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Steady states with unbounded mass of the Keller–Segel system

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We consider the boundary-value problem

$$\begin{aligned} -\Delta u + u = \lambda \mathrm{e}^u \quad & \text{in } B_{r_0}, \\ \partial_\nu u = 0 \quad & \text{on } \partial B_{r_0}, \end{aligned}$$

where B_{r_0} is the ball of radius r_0 in $\mathbb{R}^N$, $N \geqslant 2$, $\lambda > 0$ and ν is the outer normal derivative at ∂B_{r_0}. This problem is equivalent to the stationary Keller–Segel system from chemotaxis. We show the existence of a solution concentrating at the boundary of the ball as λ goes to 0.

1. Introduction

We consider a system of partial differential equations modelling chemotaxis. Chemotaxis is the movement of cells in response to the gradient of a chemical, which explains the aggregation of cells that move towards a high concentration of a chemical secreted by themselves. The basic model was introduced by Keller and Segel [16], and a simplified form of it reads as

$$\left.\begin{aligned} v_t = \Delta v - \nabla(v \nabla u) \quad & \text{in } \Omega, \\ \tau u_t = \Delta u - u + v \quad & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 \quad & \text{on } \partial\Omega, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x),& \end{aligned}\right\} \tag{1.1}$$

where $u = u(x,t) \geqslant 0$ and $v = v(x,t) \geqslant 0$ are the concentration of the species and the chemical, respectively. Here, Ω is a bounded smooth domain in $\mathbb{R}^N$ and $N \geqslant 2$. The cases $N = 2$ or $N = 3$ are of particular interest. In (1.1), ν denotes the unit outward vector normal at $\partial\Omega$, and τ is a positive constant.

After the seminal works of Nanjudiah [20] and Childress and Percus [3], many contributions have been made to the understanding of different analytical aspects of this system and its variations. We refer the reader, for instance, to [2, 5, 6, 12–20, 22–27].

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In this paper, we study steady states of (1.1), namely, solutions to the system

$$\left.\begin{aligned}\Delta v - \nabla(v\nabla u) = 0 \quad &\text{in } \Omega,\\ \Delta u - u + v = 0 \quad &\text{in } \Omega,\\ \partial_\nu u = \partial_\nu v = 0 \quad &\text{on } \partial\Omega.\end{aligned}\right\} \tag{1.2}$$

As pointed out in [18], stationary solutions to the Keller–Segel system are of basic importance for the understanding of the global dynamics of the system.

This problem was first studied by Schaaf [21] in the one-dimensional case. In the higher-dimensional case, Biler [1] proved the existence of a non-trivial radially symmetric solution to (1.2) for the case in which Ω is a ball. In the general two-dimensional case, Wang and Wei [28] and Senba and Suzuki [22] proved that, for any $\mu \in (0, 1/|\Omega| + \mu_1) \setminus \{4\pi m \colon m \geqslant 1\}$, problem (1.2) has a non-constant solution such that $\int_\Omega v(x)\,\mathrm{d}x = \mu|\Omega|$. Here, μ_1 is the first eigenvalue of $-\Delta$ with Neumann boundary conditions. Del Pino and Wei [4] reduced system (1.2) to a scalar equation. Indeed, it is easy to check that (u, v) solves system (1.2) if and only if $v = \lambda \mathrm{e}^u$ for some positive constant λ and u solves the equation

$$\left.\begin{aligned}-\Delta u + u = \lambda \mathrm{e}^u \quad &\text{in } \Omega,\\ \partial_\nu u = 0 \quad &\text{on } \partial\Omega.\end{aligned}\right\} \tag{1.3}$$

Using this point of view, they proved, for any integers k and ℓ, that there exists a family of solutions (u_λ, v_λ) to (1.2) such that v_λ exhibits k Dirac measures inside the domain, and ℓ Dirac measures on the boundary of the domain, as $\lambda \to 0$, i.e.

$$v_\lambda \rightharpoonup \sum_{i=1}^{k} 8\pi\delta_{\xi_i} + \sum_{i=1}^{\ell} 4\pi\delta_{\eta_i} \quad \text{as } \lambda \to 0,$$

where $\xi_1, \dots, \xi_k \in \Omega$ and $\eta_1, \dots, \eta_\ell \in \partial\Omega$. In particular, the solution has bounded mass, i.e.

$$\lim_{\lambda\to 0} \int_\Omega v_\lambda(x)\,\mathrm{d}x = \lim_{\lambda\to 0} \int_\Omega \lambda \mathrm{e}^{u_\lambda(x)}\,\mathrm{d}x = 4\pi(2k + \ell).$$

In particular, when Ω is a ball their argument allows us to find a radial solution to the system (1.2) that exhibits a Dirac measure at the centre of the ball with mass 8π when λ goes to 0.

In the present paper, we find a new radial solution to the system (1.2) for the case in which Ω is a ball with unbounded mass. Our main result reads as follows.

THEOREM 1.1. *Let $\Omega = B(0, r_0)$ be a ball centred at the origin with radius r_0. There exists λ_0 such that, for any $\lambda \in (0, \lambda_0)$, the problem (1.3) has a radial solution (u_λ, v_λ) such that, as $\lambda \to 0$,*

$$\lim_{\lambda\to 0} \int_\Omega v_\lambda(x)\,\mathrm{d}x = \lim_{\lambda\to 0} \int_\Omega \lambda \mathrm{e}^{u_\lambda(x)}\,\mathrm{d}x = +\infty. \tag{1.4}$$

Moreover, for a suitable choice of positive numbers ε_λ (see (2.3)) with $\varepsilon_\lambda \to 0$ as $\lambda \to 0$, we have

$$\lim_{\lambda\to 0} \varepsilon_\lambda u_\lambda = \frac{\sqrt{2}}{\mathcal{U}'(r_0)}\mathcal{U} \quad C^0\text{-uniformly on compact sets of } \Omega. \tag{1.5}$$

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Here $\mathcal{U}$ is the positive radial solution to the problem (see also lemma 2.1)

$$\left.\begin{aligned} -\Delta\mathcal{U}+\mathcal{U}=0 \quad &\text{in } B(0,r_0),\\ \mathcal{U}=1 \quad &\text{on } \partial B(0,r_0). \end{aligned}\right\} \tag{1.6}$$

To find the solution u_λ to (1.3), we use a fixed-point argument. More precisely, we look for a solution to (1.3) of the form $u_\lambda = \bar{u}_\lambda + \phi_\lambda$, where the leading term $\bar{u}_\lambda$ must be accurately defined. Once one has a good approximating solution $\bar{u}_\lambda$, a simple contraction mapping argument leads to the higher-order term ϕ_λ.

The difficulty with the construction of the approximate solution $\bar{u}_\lambda$ is due to the fact that $\bar{u}_\lambda$ shares the behaviour of $\mathcal{U}$ (which solves (1.6)) in the inner part of the ball, but shares the behaviour of the function w_ε (see (1.7)) near the boundary of the ball. Here

$$w_\varepsilon(r) = \ln\frac{4}{\varepsilon^2}\frac{\mathrm{e}^{\sqrt{2}(r-r_0)/\varepsilon}}{(1+\mathrm{e}^{\sqrt{2}(r-r_0)/\varepsilon})^2}, \quad r\in\mathbb{R},\ \varepsilon>0, \tag{1.7}$$

which solves the one-dimensional limit problem

$$-w'' = \mathrm{e}^w \quad \text{in } \mathbb{R}. \tag{1.8}$$

We have to expend a lot of effort, therefore, to glue the two functions together.

It is important remark on the analogy existing between our result and some recent results obtained by Grossi [7–10]. In particular, Grossi and Gladiali [10] studied the asymptotic behaviour as λ goes to 0 of the radial solution z_λ to the Dirichlet problem

$$\begin{aligned} -\Delta z = \lambda \mathrm{e}^z \quad &\text{in } \Omega,\\ z=0 \qquad &\text{on } \partial\Omega, \end{aligned}$$

when $\Omega = \{x\in\mathbb{R}^n : a<|x|<b\}$ is the annulus in $\mathbb{R}^n$. In particular, they proved that, for a suitable choice of positive numbers δ_λ with $\delta_\lambda\to 0$ as $\lambda\to 0$, z_λ satisfies

$$\lim_{\lambda\to 0}\delta_\lambda z_\lambda(r) = 2\sqrt{2}G(r,r^*) \quad C^0\text{-uniformly on compact sets of } (a,b),$$

where $G(\cdot,r^*)$ is the Green function of the radial Laplacian with Dirichlet boundary condition, and r^* is suitably chosen in (a,b). Moreover, a suitable scaling of z_λ in a neighbourhood of r^* converges (as λ goes to 0) at a solution of the one-dimensional limit problem (1.8).

The paper is organized as follows. The definition of $\bar{u}_\lambda$ is given in § 2, while the construction of a good approximation near the boundary of the ball is carried out in § 3. In § 4 we estimate the error term and in § 5 we apply the contraction mapping argument.

2. The approximated solution

We look for a radial solution to (1.3), so we are led to consider the ordinary differential equation problem

$$\left.\begin{aligned} -u''-\frac{N-1}{r}u'+u=\lambda\mathrm{e}^u \quad \text{in } (0,r_0),\\ u'(r_0)=0, \quad u'(0)=0. \end{aligned}\right\} \tag{2.1}$$

We construct a solution to (2.1) as $\bar{u}_\lambda + \phi_\lambda$, where the leading term $\bar{u}_\lambda$ is defined as

$$\bar{u}_\lambda(r) := \begin{cases} u_1(r) & \text{in } (r_0-\delta, r_0), \\ u_2(r) & \text{in } [r_0-2\delta, r_0-\delta], \\ u_3(r) & \text{in } (0, r_0-2\delta), \end{cases} \tag{2.2}$$

and u_1, u_2 and u_3 are defined as follows.

Basic cells in the construction of the approximate solution u_1 near r_0 are the functions w_ε defined in (1.7). The rate of the concentration parameter $\varepsilon := \varepsilon_\lambda$ with respect to λ is deduced by the relation

$$\lambda = \frac{4}{\varepsilon_\lambda^2} \mathrm{e}^{-(a_1/\varepsilon_\lambda + a_2 + a_3\varepsilon_\lambda)}, \quad \text{i.e. } \ln\frac{4}{\varepsilon_\lambda^2} - \ln\lambda = \frac{a_1}{\varepsilon_\lambda} + a_2 + a_3\varepsilon_\lambda, \tag{2.3}$$

where a_1, a_2 and a_3 are positive constants given in (4.2).

The correct expression of u_1 is given in (3.1). The construction of u_1 is quite involved and it will be carried out in § 3.

The approximate solution u_3 far away from r_0 is built from the function $\mathcal{U}$ that solves (1.6) and whose properties are stated in lemma 2.1.

More precisely,

$$u_3(r) = \left(\frac{A_1}{\varepsilon_\lambda} + A_2 + A_3\varepsilon_\lambda\right)\mathcal{U}(r), \tag{2.4}$$

where A_1, A_2 and A_3 are positive constants given in (4.2).

We point out that the choice of λ in (2.3) is strictly related to the value of u_3 on r_0. Indeed, since u_3 is written as a third-order expansion in ε_λ, we need an expansion of ϵ_λ with respect to λ up to the third term.

Finally, the approximate solution u_2 in the interspace is simply given by

$$u_2(r) := \chi(r)u_1(r) + (1-\chi(r))u_3(r), \tag{2.5}$$

where $\chi \in C^2([0, r_0])$ is a cut-off such that

$$\left.\begin{gathered} \chi \equiv 1 \text{ in } (r_0-\delta, r_0), \qquad \chi \equiv 0 \text{ in } (0, r_0-2\delta), \\ |\chi(r)| \leqslant 1, \quad |\chi'(r)| \leqslant \frac{c}{\delta}, \quad |\chi''(r)| \leqslant \frac{c}{\delta^2}, \end{gathered}\right\} \tag{2.6}$$

where the size of the interface $\delta := \delta_\lambda$ goes to 0 with respect to ε (or, equivalently, with respect to λ) as

$$\delta_\lambda = \varepsilon_\lambda^\eta, \quad \eta \in (\tfrac{2}{3}, 1). \tag{2.7}$$

The choice of η will be made such that lemma (4.2) holds. We remark that u_2 is a good approximation of the solution in the interspace if u_1 and u_3 perfectly glue in a left neighbourhood of r_0; this motivates the choice of the constants A_1, A_2, A_3, a_1, a_2 and a_3 made in lemma 4.1.

Lemma 2.1. *There exists a unique solution to the problem*

$$\left.\begin{gathered} -\mathcal{U}'' - \frac{N-1}{r}\mathcal{U}' + \mathcal{U} = 0 \quad \text{in } (0, r_0), \\ \mathcal{U}'(0) = 0, \quad \mathcal{U}(r_0) = 1. \end{gathered}\right\} \tag{2.8}$$

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Moreover,

$$0 \leqslant \mathcal{U}(r) \leqslant 1 \quad \textit{and} \quad \mathcal{U}'(r) > 0 \quad \textit{for any } r \in (0, r_0].$$

Proof. The proof relies on standard arguments, so we omit it. □

3. The approximation near the boundary

The function $w_\varepsilon - \ln \lambda$ is not a good approximation for our solution near r_0. We build some additional correction terms that improve the approximation near r_0. More precisely, we define the approximation near the point r_0. We define

$$u_1(r) = \underbrace{w_\varepsilon(r) - \ln \lambda + \alpha_\varepsilon(r)}_{\substack{\text{1st-order}\\ \text{approx.}}} + \underbrace{v_\varepsilon(r) + \beta_\varepsilon(r)}_{\substack{\text{2nd-order}\\ \text{approx.}}} + \underbrace{z_\varepsilon(r)}_{\substack{\text{3rd-order}\\ \text{approx.}}}, \tag{3.1}$$

where α_ε is defined in lemma 3.1, v_ε is defined in lemma 3.2, β_ε is defined in lemma 3.4 and z_ε is defined in lemma 3.5.

The first term we have to add is a sort of projection of the function w_ε, namely, the function α_ε given in the next lemma.

LEMMA 3.1.

(i) *The Cauchy problem*

$$\left.\begin{aligned} -\alpha''_{\varepsilon,N} - \frac{N-1}{r}\alpha'_{\varepsilon,N} = \frac{N-1}{r}w'_\varepsilon(r) - w_\varepsilon(r) + \ln\lambda \quad \textit{in } (0, r_0), \\ \alpha_\varepsilon(r_0) = \alpha'_\varepsilon(r_0) = 0 \end{aligned}\right\} \tag{3.2}$$

has the solution

$$\alpha_\varepsilon(r) := -\int_{r_0}^{r} \frac{1}{t^{N-1}} \int_{r_0}^{t} \tau^{N-1}\left[\frac{N-1}{\tau}w'_\varepsilon(\tau) - w_\varepsilon(\tau) + \ln\lambda\right] \mathrm{d}\tau\, \mathrm{d}t.$$

(ii) *The following expansion holds:*

$$\alpha_\varepsilon(\varepsilon s + r_0) = \varepsilon\alpha_1(s) + \varepsilon^2\alpha_2(s) + O(\varepsilon^3 s^4), \tag{3.3}$$

where

$$\alpha_1(s) := -\frac{N-1}{r_0}\int_0^s w(\sigma)\,\mathrm{d}\sigma + \tfrac{1}{2}a_1 s^2 \tag{3.4}$$

and

$$\begin{aligned} \alpha_2(s) := \int_0^s \int_0^\sigma [w(\rho) - \ln 4]\,\mathrm{d}\rho\,\mathrm{d}\sigma + \frac{(N-1)(N-2)}{r_0^2}\int_0^s \int_0^\sigma w(\rho)\,\mathrm{d}\rho\,\mathrm{d}\sigma \\ + \frac{N-1}{r_0^2}\int_0^s \sigma w(\sigma)\,\mathrm{d}\sigma - \frac{N-1}{6r_0}a_1 s^3 + \tfrac{1}{2}a_2 s^2. \end{aligned} \tag{3.5}$$

(iii) *For any* $r \in (0, r_0 - \delta)$,

$$\begin{aligned}
\alpha_\varepsilon(r) = &-\frac{(N-1)\ln 4}{r_0}(r-r_0)\\
&+\left[\frac{(N-1)^2\ln 4}{r_0^2} - \frac{\sqrt{2}(N-1)}{\varepsilon r_0} + \ln\frac{4}{\varepsilon^2} - \ln\lambda\right]\frac{(r-r_0)^2}{2}\\
&+\left[\frac{N(N-1)\sqrt{2}}{\varepsilon r_0^2} + \frac{\sqrt{2}}{\varepsilon} - \frac{N-1}{r_0}\left(\ln\frac{4}{\varepsilon^2} - \ln\lambda\right)\right]\frac{(r-r_0)^3}{6}\\
&+O\left(\frac{(r-r_0)^4}{\varepsilon}\right) + O((r-r_0)^3). \qquad (3.6)
\end{aligned}$$

Proof. (i) It is just a straightforward computation.

(ii) Setting $t = \varepsilon\sigma + r_0$ and $\tau = \varepsilon\rho + r_0$, we find

$$\begin{aligned}
\alpha_\varepsilon(\varepsilon s + r_0) &= -\varepsilon^2 \int_0^s \frac{1}{(\varepsilon\sigma + r_0)^{N-1}} \int_0^\sigma (\varepsilon\rho + r_0)^{N-1}\\
&\qquad \times \left[\frac{N-1}{\varepsilon\rho + r_0}\frac{1}{\varepsilon}w'(\rho) - [w(\rho) - \ln 4] + \ln\lambda - \ln\frac{4}{\varepsilon^2}\right] \mathrm{d}\sigma\,\mathrm{d}\rho\\
&= -\varepsilon^2 \int_0^s \left(\frac{1}{r_0^{N-1}} - \frac{N-1}{r_0^N}\varepsilon\sigma\right) \int_0^\sigma (r_0^{N-1} + (N-1)r_0^{N-2}\varepsilon\rho)\\
&\qquad \times \left[(N-1)\left(\frac{1}{r_0} - \frac{1}{r_0^2}\varepsilon\rho\right)\frac{1}{\varepsilon}w'(\rho)\right.\\
&\qquad\qquad \left. - [w(\rho) - \ln 4] + \ln\lambda - \ln\frac{4}{\varepsilon^2}\right] \mathrm{d}\sigma\,\mathrm{d}\rho + O(\varepsilon^3 s^4).
\end{aligned}$$

Here we used that

$$w_\varepsilon(r) = \ln\frac{4}{\varepsilon^2} + w\left(\frac{r-r_0}{\varepsilon}\right) - \ln 4 \quad \text{and} \quad w'_\varepsilon(r) = \frac{1}{\varepsilon}w'\left(\frac{r-r_0}{\varepsilon}\right).$$

The claim follows by (2.3).

(iii) Set $\bar{w}_\varepsilon(r) := w_\varepsilon(r) - \ln(1/\varepsilon^2)$. We have

$$\begin{aligned}
\alpha_\varepsilon(r) &= -\int_{r_0}^r \frac{1}{t^{N-1}} \int_{r_0}^t \tau^{N-1}\left[\frac{N-1}{\tau}w'_\varepsilon(\tau) - w_\varepsilon(\tau) + \ln\lambda\right] \mathrm{d}\tau\,\mathrm{d}t\\
&= -\int_{r_0}^r \frac{1}{t^{N-1}} \int_{r_0}^t \tau^{N-1}\left[\frac{N-1}{\tau}\bar{w}'_\varepsilon(\tau) - \bar{w}_\varepsilon(\tau) + \left(\ln\lambda - \ln\frac{1}{\varepsilon^2}\right)\right] \mathrm{d}\tau\,\mathrm{d}t\\
&= -(N-1)\int_{r_0}^r \frac{\bar{w}_\varepsilon(t)}{t}\,\mathrm{d}t\\
&\qquad + \int_{r_0}^r \frac{1}{t^{N-1}} \int_{r_0}^t [(N-1)(N-2)\tau^{N-3} + \tau^{N-1}]\bar{w}_\varepsilon(\tau)\,\mathrm{d}\tau\,\mathrm{d}t
\end{aligned}$$

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$$+\left(\ln\lambda-\ln\frac{1}{\varepsilon^2}\right)\begin{cases}\dfrac{1}{2N}(r_0^2-r^2)+\dfrac{r_0^2}{2}\ln\dfrac{r}{r_0} & \text{if } N=2,\\ \dfrac{1}{2N}(r_0^2-r^2) \\ \qquad+\dfrac{r_0^N}{N(N-2)}\left(\dfrac{1}{r_0^{N-2}}-\dfrac{1}{r^{N-2}}\right) & \text{if } N\geqslant 3.\end{cases}$$

Now, we observe that in $[r_0-2\delta, r_0-\delta]$ we have

$$\ln\frac{r}{r_0}=\ln\left(1+\frac{r-r_0}{r_0}\right)=\frac{r-r_0}{r_0}-\frac{(r-r_0)^2}{2r_0^2}+\frac{(r-r_0)^3}{3r_0^3}+O((r-r_0)^4) \quad (3.7)$$

$$\begin{aligned}\frac{1}{r^{N-2}}&=\frac{1}{r_0^{N-2}}-\frac{N-2}{r_0^{N-1}}(r-r_0)+\frac{(N-2)(N-1)}{r_0^N}\frac{(r-r_0)^2}{2}\\&\quad-\frac{N(N-1)(N-2)}{r_0^{N+1}}\frac{(r-r_0)^3}{6}+O((r-r_0)^4)\end{aligned} \quad (3.8)$$

and also

$$\bar w_\varepsilon(s)=\ln 4+\frac{\sqrt{2}}{\varepsilon}(s-r_0)+O(\mathrm{e}^{-|s-r_0|/\varepsilon}). \quad (3.9)$$

A tedious but straightforward computation proves our claim. □

The function $w_\varepsilon(r)-\ln\lambda+\alpha_\varepsilon(r)$ is still a bad approximation of the solution near the boundary point r_0. We have to add a correction term v_ε (given in the next lemma) that solves a linear problem and *kills* the ε-order term in (3.3).

Lemma 3.2.

(i) *There exists a solution v of the linear problem (see (3.4))*

$$\left.\begin{aligned}-v''-\mathrm{e}^w v=\mathrm{e}^w\alpha_1 \quad \text{in } \mathbb{R},\\ v(0)=v'(0)=0\end{aligned}\right\} \quad (3.10)$$

such that

$$v(s)=\nu_1 s+\nu_2+O(\mathrm{e}^s) \quad \text{and} \quad v'(s)=\nu_1+O(\mathrm{e}^s) \quad \text{as } s\to-\infty,$$

where

$$\nu_1:=-\frac{2(N-1)}{r_0}(1-\ln 2)+a_1\sqrt{2}\ln 2, \quad (3.11)$$

$\nu_2\in\mathbb{R}$ is a constant that only depends on a_1, and a_1 is given in (3.5).

(ii) *In particular, the function $v_\varepsilon(r):=\varepsilon v((r-r_0)/\varepsilon)$ is a solution of the linear problem*

$$-v_\varepsilon''-\mathrm{e}^{w_\varepsilon}v_\varepsilon=\varepsilon\mathrm{e}^{w_\varepsilon(r)}\alpha_1\left(\frac{r-r_0}{\varepsilon}\right) \quad \text{in } \mathbb{R} \quad (3.12)$$

such that, if $r\in[0,r_0-\delta]$, it satisfies

$$v_\varepsilon(r)=\nu_1(r-r_0)+\nu_2\varepsilon+O(\mathrm{e}^{-|r-r_0|/\varepsilon}) \quad \text{and} \quad v_\varepsilon'(r)=\nu_1+O(\mathrm{e}^{-|r-r_0|/\varepsilon}) \quad (3.13)$$

as $\varepsilon\to 0$.

Proof. The result immediately follows by lemma 3.3. In our case

$$\nu_1 := \frac{1}{\sqrt{2}}\int_{-\infty}^{0}\biggl(-\frac{N-1}{r_0}\int_0^r w(y)\,\mathrm{d}y + a_1\frac{r^2}{2}\biggr)w'(r)\mathrm{e}^{w}(r)\,\mathrm{d}r$$

and

$$\nu_2 := -\int_{-\infty}^{0}\biggl(\frac{2}{1-\mathrm{e}^{\sqrt{2}r}} + \frac{r}{\sqrt{2}}\biggr)\biggl(-\frac{N-1}{r_0}\int_0^r w(y)\,\mathrm{d}y + a_1\frac{r^2}{2}\biggr)w'(r)\mathrm{e}^{w}(r)\,\mathrm{d}r.$$

A straightforward computation proves (3.11). □

Lemma 3.3 (Grossi [9, lemma 4.1]). *Let $h\colon \mathbb{R}\to\mathbb{R}$ be a continuous function. The function*

$$Y(t) = w'(t)\int_0^t \frac{1}{w'(s)^2}\biggl(\int_s^0 h(z)w'(z)\mathrm{e}^{w}\,\mathrm{d}z\biggr)\mathrm{d}s \tag{3.14}$$

is a solution to

$$\left.\begin{aligned} -Y'' - \mathrm{e}^{w} &= h\mathrm{e}^{w} \quad \text{in } \mathbb{R},\\ Y(0) &= Y'(0) = 0. \end{aligned}\right\} \tag{3.15}$$

Moreover,

$$Y(t) = t\upsilon_1^- + \upsilon_2^- + O(\mathrm{e}^{t}) \quad \text{and} \quad Y'(t) = \upsilon_1^- + O(\mathrm{e}^{t}) \quad \text{as } t\to-\infty,$$

where

$$\upsilon_1^- := \frac{1}{\sqrt{2}}\int_{-\infty}^{0} h(r)w'(r)\mathrm{e}^{w}\,\mathrm{d}r,$$

$$\upsilon_2^- := -\int_{-\infty}^{0}\biggl(\frac{2}{1-\mathrm{e}^{\sqrt{2}s}} + \frac{s}{\sqrt{2}}\biggr)h(s)w'(s)\mathrm{e}^{w}\,\mathrm{d}s,$$

and

$$Y(t) = t\upsilon_1^+ + \upsilon_2^+ + O(\mathrm{e}^{-t}), \quad \text{and} \quad Y'(t) = \upsilon_1^+ + O(\mathrm{e}^{-t}) \quad \text{as } t\to+\infty,$$

where

$$\upsilon_1^+ := \frac{1}{\sqrt{2}}\int_0^{+\infty} h(r)w'(r)\mathrm{e}^{w}\,\mathrm{d}r$$

and

$$\upsilon_2^+ := -\int_0^{+\infty}\biggl(\frac{2}{1-\mathrm{e}^{\sqrt{2}s}} + \frac{s}{\sqrt{2}}\biggr)h(s)w'(s)\mathrm{e}^{w}\,\mathrm{d}s.$$

As we have done for the function w_ε, we have to add the projection of the function v_ε, namely, the function β_ε given in the next lemma.

Lemma 3.4.

(i) *The Cauchy problem*

$$\left.\begin{aligned} -\beta_\varepsilon'' - \frac{N-1}{r}\beta_\varepsilon' &= \frac{N-1}{r}v_\varepsilon'(r),\\ \beta_\varepsilon(r_0) &= \beta_\varepsilon'(r_0) = 0 \end{aligned}\right\} \tag{3.16}$$

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has the solution

$$\beta_\varepsilon(r) = -(N-1)\int_r^{r_0} \frac{1}{t^{N-1}} \int_t^{r_0} \tau^{N-2} v_\varepsilon'(\tau)\,\mathrm{d}\tau\,\mathrm{d}t.$$

(ii) *The following expansion holds:*

$$\beta_\varepsilon(\varepsilon s + r_0) = \varepsilon^2 \beta_1(s) + O(\varepsilon^3 s^3), \qquad \beta_1(s) := -\frac{N-1}{r_0}\int_0^s \int_0^\sigma v'(\rho)\,\mathrm{d}\rho\,\mathrm{d}\sigma. \tag{3.17}$$

(iii) *For any* $r \in (0, r_0 - \delta)$,

$$\beta_\varepsilon(r) = -\frac{(N-1)\nu_1}{r_0}\frac{(r-r_0)^2}{2} + O((r-r_0)^3). \tag{3.18}$$

Proof. We argue as in lemma 3.1. □

Unfortunately, the function $w_{\varepsilon,r_0}(r) - \ln\lambda + \alpha_\varepsilon(r) + v_\varepsilon(r) + \beta_\varepsilon(r)$ is still a bad approximation of the solution near the boundary point r_0. We have to add an extra correction term z_ε, given in the next lemma, that solves a linear problem and *kills* all the ε^2-order terms (in particular, those in (3.3) and in (3.17)).

Lemma 3.5.

(i) *There exists a solution* z *of the linear problem (see equations (3.4), (3.5), (3.17) and (3.10))*

$$\left.\begin{aligned} -z'' - \mathrm{e}^w z = \mathrm{e}^z[\alpha_2(s) + \beta_1(s) + \tfrac{1}{2}(\alpha_1(s) + v(s))^2] \quad \text{in } \mathbb{R}, \\ z(0) = z'(0) = 0 \end{aligned}\right\} \tag{3.19}$$

such that

$$z(s) = \zeta_1 s + \zeta_2 + O(\mathrm{e}^s) \quad \text{and} \quad z'(s) = \zeta_1 + O(\mathrm{e}^s) \quad \text{as } s \to -\infty,$$

where $\zeta_1, \zeta_2 \in \mathbb{R}$ *are constants that only depend on* a_1 *and* a_2.

(ii) *In particular, the function* $z_\varepsilon(r) := \varepsilon^2 z((r - r_0)/\varepsilon)$ *is a solution of the linear problem*

$$\begin{aligned} -z_\varepsilon'' - \mathrm{e}^{w_\varepsilon} z_\varepsilon = \varepsilon^2 \mathrm{e}^{w_\varepsilon}\bigg\{&\alpha_2\left(\frac{r-r_0}{\varepsilon}\right) + \beta_1\left(\frac{r-r_0}{\varepsilon}\right) \\ &+ \frac{1}{2}\left[\alpha_1\left(\frac{r-r_0}{\varepsilon}\right) + v\left(\frac{r-r_0}{\varepsilon}\right)\right]^2\bigg\} \end{aligned} \tag{3.20}$$

such that, if $r \in [0, r_0 - \delta]$, *it satisfies*

$$z_\varepsilon(r) = \varepsilon\zeta_1(r - r_0) + \zeta_2\varepsilon^2 + O(\varepsilon^2 \mathrm{e}^{-|r-r_0|/\varepsilon}) \quad \text{as } \varepsilon \to 0. \tag{3.21}$$

Proof. The result immediately follows by lemma 3.3, arguing as in the proof of lemma 3.2. □

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4. The error estimate

Let us define the error term

$$\mathcal{R}_\lambda(\bar{u}_\lambda) = -\bar{u}_\lambda'' - \frac{N-1}{r}\bar{u}_\lambda' + \bar{u}_\lambda - \lambda e^{\bar{u}_\lambda}, \tag{4.1}$$

where $\bar{u}_\lambda$ is defined as in (2.2).

First of all, it is necessary to choose constants a, b and c in (2.3), and A_1, A_2 and A_3 in (2.4), such that the approximate solutions in the neighbourhood of the boundary and inside the interval join up.

Lemma 4.1. *If*

$$A_1 := \frac{\sqrt{2}}{\mathcal{U}'(r_0)}, \qquad A_2 := \frac{1}{\mathcal{U}'(r_0)}\left(\frac{\ln 4}{\mathcal{U}'(r_0)} - 2\frac{N-1}{r_0}\right), \qquad A_3 := \frac{\zeta_1}{\mathcal{U}'(r_0)} \tag{4.2}$$

$$a_1 := A_1, \qquad a_2 := A_2, \qquad a_3 =: A_3 - \nu_2 \tag{4.3}$$

(ζ_1 and ν_2 are given in lemma 3.2 and lemma 3.5, respectively), then, for any $r \in [r_0 - 2\delta, r_0 - \delta]$, we have

$$\begin{aligned} u_1(r) - u_3(r) &= O(e^{-|r-r_0|/\varepsilon}) + O(\varepsilon^2) + O(\varepsilon(r-r_0)^2) \\ &\quad + O((r-r_0)^3) + O\left(\frac{(r-r_0)^4}{\varepsilon}\right), \\ u_1'(r) - u_3'(r) &= O\left(\frac{1}{\varepsilon}e^{-|r-r_0|/\varepsilon}\right) + O(\varepsilon) + O(\varepsilon(r-r_0)) \\ &\quad + O((r-r_0)^2) + O\left(\frac{(r-r_0)^3}{\varepsilon}\right). \end{aligned}$$

Proof. Let us prove the first estimate. The proof of the second estimate is similar. By (2.3), (3.6), (3.13), (3.18) and (3.21), we deduce that if $r \in [r_0 - 2\delta, r_0 - \delta]$, then

$$\begin{aligned} u_1(r) &= \left[\ln\frac{4}{\varepsilon^2} - \ln\lambda + \nu_2\varepsilon\right] + \left[\frac{\sqrt{2}}{\varepsilon} - \frac{(N-1)\ln 4}{r_0} + \nu_1 + \zeta_1\varepsilon\right](r-r_0) \\ &\quad + \left[\frac{(N-1)^2\ln 4}{r_0^2} - \frac{\sqrt{2}(N-1)}{r_0}\frac{1}{\varepsilon} + \ln\frac{4}{\varepsilon^2} - \ln\lambda - \frac{\nu_1(N-1)}{r_0}\right]\frac{(r-r_0)^2}{2} \\ &\quad + \left[\frac{N(N-1)\sqrt{2}}{r_0^2}\frac{1}{\varepsilon} + \sqrt{2}(N-1)\frac{1}{\varepsilon} - \frac{N-1}{r_0}\left(\ln\frac{4}{\varepsilon^2} - \ln\lambda\right)\right]\frac{(r-r_0)^3}{6} \\ &\quad + O(e^{-|r-r_0|/\varepsilon}) + O(\varepsilon^2) + O\left(\frac{(r-r_0)^4}{\varepsilon}\right) + O((r-r_0)^3) \\ &= \left[\frac{a_1}{\varepsilon} + a_2 + a_3\varepsilon + \nu_2\varepsilon\right] + \left[\frac{\sqrt{2}}{\varepsilon} - \frac{2(N-1)}{r_0} + a_1\sqrt{2}\ln 2 + \zeta_1\varepsilon\right](r-r_0) \\ &\quad + \left[-\frac{(N-1)\sqrt{2}}{r_0}\frac{1}{\varepsilon} + \frac{a_1}{\varepsilon} + a_2 + 2\frac{(N-1)^2}{r_0^2}\right. \\ &\qquad\qquad \left. - \frac{a_1(N-1)\sqrt{2}\ln 2}{r_0}\right]\frac{(r-r_0)^2}{2} \end{aligned}$$

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$$+\left[\frac{N(N-1)\sqrt{2}}{r_0^2}\frac{1}{\varepsilon}+\frac{\sqrt{2}}{\varepsilon}-\frac{a_1(N-1)}{r_0}\frac{1}{\varepsilon}\right]\frac{(r-r_0)^3}{6}$$
$$+O(\mathrm{e}^{-|r-r_0|/\varepsilon})+O(\varepsilon^2)+O\left(\frac{(r-r_0)^4}{\varepsilon}\right)+O((r-r_0)^3). \tag{4.4}$$

On the other hand, by the mean-value theorem, we deduce that

$$\mathcal{U}(r)=\mathcal{U}(r_0)+\mathcal{U}'(r_0)(r-r_0)+\mathcal{U}''(r_0)\frac{(r-r_0)^2}{2}+\mathcal{U}''''(r_0)\frac{(r-r_0)^3}{6}+O((r-r_0)^4)$$

with $\mathcal{U}(r_0)=1$,

$$\begin{aligned}\mathcal{U}''(r_0)&=-\frac{N-1}{r_0}\mathcal{U}'(r_0)+\mathcal{U}(r_0)=-\frac{N-1}{r_0}\mathcal{U}'(r_0)+1\\ \mathcal{U}'''(r_0)&=-\frac{N-1}{r_0}\mathcal{U}''(r_0)+\frac{N-1}{r_0^2}\mathcal{U}'(r_0)+\mathcal{U}'(r_0)\\ &=\frac{N(N-1)}{r_0^2}\mathcal{U}'(r_0)+\mathcal{U}'(r_0)-\frac{N-1}{r_0}.\end{aligned}$$

These relations easily follow by differentiating (2.8). Therefore, if $r\in[r_0-2\delta,r_0-\delta]$, we have

$$\begin{aligned}u_3(r)&=\left(\frac{A_1}{\varepsilon}+A_2+A_3\varepsilon\right)\mathcal{U}(r)\\ &=\left(\frac{A_1}{\varepsilon}+A_2+A_3\varepsilon\right)+\left(\frac{A_1}{\varepsilon}+A_2+A_3\varepsilon\right)\mathcal{U}'(r_0)(r-r_0)\\ &\quad+\mathcal{U}''(r_0)\left(\frac{A_1}{\varepsilon}+A_2\right)\frac{(r-r_0)^2}{2}+\mathcal{U}'''(r_0)\frac{A_1}{\varepsilon}\frac{(r-r_0)^3}{6}\\ &\quad+O(\varepsilon(r-r_0)^2)+O((r-r_0)^3)+O\left(\frac{(r-r_0)^4}{\varepsilon}\right).\end{aligned} \tag{4.5}$$

If (4.2) holds, then, combining (4.4) and (4.5), we easily get the claim. □

LEMMA 4.2. *There exist $C>0$ and $\lambda_0>0$ such that, for any $\lambda\in(0,\lambda_0)$, we have*

$$\|\mathcal{R}_\lambda\|_{L^1}=O(\varepsilon_\lambda^{1+\sigma})\quad \textit{for some } \sigma>0.$$

Proof. STEP 1 (evaluation of the error in $(r_0-\delta,r_0)$). We use the estimate

$$1-\mathrm{e}^t=-t-(\tfrac{1}{2}t^2)+O(t^3)$$

and we get

$$\begin{aligned}\mathcal{R}_\lambda(u_1)&=-u_1''-\frac{N-1}{r}u_1'+u_1-\lambda\mathrm{e}^{u_1}\\ &=-w_\varepsilon''-\frac{N-1}{r_0}w_\varepsilon'+w_\varepsilon-\ln\lambda-\alpha_\varepsilon''-\frac{N-1}{r}\alpha_\varepsilon'\\ &\quad+\alpha_\varepsilon-v_\varepsilon''-\frac{N-1}{r}v_\varepsilon'+v_\varepsilon-\beta_\varepsilon''-\frac{N-1}{r}\beta_\varepsilon'+\beta_\varepsilon\\ &\quad-z_\varepsilon''-\frac{N-1}{r}z_\varepsilon'+z_\varepsilon-\lambda\mathrm{e}^{w_\varepsilon-\ln\lambda+\alpha_\varepsilon+v_\varepsilon+\beta_\varepsilon+z_\varepsilon}\end{aligned}$$

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$$
\begin{aligned}
&= \alpha_\varepsilon + v_\varepsilon + \beta_\varepsilon + z_\varepsilon - \frac{N-1}{r} z'_\varepsilon \\
&\quad + \mathrm{e}^{w_\varepsilon}\biggl\{1 - \mathrm{e}^{\alpha_\varepsilon + v_\varepsilon + \beta_\varepsilon + z_\varepsilon} + v_\varepsilon + z_\varepsilon + \varepsilon\alpha_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) \\
&\qquad + \varepsilon^2\biggl[\alpha_2\biggl(\frac{r-r_0}{\varepsilon}\biggr) + \beta_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) \\
&\qquad\qquad + \frac{1}{2}\biggl(\alpha_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) + v\biggl(\frac{r-r_0}{\varepsilon}\biggr)\biggr)^2\biggr]\biggr\} \\
&= \alpha_\varepsilon + v_\varepsilon + \beta_\varepsilon + z_\varepsilon - \frac{N-1}{r} z'_\varepsilon \\
&\quad + \mathrm{e}^{w_\varepsilon}\biggl\{-\alpha_\varepsilon - \beta_\varepsilon - \tfrac{1}{2}(\alpha_\varepsilon + v_\varepsilon)^2 + \varepsilon\alpha_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) \\
&\qquad + \varepsilon^2\biggl[\alpha_2\biggl(\frac{r-r_0}{\varepsilon}\biggr) + \beta_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) \\
&\qquad\qquad + \frac{1}{2}\biggl(\alpha_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) + v\biggl(\frac{r-r_0}{\varepsilon}\biggr)\biggr)^2\biggr]\biggr\} \\
&\quad + O(\mathrm{e}^{w_\varepsilon}|\alpha_\varepsilon + v_\varepsilon + \beta_\varepsilon + z_\varepsilon|^3) + O(\mathrm{e}^{w_\varepsilon}|\beta_\varepsilon + z_\varepsilon|^2) \\
&\quad + O(\mathrm{e}^{w_\varepsilon}|(\alpha_\varepsilon + v_\varepsilon)(\beta_\varepsilon + z_\varepsilon)|), \qquad (4.6)
\end{aligned}
$$

because α_ε solves (3.2), v_ε solves (3.12), β_ε solves (3.16) and z_ε solves (3.20).

We have

$$
\int_{r_0-\delta}^{r_0} |\alpha_\varepsilon + v_\varepsilon + \beta_\varepsilon + z_\varepsilon|(r)\,\mathrm{d}r = O\biggl(\int_{r_0-\delta}^{r_0} \frac{(r-r_0)^2}{\varepsilon}\,\mathrm{d}r\biggr) = O\biggl(\frac{\delta^3}{\varepsilon}\biggr) = O(\varepsilon^{3\eta-1}),
$$

because, by (3.3), (3.17), the properties of v_ε in lemma 3.2 and those of z_ε in lemma 3.5, we deduce that

$$
\begin{aligned}
\alpha_\varepsilon(r) &= O\biggl(\frac{(r-r_0)^2}{\varepsilon}\biggr), & \beta_\varepsilon(r) &= O((r-r_0)^2), \\
v_\varepsilon(r) &= O(|r-r_0| + \varepsilon), & z_\varepsilon(r) &= O(\varepsilon|r-r_0| + \varepsilon^2).
\end{aligned}
$$

By lemma 3.5, we also deduce that $z'_\varepsilon(r) = O(\varepsilon)$, and so

$$
\int_{r_0-\delta}^{r_0} \biggl|\frac{1}{r} z'_\varepsilon(r)\biggr|\,\mathrm{d}r = O(\varepsilon\delta) = O(\varepsilon^{1+\eta}).
$$

Moreover, we scale by $s = \varepsilon r + r_0$ and we get

$$
\begin{aligned}
\int_{r_0-\delta}^{r_0} \mathrm{e}^{w_\varepsilon}\biggl| &- \alpha_\varepsilon - \beta_\varepsilon - \tfrac{1}{2}(\alpha_\varepsilon + v_\varepsilon)^2 + \varepsilon\alpha_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) \\
&+ \varepsilon^2\biggl[\alpha_2\biggl(\frac{r-r_0}{\varepsilon}\biggr) + \beta_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) \\
&\qquad + \frac{1}{2}\biggl(\alpha_1\biggl(\frac{r-r_0}{\varepsilon}\biggr) + v\biggl(\frac{r-r_0}{\varepsilon}\biggr)\biggr)^2\biggr]\biggr|\,\mathrm{d}r
\end{aligned}
$$

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$$
\begin{aligned}
&= \frac{1}{\varepsilon}\int_{-\delta/\varepsilon}^{0} \mathrm{e}^{w(s)}\big| -\alpha_\varepsilon(\varepsilon s + r_0) - \beta_\varepsilon(\varepsilon s + r_0) \\
&\qquad\qquad - \tfrac{1}{2}(\alpha_\varepsilon(\varepsilon s + r_0) + \varepsilon v(s))^2 + \varepsilon\alpha_1(s) \\
&\qquad\qquad + \varepsilon^2[\alpha_2(s) + \beta_1(s) + \tfrac{1}{2}(\alpha_1(s) + v(s))^2]\big|\,\mathrm{d}s \\
&= O\bigg(\varepsilon^2 \int_{\mathbb{R}} \mathrm{e}^{w(s)} s^3\,\mathrm{d}s\bigg) = O(\varepsilon^2).
\end{aligned}
$$

Finally, we scale by $s = \varepsilon r + r_0$ and obtain

$$
\begin{aligned}
\int_{r_0-\delta}^{r_0} \mathrm{e}^{w_\varepsilon}|\alpha_\varepsilon + v_\varepsilon + \beta_\varepsilon + z_\varepsilon|^3\,\mathrm{d}r &= O\bigg(\int_{r_0-\delta}^{r_0} \mathrm{e}^{w_\varepsilon}(|\alpha_\varepsilon|^3 + |v_\varepsilon|^3 + |\beta_\varepsilon|^3 + |z_\varepsilon|^3)\,\mathrm{d}r\bigg) \\
&= O\bigg(\varepsilon^2\int_{\mathbb{R}} \mathrm{e}^{w(s)}s^6\,\mathrm{d}s + \varepsilon^2\int_{\mathbb{R}} \mathrm{e}^{w(s)}v^3(s)\,\mathrm{d}s \\
&\qquad + \varepsilon^5\int_{\mathbb{R}} \mathrm{e}^{w(s)}s^6\,\mathrm{d}s + \varepsilon^5\int_{\mathbb{R}} \mathrm{e}^{w(s)}z^3(s)\,\mathrm{d}s\bigg) \\
&= O(\varepsilon^2), \\
\int_{r_0-\delta}^{r_0} \mathrm{e}^{w_\varepsilon}|\beta_\varepsilon + z_\varepsilon|^2\,\mathrm{d}r &= O\bigg(\varepsilon^3\int_{\mathbb{R}} \mathrm{e}^{w(s)}s^4\,\mathrm{d}s + \varepsilon^3\int_{\mathbb{R}} \mathrm{e}^{w(s)}z^2\,\mathrm{d}s\bigg) = O(\varepsilon^3), \\
\int_{r_0-\delta}^{r_0} \mathrm{e}^{w_\varepsilon}|(\alpha_\varepsilon + v_\varepsilon)(\beta_\varepsilon + z_\varepsilon)|\,\mathrm{d}r &= O\bigg(\varepsilon^2\int_{\mathbb{R}} \mathrm{e}^{w(s)}(s^2 + |v|)(s^2 + |z|)\,\mathrm{d}s\bigg) = O(\varepsilon^2),
\end{aligned}
$$

because, by (3.3) and (3.17), we deduce that

$$
\alpha_\varepsilon(\varepsilon s + r_0) = O(\varepsilon s^2), \qquad \beta_\varepsilon(\varepsilon s + r_0) = O(\varepsilon^2 s^2).
$$

Collecting all the previous estimates and taking into account the choice of η in (2.7), we get

$$
\|\mathcal{R}_\lambda\|_{L^1((r_0-\delta,r_0))} = O(\varepsilon^{1+\sigma}) \quad \text{for some } \sigma > 0. \tag{4.7}
$$

Step 2 (evaluation of the error in $(0, r_0 - 2\delta)$). First of all, if δ is small enough (that is, ε is small enough), we have

$$
\mathcal{U}(r) \leqslant \mathcal{U}(r_0 - 2\delta) = \mathcal{U}(r_0) + \mathcal{U}'(r_0)(-2\delta) + \tfrac{1}{2}\mathcal{U}''(r_0 - 2\theta\delta)(2\delta)^2 \leqslant 1 - 2\mathcal{U}'(r_0)\delta
$$

because $\mathcal{U}$ is increasing (see lemma 2.1) and the mean-value theorem applies for some $\theta \in (0, 1)$.

Therefore, by (2.3), (2.7) and (4.2), we get

$$
\begin{aligned}
\mathcal{R}_\lambda(u_3) &= -u_3'' - \frac{N-1}{r}u_3' + u_3 - \lambda\mathrm{e}^{u_3} \\
&= -\lambda\mathrm{e}^{(A_1/\varepsilon + A_2 + A_3\varepsilon)\mathcal{U}(r)} \\
&= -\frac{4}{\varepsilon^2}\mathrm{e}^{(A_3 - a_3)\varepsilon}\mathrm{e}^{(A_1/\varepsilon + A_2 + A_3\varepsilon)[\mathcal{U}(r) - 1]} \\
&= O\bigg(\frac{1}{\varepsilon^2}\mathrm{e}^{-2A_1\mathcal{U}'(r_0)(\delta/\varepsilon)}\bigg) \\
&= O\bigg(\frac{1}{\varepsilon^2}\mathrm{e}^{-2\sqrt{2}(1/\varepsilon^{1-\eta})}\bigg).
\end{aligned}
$$

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This implies that

$$\|\mathcal{R}_\lambda(u_3)\|_{L^1((0,r_0-2\delta))} = O(\varepsilon^{1+\sigma}) \quad \text{for any } \sigma > 0. \tag{4.8}$$

STEP 3 (evaluation of the error in $[r_0-2\delta, r_0-\delta]$). We recall that

$$u_2 = \chi u_1 + (1-\chi)u_3$$

and hence,

$$\begin{aligned}
\mathcal{R}_\lambda(u_2) &= \chi\biggl[-u_1'' - \frac{N-1}{r}u_1' + u_1\biggr] + (1-\chi)\biggl[-u_3'' - \frac{N-1}{r}u_3' + u_3\biggr] \\
&\quad - 2\chi'(u_1'-u_3') + \biggl[-\chi'' - \frac{N-1}{r}\chi' + \chi\biggr](u_1-u_3) - \lambda e^{\chi(u_1-u_3)+u_3} \\
&= \chi\mathcal{R}_\lambda(u_1) + (1-\chi)\mathcal{R}_\lambda(u_3) - \lambda\chi e^{u_1}[e^{(\chi-1)(u_1-u_3)} - 1] + \lambda(1-\chi)e^{u_3} \\
&\quad - 2\chi'(u_1'-u_3') + \biggl[-\chi'' - \frac{N-1}{r}\chi' + \chi\biggr](u_1-u_3).
\end{aligned}$$

By lemma (4.1), we immediately get (taking into account the choice of η in (2.7))

$$\int_{r_0-2\delta}^{r_0-\delta} |\chi'(r)(u_1'(r) - u_3'(r))|\,\mathrm{d}r = O(\delta^2) = O(\varepsilon^{1+\sigma}),$$

$$\int_{r_0-2\delta}^{r_0-\delta} \biggl|\biggl[-\chi''(r) - \frac{N-1}{r}\chi'(r) + \chi(r)\biggr](u_1(r)-u_3(r))\biggr|(r)\,\mathrm{d}r = O(\delta^2) = O(\varepsilon^{1+\sigma})$$

and

$$\begin{aligned}
\int_{r_0-2\delta}^{r_0-\delta} |\lambda\chi e^{u_1(r)}[e^{(\chi(r)-1)(u_1(r)-u_3(r))} - 1]|\,\mathrm{d}r \\
&= O\biggl(\int_{r_0-2\delta}^{r_0-\delta} \lambda e^{u_1(r)}|u_1(r)-u_3(r)|\,\mathrm{d}r\biggr) \\
&= O(\lambda\varepsilon^2)
\end{aligned}$$

because $e^t - 1 = O(t)$. Arguing exactly as in step 1, one proves that

$$\int_{r_0-2\delta}^{r_0-\delta} \chi(r)|\mathcal{R}_\lambda(u_1)(r)|\,\mathrm{d}r = O(\varepsilon^{1+\sigma}),$$

and arguing exactly as in step 2, one proves that

$$\int_{r_0-2\delta}^{r_0-\delta} (1-\chi(r))|\mathcal{R}_\lambda(u_3)(r)|\,\mathrm{d}r = O(\varepsilon^{1+\sigma})$$

and

$$\int_{r_0-2\delta}^{r_0-\delta} \lambda(1-\chi(r))e^{u_3}(r)\,\mathrm{d}r = O(\varepsilon^{1+\sigma}).$$

Collecting all the previous estimates, we get

$$\|\mathcal{R}_\lambda(u_2)\|_{L^1((r_0-2\delta,r_0-\delta))} = O(\varepsilon^{1+\sigma}) \quad \text{for some } \sigma > 0. \tag{4.9}$$

The claim follows by (4.7), (4.8) and (4.9). □

LEMMA 4.3. *It holds that*

$$\lambda\varepsilon_\lambda^2 \mathrm{e}^{u_\lambda(\varepsilon_\lambda s+r_0)} \to \mathrm{e}^{w(s)} \quad C^0\text{-uniformly on compact sets of } (-\infty,0] \text{ as } \lambda\to 0, \tag{4.10}$$

and that

$$\lambda\varepsilon_\lambda \int_0^{r_0} \mathrm{e}^{u_\lambda(r)}\,\mathrm{d}r \to \int_{\mathbb{R}} \mathrm{e}^{w(s)}\,\mathrm{d}s \quad \text{as } \lambda\to 0. \tag{4.11}$$

Proof. Let $[a,b]\subset(-\infty,0]$. If λ is small enough, then

$$u_\lambda(\varepsilon_\lambda s+r_0) = u_1(\varepsilon_\lambda s+r_0) \quad \text{for any } s\in[a,b].$$

On the other hand, by (3.3), (3.17), the properties of v_ε in lemma 3.2 and those of z_ε in lemma 3.5, we deduce that

$$\begin{aligned}\alpha_\varepsilon(\varepsilon s+r_0)+\varepsilon v(s)+\beta_\varepsilon(\varepsilon s+r_0)+\varepsilon^2 z(s)&\\ = O(\varepsilon^2)+O(\varepsilon|s|+\varepsilon)+O(\varepsilon^2 s^2)&+O(\varepsilon^2|s|+\varepsilon^2),\end{aligned}$$

and so

$$u_1(\varepsilon s+r_0) = w(s)+\ln\frac{1}{\varepsilon^2}-\ln\lambda+O(\delta|s|+\delta).$$

Therefore,

$$\lambda\varepsilon_\lambda^2 \mathrm{e}^{u_\lambda(\varepsilon_\lambda s+r_0)} = \mathrm{e}^{w(s)+O(\delta|s|+\delta)} \tag{4.12}$$

and (4.10) follows, since $s\in[a,b]$.

Moreover, since $w(s)=\sqrt{2}s+O(\mathrm{e}^{\sqrt{2}s})$ as s goes to $-\infty$, we also deduce that if λ (and also δ) is small enough, then there exist $a,b>0$ such that

$$\lambda\varepsilon^2\mathrm{e}^{u_1(\varepsilon s+r_0)} \leqslant b\mathrm{e}^{-a|s|} \quad \text{for any } s\in(-\infty,0]. \tag{4.13}$$

Now, we have (scaling by $r=\varepsilon s+r_0$ in the first integral and arguing as in step 3 of lemma 4.2 to estimate the second and third integrals)

$$\begin{aligned}\lambda\varepsilon_\lambda\int_0^{r_0}\mathrm{e}^{u_\lambda(r)}\,\mathrm{d}r &= \lambda\varepsilon_\lambda\int_{r_0-\delta}^{r_0}\mathrm{e}^{u_1(r)}\,\mathrm{d}r+\lambda\varepsilon_\lambda\int_{r_0-2\delta}^{r_0-\delta}\mathrm{e}^{u_2(r)}\,\mathrm{d}r\\ &\quad+\lambda\varepsilon_\lambda\int_0^{r_0-2\delta}\mathrm{e}^{u_3(r)}\,\mathrm{d}r\\ &= \lambda\varepsilon_\lambda^2\int_{-\delta/\varepsilon}^0\mathrm{e}^{u_1(\varepsilon_\lambda s+r_0)}\,\mathrm{d}r+O(\varepsilon^{1+\sigma})\\ &\to \int_{\mathbb{R}}\mathrm{e}^{w(s)}\,\mathrm{d}s \quad \text{as } \lambda\to 0\end{aligned}$$

because of (4.12), (4.13) and Lebesgue's dominated convergence theorem. We have now proved (4.11). □

5. A contraction mapping argument and the proof of the main theorem

First of all we point out that $u_\lambda+\phi_\lambda$ is a solution to (2.1) if and only if ϕ_λ is a solution of the problem

$$\left.\begin{aligned}\mathcal{L}_\lambda(\phi_\lambda)=\mathcal{N}_\lambda(\phi_\lambda)+\mathcal{R}_\lambda(u_\lambda) \quad \text{in } (0,r_0),\\ \phi_\lambda'(0)=\phi_\lambda'(r_0)=0,\end{aligned}\right\} \tag{5.1}$$

where $\mathcal{R}_\lambda(u_\lambda)$ is given in (4.1),

$$\mathcal{L}_\lambda(\phi_\lambda) := -\phi_\lambda'' - \frac{N-1}{r}\phi_\lambda' + \phi_\lambda - \lambda e^{u_\lambda}\phi_\lambda$$

and

$$\mathcal{N}_\lambda(\phi_\lambda) := \lambda e^{u_\lambda+\phi_\lambda} - \lambda e^{u_\lambda} - \lambda e^{u_\lambda}\phi_\lambda.$$

The next results state that the linearized operator $\mathcal{L}_\lambda$ is uniformly invertible.

Proposition 5.1. *There exist $\lambda_0 > 0$ and $C > 0$ such that, for any $\lambda \in (0, \lambda_0)$ and for any $h \in L^1((0, r_0))$, there exists a $\phi \in W^{2,2}((0, r_0))$ that is a unique solution of*

$$\mathcal{L}_\lambda(\phi) = h,$$
$$\phi'(0) = \phi'(r_0) = 0$$

that satisfies

$$\|\phi\|_\infty \leqslant C\|h\|_{L^1}.$$

Proof. Attempting a contradiction, we assume that there exist sequences $\lambda_n \to 0$, $h_n \in L^1((0, r_0))$ and $\phi_n \in W^{2,2}((0, r_0))$ that solve

$$\left.\begin{aligned} -\phi_n'' - \frac{N-1}{r}\phi_n' + \phi_n - \lambda_n e^{u_{\lambda_n}}\phi_n = h_n \quad \text{in } (0, r_0),\\ \phi_n'(0) = \phi_n'(r_0) = 0 \end{aligned}\right\} \tag{5.2}$$

and

$$\|\phi_n\|_\infty = 1, \qquad \|h_n\|_{L^1} \to 0. \tag{5.3}$$

Let $\psi_n(s) = \phi_n(\varepsilon_n s + r_0)$. Then ψ_n solves

$$\left.\begin{aligned} &-\psi_n'' - \frac{N-1}{\varepsilon_n s + r_0}\varepsilon_n\psi_n' + \varepsilon_n^2\psi_n\\ &\qquad - \lambda_n\varepsilon_n^2 e^{u_n(\varepsilon_n s + r_0)}\psi_n = \varepsilon_n^2 h_n(\varepsilon_n s + r_0) \quad \text{in } (-r_0/\varepsilon_n, 0),\\ &\qquad\psi_n'(-r_0/\varepsilon_n) = \psi_n'(0) = 0, \end{aligned}\right\} \tag{5.4}$$

which can also be written as

$$\left.\begin{aligned} &-((\varepsilon_n s + r_0)^{N-1}\psi_n')'\\ &\quad = (\varepsilon_n s + r_0)^{N-1}(-\varepsilon_n^2\psi_n + \lambda_n\varepsilon_n^2 e^{u_n(\varepsilon_n s + r_0)}\psi_n + \varepsilon_n^2 h_n(\varepsilon_n s + r_0))\\ &\qquad\qquad \text{in } s \in (-r_0/\varepsilon_n, 0),\\ &\qquad \psi_n'(-r_0/\varepsilon_n) = \psi_n'(0) = 0. \end{aligned}\right\} \tag{5.5}$$

Now, let us fix $a < 0$. By (5.5) we immediately deduce that, for any $\sigma \in [a, 0]$,

$$\begin{aligned} &(\varepsilon_n\sigma + r_0)^{N-1}\psi_n'(\sigma)\\ &\quad = \int_\sigma^0 (\varepsilon_n s + r_0)^{N-1}(-\varepsilon_n^2\psi_n + \lambda_n\varepsilon_n^2 e^{u_n(\varepsilon_n s + r_0)}\psi_n + \varepsilon_n^2 h_n(\varepsilon_n s + r_0))\,\mathrm{d}s, \end{aligned}$$

which implies that $\sup_{\sigma\in[a,0]}|\psi_n'(\sigma)| = O(\varepsilon_n)$, because ψ_n is bounded in L^∞, $\|h_n\|_{L^1} \to 0$ and (4.11) holds. The Ascoli–Arzelá theorem then implies that $\psi_n \to \psi$ uniformly on compact sets of $(-\infty, 0]$.

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Hence, we multiply the equation in (5.4) by a C_0^∞ test function, integrate and use (4.10) to deduce that ψ solves

$$\left.\begin{aligned} -\psi'' - \mathrm{e}^w\psi = 0 \quad \text{in } (-\infty, 0),\\ \|\psi\|_\infty \leqslant 1,\\ \psi'(0) = 0. \end{aligned}\right\} \tag{5.6}$$

A straightforward computation shows (see [9]) that there exist a and b such that

$$\psi(s) = a\frac{\mathrm{e}^{\sqrt{2}s} - 1}{\mathrm{e}^{\sqrt{2}s} + 1} + b\biggl(-2 + \sqrt{2}s\frac{\mathrm{e}^{\sqrt{2}s} - 1}{\mathrm{e}^{\sqrt{2}s} + 1}\biggr).$$

It is immediate to check that $b = 0$ (since $\|\psi\|_\infty \leqslant 1$), and then

$$\psi(s) = a\frac{\mathrm{e}^{\sqrt{2}s} - 1}{\mathrm{e}^{\sqrt{2}s} + 1}.$$

By using the condition $\psi'(0) = 0$ we also get that $a = 0$.

We claim that $\|\phi_n\|_\infty = o(1)$. This immediately gives a contradiction since, by assumption, $\|\phi_n\|_\infty = 1$. Let G be the Green function of the operator $-u'' - ((N-1)/r)u' + u$ with Neumann boundary condition, whose properties can be found in Grossi [9, appendix A] (see also Grossi and Noris [11]).

By (5.2), we deduce that

$$\begin{aligned} \phi_n(r) &= \int_0^{r_0} G(r,t)\lambda_n \mathrm{e}^{u_{\lambda_n}}\phi_n(t)\,\mathrm{d}t + \int_0^{r_0} G(r,t)h_n(t)\,\mathrm{d}t \\ &= \varepsilon_n\lambda_n \int_{-r_0/\varepsilon_n}^0 G(r, \varepsilon_n s + r_0)\mathrm{e}^{u_{\lambda_n}(\varepsilon_n s + r_0)}\psi_n(s)\,\mathrm{d}s + \int_0^{r_0} G(r,t)h_n(t)\,\mathrm{d}t \\ &= G(r)\varepsilon_n\lambda_n \int_{-r_0/\varepsilon_n}^0 \mathrm{e}^{u_{\lambda_n}(\varepsilon_n s + r_0)}\psi_n(s)\,\mathrm{d}s + \int_0^{r_0} G(r,t)h_n(t)\,\mathrm{d}t \\ &\quad + \varepsilon_n\lambda_n \int_{-r_0/\varepsilon_n}^0 [G(r, \varepsilon_n s + r_0) - G(r)]\mathrm{e}^{u_{\lambda_n}(\varepsilon_n s + r_0)}\psi_n(s)\,\mathrm{d}s. \end{aligned}$$

Since G is bounded, it is immediate to check that $\int_0^{r_0} G(r,t)h_n(t)\,\mathrm{d}t = o(1)$. We also want to show that

$$\varepsilon_n\lambda_n \int_{-r_0/\varepsilon_n}^0 [G(r, \varepsilon_n s + r_0) - G(r)]\mathrm{e}^{u_{\lambda_n}(\varepsilon_n s + r_0)}\psi_n(s)\,\mathrm{d}s = o(1). \tag{5.7}$$

If this is true, then

$$\phi_n(r) = G(r)K_n + o(1),$$

where

$$K_n := \varepsilon_n\lambda_n \int_{-r_0/\varepsilon_n} \mathrm{e}^{u_{\lambda_n}(\varepsilon_n s + r_0)}\psi_n(s)\,\mathrm{d}s.$$

We compute

$$G(r_0)K_n + o(1) = \phi_n(r_0) = \psi_n(0) = o(1),$$

and hence $K_n = o(1)$ since $G(r_0) \neq 0$. Then $\|\phi_n\|_\infty = o(1)$ and this gives a contradiction.

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It remains to prove (5.7). We have that

$$
\begin{aligned}
&\left|\varepsilon_n\lambda_n\int_{-r_0/\varepsilon_n}[G(r,\varepsilon_n s+r_0)-G(r)]\mathrm{e}^{u_{\lambda_n}(\varepsilon_n s+r_0)}\psi_n(s)\,\mathrm{d}s\right|\\
&\qquad\leqslant \varepsilon_n^2\lambda_n\int_{-r_0/\varepsilon_n}^{0}|s|\mathrm{e}^{u_{\lambda_n}(\varepsilon_n s+r_0)}|\psi_n(s)|\,\mathrm{d}s\\
&\qquad= \underbrace{\varepsilon_n^2\lambda_n\int_{-r_0/\varepsilon_n}^{-2\delta_n/\varepsilon_n}|s|\mathrm{e}^{u_n^3(\varepsilon_n s+r_0)}|\psi_n(s)|\,\mathrm{d}s}_{(I)}\\
&\qquad\quad+\underbrace{\varepsilon_n^2\lambda_n\int_{-2\delta_n/\varepsilon_n}^{-\delta_n/\varepsilon_n}|s|\mathrm{e}^{u_n^2(\varepsilon_n s+r_0)}|\psi_n(s)|\,\mathrm{d}s}_{(II)}\\
&\qquad\quad+\underbrace{\varepsilon_n^2\lambda_n\int_{-\delta_n/\varepsilon_n}^{0}|s|\mathrm{e}^{u_n^1(\varepsilon_n s+r_0)}|\psi_n(s)|\,\mathrm{d}s}_{(III)}.
\end{aligned}
$$

Now, arguing as in step 3 of lemma 4.2, we get that

$$
(I)=O\biggl(\int_{-r_0/\varepsilon_n}^{-2\delta_n/\varepsilon_n}|s|\mathrm{e}^{-|s|}|\psi_n(s)|\,\mathrm{d}s\biggr)=O\biggl(\int_{-\infty}^{0}|s|\mathrm{e}^{-|s|}|\psi_n(s)|\,\mathrm{d}s\biggr)=o(1)
$$

because $\psi_n\to 0$ pointwise in $(-\infty,0)$ and $\|\psi_n\|_\infty\leqslant 1$. Moreover, as in step 2 of lemma 4.2,

$$
(II)=O\biggl(\int_{-2\delta_n/\varepsilon_n}^{-\delta_n/\varepsilon_n}|s|\mathrm{e}^{-|s|}|\psi_n(s)|\,\mathrm{d}s\biggr)=O\biggl(\int_{-\infty}^{0}|s|\mathrm{e}^{-|s|}|\psi_n(s)|\,\mathrm{d}s\biggr)=o(1).
$$

By (4.13) we deduce that

$$
(III)=O\biggl(\int_{-\infty}^{0}|s|\mathrm{e}^{-a|s|}|\psi_n(s)|\,\mathrm{d}s\biggr)=o(1)
$$

for some $a>0$. □

Finally, we are in position to use a contraction mapping argument to prove theorem 1.1.

Proof of theorem 1.1. By proposition 5.1, we deduce that the linear operator $\mathcal{L}_\lambda$ is uniformly invertible, and so problem (5.1) can be rewritten as

$$
\phi=\mathcal{T}_\lambda(\phi):=\mathcal{L}_\lambda^{-1}[\mathcal{R}_\lambda(\bar{u}_\lambda)+\mathcal{N}_\lambda(\phi)]. \tag{5.8}
$$

For a given number $\rho>0$ let us consider the closed set

$$
A_\rho:=\{\phi\in L^\infty(0,r_0)\colon \|\phi\|_\infty\leqslant\rho\varepsilon^{1+\sigma}\},
$$

where $\sigma>0$ is given in lemma 4.2.

We will prove that if λ is small enough, then $\mathcal{T}_\lambda\colon A_\rho\to A_\rho$ is a contraction map.

First of all, by (4.11), we get

$$\|\mathcal{N}_\lambda(\phi)\|_{L^1} \leqslant \|\lambda e^{u_\lambda}\|_{L^1}\|\phi\|_\infty^2 \leqslant \frac{C}{\varepsilon}\|\phi\|_\infty^2 \quad \text{for any } \phi \in A_\rho$$

and also

$$\|\mathcal{N}_\lambda(\phi_1) - \mathcal{N}_\lambda(\phi_2)\|_{L^1} \leqslant \frac{C}{\varepsilon}\Big(\max_{i=1,2}\|\phi_i\|_\infty\Big)\|\phi_1 - \phi_2\|_\infty \quad \text{for any } \phi_1, \phi_2 \in A_\rho$$

for some $C > 0$.

By lemma 4.2 we deduce that, for some $\rho > 0$,

$$\|\mathcal{T}_\lambda(\phi)\|_\infty \leqslant C(\|\mathcal{R}_\lambda(u_\lambda)\|_{L^1} + \|\mathcal{N}_\lambda(\phi)\|_{L^1}) \leqslant \rho\varepsilon^{1+\sigma},$$

and so $\mathcal{T}_\lambda$ maps A_ρ into itself. Moreover,

$$\|\mathcal{T}_\lambda(\phi_1) - \mathcal{T}_\lambda(\phi_2)\|_\infty \leqslant C\|\mathcal{N}_\lambda(\phi_1) - \mathcal{N}_\lambda(\phi_2)\|_{L^1} \leqslant C\varepsilon^\sigma\|\phi_1 - \phi_2\|_\infty,$$

which proves that, for ε small enough, $\mathcal{T}_\lambda$ is a contraction mapping on A_ρ for a suitable ρ.

Therefore, a unique fixed point of $\mathcal{T}_\lambda$ has a unique fixed point in A_ρ, namely, there exists a unique solution $\phi = \phi_\lambda \in A_\rho$ of (5.8) or, equivalently, there exists a unique solution $u_m + \phi_m$ of (2.1).

Estimate (1.5) follows by the definition of u_m, which coincides with u_3 far away from r_0. Indeed, if $[a, b]$ is a compact set in $(0, r_0 - 2\delta)$, we get that, for λ small enough,

$$\begin{aligned}\varepsilon(u_\lambda(r) + \phi_m(r)) &= (A_1 + A_2\varepsilon + A_3\varepsilon^2)\mathcal{U}(r) + \varepsilon\phi_m(r) \\ &\to \frac{\sqrt{2}}{\mathcal{U}'(r_0)}\mathcal{U}(r) \quad \text{as } \lambda \to 0\end{aligned}$$

because of (4.2) and the fact that $\|\phi\|_\infty \to 0$ as $\lambda \to 0$.

Finally, (1.4) follows by (4.11), taking into account that $\|\phi\|_\infty \to 0$ as $\lambda \to 0$. □

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