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Collisional alpha transport in a weakly rippled magnetic field

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To properly treat the collisional transport of alpha particles due to a weakly rippled tokamak magnetic field the tangential magnetic drift due to its gradient (the ∇B drift) and pitch angle scatter must be retained. Their combination gives rise to a narrow boundary layer in which collisions are able to match the finite trapped response to the ripple to the vanishing passing response of the alphas. Away from this boundary layer collisions are ineffective. There the ∇B drift of the alphas balances the small radial drift of the trapped alphas caused by the ripple. A narrow collisional boundary layer is necessary since this balance does not allow the perturbed trapped alpha distribution function to vanish at the trapped–passing boundary. The solution of this boundary layer problem allows the alpha transport fluxes to be evaluated in a self-consistent manner to obtain meaningful constraints on the ripple allowable in a tokamak fusion reactor. A key result of the analysis is that collisional alpha losses are insensitive to the ripple near the equatorial plane on the outboard side where the ripple is high. As the high field side ripple is normally very small, collisional $\sqrt{\nu}$ ripple transport is unlikely to be a serious issue.

Key words: fusion plasma, plasma confinement

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1. Introduction

For birth alphas the magnetic drift due to the gradient of the magnetic field, ∇B, is more important than the electric or $\boldsymbol{E} \times \boldsymbol{B}$ drift. However, no satisfactory analytic evaluation of collisional transport is presently available to test simulations of alpha transport in the presence of ripple when ∇B drift dominates. Here, it is demonstrated that in the weak ripple limit analytic expressions for the alpha fluxes can be obtained and used to place constraints on the ripple that can be tolerated, as well as be used to validate simulations.

In the presence of ∇B drift a narrow boundary layer must form just inside the trapped–passing boundary to allow the finite trapped response to match to the vanishing response of the well-confined passing alpha. The boundary layer always results in alpha particle and energy diffusivities proportional to the square root of the appropriate collision frequency, ν. The behaviour is similar to the $\sqrt{\nu}$ regime in stellarators as approximately treated by Galeev *et al.* (1969) and Ho & Kulsrud (1987),

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and more rigorously formulated by Calvo *et al.* (2017), and recently applied to tokamaks by Catto (2018).

Physically the transport is expected to be a result of the sensitivity of trapped energetic alphas in tokamaks to ripple in the vicinity of their turning points (Goldston, White & Boozer 1981; Linsker & Boozer 1982; Yushmanov 1982, 1983; Mynick 1986; White 2001). However, these latter references do not consider the collisional boundary layer analysis associated with the ∇B drifting barely trapped alphas and so obtain diffusivities linear in ν or stochastic transport. By regarding collisions as a perturbation instead of on equal footing with the ∇B drift, they ignore the vital barely trapped alphas that their treatments are unable to properly handle.

The ripple $\delta = \delta(\psi, \vartheta)$ due to N toroidal field coils is normally defined as

$$\delta = (B_{max} - B_{min})/(B_{max} + B_{min}), \tag{1.1}$$

with B_{max} and B_{min} the maximum and minimum fields on the flux surface labelled by the poloidal flux function ψ, and ϑ the poloidal angle variable. The ripple form

$$B = B_0[1 - \varepsilon(\psi)\cos\vartheta - \delta(\psi, \vartheta)\cos(N\zeta)], \tag{1.2}$$

is often used to obtain explicit results, with ε the inverse aspect ratio, B_0 the magnetic field at the magnetic axis, ζ the toroidal angle variable and $\delta(\psi, \vartheta)$ a slow function of ϑ. To avoid introducing ripple wells along the magnetic field $\boldsymbol{B}$, $\delta \ll qN\delta < \varepsilon$ must be assumed, where q is the safety factor and $\varepsilon/q \simeq B_p/B_0$, with B_p the poloidal magnetic field. By considering the weak ripple limit, the departure from axisymmetry only matters for the radial magnetic drift.

The proportionality of the diffusivity to $\sqrt{\nu}$ in the boundary layer treatments of Galeev *et al.* (1969), Ho & Kulsrud (1987), Calvo *et al.* (2017), Catto (2018) is very different than the precession or tangential drift results of Goldston *et al.* (1981), Linsker & Boozer (1982), Yushmanov (1982, 1983) and Mynick (1986). They do not allow for the existence of a narrow collisional boundary layer that is necessary to make the perturbed trapped distribution function vanish at the trapped–passing boundary. In the $\boldsymbol{E} \times \boldsymbol{B}$ toroidal precession, weak ripple case considered by Linsker & Boozer (1982) and Mynick (1986) the heuristic particle diffusivity of their equations (27) and (4), respectively, is

$$D_{LBM}^{tp} \sim \varepsilon^{1/2}(q^2N^2\nu/\varepsilon)(\delta qN\rho_0 v_0/rN\omega\sqrt{qN})^2 = [\nu Nq^3\delta^2\rho_0^2 v_0^2/\varepsilon^{1/2}(r\omega)^2] \propto \nu, \tag{1.3}$$

where ν is the pitch angle scattering frequency of alphas off the background ions, $\omega \sim cE_r/RB_p \sim cqE_r/rB_0$ is the precession frequency with E_r the radial component of the electric field $\boldsymbol{E}$ and c the speed of light and v_0 and ρ_0 are the birth speed and birth gyroradius of the alphas. Both Linsker & Boozer and Mynick use $\varepsilon^{1/2}$ as the trapped fraction, as there is no boundary layer in their calculation, and $q^2N^2\nu/\varepsilon$ as the effective collision frequency with a step size of $\delta qN\rho_0 v_0/rN\omega\sqrt{qN}$. However, the perturbed distribution functions as given by their equations (32) and (47), respectively, do not vanish at the trapped–passing boundary as required to match onto the vanishing trapped response. In addition, Linsker & Boozer (1982) realize they are assuming that the dominant transport contribution is from banana orbits with turning points away from the equatorial plane, while for the boundary layer evaluation to be presented here the barely trapped alphas dominate. Mynick (1986) extends Linsker & Boozer to more general magnetic field perturbations, but continues to treat collisions perturbatively so no boundary layer is considered in the precession case. Yushmanov (1983) extends his earlier work (Yushmanov 1982) to find the diffusivities by a perturbation technique that treats collisions as weak and therefore ignores boundary layer effects. Finally, the map used by Goldston *et al.* (1981) uses the radial step in their equation (8) that becomes infinite for the barely trapped and ignore collisions so can only study stochastic transport without any provision for a collisional boundary layer due to the barely trapped stalling at turning points where collisions and precession matter most. It is unclear how sensitive their ripple threshold estimate of

$$\delta \leqslant (\varepsilon/\pi q N)^{3/2}(\rho_0\, \mathrm{d}q/\mathrm{d}r)^{-1}, \tag{1.4}$$

is to these approximations, but some improvements were made by White *et al.* (1996). Mynick (1986) and White (2001) provide useful summaries of these results with (and without) precession.

In the following section a phenomenological estimate of the alpha diffusivity associated with the collisional narrow boundary layer formed by the presence of tangential ∇B drift. Then § 3 gives a detailed evaluation of all the terms in the transit averaged kinetic equation to be solved in the boundary layer. The solution of the kinetic equation using the results of Catto (2018) is given in § 4 for the $\sqrt{\nu}$ regime, followed by the evaluation of the transport fluxes and diffusivities. The closing section presents a summary and discussion of key points. It stresses that collisional $\sqrt{\nu}$ alpha losses are insensitive to the ripple near the equatorial plane on the outboard side where ripple is much stronger (Redi *et al.* 1996).

2. Phenomenological estimate and comparisons

Both Clebsch and Boozer (1981) representations are employed to write $\boldsymbol{B}$ as

$$\boldsymbol{B} = B\boldsymbol{b} = \nabla\alpha \times \nabla\psi = K(\psi, \vartheta, \zeta)\nabla\psi + G(\psi)\nabla\vartheta + I(\psi)\nabla\zeta, \tag{2.1}$$

with $K(\psi, \vartheta, \zeta)$ periodic in the poloidal and toroidal angle, and

$$\alpha = \zeta - q\vartheta, \tag{2.2}$$

with $q = q(\psi)$ the safety factor. The preceding give

$$\boldsymbol{B}\boldsymbol{\cdot}\nabla\zeta = q\nabla\psi \times \nabla\vartheta \boldsymbol{\cdot} \nabla\zeta = q\boldsymbol{B}\boldsymbol{\cdot}\nabla\vartheta \tag{2.3}$$

and

$$B^2 = (G + qI)\boldsymbol{B}\boldsymbol{\cdot}\nabla\vartheta, \tag{2.4}$$

as well as $\boldsymbol{B}\boldsymbol{\cdot}\nabla\alpha = 0 = \boldsymbol{B}\boldsymbol{\cdot}\nabla\psi$, with $G/qI \sim rB_p/qRB_0 \sim \varepsilon^2/q^2 \ll 1$.

The transit averaged drift kinetic equation need only be solved for the trapped since $\overline{\boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi} = 0$ for the passing means that the perturbed passing distribution function vanishes ($f_p = 0$). Here $\boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi$ is the radial magnetic drift due to the rippled magnetic field, with the overbar indicating transit averaging over the trapped,

$$\bar{A} = \frac{\oint_\alpha \mathrm{d}\ell A/v_\parallel}{\oint_\alpha \mathrm{d}\ell/v_\parallel} = \frac{\oint_\alpha \mathrm{d}\tau A}{\oint_\alpha \mathrm{d}\tau} = \frac{\oint_\alpha \mathrm{d}\vartheta A/v_\parallel \boldsymbol{b}\boldsymbol{\cdot}\nabla\vartheta}{\oint_\alpha \mathrm{d}\vartheta/v_\parallel \boldsymbol{b}\boldsymbol{\cdot}\nabla\vartheta}, \tag{2.5}$$

with A arbitrary, $\mathrm{d}\tau = \mathrm{d}\ell/v_\parallel = \mathrm{d}\vartheta/v_\parallel \boldsymbol{b}\cdot\nabla\vartheta$ and $q\,\mathrm{d}\vartheta = \mathrm{d}\zeta$ for $\alpha = \zeta - q\vartheta$ fixed (denoted by the subscript on the integral). The integrals are over a full bounce for trapped particles. For energetic alphas, the $\boldsymbol{E}\times\boldsymbol{B}$ drift is small and can be ignored.

The usual transit averaged equation

$$\overline{\boldsymbol{v}_m\cdot\nabla\psi}\frac{\partial f_s}{\partial\psi} + \overline{\boldsymbol{v}_m\cdot\nabla\alpha}\frac{\partial f_t}{\partial\alpha} = \overline{C\{f_t\}}, \tag{2.6}$$

must be solved for the trapped correction $f_t = f_t(\psi, \alpha, v, \mu, \sigma)$ to the slowing down distribution

$$f_s(\psi, v) = \frac{S(\psi)\tau_s(\psi)H(v_0 - v)}{4\pi[v^3 + v_c^3(\psi)]}, \tag{2.7}$$

where $f = f_s + f_t$ with $f_t \ll f_s$, $\mu = v_\perp^2/2B$ is the magnetic moment, $\sigma = v_\parallel/|v_\parallel|$ is the sign of the parallel velocity $v_\parallel = \sqrt{v^2 - 2\mu B}$ and the magnetic drift is

$$\boldsymbol{v}_m = \Omega^{-1}[\mu\boldsymbol{b}\times\nabla B + v_\parallel^2\boldsymbol{b}\times(\boldsymbol{b}\cdot\nabla\boldsymbol{b})] \simeq \Omega^{-1}v_\parallel\nabla\times(v_\parallel\boldsymbol{b}), \tag{2.8}$$

where the final form is a useful approximate form for the magnetic drift as the parallel velocity correction is negligible. In addition, C is the alpha collision operator, S the alpha birth rate, τ_s is the slowing down time for the alphas, v_c is the critical speed at which the drag of the background ions and electrons on the alphas is equal and v_0 is the alpha birth speed with H a Heaviside step function that vanishes for speeds $v > v_0$. The magnetic drift term in a flux surface, $\boldsymbol{v}_m\cdot\nabla\alpha$, is dominated by the ∇B drift as curvature drift is small for the trapped.

The weak ripple limit means that the departure from axisymmetry only matters for the radial magnetic drift term. Everywhere else the axisymmetric limit

$$\boldsymbol{B}\to I(\psi)\nabla\zeta + \nabla\zeta\times\nabla\psi \tag{2.9}$$

is used with $I = RB_t$ and $R|\nabla\zeta| = 1$, where R is the major radius and $B_t \simeq B_0$ is the toroidal magnetic field. Therefore, except for a very small radial drift due to asymmetry, the alphas try to move on surfaces of constant drift kinetic canonical angular momentum

$$\bar{\psi}_* = \psi - I(\psi)v_\parallel/\Omega, \tag{2.10}$$

with $\Omega = ZeB/Mc$ the alpha gyrofrequency.

Pitch angle scattering must be the dominant collisional process to get a boundary layer narrower than $\varepsilon^{1/2}$. This balance between the strong ∇B drift of the alphas tangential to the flux surface and collisions, reduces the width w in pitch angle λ of the boundary layer by enhancing the pitch angle scattering frequency $\nu\sim v_c^3/v_0^3\tau_s$, with τ_s the slowing down time. Using

$$\overline{C\{f_t\}}\sim\nu\frac{\partial^2 f_t}{\partial\lambda^2}\sim\frac{\nu f_t}{w^2}\sim\frac{Nqv_0^2 f_t}{\Omega_0\varepsilon R^2}\sim\overline{\boldsymbol{v}_m\cdot\nabla\alpha}\frac{\partial f_t}{\partial\alpha}, \tag{2.11}$$

where $\partial f_t/\partial\alpha\sim Nf_t$ due to the N coils, gives the normalized width of the boundary layer w to be

$$w\sim(rR\nu/N\rho_0 qv_0)^{1/2}\ll\varepsilon^{1/2}, \tag{2.12}$$

indicating that the alphas must ∇B precess on a flux surface much faster than they pitch angle scatter off the ions. This condition for the boundary layer analysis to be valid is easily satisfied by alphas. The effective barely trapped fraction is estimated from this boundary layer width to be

$$F \sim w, \tag{2.13}$$

with $w \ll \varepsilon^{1/2}$ requiring $(N\rho_0 q v_0/R^2\nu)^{1/2} \gg 1$. For deuterium–tritium (D–T) with $R = 10$ m, $B = 5T$ and $T_i \simeq 10$ keV and $n_e \simeq 10^{14}$ cm^{-3}, $R/\rho_0 \sim 10^2$ and $v_0/\nu R \sim 10^6$, so a narrow boundary layer will occur. Then the effective drift decorrelation time for the alphas is

$$\tau \sim F^2/\nu \sim (rR/Nq\rho_0 v_0). \tag{2.14}$$

The radial ∇B drift speed of the alphas is

$$V = \overline{\boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi}/RB_p \sim q\rho_0 v_0 N\delta/r, \tag{2.15}$$

and small decorrelation time limits the effective radial step size $\varDelta$ to be

$$\varDelta \sim V\tau \sim R\delta. \tag{2.16}$$

Consequently, the weak ripple diffusivity is

$$D^{\text{weak}}_{\sqrt{\nu}} \sim F\varDelta^2/\tau \sim q^{1/2}\varepsilon^{-1/2}N^{1/2}\delta^2 v_0^{1/2}\rho_0^{1/2}R\nu^{1/2}. \tag{2.17}$$

This result will be justified in more detail by solving a boundary layer problem that allows the trapped distribution function f_t to vanish at the trapped–passing boundary.

The banana regime diffusivity is due to a combination of pitch angle scattering off the ions and electron drag and is found to be of order $D^{\text{ban}}_{\text{axi}} \simeq 0.25 q^2\rho_0^2/\varepsilon\tau_s \ell n(v_0/v_c) \sim q^2\rho_0^2/\varepsilon\tau_s$ for D–T (Hsu, Catto & Sigmar 1990), giving

$$\frac{D^{\text{weak}}_{\sqrt{\nu}}}{D^{\text{ban}}_{\text{axi}}} \sim \left(\frac{v_c^{3/2}}{v_0^{3/2}}\right)\left(\frac{N^{1/2}\delta^{1/2}\varepsilon^{1/2}}{q^{3/2}}\right)\frac{R\sqrt{v_0\tau_s}}{\rho_0^{3/2}} = \left(\frac{qN\delta}{\varepsilon}\right)^{1/2}\left(\frac{v_c R\,\delta}{v_0\rho_0}\right)^{3/2}\frac{\varepsilon\sqrt{v_0\tau_s/R}}{q^2}, \tag{2.18}$$

for weak ripple ($qN\delta < \varepsilon$). For this estimate, weak ripple transport will be larger than neoclassical for $\delta \sim 10^{-3}$, even though $qN\delta < \varepsilon$, because $R/\rho_0 \sim 10^2$, $v_0\tau_s/R \sim 10^6$, and $v_c^{3/2}/v_0^{3/2} \sim 1/5$. More specifically, weak ripple transport will occur and be larger than neoclassical whenever

$$\frac{\varepsilon}{qN} > \delta > \left(\frac{q\rho_0 v_0}{Rv_c}\right)^{3/4}\left(\frac{R}{N\varepsilon v_0\tau_s}\right)^{1/4}. \tag{2.19}$$

These inequalities indicate that ripple levels of $\delta \sim 10^{-3}$ are needed to keep ripple and neoclassical losses comparable. The ripple on the outboard or low field side of a tokamak is typically of this order, but it is much smaller on the high field side (Redi *et al.* 1996).

Ripple of $\delta \sim 10^{-3}$ will also avoid seriously depleting the slowing down distribution function during $\sqrt{\nu}$ regime transport as $\tau_s D^{\text{weak}}_{\sqrt{\nu}}/a^2 \ll 1$ gives the constraint on the ripple of

$$\delta < \left(\frac{v_0}{v_c}\right)^{3/4}\left(\frac{a}{R}\right)\left(\frac{rR}{Nq\rho_0 v_0\tau_s}\right)^{1/4}. \tag{2.20}$$

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The detailed boundary layer evaluation performed in the § 4 indicates that the barely trapped alphas dominate $\sqrt{\nu}$ regime collisional transport so that outboard ripple larger than $\delta \sim 10^{-3}$ is tolerable.

Catto (2018) performed a boundary layer analysis for strong ripple ($qN\delta \gg \varepsilon$) to find

$$D_{\sqrt{\nu}}^{\text{strong}} \sim (qv_c/v_0)^{3/2}(\rho_0 v_0/\omega R)^2(\omega/\tau_s)^{1/2}, \tag{2.21}$$

with ω the rotation frequency due to a radial electric field. Comparing this to the small ripple result, but using a magnetic drift estimate of $\omega R \sim v_0\rho_{p0}/R \sim q\rho_0 v_0/R$, gives

$$\frac{D_{\sqrt{\nu}}^{\text{weak}}}{D_{\sqrt{\nu}}^{\text{strong}}} \sim \frac{q^{1/2}N^{1/2}\delta^2}{\varepsilon^{1/2}} \sim \left(\frac{qN\delta}{\varepsilon}\right)^{1/2}\delta^{3/2}, \tag{2.22}$$

which is very small even if $qN\delta \sim \varepsilon$, as might be expected. Unfortunately, retaining the ∇B drift in the strong ripple limit is not an analytically tractable limit so this estimate is likely to be too crude. Moreover, the large ripple limit treats only ripple trapped alphas in wells that are poloidally localized.

3. Transit averaged kinetic equation

Only the axisymmetric forms of the collision operator and drift within a flux surface are required in the kinetic equation. Neglecting curvature drift, they may be written as

$$\overline{C\{f_t\}} \simeq \frac{2v_\lambda^3}{\tau_s v^3\left(\oint_\alpha \mathrm{d}\vartheta/\xi\right)}\frac{\partial}{\partial\lambda}\left[\lambda\left(\oint_\alpha \mathrm{d}\vartheta\,\xi\right)\frac{\partial f_t}{\partial\lambda}\right], \tag{3.1}$$

and

$$\overline{\boldsymbol{v}_m\cdot\nabla\alpha} \simeq \frac{Mcv^2}{Ze}\frac{(\partial/\partial\psi)\left(\oint_\alpha \mathrm{d}\vartheta\,\xi\right)}{\left(\oint_\alpha \mathrm{d}\vartheta/\xi\right)}, \tag{3.2}$$

where $\lambda = 2\mu B_0/v^2$, and $\xi^2 = v_\parallel^2/v^2 = 1 - 2\mu B/v^2 = 1 - \lambda B/B_0 \simeq 1 - \lambda(1-\varepsilon) - 2\lambda\varepsilon\sin^2(\vartheta/2)$. The pitch angle scattering collision frequency is $\nu = v_\lambda^3/v_0^3\tau_s \sim v_c^3/v_0^3\tau_s$, where

$$v_\lambda^3 = \frac{3\pi^{1/2}T_e^{3/2}}{(2m)^{1/2}Mn_e}\sum_i Z_i^2 n_i, \tag{3.3}$$

$$v_c^3 = \frac{3\pi^{1/2}T_e^{3/2}}{(2m)^{1/2}n_e}\sum_i \frac{Z_i^2 n_i}{M_i}, \tag{3.4}$$

and

$$\tau_s = \frac{3MT_e^{3/2}}{4(2\pi m)^{1/2}Z^2e^4 n_e \ell n\varLambda}, \tag{3.5}$$

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with the electron and bulk ion densities and temperatures denoted by n_j and T_j, and m the electron mass and M_i the mass of an ion of charge Z_i. The density of slowing down alphas is

$$n_s = \int \mathrm{d}^3 v f_s = S\tau_s \int_0^{v_0} \frac{v^2\,\mathrm{d}v}{(v^3 + v_c^3)} = \frac{S\tau_s}{3} \ell n[1 + (v_0^3/v_c^3)] \simeq S\tau_s \ell n(v_0/v_c), \tag{3.6}$$

where $v_0^3/v_c^3 \gg 1$ and $v_\lambda^3 \,/\, v_c^3 = 3/5$ for the deuterium–tritium (D–T) reaction of interest here.

To evaluate the magnetic drift the divergence of an arbitrary vector $\boldsymbol{A}$ is written as

$$\nabla \boldsymbol{\cdot} \boldsymbol{A} = \boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta \left[\frac{\partial}{\partial\psi} \left(\frac{\boldsymbol{A} \boldsymbol{\cdot} \nabla\psi}{\boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta} \right) + \frac{\partial}{\partial\vartheta} \left(\frac{\boldsymbol{A} \boldsymbol{\cdot} \nabla\vartheta}{\boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta} \right) + \frac{\partial}{\partial\alpha} \left(\frac{\boldsymbol{A} \boldsymbol{\cdot} \nabla\alpha}{\boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta} \right) \right], \tag{3.7}$$

giving

$$\begin{aligned} \boldsymbol{v}_m \boldsymbol{\cdot} \nabla\alpha &\simeq \frac{v_\parallel}{\Omega} \nabla\alpha \boldsymbol{\cdot} \nabla \times (v_\parallel \boldsymbol{b}) \\ &= \frac{v_\parallel}{\Omega} \boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta \left[\frac{\partial}{\partial\psi} \left(\frac{B v_\parallel}{\boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta} \right) + \frac{\partial}{\partial\vartheta} \left(\frac{v_\parallel \boldsymbol{b} \boldsymbol{\cdot} \nabla\alpha \times \nabla\vartheta}{\boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta} \right) \right], \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi &\simeq \frac{v_\parallel}{\Omega} \nabla\psi \boldsymbol{\cdot} \nabla \times (v_\parallel \boldsymbol{b}) \\ &= -\frac{v_\parallel}{\Omega} \boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta \left[\frac{\partial}{\partial\alpha} \left(\frac{B v_\parallel}{\boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta} \right) - \frac{\partial}{\partial\vartheta} \left(\frac{v_\parallel \boldsymbol{b} \boldsymbol{\cdot} \nabla\psi \times \nabla\vartheta}{\boldsymbol{B} \boldsymbol{\cdot} \nabla\vartheta} \right) \right]. \end{aligned} \tag{3.9}$$

As a result, neglecting curvature drift

$$\overline{\boldsymbol{v}_m \boldsymbol{\cdot} \nabla\alpha} \simeq \frac{Mcv^2}{Ze \oint_\alpha \mathrm{d}\vartheta\,\xi^{-1}} \frac{\partial}{\partial\psi} \oint_\alpha \mathrm{d}\vartheta\,\xi, \tag{3.10}$$

and

$$\overline{\boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi} \simeq -\frac{Mcv^2}{Ze \oint_\alpha \mathrm{d}\vartheta\,\xi^{-1}} \frac{\partial}{\partial\alpha} \oint_\alpha \mathrm{d}\vartheta\,\xi. \tag{3.11}$$

Ripple only matters in the radial drift. Using $2\xi\,\partial\xi/\partial\alpha|_\vartheta = -N\delta\lambda \sin[N(\alpha + q\vartheta)]$, gives

$$\overline{\boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi} = \frac{McN\lambda v^2}{2Ze} \frac{\oint_\alpha \mathrm{d}\vartheta\,\xi^{-1}\delta \sin[N(\alpha + q\vartheta)]}{\oint_\alpha \mathrm{d}\vartheta\,\xi^{-1}}. \tag{3.12}$$

Then, using $\sin[N(\alpha + q\vartheta)] = \sin(N\alpha)\cos(Nq\vartheta) + \cos(N\alpha)\sin(Nq\vartheta)$, and noting that the term odd in ϑ about the equatorial plane ($\vartheta = 0$) vanishes, leaves

$$\overline{\boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi} = \frac{McN\lambda v^2 \sin(N\alpha)}{2Ze} \frac{\oint_\alpha \mathrm{d}\vartheta\,\xi^{-1}\delta \cos(qN\vartheta)}{\oint_\alpha \mathrm{d}\vartheta\,\xi^{-1}}, \tag{3.13}$$

giving the estimate $\overline{\boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi}/RB_p \sim qN\rho_0 v_0 \delta/r$ used earlier.

Using the full bounce, large aspect ratio results

$$\oint_\alpha \mathrm{d}\vartheta/\xi = 8(2\varepsilon\lambda)^{-1/2}\int_0^{\pi/2}\mathrm{d}x/\sqrt{1-\kappa^2\sin^2 x} = 8(2\varepsilon)^{-1/2}\sqrt{1-\varepsilon+2\varepsilon\kappa^2}K(\kappa), \tag{3.14}$$

$$\oint_\alpha \mathrm{d}\vartheta\,\xi = \frac{8\sqrt{2\varepsilon}\kappa^2}{\sqrt{1-\varepsilon+2\varepsilon\kappa^2}}\int_0^{\pi/2}\frac{\mathrm{d}x\cos^2 x}{\sqrt{1-\kappa^2\sin^2 x}} = \frac{8\sqrt{2\varepsilon}[E(\kappa)-(1-\kappa^2)K(\kappa)]}{\sqrt{1-\varepsilon+2\varepsilon\kappa^2}}, \tag{3.15}$$

and

$$\begin{aligned}\oint_\alpha \frac{\mathrm{d}\vartheta}{\xi}\cos\vartheta &= \frac{8}{(2\varepsilon\lambda)^{1/2}}\int_0^{\pi/2}\mathrm{d}x\frac{(1-2\kappa^2\sin^2 x)}{\sqrt{1-\kappa^2\sin^2 x}}\\ &= \frac{8}{(2\varepsilon)^{1/2}}\sqrt{1-\varepsilon+2\varepsilon\kappa^2}[2E(\kappa)-K(\kappa)],\end{aligned} \tag{3.16}$$

where $\kappa\sin x = \sin(\vartheta/2)$ and the α subscript is a reminder that the integral is to be performed at fixed α. Here $K(\kappa)$ and $E(\kappa)$ are complete elliptic integrals of the first and second kind, respectively, with

$$\kappa^2 = [1-(1-\varepsilon)\lambda]/2\varepsilon\lambda, \tag{3.17}$$

so that $\kappa = 0$ are the deeply trapped and $\kappa = 1$ the is the barely passing boundary. Using these results along with the barely trapped limits $E(\kappa)\to 1+\cdots$ and $K(\kappa)\to \ell n(4/\sqrt{1-\kappa^2})+\cdots$, gives

$$\overline{C\{f_t\}} \simeq \frac{v_\lambda^3}{4\tau_s v^3\varepsilon\kappa K(\kappa)}\frac{\partial}{\partial\kappa}\left\{\frac{[E(\kappa)-(1-\kappa^2)K(\kappa)]}{\kappa}\frac{\partial f_t}{\partial\kappa}\right\}\underset{\kappa\to 1}{\longrightarrow}\frac{(v_\lambda^3/2\varepsilon\tau_s v^3)}{\ell n[8/(1-\kappa)]}\frac{\partial^2 f_t}{\partial\kappa^2}, \tag{3.18}$$

and

$$\overline{\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\alpha} \simeq \frac{v^2[2E(\kappa)-K(\kappa)]}{2\Omega_p R^2 K(\kappa)}\underset{\kappa\to 1}{\longrightarrow}\frac{-v^2}{2\Omega_p R^2}, \tag{3.19}$$

where $\Omega_p = ZeB_p/Mc \simeq \varepsilon\Omega_0/q$.

The integral appearing in $\overline{\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\psi}$ was evaluated in the $Nq\gg 1$ limit in Linsker & Boozer (1982), Mynick (1986) and White (2001), but their procedure is inadequate for the barely trapped ($\kappa\to 1$) as the result is singular. Appendix A gives the asymptotic expansion for $\kappa\to 1$ that is found by more carefully expanding the ϑ dependence of ξ about the turning point ϑ_t, where

$$\sin(\vartheta_t/2) = \sqrt{[1-(1-\varepsilon)\lambda]/2\varepsilon\lambda} = \kappa. \tag{3.20}$$

Using $\sin(\vartheta/2) = \kappa\sin x$ leads to the form

$$\begin{aligned}\oint_\alpha \frac{\mathrm{d}\vartheta}{\xi}\delta\cos(qN\vartheta) &= \frac{8}{(2\varepsilon\lambda)^{1/2}}\mathrm{Re}\int_0^{\pi/2}\frac{\mathrm{d}x\,\delta\,\mathrm{e}^{\mathrm{i}qN\vartheta}}{\sqrt{1-\kappa^2\sin^2 x}}\\ &\simeq \frac{8\delta}{(2\varepsilon)^{1/2}}\sqrt{1-\varepsilon+2\varepsilon\kappa^2}\,\mathrm{Re}\{\mathrm{e}^{\mathrm{i}qN\vartheta_t}\tilde{K}(\kappa)\},\end{aligned} \tag{3.21}$$

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with

$$\widetilde{K}(\kappa) \equiv \int_0^{\pi/2} \mathrm{d}x \mathrm{e}^{\mathrm{i}qN(\vartheta-\vartheta_t)} / \sqrt{1-\kappa^2 \sin^2 x}, \tag{3.22}$$

and where $\delta \simeq \delta(\psi, \vartheta = \pi)$ is used since only the limit $\kappa \to 1$ ($\vartheta \to \pi$) is of interest. Then changing to ϑ by using $\cos(\vartheta/2)\,\mathrm{d}\vartheta = 2\kappa \cos x\, \mathrm{d}x$, the result from appendix A gives

$$\widetilde{K}(\kappa) \equiv \frac{1}{2} \int_0^{\vartheta_t} \frac{\mathrm{d}\vartheta\, \mathrm{e}^{\mathrm{i}qN(\vartheta-\vartheta_t)}}{\sqrt{\kappa^2 - \sin^2(\vartheta/2)}} \simeq \ell n \left(\frac{1}{qN\sqrt{2(1-\kappa)}} \right). \tag{3.23}$$

Therefore, using $\delta = \delta(\psi, \vartheta = \pi)$,

$$\overline{\boldsymbol{v}_m \cdot \nabla \psi} \simeq \frac{N\delta B_0 v^2 \ell n[2q^2N^2(1-\kappa)]}{2\Omega_0 \ell n[(1-\kappa)/8]} \cos(\pi q N) \sin(N\alpha). \tag{3.24}$$

Inserting the preceding $\kappa \to 1$ results into the transit averaged kinetic equation yields the equation that must be solved for the trapped

$$\begin{aligned} &\frac{v_\lambda^3}{2\varepsilon \tau_s v^3 \ell n[8/(1-\kappa)]} \frac{\partial^2 f_t}{\partial \kappa^2} + \frac{q v^2}{2\Omega_0 R r} \frac{\partial f_t}{\partial \alpha} \\ &\quad = \frac{qNv^2 \delta \ell n[2q^2N^2(1-\kappa)] \cos(\pi q N)}{2\Omega_0 r \ell n[(1-\kappa)/8]} RB_p \frac{\partial f_s}{\partial \psi} \sin(N\alpha). \end{aligned} \tag{3.25}$$

In the section that follows this equation is solved for the trapped in the narrow boundary layer just inside the trapped–passing boundary.

4. Boundary layer analysis and transport fluxes

Fortunately, the weak ripple limit with the ∇B drift retained is analytically tractable, as will now be demonstrated. There is no need to assume poloidal localization since ripple trapping does not occur.

Defining

$$\eta \equiv (1-\kappa)/8, \tag{4.1}$$

$$f_t \equiv \mathrm{Im}[H(\eta)\mathrm{e}^{\mathrm{i}N\alpha}], \tag{4.2}$$

$$k \equiv \frac{32 qN \tau_s v^5}{v_\lambda^3 \Omega_0 R^2} \gg 1, \tag{4.3}$$

and

$$L \equiv \frac{32\delta\, qN\, \tau_s v^5 \cos(\pi q N)}{\Omega_0 R v_\lambda^3} RB_p \frac{\partial f_s}{\partial \psi}, \tag{4.4}$$

the boundary layer equation becomes of the form first considered by Calvo *et al.* (2017):

$$\frac{1}{\ell n(\eta)} \frac{\partial^2 H}{\partial \eta^2} - 2\mathrm{i}kH = -2L \frac{\ell n(16 q^2 N^2 \eta)}{\ell n(\eta)} \simeq -2L, \tag{4.5}$$

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where for alphas it is not unreasonable to assume $\ell n(\eta^{-1}) \sim \ell n(k^{1/2}) > \ell n(16q^2N^2)$ as $v_0^4\tau_s\rho_0/v_\lambda^3 R^2 \ggg 4q^3N^3$, and Im denotes imaginary part. Then, a boundary layer equation of the exact same form as in the strong ripple limit of Catto (2018) is obtained. There it is shown that the matched asymptotic solution vanishing at the trapped–passing boundary is

$$H = \frac{\mathrm{i}L}{k}[\mathrm{e}^{-(1-\mathrm{i})\eta\sqrt{2k\ell n(\sqrt{2k})}} - 1]. \tag{4.6}$$

Consequently,

$$f_t = B_p R^2 \frac{\partial f_s}{\partial \psi}\delta \cos(\pi q N)\,\mathrm{Re}\{[\mathrm{e}^{-(1-\mathrm{i})\eta\sqrt{2k\ell n(\sqrt{2k})}} - 1]\mathrm{e}^{\mathrm{i}N\alpha}\}, \tag{4.7}$$

where $f_t/f_s \sim R\delta/a \ll 1$ is required, with the minor radius, a, assumed to be roughly the radial scale length of the alpha density variation, and Re denoting the real part.

The alpha flux is evaluated from

$$\Gamma_d = \left\langle \int \mathrm{d}^3 v (Mv^2/2)^d f_t \boldsymbol{v}_m \boldsymbol{\cdot} \nabla\psi \right\rangle, \tag{4.8}$$

where $\langle\cdots\rangle$ is the flux surface average, with $d=0$ for the alpha particle flux and $d=1$ for the alpha energy flux. Ignoring curvature drift, using $\mathrm{d}^3v \rightarrow 2\pi(Bv^2/B_0\xi)\,\mathrm{d}v\,\mathrm{d}\lambda$, performing the transit average first by holding α fixed and noting

$$\begin{aligned}\int_{-\vartheta_t}^{\vartheta_t} \frac{\mathrm{d}\vartheta}{\boldsymbol{B}\boldsymbol{\cdot}\nabla\vartheta}\frac{B}{\xi}\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\psi &= \overline{\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\psi}\int_{-\vartheta_t}^{\vartheta_t}\frac{\mathrm{d}\vartheta}{\xi\,\boldsymbol{b}\boldsymbol{\cdot}\nabla\vartheta}\\ &\simeq \frac{2B_0}{\pi\sqrt{2\varepsilon}}\sqrt{1-\varepsilon+2\varepsilon\kappa^2}K(\kappa)\overline{\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\psi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\vartheta}{\boldsymbol{B}\boldsymbol{\cdot}\nabla\vartheta},\end{aligned} \tag{4.9}$$

gives the alternate and more useful form of the fluxes to be

$$\begin{aligned}\Gamma_d &= \left\langle \int \mathrm{d}^3 v (Mv^2/2)^d f_t \overline{\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\psi} \right\rangle\\ &\simeq 8\sqrt{2\varepsilon}\int_0^{v_0}\mathrm{d}v v^2\left(\frac{Mv^2}{2}\right)^d\left\langle\int_0^1 \mathrm{d}\kappa\, f_t\overline{\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\psi}\,\ell n\left(\frac{8}{1-\kappa}\right)\right\rangle,\end{aligned} \tag{4.10}$$

where $f_t \propto \exp(\mathrm{i}N\alpha)$is independent of ϑ and the last form is for $\kappa \rightarrow 1$. Inserting f_t and $\overline{\boldsymbol{v}_m\boldsymbol{\cdot}\nabla\psi}$, and using

$$\frac{\ell n(v_0/v_c)}{n_s}\frac{\partial n_s}{\partial\psi} \gg \frac{1}{v_c}\frac{\partial v_c}{\partial\psi} \sim \frac{1}{n_i}\frac{\partial n_i}{\partial\psi}, \tag{4.11}$$

leaves

$$\begin{aligned}\Gamma_d &\simeq -\frac{16B_p^2R^2qN\delta^2\cos^2(\pi qN)}{\pi\sqrt{2\varepsilon}\Omega_0\ell n(v_0/v_c)}\frac{\partial n_s}{\partial\psi}\int_0^{v_0}\frac{\mathrm{d}v v^4}{v^3+v_c^3}\left(\frac{Mv^2}{2}\right)^d\\ &\quad\times \mathrm{Re}\left\{\mathrm{i}\int_0^\infty \mathrm{d}\eta[\mathrm{e}^{-(1-\mathrm{i})\eta\sqrt{2k\ell n(\sqrt{2k})}} - 1]\ell n(16q^2N^2\eta)\right\},\end{aligned} \tag{4.12}$$

where $\langle \sin(N\alpha) e^{iN\alpha} \rangle \simeq i/2$ is used. Letting $\chi = \eta\sqrt{k\ell n(2k)} \propto v^{5/2}$ and defining

$$k_0 \equiv \frac{32qN\tau_s v_0^5}{v_\lambda^3 \Omega_0 R^2} \gg 1, \tag{4.13}$$

then the fraction of barely trapped particles that contribute is proportional to

$$\mathrm{Re}\left\{ i\int_0^\infty d\eta [e^{-(1-i)\eta\sqrt{2k\ell n(\sqrt{2k})}} - 1]\ell n(16q^2N^2\eta)\right\} \simeq \frac{\ell n(\sqrt{k\ell n(2k)}/16q^2N^2)}{2\sqrt{k\ell n(2k)}}, \tag{4.14}$$

with the speed weighting

$$\begin{aligned} &\int_0^{v_0} dv v^4 \left(\frac{Mv^2}{2}\right)^d \frac{\ell n[\sqrt{k\ell n(2k)}/16q^2N^2]}{(v^3+v_c^3)\sqrt{k\ell n(2k)}} \\ &\quad \simeq \frac{v_0^2 \ell n[\sqrt{k_0\ell n(2k_0)}/16q^2N^2]}{\sqrt{k_0 \ell n(2k_0)}} \begin{cases} \ell n(v_0/v_c) & d=0 \\ Mv_0^2/3 & d=1, \end{cases} \end{aligned} \tag{4.15}$$

thereby yielding the collisional particle and energy fluxes for the alphas

$$\begin{aligned} \Gamma_d \simeq &-\frac{B_p^2 R^3 q^{1/2} N^{1/2} \delta^2 v_\lambda^{3/2} \rho_0^{1/2} \cos^2(\pi qN)}{\pi\sqrt{\varepsilon\tau_s} v_0 \ell n(v_0/v_c)} \frac{\partial n_s}{\partial \psi} \frac{\ell n[\sqrt{k_0\ell n(2k_0)}/16q^2N^2]}{\sqrt{\ell n(2k_0)}} \\ &\times \begin{cases} \ell n(v_0/v_c) & d=0 \\ Mv_0^2/3 & d=1. \end{cases} \end{aligned} \tag{4.16}$$

The flux implies that the particle diffusivity of the alphas is

$$D_0^{\mathrm{weak}} = \frac{q^{1/2}N^{1/2}\delta^2 v_\lambda^{3/2}\rho_0^{1/2} R}{2\pi v_0\sqrt{\varepsilon\tau_s \ell n(2k_0)}} \ell n\left[\frac{\sqrt{k_0\ell n(2k_0)}}{16q^2N^2}\right], \tag{4.17}$$

while the alpha energy diffusivity is

$$D_1^{\mathrm{weak}} = \frac{q^{1/2}N^{1/2}\delta^2 v_\lambda^{3/2}\rho_0^{1/2} R}{3\pi v_0\sqrt{\varepsilon\tau_s \ell n(2k_0)}} \frac{\ell n[\sqrt{k_0\ell n(2k_0)}/16q^2N^2]}{\ell n(v_0/v_c)}, \tag{4.18}$$

where a coarse grain average is used to replace $\cos^2(\pi qN)$ by 1/2.

The first expression is the same as the estimate in § 2 within numerical and logarithmic factors that decrease the diffusivity since $\nu = v_\lambda^3/v_0^3\tau_s \sim v_c^3/v_0^3\tau_s$. However, the boundary layer solution technique makes it clear that the ripple that matters is at the equatorial plane on the high field side since $\delta = \delta(\psi, \vartheta = \pi)$. Consequently, the weak ripple, $\sqrt{\nu}$ transport regime seems unlikely to be an important consideration for the alphas since the details of the collisional boundary layer analysis presented here imply that these results are insensitive to the ripple near the low field side equatorial plane where the ripple is largest and typically approximately $\delta \sim 10^{-3}$. High field side ripple is substantially smaller. Therefore, adequately confined collisionless alpha orbits appears to be all that is required to keep collisional alpha confinement at axisymmetric neoclassical levels (Hsu *et al.* 1990) in weakly rippled tokamak fields.

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5. Summary and discussion

A fully self-consistent evaluation of alpha particle and energy transport fluxes for weak ripple ($\varepsilon > qN\delta$) has been performed in the $\sqrt{\nu}$ regime. The new features of this evaluation are a complete boundary layer analysis retaining collisions to enable the perturbed trapped distribution function to vanish at the trapped–passing boundary so it can properly match to the passing response and the careful treatment of the tangential magnetic ∇B drift on a flux surface so that the radial steps remain well behaved for the barely trapped alphas. The result places only mild constraints on ripple. These are necessary to satisfy to keep ripple transport comparable to neoclassical, while avoiding alpha depletion. Indeed, since the results are only sensitive to the ripple near the equatorial plane on the high field side it is likely that the losses are well below axisymmetric neoclassical transport losses (Hsu *et al.* 1990).

The $\sqrt{\nu}$ regime ripple restriction on the alpha energy loss to avoid depletion of the alphas just after birth found from $\tau_s D_1^{weak}/a^2 \ll 1$ is

$$\delta \ll \left[\frac{3\pi\ell n(v_0/v_c)\sqrt{\ell n(2k_0)}}{\ell n[\sqrt{k_0\ell n(2k_0)}/16q^2N^2]}\right]^{1/2}\left(\frac{v_0}{v_\lambda}\right)^{3/4}\left(\frac{a}{R}\right)\left(\frac{rR}{Nq\rho_0 v_0\tau_s}\right)^{1/4}$$
$$\sim 5\left(\frac{v_0}{v_\lambda}\right)^{3/4}\left(\frac{a}{R}\right)\left(\frac{\varepsilon R^2}{Nq\rho_0 v_0\tau_s}\right)^{1/4}\sim 10^{-2}, \tag{5.1}$$

with $k_0 \gg 1$ required for a narrow boundary layer, and

$$k_0 = \frac{32qN\rho_0 v_0^4\tau_s}{v_\lambda^3 R^2}\sim 10^8, \tag{5.2}$$

for $R/\rho_0 \sim 10^2$, $v_0\tau_s/R \sim 10^6$ and $v_0/v_\lambda \sim 3$. The $\sqrt{\nu}$ regime results suggest that when the ripple is weak ($\delta < \varepsilon/qN$), alpha energy depletion will be not be a issue in tokamaks because the collisional boundary layer analysis is dominated by the barely trapped particles and they are only sensitive to the very small ripple near the high field side equatorial plane (Redi *et al.* 1996). The analytic results obtained here can be used to validate a full simulation of the solution of the transit averaged equation for a more realistic model of collisional transport with strongly varying poloidal ripple. Based on the analytic results presented here, it seems likely that adequate confinement of collisionless alpha orbits will ensure that collisional $\sqrt{\nu}$ alpha losses due to ripple will be small.

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Appendix A. Asymptotic expansions for $\kappa \to 1$

To approximately expand the elliptic integral

$$K(\kappa) = \int_0^{\pi/2} \mathrm{d}x/\sqrt{1-\kappa^2\sin^2 x} = \int_0^{\pi/2}\mathrm{d}x/\sqrt{1-\kappa^2+\kappa^2\cos^2 x} \tag{A 1}$$

as $\kappa \to 1$, expand $\cos^2 x$ about $x = \pi/2$. Then letting $z = -x + \pi/2$ gives

$$K(\kappa) \simeq \int_0^{\pi/2} \mathrm{d}x/\sqrt{1-\kappa^2+\kappa^2(x-\pi/2)^2}$$
$$= \int_0^{\pi/2} \mathrm{d}z/\sqrt{1-\kappa^2+\kappa^2 z^2} = \kappa^{-1}\ell n[\kappa z + \sqrt{\kappa^2 z^2 + 1 - \kappa^2}]|_0^{\pi/2}, \quad \text{(A 2)}$$

so as $\kappa \to 1$, $\ell n[\pi/\sqrt{1-\kappa^2}]$, which is close to $K(\kappa) \to \ell n(4/\sqrt{1-\kappa^2}) + \cdots$ and recovered using

$$K(\kappa) = \int_0^{\pi/2} \mathrm{d}x/\sqrt{1-\kappa^2+\kappa^2(x-\pi/2)^2}$$
$$+ \int_0^{\pi/2} \mathrm{d}x/\sqrt{1-\kappa^2+\kappa^2\cos^2 x} - \int_0^{\pi/2} \mathrm{d}x/\sqrt{1-\kappa^2+\kappa^2(x-\pi/2)^2}. \quad \text{(A 3)}$$

Next use $\sin(\vartheta/2) = \kappa \sin x$ and $\sin(\vartheta_t/2) = \kappa$ to consider the form

$$\widetilde{K}(\kappa) \equiv \int_0^{\pi/2} \mathrm{d}x \mathrm{e}^{\mathrm{i}qN(\vartheta-\vartheta_t)}/\sqrt{1-\kappa^2\sin^2 x}. \quad \text{(A 4)}$$

Then changing to ϑ by using $\cos(\vartheta/2)\,\mathrm{d}\vartheta = 2\kappa \cos x\,\mathrm{d}x$, gives

$$\widetilde{K}(\kappa) \equiv \frac{1}{2}\int_0^{\vartheta_t} \frac{\mathrm{d}\vartheta\,\mathrm{e}^{\mathrm{i}qN(\vartheta-\vartheta_t)}}{\sqrt{\kappa^2-\sin^2(\vartheta/2)}} = \frac{1}{2}\int_0^{\vartheta_t} \frac{\mathrm{d}\vartheta\,\mathrm{e}^{\mathrm{i}qN(\vartheta-\vartheta_t)}}{\sqrt{[\kappa+\sin(\vartheta/2)][\kappa-\sin(\vartheta/2)]}}. \quad \text{(A 5)}$$

Expanding about ϑ_t using

$$\sin(\vartheta/2) = \kappa + \sqrt{1-\kappa^2}(\vartheta-\vartheta_t)/2 - \kappa(\vartheta-\vartheta_t)^2/8 + \cdots, \quad \text{(A 6)}$$

yields

$$\widetilde{K}(\kappa) \simeq \frac{1}{2}\int_0^{\vartheta_t} \frac{\mathrm{d}\vartheta\,\mathrm{e}^{-\mathrm{i}qN(\vartheta_t-\vartheta)}}{\sqrt{(\kappa+\cdots)(\vartheta_t-\vartheta)[\sqrt{1-\kappa^2}+\kappa(\vartheta_t-\vartheta)/4+\cdots]}}$$
$$= \frac{1}{\kappa}\int_0^{qN\vartheta_t} \frac{\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}z}}{\sqrt{z(\delta+z)}}, \quad \text{(A 7)}$$

where $z = qN(\vartheta_t - \vartheta)$ and $\delta = 4qN\sqrt{1-\kappa^2}/\kappa$, with $\vartheta_t \to \pi$ as $\kappa \to 1$.

Integrate by parts using

$$\frac{1}{\sqrt{z(\delta+z)}} = \frac{\mathrm{d}}{\mathrm{d}z}\left[\ell n\left(\frac{2}{\delta}\sqrt{z(\delta+z)} + \frac{2z}{\delta} + 1\right) - \ell n(\text{const.})\right], \quad \text{(A 8)}$$

to find

$$\kappa\widetilde{K}(\kappa) \simeq \int_0^{qN\vartheta_t} \frac{\mathrm{d}z\,\mathrm{e}^{-\mathrm{i}z}}{\sqrt{z(\delta+z)}}$$
$$= \int_0^{qN\vartheta_t} \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}z}\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ell n\left[\frac{2}{\delta}\sqrt{z(\delta+z)} + \frac{2z}{\delta} + 1\right]\right.$$

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$$
\begin{aligned}
&- \ell n\left[\frac{2qN\vartheta_t}{\delta}\left(\sqrt{1+\frac{\delta}{qN\vartheta_t}}+1\right)+1\right]\Bigg\} \\
&= \ell n\left[\frac{2qN\vartheta_t}{\delta}\left(\sqrt{1+\frac{\delta}{qN\vartheta_t}}+1\right)+1\right] \\
&\quad + \mathrm{i}\int_0^{qN\vartheta_t} \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}z}\left\{\ell n\left[\frac{2z}{\delta}\left(\sqrt{1+\frac{\delta}{z}}+1\right)+1\right]\right. \\
&\quad \left. - \ell n\left[\frac{2qN\vartheta_t}{\delta}\left(\sqrt{1+\frac{\delta}{qN\vartheta_t}}+1\right)+1\right]\right\}. \qquad (A\,9)
\end{aligned}
$$

Letting $\delta \to 0$ leaves

$$
\begin{aligned}
\kappa\widetilde{K}(\kappa) &\simeq \ell n\left(\frac{4qN\vartheta_t}{\delta}\right) + \mathrm{i}\int_0^{qN\vartheta_t} \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}z}\ell n\left(\frac{z}{qN\vartheta_t}\right) \\
&\simeq \ell n\left(\frac{\pi}{\sqrt{2(1-\kappa)}}\right) + \mathrm{i}\int_0^{qN\pi} \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}z}\ell n\left(\frac{z}{qN\pi}\right) \\
&\simeq \ell n\left(\frac{1}{qN\sqrt{2(1-\kappa)}}\right), \qquad (A\,10)
\end{aligned}
$$

where using $y = z/qN\pi$ yields (Gradshteyn & Ryzhik 2007)

$$
\begin{aligned}
\mathrm{i}\int_0^{qN\pi} \mathrm{d}z\,\mathrm{e}^{-\mathrm{i}z}\ell n\left(\frac{z}{qN\pi}\right) &= \mathrm{i}\pi qN \int_0^1 \mathrm{d}y \mathrm{e}^{-\mathrm{i}\pi qNy}\ell ny \\
&= -\int_0^1 \mathrm{d}y\ell ny \frac{\mathrm{d}}{\mathrm{d}y}(\mathrm{e}^{-\mathrm{i}\pi qNy}-1) \\
&= \int_0^1 \frac{\mathrm{d}y}{y}(\mathrm{e}^{-\mathrm{i}\pi qNy}-1) \\
&= \int_0^{\pi qN} \frac{\mathrm{d}z}{z}(\mathrm{e}^{-\mathrm{i}z}-1) \\
&= \int_0^{\pi qN} \frac{\mathrm{d}z}{z}(\cos z - 1) - \mathrm{i}\int_0^{\pi qN}\frac{\mathrm{d}z}{z}\sin z \\
&= [Ci(\pi qN) - C - \ell n(\pi qN)] - \mathrm{i}[Si(\pi qN)], \qquad (A\,11)
\end{aligned}
$$

where $C = 0.577215$ is Euler's constant and

$$
Si(x) = \int_0^x \frac{\mathrm{d}t \sin t}{t} = si(x) + \frac{\pi}{2}, \qquad (A\,12)
$$

and

$$
Ci(x) = C + \ell nx + \int_0^x \frac{\mathrm{d}t(\cos t - 1)}{t}. \qquad (A\,13)
$$

The preceding evaluation is very different from results (Linsker & Boozer 1982; White 2001) obtained by simply using

$$
\xi^2 \simeq 2\varepsilon\lambda[\sin(\vartheta_t/2)\cos(\vartheta_t/2)](\vartheta_t - \vartheta) = \varepsilon\lambda(\sin\vartheta_t)(\vartheta_t - \vartheta) \qquad (A\,14)
$$

to write, for example,

$$\oint_\alpha \frac{\mathrm{d}\vartheta}{\xi}\cos(Nq\vartheta) \simeq \frac{4}{\sqrt{\varepsilon\lambda\sin\vartheta_t}}\int_0^{\vartheta_t}\mathrm{d}\vartheta\,\frac{\cos(Nq\vartheta)}{\sqrt{\vartheta_t-\vartheta}} = \frac{4}{\sqrt{\varepsilon\lambda\sin\vartheta_t}}\,\mathrm{Re}\left[\mathrm{e}^{\mathrm{i}qN\vartheta_t}\int_0^{\vartheta_t}\mathrm{d}\vartheta\,\frac{\mathrm{e}^{\mathrm{i}qN(\vartheta-\vartheta_t)}}{\sqrt{\vartheta_t-\vartheta}}\right], \tag{A 15}$$

for a complete bounce. Letting $z=Nq(\vartheta_t-\vartheta)$ gives

$$\int_0^{\vartheta_t}\mathrm{d}\vartheta\,\frac{\mathrm{e}^{\mathrm{i}qN(\vartheta-\vartheta_t)}}{\sqrt{\vartheta_t-\vartheta}} = \frac{1}{\sqrt{Nq}}\int_0^{qN\vartheta_t}\mathrm{d}z\frac{\mathrm{e}^{-\mathrm{i}z}}{\sqrt{z}} \simeq \frac{1}{\sqrt{Nq}}\int_0^{\infty}\mathrm{d}z\frac{\mathrm{e}^{-\mathrm{i}z}}{\sqrt{z}}. \tag{A 16}$$

Using (Magnus, Oberhettinger & Soni 1966)

$$\int_0^\infty \mathrm{d}z\frac{\mathrm{e}^{-\mathrm{i}z}}{\sqrt{z}} = \int_0^\infty \mathrm{d}z\frac{\cos z}{\sqrt{z}} - \mathrm{i}\int_0^\infty \mathrm{d}z\frac{\sin z}{\sqrt{z}} = \Gamma(1/2)\left(\cos\frac{\pi}{4} - \mathrm{i}\sin\frac{\pi}{4}\right) = \pi^{1/2}\mathrm{e}^{-\mathrm{i}\pi/4}, \tag{A 17}$$

yields

$$\oint_\alpha \frac{\mathrm{d}\vartheta}{\xi}\cos(Nq\vartheta) \simeq \frac{2\sqrt{2\pi}}{\sqrt{\varepsilon Nq\lambda\kappa\sqrt{1-\kappa^2}}}\cos\left(Nq\vartheta_t - \frac{\pi}{4}\right), \tag{A 18}$$

where $2\kappa\sqrt{1-\kappa^2} = 2\sin(\vartheta_t/2)\cos(\vartheta_t/2) = \sin\vartheta_t$ and $\vartheta_t \to \pi - 2\sqrt{2(1-\kappa)}$. Then (A 18) gives

$$\frac{\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}}{RB_p} \simeq \frac{\sqrt{\pi Nq\lambda}v^2\delta\cos\left[(Nq-\frac{1}{4})\pi\right]}{4\Omega_0 rK(\kappa)\sqrt{\kappa\sqrt{1-\kappa^2}}}\sin(N\alpha) \sim \frac{q\rho_0 v_0\delta\sqrt{Nq}}{r}. \tag{A 19}$$

Taking the $\kappa\to 1$ $(\vartheta_t\to\pi)$ limit yields

$$\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi} \underset{\kappa\to 1}{\longrightarrow} -RB_p\frac{\sqrt{\pi qN}\,v^2\delta\cos[(Nq-1/4)\pi]}{2^{5/4}\Omega_0 r(1-\kappa)^{1/4}\,\ell n[8/(1-\kappa)]}\sin(N\alpha), \tag{A 20}$$

which is singular! However, the result for $\overline{\boldsymbol{v}_d\boldsymbol{\cdot}\nabla\psi}$ should not be singular as $\kappa\to 1$.

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