In § 4 we describe some results obtained for nonlinear inverse source problems.

2 Statement of direct and inverse source problems

2.1 Statement of a direct sound radiation problem

Let D be an unbounded domain of the space $\mathbb{R}^n$ ($n = 3$ or 2), occupied by the acoustic medium, with a reflecting boundary $S = \partial D$. We want to study acoustic wave propagation in D in the 'frequency domain'. Let sound volume sources be distributed with a density f. The direct problem of sound radiation consists of finding an acoustic field potential Φ satisfying Helmholtz equation with variable coefficients, or

$$L\Phi \equiv \rho \operatorname{div}\left(\frac{1}{\rho}\operatorname{grad}\Phi\right) + k^2\Phi = -f \quad \text{in } D, \tag{2.1}$$

as well as the following boundary condition on S:

$$\mathscr{B}\Phi \equiv a(\mathbf{x})\Phi + b(\mathbf{x})\frac{\partial \Phi}{\partial n} = 0 \text{ on } S, \tag{2.2 a}$$

and the radiation condition as $|\mathbf{x}| \to \infty$,

$$\Phi \in \mathscr{R}(D), \quad |\mathbf{x}| \to \infty; \tag{2.2 b}$$

here $a(\mathbf{x})$ and $b(\mathbf{x})$ are given functions describing the acoustic properties of the surface S, $\partial\Phi/\partial n$ denotes the outward normal derivative and $\mathscr{R}(D)$ is the set of functions $D \to \mathbb{C}$, i.e. complex-valued functions in D, satisfying a suitable radiation condition as $|\mathbf{x}| \to \infty$, to be discussed further below: the function $\rho(\mathbf{x})$ describes the density of the medium, and $k(\mathbf{x}) = \omega/c(\mathbf{x})$ is a variable wave number, where ω is the frequency and $c(\mathbf{x})$ is the sound speed.

Depending on the behaviour of the functions a and b, the reflecting boundary S can be divided into three parts: S_1 is the *soft boundary* where $a = 1,\ b = 0$, S_2 is the *hard boundary* where $a = 0,\ b = 1$ and S_3 is the *impedance boundary* where $a(\mathbf{x}) \neq 0,\ b(\mathbf{x}) \neq 0$; the so-called acoustic impedance $Z(\mathbf{x})$ which, by condition (2.2a), is equal to $b(\mathbf{x})/i\omega\rho(\mathbf{x})a(\mathbf{x})$, does not equal 0 or ∞ on S_3. In fact $Z = 0$ on S_1, $Z = \infty$ on S_2 and $0 \neq Z \neq \infty$ on S_3.

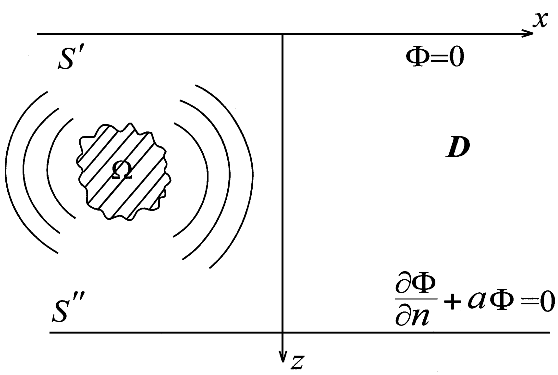

FIGURE 1. Geometry of a regular waveguide.

The problem (2.1)–(2.2) describes, for example, radiation and propagation of sound in an oceanic waveguide, and now we describe some of the relevant configurations. To this end we present the data for problem (2.1)–(2.2), i.e. the domain D, the coefficients ρ and k and the boundary operator $\mathscr{B}$ as a quadruple $\mathscr{W} = (D, \rho, k, \mathscr{B})$. Further on we shall refer to $(D, \rho, k, \mathscr{B})$ as the *waveguide quadruple*, or simply *waveguide*.

The simplest model of an infinitely deep homogeneous ocean is characterized by the relations

$$D = \mathbb{R}^n, \quad n = 3 \text{ or } 2, \quad \rho = \rho_0 = \text{const} > 0, \quad k = k_0 = \text{const} > 0. \tag{2.3}$$

For this model, equation (2.1) is the Helmholtz equation with constant coefficients

$$L\Phi \equiv \Delta\Phi + k_0^2\Phi = -f, \tag{2.4}$$

the boundary condition (2.2 a) is absent, and the set $\mathscr{R}(D)$ in (2.2 b) consists of functions $\Phi : \mathbb{R}^n \to \mathbb{C}$ satisfying the usual Sommerfeld radiation condition of the form

$$\frac{\partial\Phi(\mathbf{x})}{\partial|\mathbf{x}|} - ik_0\Phi(\mathbf{x}) = o(|\mathbf{x}|^{\frac{1-n}{2}}), \ |\mathbf{x}| \to \infty. \tag{2.5}$$

It is well-known (e.g. see Rellich [15]) that the radiation condition (2.5) indeed selects a unique solution of equation (2.4), at least if the support supp$f$ of the density f in equation (2.1) is bounded.

Another model of an ocean of finite depth corresponds to the domain D being unbounded in horizontal directions and bounded in the vertical direction under the condition that the upper surface S' is soft and the lower surface S'' is hard or of impedance-type. In the case where coefficients ρ and k are constant or depend only upon the vertical coordinate z, and both surfaces S' and S'' are plane-parallel (see Figure 1), this model is called a *regular* or *stratified* waveguide. In the more common case where ρ and k are constant or depend upon z only outside a compact set in $\overline{D}$ and the boundaries S' and S'' are locally curvilinear, as in Figure 2, we have a *locally non-regular* waveguide.

For both models, the space $\mathscr{R}(D)$ in equation (2.2 b) consists of functions satisfying the partial Sveshnikov radiation conditions (see Sveshnikov [16] for three dimensions and Alekseev *et al.* [17] for two dimensions). In contrast to equation (2.5), they require the additional condition that the frequency ω does not belong to the discrete set $\{\omega_1, \omega_2, \ldots\}$

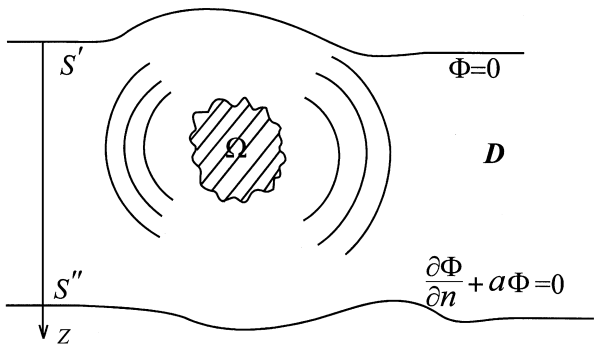

FIGURE 2. Geometry of a locally non-regular waveguide.

of *eigenfrequencies* for the considered waveguide quadruple $(D, \rho, k, \mathscr{B})$. Otherwise the respective homogeneous problem (2.1)–(2.2) with $f = 0$ has a nontrivial eigenfunction Φ.

The study of the solvability of the direct problem (2.1)–(2.2) poses serious difficulties. They are connected on the one hand with the unboundedness of the domain D, as well as its boundary S, and presence of the variable coefficients in equation (2.1) and the boundary operator $\mathscr{B}$ in equation (2.2 a) on the other hand. A series of papers is devoted to the study of this problem in the general case. Among them, one can mention [18, 19, 20]. The unique solvability of the Dirichlet problem for equation (2.4) in the domain D for which the boundary is only locally non-planar is studied in the first paper. In Zhang [19] the unique solvability of the respective transmission problem for the Helmholtz equation (2.1) with a locally non-planar interface is proved. Finally, the unique solvability of the boundary-value problem (2.1)–(2.2) for a locally non-planar waveguide in $\mathbb{R}^2$ is established by Alekseev & Komarov [20] in the absence of acoustic resonance, i.e. when ω is not an eigenfrequency of the waveguide considered. The importance of non-resonance condition is stressed by the fact that in a number of papers (see, for example, Evans *et al.* [21] and the references therein), it has been shown that acoustic resonances in locally non-planar waveguides are proved to exist. We assume below that ω is not an eigenfrequency for the wavequide $\mathscr{W}$ considered.

Let Ω be a bounded open subset of D, such that $\mathrm{supp} f \subseteq \Omega$. Then, if it exists, the solution Φ of the direct problem (2.1)–(2.2) can be formally represented as

$$\Phi(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y}, \tag{2.6}$$

where $G(\mathbf{x}, \mathbf{y}) \equiv G_{\mathscr{W}}(\mathbf{x}, \mathbf{y})$ is the Green's function of the direct problem (2.1)–(2.2). The free-space Green's function G has the form

$$G(\mathbf{x}, \mathbf{y}) = \frac{\exp(ik|\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|} \ \text{for}\ n = 3, \ G(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|) \ \text{for}\ n = 2, \tag{2.7}$$

where $H_0^{(1)}$ is the Hankel function of the first kind and order zero.

The formula (2.6) shows that the sound field is generated by the pair (Ω, f). We call the pair (Ω, f) a *volume radiating system*, and the set Ω itself a *volume antenna*. Thus the direct sound radiation problem consists of finding a potential $\Phi : D \to \mathbb{C}$ from the conditions (2.1)–(2.2), and it can be represented in a schematic form as $(\mathscr{W}; \Omega, f) \to \Phi$.

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2.2 Inverse source problems

Inverse source problems play important role in applications for obtaining information, for example, on the environment. These problems consist of finding an unknown radiating system (Ω, f) or, for example, a density f for a known Ω, given some information about the radiated acoustic field. This information may take many different forms. For example, one can choose the set of values $\Phi_Q(\mathbf{x})$ of the potential Φ at the points of some set Q located away from Ω. In the model case when D is the space $\mathbb{R}^3$ or half-space $\mathbb{R}^3_+$ the sphere S_∞ or hemisphere S^+_∞ of the radius R_∞ located far from Ω, is often taken as the set Q. In this case prescribing the potential Φ_∞ on S_∞ is equivalent, in fact, to prescribing the far-field pattern generated by the pair (Ω, f). From a theoretical point of view, the values of the potential Φ can be given in the domain outside the sphere S_∞ or in the domain $\Omega_e \equiv D \setminus \overline{\Omega}$. Inverse source problems in which the unknown sources are found from the measurements of the acoustic potential Φ at some set of points of the domain D are called *inverse source problems for the acoustic potential* by analogy with inverse source problems for the gravitational potential (sf. Cherednichenko [2]).

Information of another type is used in *extremum inverse source problems*. In these problems, the field $-\Phi_0$ which is generated by a primary source is given, and it is required to suppress it completely or to minimize it by the action of *secondary sources*. To formulate extremum problems mathematically, we introduce a cost functional J. Usually this depends upon the sum $\Phi - \Phi_0$, where Φ is the field created by the secondary sources sought. For example, one may take this quantity to be the total or potential sound energy of $\Phi - \Phi_0$ in some subset Q of the domain D, or the total acoustic power radiated by all sources (primary and secondary) to the waveguide far zone. The secondary sources are usually assumed to lie in some restricted set, for example, a ball in D. Minimizing this functional, one can determine the desired optimal distribution of the secondary sources and the minimal value J_{opt} of the functional J. This minimal value J_{opt} determines the *suppression level* (magnitude of the suppressed power) of the primary sound field by optimal secondary source [3, 4].

Now we formulate our inverse source problems, beginning with linear inverse source problems. Let Φ_e be the external acoustic potential in the complement Ω_e of Ω. For every function f defined in Ω (or even in D) let

$$f_\Omega(\mathbf{x}) = \begin{cases} f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \notin \Omega. \end{cases} \tag{2.8}$$

Problem Pr.1.1 (Inverse Source Problem). *Given a waveguide $\mathscr{W}$, a set Ω and a function Φ_e in Ω_e, find a function f such that the solution Φ of the direct sound radiation problem*

$$L\Phi = -f_\Omega \ \text{ in } \ D, \quad \mathscr{B}\Phi = 0 \ \text{ on } \ S, \quad \Phi \in \mathscr{R}(D) \tag{2.9}$$

satisfies the condition $\Phi = \Phi_e$ on Ω_e.

Further, let $Q \subset \Omega_e$ be a bounded open subset, Φ_0 be a given function in Q which represents a sound field generated in Q by a primary source as in Figure 3. Suppose also that X and Y are normed spaces of densities f in Ω and potentials Φ in Q, respectively. Let us introduce a cost functional J on X by $J(g) = \|\Phi_g - \Phi_0\|^2_Y$, where $\|\cdot\|_Y$ is a suitable

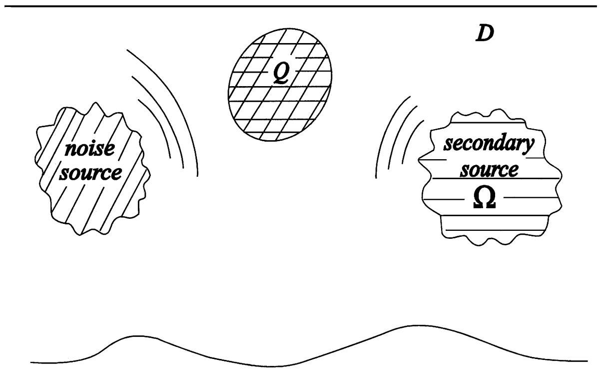

FIGURE 3. Geometry of primary and secondary sources.

norm on Y and let $\Phi_g \in Y$ be the unique solution of equation (2.9) with f replaced by g, where $g \in X$.

Problem Pr.1.2 (Inverse Extremum Source Problem). *Given a waveguide $\mathscr{W}$, sets Ω, Q and a function Φ_0 in Q, find $f \in X$ such that $J(f)$ is minimized on X.*

Now we formulate the nonlinear version of these problems in which we have find Ω rather than f. Let $B = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| < R\} \subset D$ be a ball in D and f be a given function defined in B; let Φ_e be the external potential defined in $B_e = D \setminus \overline{B}$.

Problem Pr.2.1 (Inverse Shape Problem). *Given a waveguide $\mathscr{W}$, a ball B and functions f, Φ_e defined in B, B_e, respectively, find a bounded open subset $\Omega \subset B$ such that the solution Φ of the problem* (2.9) *satisfies the condition $\Phi = \Phi_e$ in B_e.*

Again, let f be a given function defined in the ball B and denote by T a set of all bounded open subsets of B, and introduce a cost functional J on T by $J(\Omega) = \|\Phi_\Omega - \Phi_0\|_Y^2$, where Φ_Ω is a unique solution of equation (2.9) for a subset $\Omega \in T$.

Problem Pr.2.2 (Inverse Extremum Shape Problem). *Given a waveguide $\mathscr{W}$, sets B, Q and functions f, Φ_0 defined in B, Q, respectively, find a bounded open subset $\Omega \subset B$ such that the solution Φ_Ω of the problem* (2.9) *satisfies*: $J(\Omega) \to \inf$ on T.

Remark 2.1 Problems 1.1 and 2.1 are the acoustic analogues of the corresponding inverse source problems for the gravitational potential (sf. [1, 2]). Also, Problem 1.2 is an infinite-dimensional analogue of the linear problem of active sound minimization with a discrete antenna in a regular waveguide that has been studied numerically by Alekseev & Komarov [22, 23] and Stell & Bernhard [24, 25]. Similarly, Problem 2.2 is an infinite-dimensional analogue of the problem studied by Alekseev & Martynenko [4], Alekseev & Komarov [26] and Alekseev [27].

Remark 2.2 Let us denote by Ω_{opt} a solution of Pr.2.2, and assume that $J_{opt} = J(\Omega_{opt}) = 0$. Physically, this case corresponds to the complete suppression of the sound field of a primary source in the domain Q. Mathematically, this case implies the solution Ω_{opt} of Pr.2.2 is a solution of Pr.2.1. This of course holds under the condition that a given field Φ_0 in Q can be extended to the domain B_e as the exterior potential Φ_e satisfying $L\Phi_e = 0$. Thus, for the proof of the existence theorem for Pr.2.1, it suffices to prove the existence of the solution Ω_{opt} of the extremum Problem 2.2 for which $J(\Omega_{opt}) = 0$. We shall make use of this fact when studying Pr.2.1 in §4.

3 Inverse source problems

In what follows we deal both with functions defined on the whole domain D (the potential Φ is an example of such a function) and with functions defined on certain subsets of D, for instance, in Ω. Sometimes, it will be necessary to extend a function f defined in Ω by zero outside Ω as in equation (2.8). Equally, if a function Φ is defined in D, we consider its restrictions to certain subsets of D, for example, to $\Omega = \Omega_i$ and Ω_e. These restrictions are denoted by $S_i\Phi$, $S_e\Phi$, respectively, where the symbols S_i and S_e have the meanings of the restriction operators to the subsets Ω_i, Ω_e, respectively. Let $L_i = S_i \circ L$, $L_e = S_e \circ L$.

We use the function spaces $L^2(\Omega), W^{2,2}(\Omega), W^{2,2}_{loc}(D)$ and subspaces $\dot{W}^{2,2}(\Omega) = \{f \in W^{2,2}(\Omega) : f_\Omega \in W^{2,2}(D)\}, W^{2,2}_0(\Omega) =$ closure of $\mathscr{D}(\Omega)$ in $W^{2,2}(\Omega)$. Here $\mathscr{D}(\Omega)$ is the space of functions in $\mathbb{R}^n$ infinitely differentiable with compact supports in Ω, the function f_Ω being defined by equation (2.8). It is well-known that $W^{2,2}_0(\Omega) \subset \dot{W}^{2,2}(\Omega) \subset W^{2,2}(\Omega)$ for any open subset $\Omega \subset \mathbb{R}^n$. Furthermore, if $\Gamma \equiv \partial\Omega \in C^{0,1}$, i.e. Γ is Lipschitz-continuous, then

$$W^{2,2}_0(\Omega) = \dot{W}^{2,2}(\Omega). \tag{3.1}$$

We assume that the following conditions are imposed on $\mathscr{W}$ and Ω:

(1) $S \in C^2, k \in C^\infty(\overline{D}), \rho \in C^\infty(\overline{D});\ \rho \geqslant \rho_0 = const > 0; \mathrm{Im} k = 0$.

(2) the homogeneous direct problem (2.9) has only a trivial solution $\Phi = 0$ (i.e. ω is not an eigenfrequency of the waveguide $\mathscr{W} = (D, \rho, k, \mathscr{B})$ considered).

(3) a bounded open subset $\Omega \subset D$ is such that dist $(\overline{\Omega}, \partial D) > 0$, and $\Gamma \in C^{0,1}$.

This estimate is the consequence of the regularity conditions in point (1):

$$\| L\Phi \|_{L^2(\Omega')} \leqslant C_1 \| \Phi \|_{W^{2,2}(\Omega')} \quad \forall \Phi \in W^{2,2}_{loc}(D), \tag{3.2}$$

where Ω' is an arbitrary bounded open subset of D and the constant C_1 is independent of Ω', Φ. To impose additional conditions on the data let us consider the linear subset

$$V^{2,2}_\Omega(D) = \{\Psi \in W^{2,2}_{loc}(D) \cap \mathscr{R}(D) : L_e\Psi = 0 \ \text{ in } \ \Omega_e, \ \mathscr{B}\Psi = 0 \ \text{ on } \ S\}. \tag{3.3}$$

It follows from point (2) that the operator $L_i : V^{2,2}_\Omega(D) \to L^2(\Omega)$ is invertible. Therefore, there exists an inverse operator L_i^{-1} defined on the set $L_i[V^{2,2}_\Omega(D)] \subseteq L^2(\Omega)$. Every function $\Psi \in V^{2,2}_\Omega(D)$ is just the potential generated by the pair (Ω, f) with the density $f = -L_i\Psi$. Let $A = -L_i^{-1}$, $A_i = S_i \circ A$, $A_e = S_e \circ A$. One can easily formulate sufficient conditions for the data $(S, \rho, k, \mathscr{B})$ which provide:

(4) The operator L_i is surjective, so that $L_i[V^{2,2}_\Omega(D)] = L^2(\Omega)$, and the operator A is

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represented on the space $L^2(\Omega)$ by Green's function G of the direct problem (2.9) as

$$[Af](\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{y})\mathrm{d}\mathbf{y},\ \mathbf{x} \in D. \tag{3.4}$$

Also, by elliptic regularity theory, $A_i[C^\infty(\overline{\Omega})] \subset C^\infty(\Omega)$ and the following estimate holds:

$$\|Af\|_{W^{2,2}(\Omega')} \leqslant C_2\|f\|_{L^2(\Omega)} \quad \forall f \in L^2(\Omega), \tag{3.5}$$

where Ω' is an arbitrary bounded open subset of D, and the constant C_2 depends only upon $n, \mathscr{W}, \Omega \cup \Omega'$. When conditions (2.3) hold, the free-space Green's function G has the form of equation (2.7). In this case, all the above-mentioned properties of the operator A defined by equation (3.4) are well-known from the acoustic potential theory (see, for example, Alekseev [12]).

Consider the spaces $L^2(\Omega), W_0^{2,2}(\Omega), \dot{W}^{2,2}(\Omega), \mathscr{D}(\Omega)$. By extending functions in these spaces to be zero outside Ω, we obtain (closed) subspaces of the spaces $L^2(D), W^{2,2}(D)$, and $\mathscr{D}(D)$, respectively, which are denoted by $L^2_\Omega\ (D),\ W^{2,2}_{0,\Omega}(D), \dot{W}^{2,2}_\Omega(D), \mathscr{D}_\Omega(D)$. It follows from their definitions, (3.1) and point (3) above that $\mathscr{D}_\Omega(D) \subset W^{2,2}_{0,\Omega}(D) = \dot{W}^{2,2}_\Omega(D) \subset V^{2,2}_\Omega(D)$. Since the restriction of any element of such spaces to Ω_e equals to zero, we shall call them the zero external potential spaces. Now we study L_i, the images of $\dot{W}^{2,2}_\Omega(D)$ and $W^{2,2}_{0,\Omega}(D)$ in the space $L^2(\Omega)$. To this end, we introduce in $L^2(\Omega)$ two closed subspaces: $\dot{\mathscr{N}}^2(\Omega) =$ Ker A_e in $L^2(\Omega)$ and $\mathscr{N}^2_0(\Omega) =$ closure of $\mathscr{D}(\Omega) \cap \dot{\mathscr{N}}^2(\Omega)$ in $L^2(\Omega)$.

Lemma 3.1 *Assume that conditions* (1)–(4) *hold. Then*

$$L_i[\mathscr{D}_\Omega(D)] \equiv L[\mathscr{D}(\Omega)] = \mathscr{D}(\Omega) \cap \dot{\mathscr{N}}^2(\Omega),\ A[\mathscr{D}(\Omega) \cap \dot{\mathscr{N}}^2(\Omega)] = \mathscr{D}_\Omega(D). \tag{3.6}$$

Proof Let $g \in L_i[\mathscr{D}_\Omega(D)]$, i.e. $g = L_i\Phi_\Omega = L\Phi$, where $\Phi \in \mathscr{D}(\Omega)$. Since the supports of functions are not increased by differentiating, by (1), we have $g \in \mathscr{D}(\Omega)$. Besides, $g \in \dot{\mathscr{N}}^2(\Omega)$ as $Ag = -\Phi_\Omega$ by $A = -L_i^{-1}$. Therefore, $A_e g = 0$. This implies $g \in \mathscr{D}(\Omega) \cap \dot{\mathscr{N}}^2(\Omega)$.

Assume next that $g \in \mathscr{D}(\Omega) \cap \dot{\mathscr{N}}^2(\Omega)$ and $\Phi = -Ag$. Since $g \in C^\infty(\overline{\Omega})$ we have, by elliptic regularity theory, that $S_i\Phi \equiv A_i g \in C^\infty(\Omega)$. Also, there exists a strictly interior subset Ω' of Ω such that $g = 0$ outside of Ω'. The latter implies that the function Φ is a potential of Ω' and therefore, by (4), $L\Phi = 0$ in $\Omega'_e = D \setminus \overline{\Omega'}$. Finally, $\Phi = 0$ outside of Ω as $g \in \dot{\mathscr{N}}^2(\Omega)$. It follows from the unique extension principle (sf. Colton & Kress [28]) that $\Phi = 0$ outside of $\overline{\Omega'}$. Therefore, $\Phi \in \mathscr{D}_\Omega(D)$, and Lemma follows from $A = -L_i^{-1}$. □

Theorem 3.1 *Assume that the conditions of Lemma* 3.1 *hold. Then*

$$L_i[\dot{W}^{2,2}_\Omega(D)] \equiv L[\dot{W}^{2,2}(\Omega)] = \dot{\mathscr{N}}^2(\Omega),\ \ L_i[W^{2,2}_{0,\Omega}(D)] \equiv L[W^{2,2}_0(\Omega)] = \mathscr{N}^2_0(\Omega). \tag{3.7}$$

Proof We restrict ourselves to the proof of the second relation in equation (3.7) since the correctness of the former follows from the definitions of $\dot{W}^{2,2}_\Omega$ and $\dot{\mathscr{N}}^2(\Omega)$.

Let $g \in L_i[W^{2,2}_{0,\Omega}(D)]$, i.e. $g = L_i\Phi_\Omega \equiv L\Phi$, where $\Phi \in W^{2,2}_0(\Omega)$. By definition of $W^{2,2}_0(\Omega)$, there exists a sequence $\{\Phi_m\} \equiv \{\Phi_m\}_{m=1}^\infty$ such that

$$\Phi_m \in \mathscr{D}(\Omega),\ \Phi_m \to \Phi \text{ in } W^{2,2}(\Omega) \ \text{ as } \ m \to \infty. \tag{3.8}$$

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Let $g_m = L\Phi_m$. By equations (3.6) and (3.2), we have

$$g_m \in \mathscr{D}(\Omega) \cap \dot{\mathscr{N}}^2(\Omega),\ g_m \to g \text{ in } L^2(\Omega) \text{ as } m \to \infty. \tag{3.9}$$

This implies $g \in \mathscr{N}^2_0(\Omega)$. Now, let $g \in \mathscr{N}^2_0(\Omega)$. Then there exists a sequence $\{g_m\}$ satisfying equation (3.9). Let $\Psi = -Ag$, $\Psi_m = -Ag_m$. By equations (3.6), (3.5), $\Psi_m \in \mathscr{D}_\Omega(D)$, $S_e\Psi = 0$ and the functions $\Phi_m = S_i\Psi_m$, $\Phi = S_i\Psi$ satisfy equation (3.8). This means that $\Phi \in W_0^{2,2}(\Omega)$, $\Psi = \Phi_\Omega \in W^{2,2}_{0,\Omega}(D)$ and, therefore, $g = L_i\Psi \in L_i[W^{2,2}_{0,\Omega}(D)]$. □

It follows from Theorem 3.1 that $\dot{\mathscr{N}}^2(\Omega)$ and $\mathscr{N}^2_0(\Omega)$ are the density spaces generating a sound field with a zero external potential and, by equation (3.1), $\mathscr{N}^2_0(\Omega) = \dot{\mathscr{N}}^2(\Omega)$.

Remark 3.1 Using the terminology of Bleistein & Cohen [5], the pair (Ω, f) with $A_e f = 0$ is called a non-radiating system. It follows from Theorem 3.1 that the pair (Ω, f) is nonradiating iff $f \in \dot{\mathscr{N}}^2(\Omega) \equiv L[\dot{W}^{2,2}(\Omega)]$ provided that $f \in L^2(\Omega)$ and points (1)–(4) hold.

To introduce one more non-radiating criterion, denote by $\mathscr{K}^2(\Omega)$ the closed subspace in $L^2(\Omega)$ which consists of functions f satisfying $Lf = 0$ in Ω in the distributional sense:

$$\left\langle Lf, \frac{1}{\rho}\overline{\varphi}\right\rangle_{\mathscr{D}'(\Omega)\times\mathscr{D}(\Omega)} \equiv \int_\Omega \frac{1}{\rho} f L\overline{\varphi}\mathrm{d}\mathbf{x} = 0 \quad \forall\varphi \in \mathscr{D}(\Omega), \tag{3.10}$$

where '-' denotes the complex conjugate. We call an annihilator of a set $\mathscr{M} \subset L^2(\Omega)$ with power $1/\rho$ the closed subspace $\mathscr{M}^\perp_\rho$ of $L^2(\Omega)$ every element f of which satisfies

$$\int_\Omega \frac{1}{\rho} f(\mathbf{x})\overline{g}(\mathbf{x})\mathrm{d}\mathbf{x} = 0 \quad \forall g \in \mathscr{M}. \tag{3.11}$$

Theorem 3.2 *Assume that conditions of Lemma* 3.1 *hold. Then* $[\mathscr{N}^2_0(\Omega)]^\perp_\rho = \mathscr{K}^2(\Omega)$.

Proof Let $f \in \mathscr{K}^2(\Omega)$. We show that f satisfies equation (3.11) for any $g \in \mathscr{N}^2_0(\Omega)$. By definition of $\mathscr{N}^2_0(\Omega)$, there exists a sequence $\{g_m\}$ satisfying equation (3.9). Let $\varphi_m = -A_i g_m$. Then, by equation (3.6), $\varphi_m \in \mathscr{D}(\Omega)$ and $L\varphi_m = g_m$. Since $f \in \mathscr{K}^2(\Omega)$ we have

$$\int_\Omega \frac{1}{\rho} f(\mathbf{x})\overline{g}_m(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Omega \frac{1}{\rho} f(\mathbf{x}) L\overline{\varphi}_m(\mathbf{x})\mathrm{d}\mathbf{x} = 0, \quad m = 1, 2, \ldots. \tag{3.12}$$

Passing on here to the limit as $m \to \infty$ we obtain equation (3.11) for any $g \in \mathscr{N}^2_0(\Omega)$.

Now, let $f \in [\mathscr{N}^2_0(\Omega)]^\perp_\rho$, i.e. $f \in L^2(\Omega)$ and f satisfies equation (3.11) for $\mathscr{M} = \mathscr{N}^2_0(\Omega)$. As for any $\varphi \in \mathscr{D}(\Omega)$, by Lemma 3.1, $L\varphi \in \mathscr{D}(\Omega) \cap \dot{\mathscr{N}}^2(\Omega) \subset \mathscr{N}^2_0(\Omega)$ then it follows from (3.11) that f satisfies equation (3.10). This implies that $f \in \mathscr{K}^2(\Omega)$. □

Corollary 3.1 *Under conditions* (1)–(4)*, the pair* (Ω, f) *is non-radiating iff* $f \in [\mathscr{K}^2(\Omega)]^\perp_\rho$. *Also, the following orthogonal decomposition of* $L^2(\Omega)$ *with power* $1/\rho$ *exists*

$$L^2(\Omega) = \mathscr{K}^2(\Omega) \oplus \mathscr{N}^2_0(\Omega) \equiv \mathscr{K}^2(\Omega) \oplus \dot{\mathscr{N}}^2(\Omega). \tag{3.13}$$

Remark 3.2 Formula (3.13) is an analogue for an acoustic case of the well-known relation in gravitational potential theory concerning the decomposition of the space $L^2(\Omega)$ into orthogonal complements: $L^2(\Omega) = H \oplus H^{\perp}$. Here H consists of functions in $L^2(\Omega)$ harmonic in Ω. In our case the space H consists of functions $\Phi \in L^2(\Omega)$ satisfying $L\Phi = 0$ in the distributional sense, and the spaces $H = \mathscr{K}^2(\Omega)$ and $H^{\perp} = \mathscr{N}_0^2(\Omega)$ are orthogonal with power $1/\rho$, where ρ is a variable density of a medium. Also, instead of the analytic extension principle used in inverse source problems for the gravitational potential, we make here use of the *unique extension principle* which is valid for elliptic equations with smooth coefficients. Using equation (3.13) and other results obtained, we construct below the theory of solvability of Problem 1.1, which is analogous to the corresponding problem for the gravitational potential.

Let Φ_e be a given function in the set Ω_e. It follows from § 2 and definition of A_e that solving Problem 1.1 is reduced to finding a density $f \in L^2(\Omega)$ from the equation

$$[A_e f](\mathbf{x}) = \Phi_e(\mathbf{x}), \quad \mathbf{x} \in \Omega_e. \tag{3.14}$$

Denote by $V_{\Omega}^{2,2}(\Omega_e)$ a linear subset of $W_{loc}^{2,2}(\Omega_e)$ which consists of restrictions to Ω_e of functions from $V_{\Omega}^{2,2}(D)$. It follows from the definition of $V_{\Omega}^{2,2}(D)$ that each function $\Phi_e \in V_{\Omega}^{2,2}(\Omega_e)$ has the physical meaning of the external potential of the set Ω. So we have

Lemma 3.2 *Under conditions* (1)–(4) *a solution $f \in L^2(\Omega)$ of equation* (3.14) *exists if and only if $\Phi_e \in V_{\Omega}^{2,2}(\Omega_e)$, i.e. Φ_e admits an extension into D as a function Φ from $V_{\Omega}^{2,2}(D)$. Besides, a particular solution f of equation* (3.14) *is defined by $f = -L_i\Phi$.*

Following Stein [29], one can check that it is possible to extend Φ_e as a function Φ from $V_{\Omega}^{2,2}(D)$ due to condition (3), when the following conditions hold:

(5) $\Phi_e \in W_{loc}^{2,2}(\Omega_e) \cap \mathscr{R}(\Omega_e), \quad L\Phi_e = 0 \quad \text{in} \quad \Omega_e, \quad \mathscr{B}\Phi_e = 0$ on S

where the condition $\mathscr{B}\Phi_e = 0$ is absent when $D = \mathbb{R}^n$. As a result we obtain

Theorem 3.3 *Under conditions* (1)–(5)*, there exists an infinite set of solutions of equation* (3.14) *from the space $L^2(\Omega)$. These solutions and only these can be represented as*

$$f = -L_i\Phi - f_0 = -L(\Phi_i + \Phi_0), \tag{3.15}$$

where f_0 (or Φ_0) is an arbitrary function of the space $\dot{\mathscr{N}}^2(\Omega)$ (or $\dot{W}^{2,2}(\Omega)$), $\Phi_i = S_i\Phi$, where $\Phi \in V_{\Omega}^{2,2}(D)$ is an extension of a given function Φ_e into D.

Theorem 3.3, along with equation (3.15), provides a simple way for constructing the set of all solutions to the equation (3.14) (or Pr.1.1). The cardinality of this set is determined by the cardinality of the class $\dot{\mathscr{N}}^2(\Omega) = L[\dot{W}^{2,2}(\Omega)]$ of densities generating the sound field with a zero external potential. This is a consequence of the statement of Pr.1.1. In fact, the crucial property for the sound radiation problems is that the sources are located in a limited part of the waveguide while the radiated field of interest is located outside the domain occupied by sources and, as a rule, at large distances from the sources. Therefore, if only a far field is of interest, then the potentials of the class $\dot{W}_{\Omega}^{2,2}(D)$ do not differ from each other as they describe a 'zero' radiated field while all the densities $f = -L_i\Psi$, where Ψ belongs to $\dot{W}_{\Omega}^{2,2}(D)$, are different and form the set $\dot{\mathscr{N}}^2(\Omega)$. Thus, the same zero radiated field corresponds to different densities of the class $\dot{\mathcal{N}}^2(\Omega)$ and this fact alone implies non-uniqueness of the solution of Pr.1.1. However, if we were to consider a radiated field in the whole domain D including the set Ω where sources are located, then, by condition (2), every radiated field $\Psi \in \dot{W}^{2,2}_{\Omega}(D)$ would correspond to a single density $f = -L_i\Psi \in \dot{\mathcal{N}}^2(\Omega)$.

Denote by $S^2(\Omega, \Phi_e)$ the set all solutions of Pr.1.1 which correspond to given Φ_e. This set is empty if $\Phi_e \notin V^{2,2}_{\Omega}(\Omega_e)$ or is a translation of the closed subspace $\dot{\mathcal{N}}^2(\Omega)$ of the space $L^2(\Omega)$ by an element $L_i\Phi$ (see Theorem 3.3). In the latter case $S^2(\Omega, \Phi_e)$ is a convex closed subset of $L^2(\Omega)$ and therefore it contains a unique element f^+ with a minimum norm $\|f^+/\sqrt{\rho}\|_{L^2(\Omega)}$. This element f^+ is called a normal solution of equation (3.14) (or Pr.1.1).

By virtue of properties of normal solutions of linear equations in Hilbert spaces, we have that $f^+ \in [\dot{\mathcal{N}}^2(\Omega)]^{\perp}_{\rho}$. By equation (3.13), it follows from this fact that $f^+ \in \mathcal{K}^2(\Omega)$, i.e. f^+ satisfies the homogeneous Helmholtz equation $Lf^+ = 0$ in Ω. Conversely, if $f \in S^2(\Omega, \Phi_e)$ is an element of $\mathcal{K}^2(\Omega)$, then by equation (3.13), f is a normal solution of Pr.1.1. Let $\Phi_e \in V^{2,2}_{\Omega}(\Omega_e) \cap W^{2,2}(\Omega_e)$, and let $\Phi_i = S_i\Phi$, where Φ is an extension of Φ_e into D. Then, by continuity of the extension operator [29], $\| \Phi_i \|_{W^{2,2}(\Omega)} \leqslant C_3 \| \Phi_e \|_{W^{2,2}(\Omega_e)}$, where a constant C_3 is independent of Φ_e. Since $L\Phi_i$ is a particular solution of Pr.1.1, then we obtain, by definition of the normal solution and equation (3.2), the estimate

$$\| \frac{1}{\sqrt{\rho}} f^+ \|_{L^2(\Omega)} \leqslant \| \frac{1}{\sqrt{\rho}} L\Phi_i \|_{L^2(\Omega)} \leqslant \frac{C_1}{\sqrt{\rho_0}} \| \Phi_i \|_{W^{2,2}(\Omega)} \leqslant C \|\Phi_e\|_{W^{2,2}(\Omega_e)}, \; C = \frac{C_1 C_3}{\sqrt{\rho_0}}, \tag{3.16}$$

which means the stability of f^+ in $L^2(\Omega)$-norm. Now we formulate these results as

Theorem 3.4 *Assume that conditions* (1)–(4) *hold. Then:*

(1) *the function $f \in S^2(\Omega, \Phi_e)$ belongs to the space $\mathcal{K}^2(\Omega)$, i.e. f satisfies the homogeneous Helmholtz equation $Lf = 0$ if and only if f is the normal solution of Pr.1.1;*

(2) *the normal solution $f^+ \in L^2(\Omega)$ of Pr.1.1 for any $\Phi_e \in V^{2,2}_{\Omega}(\Omega_e)$ exists, is unique and is stable with respect to small perturbations of Φ_e in the $W^{2,2}(\Omega_e)$-norm.*

Now we turn to Pr.1.2. Consider the case where the domain Q is $\Omega_e \cap B_R$. Here B_R is a ball of a radius R such that $\overline{\Omega} \subset B_R$. Let $X = L^2(\Omega)$, $Y = W^{2,2}(Q)$, and $\Phi_0 \in Y$ be a given function defined in $Q, J(g) = \|A_e g - \Phi_0\|^2_Y$. It follows from the statement of Pr.1.2 that solving Pr.1.2 for this case is equivalent to finding a minimizer of J.

Theorem 3.5 *Under conditions* (1)–(4), *for any $\Phi_0 \in Y$ there exists at least one solution $f \in X$ of Pr.1.2. The normal solution $f^+ \in X$ of Pr.1.2 exists and is unique.*

Proof Let $J_* = \inf_{g\in X} J(g) \equiv \inf_{g\in X} \|A_e g - \Phi_0\|^2_Y$. Denote by $\{g_m\} \equiv \{g_m\}^{\infty}_{m=1}$ a minimizing sequence for J. Let $\Phi_m = A_e g_m$. Consider for any m the set $S^2(\Omega, \Phi_m)$ of all solutions of Pr.1.1 corresponding to Φ_m. It is clear that $S^2(\Omega, \Phi_m) \neq \emptyset$ and for any $f_m \in S^2(\Omega, \Phi_m)$ (in particular, for the normal solution f^+_m) we have $\|A_e f^+_m - \Phi_0\|_Y = \|A_e g_m - \Phi_0\|_Y = \|\Phi_m - \Phi_0\|_Y \to J_*$ as $m \to 0$. Hence it follows that $\{f^+_m\}$ as well as $\{g_m\}$ are minimizing sequences for J and, besides, $\|\Phi_m\|_Y \leqslant C_4 = \text{const}$. Proceeding as in the proof of the

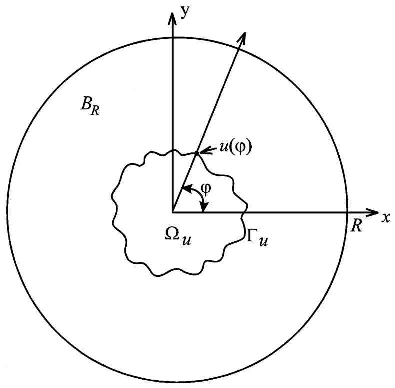

FIGURE 4. Geometry of an unknown domain in inverse shape problem.

estimate (3.16), we deduce that $\|f_m^+/\sqrt{\rho}\|_X \leqslant C_R\|\Phi_m\|_Y \leqslant C_R C_4$, where C_R is a constant. Thus, there exists a subsequence of $\{f_m^+\}$, which is denoted by $\{f_m^+\}$, and a function $f \in X$ such that $f_m^+ \to f$ weakly in X as $m \to \infty$. By (3.5) for $\Omega' = Q$, we have the continuity of A_e under conditions (i)-(iv). Then $A_e f_m - \Phi_0 \to A_e f - \Phi_0$ weakly in Y as $m \to \infty$. Finally, by the weak lower-semicontinuity of the norm we have $\|A_e f - \Phi_0\|_Y \leqslant \liminf \|A_e f_m - \Phi_0\|_Y = J_*$. This implies that f is a solution of Pr.1.2. The existence and uniqueness of the normal solution $f^+ \in X$ of Pr.1.2 are obvious. □

4 Inverse shape problems

For simplicity, we consider here the case of two dimensions. Three-dimensional case can be studied in a similar way. Let $\Sigma = [0, 2\pi)$. Denote by $H^s(\Sigma)$ the function space which consists of the restrictions to Σ of the Sobolev space $H^s_{2\pi}(\mathbb{R})$ with a norm $\|\cdot\|_s$ periodic with a period 2π functions (these restrictions are continuous for $s > 1/2$). Introduce in $H^s(\Sigma)$ a convex closed subset $K = K^s_M = \{v \in H^s : v \geqslant 0, \|v\|_s \leqslant M\}$, where M is a constant such that $\|v\|_{C(\Sigma)} \leqslant C_s\|v\|_s \leqslant C_s M < R$, and, by the Sobolev imbedding theorem, the constant C_s is independent of v. Every function $u \in K$ defines a continuous (and invertible at $u > 0$ on Σ) map $\mathbf{y}^u : \Sigma \to \mathbb{R}^2$. It acts as $\Sigma \ni \varphi \to \mathbf{y}^u(\varphi) = (u(\varphi), \varphi)_p \equiv \mathbf{y}^u_\varphi$, where $(r, \varphi)_p$ are polar coordinates of a point $\mathbf{y} \in \mathbb{R}^2$. Let $\Gamma_u = \left\{\mathbf{y} = \mathbf{y}^u_\varphi \equiv (u(\varphi), \varphi)_p, \varphi \in \Sigma\right\}$ be the $\mathbf{y}^u$-image of Σ and let $\Omega_u = \mathrm{int}\Gamma_u$ (see Figure 4).

We assume that the following conditions hold:

Q is a bounded open subset of D such that $\overline{B} \cap \overline{Q} = \emptyset, f \in C(\overline{B}), \Phi_0 \in L^2(Q)$. (4a)

The Green's function $G(\mathbf{x}, \mathbf{y}) \equiv G_{\mathscr{W}}(\mathbf{x}, \mathbf{y})$ of the direct problem (2.1)–(2.2) exists, is unique, symmetric and is continuous on $\overline{B} \times \overline{Q}$. (4b)

Denote by T_K a set of bounded open subsets of B which boundaries are described by functions $v \in K$. In what follows we shall study the both nonlinear problems: Pr.2.1 and Pr.2.2 under additional condition that an unknown subset Ω belongs to the set T_K. It follows from (4b) that Pr.2.1 on the set T_K is reduced to finding a solution $u \in T_K$ of the nonlinear first-kind integral equation

$$[Fu](\mathbf{x}) \equiv \int_0^{2\pi} \int_0^{u(\varphi)} G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) r \mathrm{d}r \mathrm{d}\varphi = \Phi_e(\mathbf{x}), \quad \mathbf{x} \in B_e \tag{4.1}$$

and Pr.2.2 on T_K is reduced to finding a quasisolution $u \in K$, i.e. a least-square solution, on the set K of

$$[Fu](\mathbf{x}) = \Phi_0(\mathbf{x}), \quad \mathbf{x} \in Q. \tag{4.2}$$

The last problem is equivalent to finding a function u such that

$$u \in K \text{ and } J(u) \leqslant J(v) \equiv \int_Q |Fv - \Phi_0|^2 d\mathbf{x} \ \ \forall v \in K. \tag{4.3}$$

Note that the set Ω_u associated with the solution u of (4.1) or (4.3) is reconstructed from u by $\overline{\Omega}_u = \{\mathbf{y} = (r, \varphi)_p : 0 \leqslant r \leqslant u(\varphi), \varphi \in \Sigma\}$.

Denote by $(,)$ the inner product in $L^2(Q)$ and by $<,>$ the duality pairing on $H^{-s}(\Sigma) \times H^s(\Sigma)$, where $H^{-s}(\Sigma) = (H^s(\Sigma))'$ for $s \geqslant 0$. Let

$$V[u,h](\mathbf{x}) = \int_\Sigma G(\mathbf{x}, \mathbf{y}^u_\varphi) f(\mathbf{y}^u_\varphi) u(\varphi) h(\varphi) \mathrm{d}\varphi, \ \ U[Q,g](\mathbf{y}) = \int_Q G(\mathbf{x}, \mathbf{y}) g(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{4.4}$$

Lemma 4.1 *Under conditions* (4a) *and* (4b) *the operator* $F : H^s(\Sigma) \to L^2(Q)$ *in equation* (4.1) *and the functional* $J : H^s(\Sigma) \to \mathbb{R}$ *in equation* (4.3) *are continuous and Frechét-differentiable on the set* K *in the relative topology of* $H^0(\Sigma)$. *Besides, for every* $u \in K, h \in H^s(\Sigma)$

$$F'(u)h = V[u,h], \ \langle J'(u), h \rangle = 2\mathrm{Re} \int_\Sigma U[Q, \overline{Fu - \Phi_0}](\mathbf{y}^u_\varphi) f(\mathbf{y}^u_\varphi) u(\varphi) h(\varphi) \mathrm{d}\varphi. \tag{4.5}$$

Proof The statement of Lemma 4.1 is a direct consequence of equations (4.1), (4.3) and conditions (4a), (4b) (see also Alekseev & Chebotarev [13] and Alekseev [14] in model cases). □

Theorem 4.1 *Under conditions* (4a), (4b) *there exists at least one solution* u *of problem* (4.3).

Proof To prove this result it suffices to consider a minimizing sequence $\{v_m\} \in K$ which is bounded in $H^s(\Sigma)$ and to take into account the lower-semicontinuty of J. □

Let $u \in K$ be an arbitrary solution of equation (4.3), Γ_u be the curve associated with u and $\Omega_u = \mathrm{int}\Gamma_u$ be a corresponding solution of Pr.2.2. Consider the function $g \equiv \overline{Fu - \Phi_0} \in L^2(Q)$ which is the residual of equation (4.2) for the solution u. Let

$$\Psi(\mathbf{y}) \equiv \Psi_1(\mathbf{y}) + i\Psi_2(\mathbf{y}) = U[Q,g](\mathbf{y}) \equiv \int_Q G(\mathbf{x}, \mathbf{y}) g(\mathbf{x}) \mathrm{d}\mathbf{x}. \tag{4.6}$$

It follows from properties of G that Ψ is the potential generated in D by the pair (Q, g) and

$$L\Psi = -g \text{ in } Q, \ L\Psi = 0 \text{ in } Q_e = D \setminus \overline{Q}, \ \mathscr{B}\Psi = 0 \text{ on } \partial D \text{ and } \Psi \in W^{2,2}_{loc}(D) \cap \mathscr{R}(D). \tag{4.7}$$

Since J is a differentiable functional we have $\langle J'(u), v - u \rangle \geqslant 0 \ \forall v \in K$. Furthermore, if

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the solution u of equation (4.3) is an interior point of K, i.e. u satisfies the conditions

$$u(\varphi) > 0 \text{ on } \Sigma, \quad \|u\|_s < M \tag{4.8}$$

then, by equation (4.5), we have

$$\mathrm{Re}[\Psi f](y) \equiv \mathrm{Re}\{U[Q, \overline{Fu - \Phi_0}](\mathbf{y}) f(\mathbf{y})\} = 0 \text{ on } \Gamma_u. \tag{4.9}$$

Based on equation (4.9) we formulate and prove a conditional existence theorem of a solution of Pr.2.1. For this purpose, we need Theorem 3.4 on properties of the normal solution of Pr.1.1 from the class $\mathscr{K}^2(\Omega)$ which was proved in § 3 under conditions (1)–(4) imposed on the pair $(\mathscr{W}, \Omega)$. Also we need the additional condition on $\mathscr{W}$ concerning some properties of solutions of the direct problem in D. Let us denote by Q_e^∞ the unbounded connected component of the compliment $Q_e = D \setminus \overline{Q}$ of the bounded open subset Q of D. (We note that $Q_e^\infty = Q_e$ in the particular case when Q is a simply-connected domain). The condition is:

the waveguide $\mathscr{W}$ is such that every function $\Psi \in W^{2,2}_{loc}(D)$ satisfying

$$L\Psi = 0 \text{ in } Q_e^\infty, \ \mathscr{B}\Psi = 0 \text{ on } \partial D, \ \Psi \in \mathscr{R}(D) \text{ and } \mathrm{Re}\Psi \text{ (or } \mathrm{Im}\Psi) = 0 \text{ in } Q_e^\infty$$

vanishes identically in Q_e^∞. (4c)

It has been shown by Alekseev *et al.* [17] that condition (4c) (which means, in fact, non-existence of a pure imaginary or pure real external potential Ψ_e in the unbounded subdomain Q_e^∞ of D) is true for $k^2 = \mathrm{const} > 0$, $\rho = \mathrm{const} > 0$ in the case where D is an unbounded domain with a smooth compact boundary or a half-plane $y > 0$ or the plane $D = \mathbb{R}^2$. Similar results for $\mathbb{R}^3$ follow from Rellich [15] (see also Alekseev [14]). There are also examples of a waveguide type domain D for which there exists a pure imaginary (or real) external potential Ψ_e (see Alekseev [14] and Alekseev *et al.* [17]) and therefore condition (4c) is not valid. So one can assume that the breakdown of (4c) is a manifestation of the effects of waveguide properties of the domain D.

By analogy with § 3, denote by $\mathscr{K}^2(Q)$ the closed subspace in $L^2(Q)$ which consists of functions g satisfying the equation $Lg = 0$ in the distribution sense.

Theorem 4.2 *In addition to* (1), (2), (3) *for* Q, (4) *and* (4a)–(4c), *let the following conditions hold:*

$f = Ref > 0$ *on* $\overline{B}$; (4d)

Q_e *is a connected subdomain of* D *and* $\partial Q \in C^{0,1}$, $\Phi_0 \in \mathscr{K}^2(Q)$; (4e)

there exist a solution $u \in H^s(\Sigma), s > 1/2$, *of* (4.3) *satisfying relations* (4.8) *such that zero is not a Dirichlet eigenvalue of the operator* L *in the set* Ω_u. (4f)

Then the function u *is a solution of the equation* (4.2).

Proof Let $u \in K$ be the solution of problem (4.3) satisfying all the conditions in condition (4f). It is obvious from definition of K that $L(Fu) = 0$ in B_e so that, by condition (4a), we have $Fu\,|_Q \in \mathscr{K}^2(Q)$. Hence it follows from condition (4e) and the condition $\mathrm{Im}k^2 = 0$ in (3) that $g \equiv \overline{Fu - \Phi_0} \in \mathscr{K}^2(Q)$. Consider the function $\Psi = U[Q, g]$ in equation (4.6). As $\Omega_u \subset Q_e$ and $\mathrm{Im}k^2 = 0$, we deduce from equations (4.7) and (4.9) that

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the restriction $\Psi_1 |_{\Omega_u}$ of the real part Ψ_1 of Ψ to the domain Ω_u is a solution of the homogeneous Dirichlet problem $L\Psi_1 = 0$ in Ω_u, $\Psi_1 = 0$ on Γ_u. Using the condition (4f) we conclude that $\Psi_1 = 0$ on $\overline{\Omega}_u$. Since $L\Psi_1 = 0$ in Q_e and, by (4e), the set $Q_e \equiv Q_e^\infty$ is connected, it follows from the unique extension principle that $\Psi_1(\mathbf{y}) \equiv 0$ in Q_e. Together with equations (4.7), this means that the function Ψ satisfies all the conditions in (4c). Taking into consideration condition (4c), we deduce that $\Psi = 0$ in Q_e. Thus the function Ψ is a zero external potential of the field generated by the pair (Q, g) with the density $g \in \mathscr{K}^2(Q)$. It follows from Theorem 3.4 that $g = 0$ in Q. Then we have $Fu - \Phi_0 = 0$ in $Q \Rightarrow Fu = \Phi_0$ in Q. □

We end this section with the following consequence to Theorem 4.2, which supplements Remark 2.2.

Corollary 4.1 *Under conditions of Theorem* 4.2, *the given function* Φ_0 *in equation* (4.2) *permits the (unique) extension* Φ_e *to the set* B_e *which satisfies the conditions*

$$L\Phi_e = 0 \text{ in } B_e, \quad \mathscr{B}\Phi_e = 0 \text{ on } D, \quad \Phi_e \in \mathscr{R}(D).$$

Moreover the function u mentioned in Theorem 4.2 *is simultaneously a solution of equation* (4.1), *and the set* Ω_u *corresponding to u is a solution of the inverse shape Problem 2.1 with the external potential* Φ_e.

5 Conclusions

In this paper we have studied inverse source and shape problems of sound radiation in an unbounded domain D with a reflecting boundary. We have given the exact formulation of these inverse problems and developed, under some assumptions on the data (i.e. the domain D and variable coefficients of differential and boundary operators), the theory of the solvability of the inverse source problem for the acoustic potential (Problem 1.1). We have also deduced the formula (3.13) of the orthogonal decomposition with power $1/\rho$ of the space $L^2(\Omega)$ into two orthogonal subspaces, the first one of which, $\mathscr{K}^2(\Omega)$, consists of functions Φ satisfying the homogeneous Helmholtz equation $L\Phi = 0$ in Ω. Based on the decomposition (3.13), we have proved the existence, uniqueness and stability of the normal solution of Problem 1.1, i.e. a solution (density) f with the minimal $L^2(\Omega)$ norm $f/\sqrt{\rho}$. These results can be considered as the generalization to acoustics of the well-known theory of the solvability for the gravitational inverse source problem (cf. Cherednichenko [2]).

Also we have proved the existence theorems for extremum Problems 1.2 and 2.2 which have no gravitational analogues but are infinite-dimensional analogues of the inverse extremal problems of the active minimization of sound fields in regular acoustic waveguides. Finally we have established sufficient conditions for the data and a solution of the nonlinear extremum Problem 2.2 under which it is a solution of the corresponding nonlinear inverse shape Problem 2.1. Using this result, we intend to develop further the theory of the local solvability for inverse shape problems in acoustics.

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