Ergod. Th. & Dynam. Sys. (2000), **20**, 1627–1637
Printed in the United Kingdom

Symmetric and reversible families of vector fields near a partially hyperbolic singularity

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(*Received* 16 *October* 1998 *and accepted* 18 *November* 1998)

Abstract. We study smooth local models of families of symmetric and reversible vector fields near a partially hyperbolic singularity. Special attention is given to the question of whether the involved changes of variables commute with the symmetry.

1. *Introduction*

For the investigation of nonlocal bifurcations, such as near a homoclinic orbit, it is not enough to perform center manifold reduction [**12**] near a singularity: a simple smooth model in a full neighborhood is often needed, for example in order to compute Poincaré maps. The dependence of the smooth local model on the bifurcation parameter is important, see [**2, 10**]. A special case of what we shall present in §2 was used by Rychlik [**14**, Theorem 1.1] in the construction of geometric Lorenz attractors within the class of vector fields on $\mathbb{R}^3$ having the involution $R(x_1, x_2, x_3) = (-x_1, -x_2, x_3)$ as a symmetry. See also [**1, 6, 9, 11, 13**] and many others. A particularly motivating example for this paper has been studied by Champneys and Härterich in [**6**] and concerns a smooth p-parameter family X_μ of vector fields defined near the origin of $\mathbb{R}^4$ having a singularity at zero for all μ near zero such that the eigenvalues of the linearization $dX_0(0)$ of X_0 at $x = 0$ are $\alpha, -\alpha, i\omega$ and $-i\omega$, with α, ω nonzero real numbers. It is, moreover, assumed that these vector fields have a homoclinic orbit at $\mu = 0$ and are time reversible for the linear involution $R(x_1, x_2, x_3, x_4) = (x_2, x_1, x_4, x_3)$, that is: $R_* X_\mu = -X_\mu$ for all parameters μ.

We consider a p-parameter family of vector fields X_μ near a singularity x_0 where (possibly) not all eigenvalues have their real part different from zero for some value $\mu = \mu_0$ of the parameter. Such a singularity x_0 is called partially hyperbolic [**17**]. In this paper we are interested in symmetric and reversible vector fields.

In the case of extra constraints, such as symmetry or reversibility, or in high codimension, it may be unavoidable that resonances and nonhyperbolicities of the eigenvalues at an equilibrium point persist. In such circumstances linearizability near the singularity becomes ungeneric, so a more complicated form is needed [**15**].

A first simplification, which we shall not describe as it is widely discussed in the literature ([**5, 7, 8, 11**] and many others), is the formal normal form, that is: use successive polynomial changes of variables in order to simplify the Taylor series of the vector field at the singularity. The preservation of extra structure in this formal context is also well understood, using the theory of graded Lie algebras (see the cited references).

The result of the formal normal form procedure is a family of the form $X_\mu(x) = P_N(x, \mu) + R_N(x, \mu)$ where P_N is a 'simplified' polynomial system of degree at most N and where $R_N(x, \mu) = \mathrm{O}(|(x, \mu)|^{N+1})$.

The question of simplification of the remainder term R_N is of a totally different nature, especially in the partially hyperbolic case. This is the aspect we want to restrict our attention to in this paper.

Remark 1. In what follows we will consider a p-parameter family of vector fields X_μ near the origin of $\mathbb{R}^n$, with $X_0(0) = 0$, as being a vector field $X(x, \mu) = X_\mu(x)\partial/\partial x + 0.\partial/\partial\mu$ on a neighborhood of $(x, \mu) = (0, 0) \in \mathbb{R}^n \oplus \mathbb{R}^p$ having a partially hyperbolic singular point there; from the methods further on it will follow that all changes of variables respect this 'familial character', and are of the form $h(x, \mu) = (h_\mu(x), \mu)$.

So let, in general, X be a smooth vector field on a neighborhood of $0 \in \mathbb{R}^n$ with $X(0) = 0$ such that for some linear transformation R of $\mathbb{R}^n$ we have $R_*X = X$, that is $R \circ X = X \circ R$. One says that X is R-symmetric. At a partially hyperbolic fixed point we consider the so-called quasipolynomial form [**15**]; we will explain this in the next section. When there are no resonances on the real parts of the eigenvalues in the 'hyperbolic' directions (see further on) this is what Takens [**17**] called the standard form, which means: normally linear along the center manifold; see also [**3**].

A similar subject is that the R-reversible vector fields, that is vector fields X satisfying $R_*X = -X$; in that case R is called a time reversing symmetry. Note that this last relation entails resonances when there are nonzero eigenvalues: if λ is an eigenvalue then $-\lambda$ is also one, giving the relations $\lambda = (m + 1)\lambda + m(-\lambda)$, $m = 1, 2, \ldots$.

2. *The symmetric case*

For the simplicity of the exposition we start with a C^∞ vector field X defined on a neighborhood of a partially hyperbolic fixed point, although many facts can still be formulated in the C^{finite} context.

Notation 1. Let $0 \in \mathbb{R}^n$ be a partially hyperbolic singular point of the C^∞ vector field X and let its linear part at zero be $A := dX(0)$. Let R be a linear transformation of $\mathbb{R}^n$ such that $R_*X = X$. This implies $RA = AR$. Up to a linear change of variables P we can assume that

$$A = \text{diagonal}[A_0, A_1, \ldots, A_u, A_{u+1}, \ldots, A_{u+s}]$$

(i.e. A consists of the blocks A_j on the diagonal; all other blocks are zero), where the A_j are square $d_j \oplus d_j$ matrices such that the spectrum of A_0 lies entirely in the imaginary axis and such that for $j \in \{1, \ldots, u + s\}$ the eigenvalues of A_j have the same real part $\lambda_j \neq 0$ and $\lambda_{u+1} < \cdots < \lambda_{u+s} < 0 < \lambda_1 < \cdots < \lambda_u$. The chosen letters '$u$' and '$s$' refer to 'unstable' and 'stable', respectively. This partition of A into blocks corresponds to

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a splitting $\mathbb{R}^n = E_0 \oplus E_1 \oplus \cdots \oplus E_{u+s}$, where the dimension of the linear subspace E_j is d_j. As mentioned in Remark 1, possible parameter directions lie in a subspace of E_0.

Note that in general P will not commute with R; we continue our study renaming PRP^{-1} again R. Let us perform a corresponding partition for R, that is $R = [R_{ij}]_{i,j=0}^{u+s}$. The relation $RA = AR$ yields $R_{ij}A_j = A_i R_{ij}$ and, together with the spectral assumptions, this implies $R_{ij} = 0$ for $i \neq j$. So $R = \text{diagonal}[R_{00}, \dots, R_{u+s,u+s}]$.

Let us denote $\lambda = (\lambda_1, \dots, \lambda_{u+s})$, $E^u = E_1 \oplus \cdots \oplus E_u$, $E^s = E_{u+1} \oplus \cdots \oplus E_{u+s}$, $E^h = E^u \oplus E^s$,

$$R^u = \text{diagonal}[R_{11}, \dots, R_{uu}],$$
$$R^s = \text{diagonal}[R_{u+1,u+1}, \dots, R_{u+s,u+s}].$$

Let x, $y = (y_1, \dots, y_u)$ and $z = (y_{u+1}, \dots, y_{u+s})$ denote coordinate functions on E_0, E^u, E^s, respectively, and write

$$X(x, y, z) = X^c(x, y, z)\frac{\partial}{\partial x} + X^u(x, y, z)\frac{\partial}{\partial y} + X^s(x, y, z)\frac{\partial}{\partial z}.$$

We consider the normally linear part NX of X along E_0, that is

$$NX(x, y, z) = X^c(x, 0, 0)\frac{\partial}{\partial x} + \frac{\partial X^u}{\partial y}(x, 0, 0).y\frac{\partial}{\partial y} + \frac{\partial X^s}{\partial z}(x, 0, 0).z\frac{\partial}{\partial z} \tag{1}$$

which is called the 'standard form' in [**17**]. We develop X in the form

$$\begin{aligned} X(x, y, z) = NX(x, y, z) &+ \sum_{|p|+|q|=1}^{M} a^c_{pq}(x) y^p z^q \frac{\partial}{\partial x} \\ &+ \sum_{|\alpha|+|\beta|=0}^{M} a^u_{\alpha\beta}(x) y^\alpha z^\beta \frac{\partial}{\partial y} + \sum_{|\gamma|+|\delta|=0}^{M} a^s_{\gamma\delta}(x) y^\gamma z^\delta \frac{\partial}{\partial z} \\ &+ \mathrm{O}(|(y, z)|^{M+1}) \end{aligned} \tag{2}$$

for given $M \in \mathbb{N}$, where $p, q, \alpha, \beta, \gamma, \delta$ are multi-indices. The quasipolynomial form procedure consists of eliminating inductively the terms in these three summations by means of C^k changes of variables, also called conjugacies. In general, k can only be expected to be finite, due to the 'central' behavior: see [**4, 15, 17, 18**]. We say that a function $a(x)$ is N-flat at the origin if it is $\mathrm{O}(|x|^{N+1})$. Using formal normal form theory [**5, 7, 8, 11**] we can assume that, up to a polynomial change of variables *commuting with* R, the functions $a^c_{pq}(x)$ etc. are as flat as needed.

Remark 2. If X is R-symmetric then so is $\partial^{|p|+|q|}X/\partial y^p \partial z^q(x, 0, 0).y^p z^q$ for all multi-indices (p, q): we obtain this by taking the derivatives of the equality

$$R(X(x, y, z)) = X(R_{00}x, R^u y, R^s z)$$

with respect to y and z. So we may, and do, assume that the terms in the development (2) are R-symmetric.

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We shall assume that the symmetry R is similar to an orthogonal matrix, that is PRP^{-1} is orthogonal for some invertible P; this includes the, often encountered, case of an involution.

A first step in the quasipolynomial form procedure is the following.

THEOREM 1. *Suppose that R is similar to an orthogonal matrix. Let $r \in \mathbb{N}$ be given and let Z be of the form*

$$Z(x, y, z) = a(x)y^p z^q \frac{\partial}{\partial x} \quad \text{with } \langle \lambda, (p, q)\rangle \neq 0 \tag{3}$$

or

$$Z(x, y, z) = a(x)y^p z^q \frac{\partial}{\partial y_j} \quad \text{with } \langle \lambda, (p, q)\rangle \neq \lambda_j \tag{4}$$

with $(p, q) \in \mathbb{N}^u \times \mathbb{N}^s$, $j = 1, \dots, u+s$ and where a is a C^r function defined near $0 \in E_0$. Suppose that both NX and Z are R-symmetric (cf. Remark 2). If a is r-flat at the origin, then the vector fields $NX + Z$ and NX are locally C^r conjugate up to terms of order more than $\mathrm{O}(|y|^p|z|^q)$ by means of a change of variables h commuting with R.

Proof. We want to work with globally defined objects by multiplying the occurring vector fields with a C^∞ 'cut off' function $\varphi : \mathbb{R}^n \to [0, 1]$ which is equal to zero outside a given 'small' neighborhood U of the origin and equal to one on a smaller neighborhood. One has to take care that this does not destroy the symmetry. Using the assumption that R is similar to an orthogonal matrix S, say $PRP^{-1} = S$ for some invertible P, we can proceed as follows.

Let $\psi : \mathbb{R} \to [0, 1]$ be a C^∞ function which is zero outside an interval $[-\delta, \delta]$ and equal to one on $[-\varepsilon, \varepsilon]$ for $\varepsilon < \delta$. For $v \in \mathbb{R}^n$ we define $\varphi(v) = \psi(|Pv|)$. Suppose that a vector field X on $\mathbb{R}^n$ is R-symmetric. We check that $Y := \varphi.X$ is also R-symmetric using the fact that $|Sw| = |w|$ for all $w \in \mathbb{R}^n$:

$$\begin{aligned} R.Y(v) &= R.\varphi(v)X(v) \\ &= \psi(|Pv|)R.X(v) \\ &= \psi(|SPv|)X(Rv) \\ &= \varphi(P^{-1}SPv)X(Rv) \\ &= \varphi(Rv)X(Rv) \\ &= Y(Rv). \end{aligned}$$

Using such a cut off function we can replace the locally defined NX and Z by globally defined vector fields, which we give the same name. Moreover, the flows of (new) NX and $NX + Z$ are complete, i.e. defined for all times $t \in \mathbb{R}$. Let NX_t and $(NX + Z)_t$ denote the time t map of NX and $NX + Z$, respectively. We remark that these maps commute with R.

Up to a possible multiplication of both vector fields by -1, we can take care that $\langle \lambda, (p, q)\rangle < 0$ in case (3), respectively $\langle \lambda, (p, q)\rangle - \lambda_j < 0$ in case (4). Consider for each $t \in \mathbb{R}$ the diffeomorphism $h_t = (NX + Z)_{-t} \circ NX_t$, which commutes with R.

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Denote $j_{p,q}h_t(x, y, z) = (1/p!q!)\partial^{|p|+|q|}h_t/\partial y^p \partial z^q(x, 0, 0).y^p z^q$. As in Remark 2, this still commutes with R.

From [**4**] it follows that, provided a suitable cut off function is chosen, the limit

$$h(x, y, z) := (x, y, z) + \lim_{t\to\infty} j_{p,q}h_t(x, y, z) \tag{5}$$

exists for all $(x, y, z) \in \mathbb{R}^n$ and is a C^r diffeomorphism conjugating NX to $NX + Z$ up to terms of order more than $\mathrm{O}(|y|^p|z|^q)$; it commutes with R. □

Terms in the development (2) for which (3) or (4) become equalities cannot be removed by this method, and we call them *unremovable*; such equalities are called resonances on the real parts of the eigenvalues. The process of eliminating terms like (3) or (4) is inductive and leaves a development with only unremovable terms. This is the quasipolynomial form:

$$\begin{aligned} X^1(x, y, z) := NX(x, y, z) &+ \sum_{\substack{|p|+|q|=1\\(p,q)\in S^c}}^{M} a^c_{pq}(x) y^p z^q \frac{\partial}{\partial x} \\ &+ \sum_{j=1}^{u+s} \sum_{\substack{|\alpha|+|\beta|=0\\(\alpha,\beta,j)\in S^h_j}}^{M} a^h_{\alpha,\beta,j}(x) y^\alpha z^\beta \frac{\partial}{\partial y_j} + T(x, y, z) \end{aligned} \tag{6}$$

where $(p, q) \in S^c$ iff $\langle\lambda, (p, q)\rangle = 0$ and $(\alpha, \beta, j) \in S^h_j$ iff $\langle\lambda, (\alpha, \beta)\rangle = \lambda_j$, $j = 1, \ldots, u + s$, and where $T(x, y, z) = \mathrm{O}(|(y, z)|^{M+1})$. We denote $X^0 = X^1 - T$.

In order to fix the ideas of the reader we give a simple example on $\mathbb{R}^3$.

Example 1. Let X_μ be a family of vector fields defined near $0 \in \mathbb{R}^3$ with linear part $A(\mu) = dX_\mu(0) = (A_{ij}(\mu))_{0\le i,j\le 2}$ such that $A(0) = \text{diagonal}[0, \lambda, -\lambda]$ and $\lambda > 0$. Suppose that the symmetry for each X_μ is $R(x, y, z) = (-x, y, -z)$. The only resonances here are $\langle(\lambda, -\lambda), (p, q)\rangle = 0$, $\langle(\lambda, -\lambda), (p, q)\rangle = \lambda$ and $\langle(\lambda, -\lambda), (p, q)\rangle = -\lambda$, which are equivalent to $p = q$, $p = q+1$, and $p+1 = q$, respectively. Hence the quasipolynomial form in (6) is, for this example, up to a flat term T:

$$NX_\mu(x, y, z) + \sum_{p=1}^{M} (yz)^p \left\{ a^c_p(x, \mu)\frac{\partial}{\partial x} + a^u_p(x, \mu) y \frac{\partial}{\partial y} + a^s_p(x, \mu) z \frac{\partial}{\partial z} \right\}. \tag{7}$$

Note that, by Remark 2, the coefficient functions satisfy certain relations:

$$\begin{aligned} a^c_p(-x, \mu) &= (-1)^p a^c_p(x, \mu), \\ a^u_p(-x, \mu) &= (-1)^p a^u_p(x, \mu), \\ a^s_p(-x, \mu) &= (-1)^p a^s_p(x, \mu). \end{aligned}$$

PROPOSITION 1. *X^0 leaves $y = 0$ and $z = 0$ invariant.*

Proof. Terms of the form $a(x)y^p\partial/\partial y_j$ with $u + 1 \le j \le u + s$ satisfy $\langle\lambda, (p, 0)\rangle \ne \lambda_j$, since $\langle\lambda, (p, 0)\rangle = \lambda_1 p_1 + \cdots + \lambda_u p_u \ge 0$ and $\lambda_j < 0$ (see Notation 1). Hence they can be removed by Theorem 1. We can make the same observation for the terms $a(x)z^q\partial/\partial y_j$ with $1 \le j \le u$. □

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The next step is to 'cut off the tail' T. We split this tail as a sum $T = T_1 + T_2$ with T_1 respectively T_2 sufficiently flat in y respectively z, and take care that this does not destroy the symmetry, i.e. both terms should commute with R. We treat two cases.

Case 1. Suppose that the initial vector field X in Notation 1 leaves (at least) one of the manifolds $y = 0$ and $z = 0$ invariant. Then we can proceed as follows. By the construction of h in Theorem 1 we can take the $(M+1)$th partial derivative of T with respect to (y, z), and write T as

$$T(x, y, z) = \frac{1}{M!} \int_0^1 (1-\xi)^M \, d^{M+1}_{(y,z)} T(x, \xi y, \xi z) \, d\xi . (y, z)^{M+1} \tag{8}$$

$$= \frac{1}{M!} \int_0^1 (1-\xi)^M \sum_{|p|+|q|=M+1} \binom{M+1}{p, q} \frac{\partial^{M+1} T}{\partial y^p \partial z^q}(x, \xi y, \xi z) \, d\xi . y^p z^q . \tag{9}$$

By taking the derivatives of the relation $RT = TR$ we get, for $|p| + |q| = M + 1$,

$$R . \frac{\partial^{|p|+|q|} T}{\partial y^p \partial z^p} = \frac{\partial^{|p|+|q|} T}{\partial y^p \partial z^q}(R_{00} x, R^u y, R^s z)(R^u)^p (R^s)^q ; \tag{10}$$

let us denote

$$T_{pq}(x, y, z) = \frac{1}{M!} \int_0^1 (1-\xi)^M \binom{M+1}{p, q} \frac{\partial^{M+1} T}{\partial y^p \partial z^q}(x, \xi y, \xi z) \, d\xi . y^p z^q , \tag{11}$$

then $T = \sum_{|p|+|q|=M+1} T_{pq}$. Using (10) and (11) it is straightforward to check that $RT_{pq} = T_{pq} R$. We set

$$T_1 = \sum_{|p| \geq (M+1)/2} T_{pq}, \quad T_2 = \sum_{|p| < (M+1)/2} T_{pq} \tag{12}$$

and $M_1 = (M+1)/2$. One has $T_1(x, y, z) = \mathrm{O}(|y|^{M_1})$ and $T_2(x, y, z) = \mathrm{O}(|z|^{M_1})$.

Let us suppose that X leaves $y = 0$ invariant; the case that X leaves $z = 0$ invariant is treated similarly.

THEOREM 2. *Suppose that R is similar to an orthogonal matrix. Let $r \in \mathbb{N}$ be given. If M_1 is large enough then there exists a C^r diffeomorphism h commuting with R conjugating X^0 to $X^0 + T$ near the origin, that is $h_* X^0 = X^0 + T$.*

Proof. We first conjugate X^0 to $X^0 + T_2$. In the same way as in the proof of Theorem 1, we use cut off functions and replace the locally defined vector fields X^0 and $X^0 + T_2$ by globally defined vector fields, which we give the same name, and the time t maps of these new vector fields are defined for all $t \in \mathbb{R}$. Moreover, these time t maps commute with R. Note that, by Proposition 1, $z = 0$ is invariant for both vector fields. Hence we can apply [**4**] and it follows that for M_1 large enough the limit $g(x, y, z) := \lim_{t\to\infty} (X^0 + T_2)_{-t} \circ X^0_t(x, y, z)$ exists for all $(x, y, z) \in \mathbb{R}^n$ and is a C^r diffeomorphism conjugating X^0 to $X^0 + T_2$, i.e. $g_* X^0 = X^0 + T_2$. Clearly g commutes with R.

Next we reduce the elimination of T_1 to a completely similar problem. Note that $g_*^{-1}(X^0 + T) = g_*^{-1}(X^0 + T_1 + T_2) = X^0 + g_*^{-1} T_1$. As X^0 and T_1 leave $y = 0$ invariant and the same is assumed for X, we get that T_2 also leaves $y = 0$ invariant. Hence also g

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and $g_*^{-1}T_1$ leave $y = 0$ invariant, considering the way g is defined. It then also follows that $g_*^{-1}T_1$ is $O(|y|^{M_1})$, and by reversing the time we are led back to a problem analogous to the one we have just solved.

Case 2. Here we do not assume that $y = 0$ or $z = 0$ are invariant for the initial vector field X. A difficulty is that the usual change of variables making the center-stable and center-unstable invariant manifolds straight might not commute with the symmetry R. This aspect is sometimes overlooked when studying structure preserving normal forms.

The following assumption is made about the symmetry $R = \text{diagonal}[R_{00}, R^u, R^s]$.

Assumption A. R_{00}, R^u and R^s are similar to an orthogonal matrix.

PROPOSITION 2. *Suppose that Assumption A is satisfied. There exists a C^∞ mapping $\lambda : E^u \times E^s \to [0, 1]$ with the following properties. For each $M \in \mathbb{N}$ and for each C^M mapping $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(x, y, z) = O(|(y, z)|^{M+1})$ near $(y, z) = (0, 0)$ one has:*
(i) $T_1(x, y, z) := \lambda(y, z)T(x, y, z)$ *is* C^M *and* $\lambda(y, z)T(x, y, z) = O(|y|^{M+1})$*;*
(ii) $T_2(x, y, z) := (1 - \lambda(y, z))T(x, y, z) = O(|z|^{M+1})$*;*
(iii) *if* $RT = TR$ *then* $RT_1 = T_1R$ *and* $RT_2 = T_2R$.

Proof. Fix a C^∞ function $\varphi : [0, \infty[\to [0, 1]$ that is equal to one on $[0, 1]$ and equal to zero on $[2, \infty[$. By Assumption A we can choose norms on E^u and E^s such that R^u and R^s are norm-preserving. We define $\lambda(y, z) = \varphi(|z|/|y|)$ for $y \neq 0$ and $\lambda(0, z) = 0$. Observe that $\lambda(y, z) = \lambda(\alpha y, \alpha z)$ for all $\alpha \in \mathbb{R}$, $\alpha \neq 0$.

The facts (i) and (ii) are now checked as in [**4**, Lemma 3.8]. We check (iii):

$$\lambda(R^u y, R^s z) = \varphi(|R^s z|/|R^u y|) = \varphi(|z|/|y|) = \lambda(y, z). \tag{13}$$

It follows from $RT = TR$ and (13) that

$$\begin{aligned} R.T_1(x, y, z)) &= R.\lambda(y, z)T(x, y, z) \\ &= \lambda(y, z).T(R(x, y, z)) \\ &= \lambda(R^u y, R^s z).T(R_{00}x, R^u y, R^s z) \\ &= T_1(R_{00}x, R^u y, R^s z). \end{aligned} \tag{14}$$

□

We now come to the conjugacy of X^0 to $X^1 = X^0 + T$. Using Proposition 2 we write $T = T_1 + T_2$. As X^0 and T_2 leave $z = 0$ invariant, the conjugacy of X^0 and $X^0 + T_2$ is handled precisely as in the first part of Theorem 2, that is if M in (6) is large enough there is a C^r diffeomorphism g, commuting with R, such that $g_*X^0 = X^0 + T_2$; hence $g_*^{-1}(X^0 + T) = X^0 + g_*^{-1}T_1$. By the construction in Proposition 2 the vector field T_1 is identically zero in a cone containing $y = 0$ and M times flat. Hence $g_*^{-1}T_1 = O(|y|^{M+1})$ and by reversing the time we are reduced to a problem analogous to the one that we have just solved. We can summarize this in the following.

THEOREM 3. *Suppose that the symmetry R satisfies Assumption A. Let $r \in \mathbb{N}$ be given. If M is large enough in (6) then there exists a C^r diffeomorphism h commuting with R such that $h_*X^0 = X^1$ near the origin.*

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So concerning Example 1 we can conclude that, for M large enough in (7), there exists a C^r change of variables $h(x, y, z, \mu) = (h_\mu(x, y, z), \mu)$ such that h_μ conjugates the original family X_μ to (7) and such that, moreover, h_μ commutes with the symmetry R.

Remark 3. Theorem 3 implies that we can find a change of variables commuting with R that makes the center-stable and center-unstable manifolds straight.

3. *The reversible case*

As observed in the introduction, if R is an invertible linear transformation and if X is a vector field satisfying $R_* X = -X$ then an eigenvalue $\lambda \neq 0$ of $A = dX(0)$ is accompanied by the eigenvalue $-\lambda$. Let us recall quickly some facts from linear algebra. We put A in a block-diagonal form $A = \text{diagonal}[A_0, A_1, \dots, A_p]$ such that A_0 has eigenvalues with real part zero and A_i has eigenvalues with nonzero real part $\lambda_i, -\lambda_i$ and $\lambda_i \neq \lambda_j$ for $i \neq j$. We perform a corresponding partition $R = [R_{ij}]_{i,j=0}^p$ for R. The relation $AR = -RA$ yields $A_i R_{ij} = -R_{ij} A_j$, and as the spectra of A_i and $-A_j$ are disjoint we get $R_{ij} = 0$ for $i \neq j$. We write, for $i \neq 0$, $A_i = \text{diagonal}[A_i^+, A_i^-]$, where A_i^+ respectively A_i^- has eigenvalue with real part λ_i respectively $-\lambda_i$, and we have a corresponding partition

$$R_{ii} = \begin{bmatrix} R_{ii}^{++} & R_{ii}^{+-} \\ R_{ii}^{-+} & R_{ii}^{--} \end{bmatrix}.$$

The relation $A_i R_{ii} = -R_{ii} A_i$ easily implies $R_{ii}^{++} = R_{ii}^{--} = 0$ for $i \neq 0$. According to the particular Jordan form of $A_i^\pm$, one could then further specify $R_{ii}^{\pm\pm}$ using Toeplitz matrices. As R is assumed to be invertible, the Jordan block structures of A_i^+ and A_i^- are the same.

Remark 4. The often studied case that R is an involution means here that $R_{ii}^{+-} = (R_{ii}^{-+})^{-1}$; in this case we can assume, up to the linear change of variables

$$P = \text{diagonal}[I, I, R_{11}^{+-}, \dots I, R_{pp}^{+-}],$$

that $R_{ii}^{\pm\mp} = I$, where I denotes the appropriate identity.

We conclude that $A = \text{diagonal}[A_0, A_1^+, A_1^-, \dots, A_p^+, A_p^-]$ and that R has the form

$$R = \begin{bmatrix} R_{00} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & R_{11}^{+-} & \dots & 0 & 0 \\ 0 & R_{11}^{-+} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & R_{pp}^{+-} \\ 0 & 0 & 0 & \dots & R_{pp}^{-+} & 0 \end{bmatrix}.$$

This partition corresponds to a splitting $\mathbb{R}^n = E_0 \oplus E_1^+ \oplus E_1^- \oplus \dots \oplus E_p^+ \oplus E_p^-$. Let $(x, y) = (x, y_1^+, y_1^-, \dots, y_p^+, y_p^-)$ denote coordinate functions on it. Write

$$X(x, y) = X^c(x, y)\frac{\partial}{\partial x} + X^h(x, y)\frac{\partial}{\partial y}.$$

The normally linear part of X along E_0 is

$$NX(x, y) = X^c(x, 0)\frac{\partial}{\partial x} + \frac{\partial X^h}{\partial y}(x, 0).y\frac{\partial}{\partial y}.$$

We develop X, abbreviating $y^+ = (y_1^+, \ldots, y_p^+)$ and $y^- = (y_1^-, \ldots, y_p^-)$:

$$X(x, y) = NX(x, y) + \sum_{|\alpha|+|\beta|=1}^{M} a^c_{\alpha\beta}(x)(y^+)^\alpha (y^-)^\beta \frac{\partial}{\partial x}$$
$$+ \sum_{|\gamma|+|\delta|=0}^{M} a^h_{\gamma\delta+}(x)(y^+)^\gamma (y^-)^\delta \frac{\partial}{\partial y^+} + \sum_{|\gamma|+|\delta|=0}^{M} a^h_{\gamma\delta-}(x)(y^+)^\gamma (y^-)^\delta \frac{\partial}{\partial y^-}$$
$$+ \mathrm{O}(|y|^{M+1}). \tag{15}$$

Let us examine what R-reversibility here means for the terms in this development. Denote

$$X_{\alpha\beta}(x, y^+, y^-) = \frac{\partial^{|\alpha|+|\beta|} X}{\partial (y^+)^\alpha (y^-)^\beta}(x, 0, 0)(y^+)^\alpha (y^-)^\beta.$$

The relation $R_* X = -X$ is, in detail,

$$R.X(x, y^+, y^-) = -X(R_{00}, R^{+-} y^-, R^{-+} y^+)$$

where $R^{\pm\mp} = \text{diagonal}[R^{\pm\mp}_{11}, \ldots, R^{\pm\mp}_{pp}]$. By taking the derivatives the foregoing relation with respect to y and z we find

$$R.\frac{\partial^{|\alpha|+|\beta|} X}{\partial (y^+)^\alpha (y^-)^\beta}(x, y^+, y^-)$$
$$= -\frac{\partial^{|\beta|+|\alpha|} X}{\partial (y^-)^\alpha \partial (y^+)^\beta}(R_{00}x, R^{+-}y^-, R^{-+}y^+).(R^{-+})^\alpha (R^{+-})^\beta \tag{16}$$

so

$$R.X_{\alpha\beta}(x, y^+, y^-) = -X_{\beta\alpha}(R_{00}x, R^{+-}y^-, R^{-+}y^+); \tag{17}$$

we conclude that

$$R_* X_{\alpha\beta} = -X_{\beta\alpha}. \tag{18}$$

Because of this equality the terms in the development (15) satisfy certain relations. Let us be more specific in the case when R is an involution, where by Remark 4 we may, and do, assume that $R(x, y^+, y^-) = (R_{00}x, y^-, y^+)$. A straightforward computation shows that

$$R_{00} a^c_{\alpha\beta}(x) = -a^c_{\beta\alpha}(R_{00}x) \tag{19}$$
$$a^h_{\alpha\beta-}(x) = -a^h_{\beta\alpha+}(R_{00}x) \tag{20}$$
$$a^h_{\alpha\beta+}(x) = -a^h_{\beta\alpha-}(R_{00}x). \tag{21}$$

We try to proceed as in Theorem 1. Write $\lambda = (\lambda_1, \ldots, \lambda_p)$. The equivalent of condition (3) here is $\langle(\lambda, -\lambda), (\alpha, \beta)\rangle \neq 0$. This is obviously violated for all $\alpha = \beta$. Condition (4) becomes $\langle(\lambda, -\lambda), (\gamma, \delta)\rangle \neq \pm\lambda_i$, which is violated when $\gamma_j = \delta_j$ for all

https://doi.org/10.1017/S0143385700000900 Published online by Cambridge University Press

$j \neq i$ and $|\gamma_i| = |\delta_i| \pm 1$. Thus the terms of the form

$$\sum_{|\alpha|=1}^{[M/2]} a_\alpha^c(x)(y_1^+)^{\alpha_1}(y_1^-)^{\alpha_1} \dots (y_p^+)^{\alpha_p}(y_p^-)^{\alpha_p} \frac{\partial}{\partial x}$$
$$+ \sum_{i=1}^{p} \sum_{|\gamma|=0}^{[(M-1)/2]} (y_1^+)^{\gamma_1}(y_1^-)^{\gamma_1} \dots (y_p^+)^{\gamma_p}(y_p^-)^{\gamma_p}$$
$$\times \left(a_{\gamma i+}^h(x) y_i^+ \frac{\partial}{\partial y_i^+} + a_{\gamma i-}^h(x) y_i^- \frac{\partial}{\partial y_i^-} \right) \tag{22}$$

are unremovable in the sense of §2. In the case when R is an involution, (22) becomes by (20)

$$\sum_{|\alpha|=1}^{[M/2]} a_\alpha^c(x)(y_1^+)^{\alpha_1}(y_1^-)^{\alpha_1} \dots (y_p^+)^{\alpha_p}(y_p^-)^{\alpha_p} \frac{\partial}{\partial x}$$
$$+ \sum_{i=1}^{p} \sum_{|\gamma|=0}^{[(M-1)/2]} (y_1^+)^{\gamma_1}(y_1^-)^{\gamma_1} \dots (y_p^+)^{\gamma_p}(y_p^-)^{\gamma_p}$$
$$\times \left(a_{\gamma i+}^h(x) y_i^+ \frac{\partial}{\partial y_i^+} - a_{\gamma i+}^h(R_{00}x) y_i^- \frac{\partial}{\partial y_i^-} \right). \tag{23}$$

Assume that these are *the only* violations of the 'nonresonance' conditions (3) and (4). In precisely the same way as in the proof of Theorem 1 we can eliminate all the other terms in the development (15). In this case the quasipolynomial form of X is NX + (22), and when R is an involution it is more specifically NX + (23); all this up to terms of order $O(|y|^{M+1})$.

Question. A question now is whether the changes of variables, obtained by the method in Theorem 1, and eliminating successively the terms in the development (15) of X, is commuting with R. We ignore this. Consider indeed a vector field of the form

$$Z(x, y^+, y^-) = a(x)(y^+)^\alpha (y^-)^\beta \frac{\partial}{\partial y_i^+}$$

with $a \neq 0$. This vector field is not R-reversible nor R-symmetric. On the other hand, we could try to eliminate in *one* step the R-reversible vector field (for simplicity we describe the case when $R^{+-} = R^{-+} = I$, but this is not essential)

$$Z(x, y^+, y^-) = a(x)(y^+)^\alpha (y^-)^\beta \frac{\partial}{\partial y_i^+} - a(R_{00}x)(y^-)^\alpha (y^+)^\beta \frac{\partial}{\partial y_i^-}. \tag{24}$$

For the first term on the right-hand side of (24), the condition in (4) is

$$\langle (\lambda, -\lambda), (\alpha, \beta) \rangle - \lambda_i \neq 0, \tag{25}$$

while for the second term it is

$$\langle (\lambda, -\lambda), (\beta, \alpha) \rangle + \lambda_i \neq 0. \tag{26}$$

As the left-hand sides of the inequalities (25) and (26) have opposite sign, the convergence of the limit in (5) is problematic (cf. [**4**]).

We meet a similar question in the 'cut of the flat tail' process (cf. Theorems 2 and 3), even in the simpler case of a 'purely' hyperbolic singularity: from the usual methods of proof of Sternberg-like theorems [**2, 15, 16**] it does not follow that one can find a change of variables which eliminates flat terms and which, moreover, commutes with the reversing symmetry. We did not find literature about this, although recent calculations of G. Belitskii in a specific case (private communication) indicate that it might be true.

REFERENCES

[1] D. C. Aronson, S. A. van Gils and M. Krupa. Homoclinic twist bifurcations with $\mathbb{Z}_2$ symmetry. *J. Nonlinear Sci.* **4** (1994), 195–219.

[2] P. Bonckaert. On the continuous dependence of the smooth change of coordinates in parametrized normal form theorems. *J. Diff. Eq.* **106** (1993), 107–120.

[3] P. Bonckaert. Partially hyperbolic fixed points with constraints. *Trans. Amer. Math. Soc.* **348** (1996), 997–1011.

[4] P. Bonckaert. Conjugacy of vector fields respecting additional properties. *J. Dynam. Control Syst.* **3** (1997), 419–432.

[5] H. Broer. Formal normal forms for vector fields and some consequences for bifurcations in the volume preserving case. *Dynamical Systems and Turbulence (Warwick 1980) (Lecture Notes in Mathematics, 898)*. Springer, 1981.

[6] A. R. Champneys and J. Härterich. Cascades of homoclinic orbits to a saddle-centre for reversible and perturbed Hamiltonian systems. *Preprint*, 1998.

[7] S.-N. Chow, C. Li and D. Wang. *Normal Forms and Bifurcations of Planar Vector Fields*. Cambridge University Press, 1994.

[8] G. Gaeta. Normal forms of reversible dynamicle systems. *Int. J. Theor. Phys.* **33** (1994), 1917–1928.

[9] P. Hirschberg and E. Knobloch. Šil'nikov-hopf bifurcation. *Physica* D **62** (1993), 202–216.

[10] A. J. Homburg, H. Kokubu and V. Naudot. Homoclinic-doubling cascades. *Preprint*, 1998.

[11] J. S. Lamb. Local bifurcations in k-symmetric dynamical systems. *Nonlinearity* **9** (1996), 537–557.

[12] C. Robinson. *Dynamical Systems, Stability, Symbolic Dynamics and Chaos*. CRC Press, Boca Raton, FL, 1995.

[13] R. Roussarie and C. Rousseau. Almost planar homoclinic loops in $\mathbb{R}^3$. *J. Diff. Eq.* **126** (1996), 1–47.

[14] M. R. Rychlik. Lorenz attractors through Šil'nikov-type bifurcation. Part I. *Ergod. Th. & Dynam. Sys.* **10** (1990), 793–821.

[15] V. S. Samovol. Equivalence of systems of differential equations in a neighbourhood of a singular point. *Trans. Moscow Math. Soc.*, 1983, pp. 217–237.

[16] S. Sternberg. On the structure of local homeomorphisms of euclidean n-space, ii. *Amer. J. Math.* **80** (1958), 623–631.

[17] F. Takens. Partially hyperbolic fixed points. *Topology* **10** (1971), 133–147.

[18] S. van Strien. Center manifolds are not C^∞. *Math. Z.* **166** (1979), 143–145.