ON THE ADAMS SPECTRAL SEQUENCE

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1. Introduction. One of the really significant advances in stable homotopy theory has been the Adams spectral sequence (see **(1)** for a general discussion). To date there has been no useful general way to obtain differentials in this spectral sequence. There is a general feeling that these differentials come about because of some geometric fact which forces a difference between the ring structure imposed on $\pi_*(S^0)$, the stable homotopy ring, by the spectral sequence and its own natural one. The purpose of this note is to show how such a difference produces a differential and to give an exposition to the general ideas behind the calculations in **(4)**. The particular example which is exploited is the observation that in $\mathrm{Ext}_A(Z_2, Z_2)$, the E_2 term of the Adams spectral sequence (which is often abbreviated $H^{*,*}(A)$), $h_1^2h_3 + h_2^3 = 0$. The related equation in homotopy is false. Indeed $\eta^2\sigma = \eta\bar{\nu} + \eta\epsilon$ while $\nu^3 = \eta\bar{\nu}$ **(7)**. In an appropriate complex this difference gives rise to an Adams differential (Proposition 2.2). This differential is then used to obtain an Adams differential in the spectral sequence for a sphere (Theorem 3.1). Most of the calculations of **(4)** are done in just this way.

2. In this section we shall use the phenomenon mentioned in the Introduction to display a non-zero differential in the Adams spectral sequence for the space $X = S^0 \cup_\nu e^4 \cup_\sigma e^8$. We must begin by calculating the E_2 term of this spectral sequence, that is, $\mathrm{Ext}_A^{**}(H(X), Z_2)$. (We shall write $H(X)$ for $\tilde{H}^*(X; Z_2)$.) The cofibring

$$S^0 \xrightarrow{i} X \xrightarrow{p} S^4 \vee S^8$$

yields the following short exact sequence of cohomology groups:

$$0 \leftarrow H(S^0) \xleftarrow{i^*} H(X) \xleftarrow{p^*} H(S^4) \oplus H(S^8) \leftarrow 0.$$

This yields the following long exact sequence of Ext groups:

$$\ldots \to H^{s,t}(A) \xrightarrow{i*} \mathrm{Ext}_A^{s,t}(H(X), Z_2) \xrightarrow{p*} H^{s,t-4}(A) \oplus H^{s,t-8}(A) \to \ldots .$$

(We shall write $H^{s,t}(A)$ for $\mathrm{Ext}_A{}^{s,t}(Z_2, Z_2)$.) Here the coboundary is given by 2.6.1 of **(2)**; we have

$$\delta(\alpha, \beta) = h_2\alpha + h_3\beta.$$

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In particular, we have

$$\delta(h_2^2, h_1^2) = h_2^3 + h_1^2 h_3 = 0,$$

and therefore

$$(h_2^2, h_1^2) = p_* x$$

for some x in $\mathrm{Ext}_A^{2,12}(H(X), Z_2)$.

PROPOSITION 2.1. *A Z_2 base of $\sum \mathrm{Ext}_A^{s,t}(H(X), Z_2)$, where the sum runs over $t - s = 9$, is given by*

$$i_*(h_1 c_0) \in \mathrm{Ext}^{4,13} \quad \textit{and} \quad i_*(P^1 h_1) \in \mathrm{Ext}^{5,14}.$$

A Z_2 base of $\mathrm{Ext}_A^{s,t}(H(X), Z_2)$, where the sum runs over $t - s = 10$, is given by

$$x \in \mathrm{Ext}^{2,12} \quad \textit{and} \quad i_*(P^1 h_1^2) \in \mathrm{Ext}^{6,16}.$$

In this proposition, the notation for elements of $H^{**}(A)$ is as in Appendix A. This proposition follows by an easy calculation from the known values of $H^{s,t}(A)$ (see **(1)** or the Appendix).

PROPOSITION 2.2. *In the Adams spectral sequence for $X = S^0 \cup_\nu e^4 \cup_\sigma e^8$ we have*

$$d_2(x) = i_*(h_1 c_0).$$

Proof. In the Adams spectral sequence for S^0, c_0 corresponds to some homotopy element y, which must be a linear combination of $\bar{\nu}$ and ϵ **(7)**. Thus $h_1 c_0$ corresponds to the homotopy element ηy. which must be a linear combination of $\eta\bar{\nu}$ and $\eta\epsilon$. But the equations **(7)**

$$\nu^3 = \eta\bar{\nu}, \qquad \eta^2\sigma = \eta\bar{\nu} + \eta\epsilon$$

show that in X we have $i_*(\eta\bar{\nu}) = 0$, $i_*(\eta\epsilon) = 0$. Hence in the spectral sequence for X, $i_*(h_1 c_0)$ is zero in E_∞. This can only happen if $d_2(x) = i_*(h_1 c_0)$.

3. In this section we shall use § 2 to calculate a non-zero differential in the Adams spectral sequence for a sphere.

THEOREM 3.1. *In the Adams spectral sequence for S^0, $d_2 e_0 = h_1^2 d_0$.*

(The notation for elements of $H^{**}(A)$ is as in Appendix A.)

Remark. In **(1)** there is an erroneous evaluation of the differentials in this range of the spectral sequence. May **(5)** proved Theorem 3.1 by simply comparting his calculations with those of Toda in the 16 and 17 stem and found this as the only consistent possibility. Our proof also uses calculations from Toda but only compositions which involve stems in dimension less than 16. Also all of these calculations could be made in the framework of the Adams spectral sequence rather easily. Doing this would not add to the clarity.

The idea of the proof is to transfer the result of § 2 to the sphere by naturality. We shall need the following maps:

$$S^7 \cup_\nu e^{11} \cup_\sigma e^{15} \xrightarrow{f} S^0 \cup_\nu e^4,$$

$$S^0 \xrightarrow{j} S^0 \cup_\nu e^4.$$

We begin by constructing f.

PROPOSITION 3.2. *There exists a map*

$$f\colon S^7 \cup_\nu e^{11} \cup_\sigma e^{15} \to S^0 \cup_\nu e^4$$

such that (i) *f has filtration $\geqslant 2$, and* (ii) *the following diagram is commutative*:

$$\begin{array}{ccc} S^7 \cup_\nu e^{11} \cup_\sigma e^{15} & \xrightarrow{f} & S^0 \cup_\nu e^4 \\ i\uparrow & & \downarrow q \\ S^7 & \xrightarrow{2\nu} & S^4 \end{array}$$

The proof will be given in § 4.

In order to exploit the map f, we need the following proposition.

PROPOSITION 3.3 **(6)**. *Let $f\colon X \to Y$ be a map of filtration $\geqslant \phi$, and let $F \in \mathrm{Ext}_A^{\phi,\phi}(H(Y), H(X))$ be a representative in E_2 for the class of f in E_∞. Then there exists an induced homomorphism f_* from the Adams spectral sequence of X to the Adams spectral sequence of Y with the following properties:*

(i) *f_* maps $E_r^{s,t}(X)$ in to $E_r^{s+\phi,t+\phi}(Y)$.*

(ii) *On $E_2^{**}(X)$, f_* coincides with Yoneda multiplication by F.*

(iii) *f_* commutes with the differentials.*

The behaviour of our classes in Ext under the maps f, j is given by the following proposition, which is proved in § 4.

LEMMA 3.4. *We have*

$$f_* i_* h_1 c_0 = j_*(h_1^2 d_0) \neq 0 \in \mathrm{Ext}_A^{6,22}(H(S^0 \cup_\nu e^4), Z_2).$$

Proof of Theorem 3.1. We have $f_* d_2 x = f_* i_* h_1 c_0 = j_* h_1^2 d_0$. Hence $f_* x \neq 0$ and $d_2 f_* x = j_* h_1^2 d_0$. The only class in $\mathrm{Ext}^{4,21}(H(S^0 \cup_\nu e^4), Z_2)$ is $j_* e_0$. Now 3.1 follows since d_2 commutes with j_*.

4. Proof of Proposition 3.2. First we shall construct the map f with the desired properties. By a calculation similar to that of § 2 we see that $\mathrm{Ext}^{2,9}(H(S^0 \cup_\nu e^4), Z_2)$ has a basis consisting of two classes, $j_* h_0 h_3$ and a

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class y such that $q_* y = h_0 h_2$. Also $E_\infty^{2,9}$ has an isomorphic basis. Let $f': S^7 \to S^0 \cup_\nu e^4$ be a representation of the class determined by y. We need to show that $\nu[f'] = \sigma[f'] = 0$. Since either y or $y + j_* h_0 h_3$ would be a suitable choice for "y", we can, if need be, alter $[f']$ by 2σ. Now $h_2 y = 0$. Indeed, if $\nu[f'] \neq 0$, it would have to be $j_* \eta\mu$, but since $\eta j_*(\eta\mu) \neq 0$ and $\eta\nu[f'] = 0$ we see that $\nu[f'] = 0$. If $h_3 y \neq 0$, it would have to be $j_* h_0 h_3^2$ and thus we can change the choice of y (and f) so $h_3 y = 0$. Thus if $\sigma[f'] \neq 0$ it must be a class of filtration at least four. Since $\sigma\sigma$ has filtration 2, either $\sigma\sigma + \kappa$ or κ has filtration 4. Hence if $\sigma[f] \neq 0$, $\eta\sigma[f'] \neq 0$. Thus $\eta[f'] \neq 0$. But $h_1 y = j_* c_0$ or zero and if $\eta[f'] \neq 0$, then $h_1 y = j_* c_0$. But $\sigma\epsilon = \sigma\bar{\nu} = \sigma\sigma\eta = 0$ by **(7)**. Thus $\sigma[f'] = 0$. Hence f exists.

Proof of Proposition 3.4. Using the map f we can construct the following commutative diagram:

$$\begin{array}{ccc} S^3 \cup_\nu e^7 & \xrightarrow{(2\nu)^-} & S^0 \\ \downarrow j_1 & & \downarrow j \\ S^7 \cup_\nu e^{11} \cup_\sigma e^{15} & \xrightarrow[f]{} & S^0 \cup_\nu e^4 \end{array}$$

The map j_1 is the composite ij_2 where $j_2: S^3 \cup e^7 \to S^7$ is the obvious projection. The map $(2\nu)^-$ is an extension of a map in the homotopy class of 2ν. It has filtration $\geqslant 2$.

LEMMA 4.1. *A basis for* $\mathrm{Ext}^{3,18}(H(S^3 \cup_\nu e^7), Z_2)$ *is given by a single class* $\bar{c}_0$ *such that* $j_2 \frac{1}{4} \bar{c}_0 = c_0$.

Proof. The calculation is made just as the calculation in § 2. One uses 2.6.1 of **(2)** and the conclusion follows easily.

Next observe that among the three non-zero homotopy classes in $\pi_8{}^S$, either ϵ or $\bar{\nu}$ has filtration 3. Clearly $\eta\sigma$ does not. By Toda **(7)**

$$\langle \epsilon, \nu, 2\nu \rangle = \langle \bar{\nu}, \nu, 2\nu \rangle = \eta\kappa \neq 0.$$

Hence if $f: S^{15} \to S^3 \cup_\nu e^7$ has filtration at least 3 and is essential, $(2\nu)^- f$ is essential. Therefore either $(2\nu)^-_* (\bar{c}_0) = h_1 d_0$ or $\eta\kappa$ has filtration > 5. The second possibility is ruled out since then $\eta\kappa$ could be divided by 2 because each class in $H^{s,t}(A)$ for $t - s = 15$, $s > 5$, is h_0 time another class and they all must project to E_∞. Toda's calculations rule this out. Thus

$$(2\nu)^-_* \bar{c}_0 = h_1 d_0.$$

Therefore $f_* j_{1*}(h_1 \bar{c}_0) = f_* i_* h_1 c_0 = j_* h_1^2 d_0$. Again we call on 2.6.1 of **(2)** to show that $j_* h_1^2 d_0 \neq 0$.

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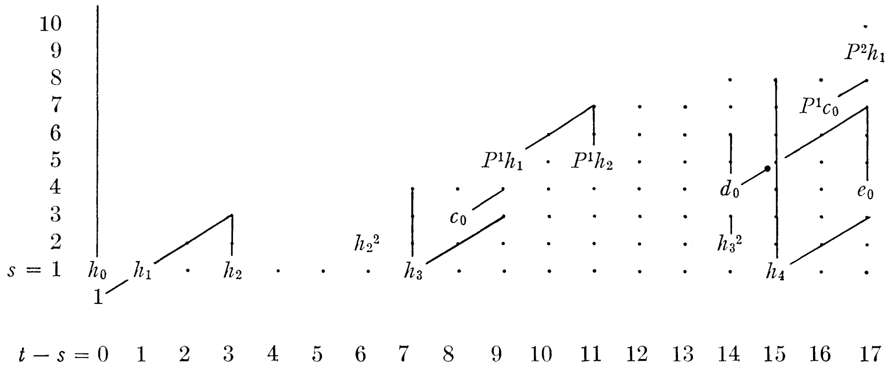
APPENDIX

TABLE OF $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$ FOR $t - s \leqslant 17$

(Vertical and horizontal lines indicate multiplication by h_0 and h_1, respectively)



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