comes immediately to mind. Accordingly, if $\alpha = \tan^{-1} a$ and $\beta = \tan^{-1} b$, then we can write

$$\text{area}(\mathbf{R}) = \frac{\tan\beta - \tan\alpha}{1 + \tan\alpha\,\tan\beta} = \tan(\beta - \alpha) \simeq \beta - \alpha$$

since $\beta - \alpha$ is also 'incomparably small'. Thus the area under any section of the curve can be approximated by a *sum of differences* and the result follows. The formal proof makes these approximations precise.

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95.33 Integrating $\int_0^\infty \frac{|\sin^a x \cos^b x|}{x^p}\,dx$

Introduction

Integrals of the type $\int_0^\infty \frac{\sin^a x \cos^b x}{x^p}\,dx$ (where $a \geqslant 2$, $b \geqslant 0$) have featured in the *Gazette* fairly recently [1] and results for $\int_0^\infty \frac{\sin^a x \cos^b x}{x^p}\,dx$ (where $a \geqslant p$) can be found in [2]. Here we consider the related integrals

$$I_p(a, b) = \int_0^\infty \frac{|\sin^a x \cos^b x|}{x^p}\,dx \text{ for integers } a \geqslant p \geqslant 2,\ b \geqslant 0.$$

Using a series approach, we reduce these to standard integrals when p is even, but when $p \geqslant 3$ is odd we use a spreadsheet to find numerical estimates.

For integers $a \geqslant 2$, $b \geqslant 0$,

$$\int_0^\infty \frac{|\sin^a x \cos^b x|}{x^p}\,dx = \sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} \frac{|\sin^a x \cos^b x|}{x^p}\,dx$$

$$= \sum_{k=0}^{\infty} \int_0^{\pi} \frac{|\sin^a t \cos^b t|}{(t + k\pi)^p}\,dt \qquad (\text{set } t = x - k\pi).$$

The elementary result $\int_0^\pi f(t)dt = \int_0^{\pi/2} f(x)dx + \int_0^{\pi/2} f(\pi - x)dx$ then reduces this to

$$I_p(a,b) = \int_0^{\pi/2} \left(\sin^a x \cos^b x\right)\sigma_p(x)dx \tag{1}$$

where $\sigma_p(x) = \frac{1}{x^p} + \sum_{k=1}^{\infty}\left\{\frac{1}{(k\pi + x)^p} + \frac{1}{(k\pi - x)^p}\right\}$ or, when p is even, $\sigma_p(x) = \sum_{k=-\infty}^{\infty} \frac{1}{(x - k\pi)^p}$.

Exact evaluation

Using the identity $\operatorname{cosec}^2 x \equiv \sum_{k=-\infty}^{\infty} \frac{1}{(x - k\pi)^p}$, which is usually derived by differentiating the expansion

$$\cot x = \sum_{-\infty}^{\infty} \frac{1}{x - k\pi}, \tag{2}$$

which in turn follows from the logarithmic derivative of Euler's product $\sin x = x \prod_{k \neq 0}\left(1 - \frac{x}{k\pi}\right)$, we find that

$$I_2(a, b) = \int_0^\infty \frac{|\sin^a x \cos^b x|}{x^2}dx = \int_0^{\pi/2} \sin^{a-2} x \cos^b x\, dx, \tag{3}$$

which is a standard integral that may be tackled by repeated integration by parts.

In particular, $\int_0^\infty \frac{|\sin^a x|}{x^2}dx = \int_0^{\pi/2} \sin^{a-2} x\, dx$, which confirms the standard result $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2}$ and proves that $\int_0^\infty \frac{|\sin^3 x|}{x^2}dx = 1$.

When both a and b are odd the result is

$$I_2(2m + 1, 2n + 1) = \int_0^{\pi/2} \sin^{2m-1} x \cos^{2n+1} x\, dx = \frac{1}{2}\frac{(m-1)!\,n!}{(m-n)!},$$

but when both are even

$$I_2(2m, 2n) = \int_0^{\pi/2} \sin^{2m-2} x \cos^{2n} x\, dx = \frac{\pi(2n)!\,(2m-2)!}{2^{2m+2n-1} n!\,(m-1)!\,(m+n-1)!}.$$

These are both instances of the standard result

$$\int_0^{\pi/2} \sin^{a-2} x \cos^b x\, dx = \frac{1}{2}\frac{\Gamma\left(\frac{a-1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a+b}{2}\right)}.$$

Since $\int_0^{\pi/2} \sin^\alpha x \cos^\beta x\, dx = \int_0^{\pi/2} \sin^\beta x \cos^\alpha x\, dx$ a further general result is

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that $I_2(a,b) = I_2(b+2, a-2)$ meaning that $\int_0^\infty \frac{|\sin^2 x \cos x|}{x^2} dx = \int_0^\infty \frac{|\sin^3 x|}{x^2} dx$ whereas (by parts) $\int_0^\infty \frac{\sin^2 x \cos x}{x^2} dx = \frac{2}{3} \int_0^\infty \frac{\sin^3 x}{x^3} dx$.

Since $\cot\frac{x}{2} = \sum_{-\infty}^{\infty} \frac{2}{x - 2k\pi}$, $\tan\frac{x}{2} = \cot\left(\frac{\pi}{2} - \frac{x}{2}\right) = -\sum_{-\infty}^{\infty} \frac{2}{x - (2k-1)\pi}$ and $\frac{\cot\frac{x}{2} + \tan\frac{x}{2}}{2} = \frac{1}{\sin x}$ equation (2) also provides the identity $\operatorname{cosec} x \equiv \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(x - k\pi)}$.

If $a = 2m - 1$ $(m \geqslant 1)$ is odd and $b = 2n$ $(n \geqslant 0)$ is even, this identity, used as in the introduction, gives

$$\int_0^\infty \frac{\sin^{2m-1} x \cos^{2n} x}{x} dx = \int_0^{\pi/2} \left(\sin^{2m-1} x \cos^{2n} x \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{x - k\pi} \right) dx$$
$$= \int_0^{\pi/2} \sin^{2m-2} x \cos^{2n} x \, dx.$$

The particular case $m = 1$, $n = 0$ allowed Lord [3] to confirm that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$. Hence, from (3) $\int_0^\infty \frac{\sin^{2m} x \cos^{2n} x}{x^2} dx = \int_0^\infty \frac{\sin^{2m-1} x \cos^{2n} x}{x} dx$, which is a generalisation of a result included in [1].

By differentiating $\operatorname{cosec}^2 x \equiv \sum_{k=-\infty}^{\infty} \frac{1}{(x - k\pi)^2}$ twice we get $\operatorname{cosec}^4 x - \frac{2}{3} \operatorname{cosec}^2 x \equiv \sum_{k=-\infty}^{\infty} \frac{1}{(x - k\pi)^4}$, which may be used similarly to show that $\int_0^\infty \frac{|\sin^a x \cos^b x|}{x^4} dx = \int_0^{\pi/2} \sin^{a-4} x \cos^b x \, dx - \frac{2}{3} \int_0^{\pi/2} \sin^{a-2} x \cos^b x \, dx$ for $a \geqslant 4$, $b \geqslant 0$. In principle, we can keep going in this way to express an integral, of the form $I_p(a, b)$ with p even, in terms of standard integrals.

Numerical approach

Despite (from [2]) exact results like

$$I_3(4,0) = \int_0^\infty \frac{\sin^4 x}{x^3} dx = \int_0^{\pi/2} \sin^4 x \, \sigma_3(x) \, dx = \ln 2$$

and, for $p > 1$,

$$\sigma_p\left(\frac{\pi}{4}\right) = \frac{2^p(2^p - 1)}{\pi^p} \zeta(p), \sigma_p\left(\frac{\pi}{3}\right) = \frac{(3^p - 1)}{\pi^p} \zeta(p) \text{ and } \sigma_p\left(\frac{\pi}{2}\right) = \frac{2(2^p - 1)}{\pi^p} \zeta(p),$$

there is no known simple expression for $\sigma_p(x)$ when p is odd: numerical evaluation is needed.

If S_N denotes the sum of the first N terms of $\sigma_p(x) - \frac{1}{x^p}$, the remaining terms, on expanding by the binomial theorem, equal

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$$C_N(x) = \frac{2}{\pi^p} \sum_{k=N+1}^{\infty} \frac{1}{k^p} \left\{ 1 + \frac{p(p+1)}{2!}\left(\frac{x}{k\pi}\right)^2 + \frac{p(p+1)(p+2)(p+3)}{4!}\left(\frac{x}{k\pi}\right)^4 + \dots \right\}.$$

Defining $\zeta_N(p) = \sum_{k=1}^{N} k^{-p}$ and $\zeta'_N(p) = \sum_{k=N+1}^{\infty} k^{-p} = \zeta(p) - \zeta_N(p)$, $C_N(x)$ may be written as a series of correction terms

$$\frac{2}{\pi^p}\left\{ \zeta'_N(p) + \frac{p(p+1)}{2!}\left(\frac{x}{\pi}\right)^2 \zeta'_N(p+2) + \frac{p(p+1)(p+2)(p+3)}{4!}\left(\frac{x}{\pi}\right)^4 \zeta'_N(p+4) + \dots \right\}. \quad (4)$$

The values of $\zeta(p)$ are known exactly or can be readily calculated [4] and a spreadsheet can be used to calculate S_N and $\zeta_N(p)$. The values of $\zeta'_N(p)$ then can be deduced and the integrand in (1) computed for any non-zero value of x, ready for numerical integration (the $x = 0$ value being 1 if $a = p$ and zero if $a > p$).

Choosing $N = 50$, and using an *Excel* spreadsheet, one finds that no more than four of the correction terms of (4) are needed for maximum accuracy (15 significant figures), even when $p = 2$.

Tables 1 and 2 show evaluations of $I_p(a, b)$ for odd values of p, with $b = 0$ and $b = 1$ obtained by combining the trapezium rule estimates using 2, 3, 4, 6, 8, 9, 12, 18, 24, 36 and 72 intervals as described in [5]. With only one exception, these results for odd p, even a, and $b = 0$ agree, to 15 decimal places, with the result from [2] in the form

$$I_{2k+1}(2m,0) = \frac{(-1)^{m-k-1}}{2^{2m-2k-1}(2k)!}\left\{ \sum_{i=0}^{m-2} (-1)^i \binom{2m}{i} (m-i)^{2k} \ln(m-i) \right\} \quad \text{for } m > k \geqslant 1.$$

$b = 0$	$p = 3$	5	7	9
$a = 3$	1.20844420949041			
4	0.693147180559945			
5	0.514649194262908	0.941887861102954		
6	0.421751358464106	0.467613876007544		
7	0.363777196180142	0.315848628851268	0.802756911753731	
8	0.323642331508254	0.242080571455144	0.356440682776443	
9	0.293928671969672	0.198644641285563	0.221696177772022	0.711448343691695
10	0.270874295806716	0.170031687922601	0.159469421290094	0.288084273586973

TABLE 1

$b=1$	$p = 3$	5	7	9
$a=3$	0.845237493853565			
4	0.384342908662528			
5	0.241015389108362	0.730912531967594		
6	0.173186372158086	0.298909128214203		
7	0.134232907817172	0.172874563057750	0.661481242190141	
8	0.109168841817002	0.116542930676375	0.250300558465676	
9	0.091781575808846	0.085780524699383	0.135966634299365	0.609192606398828
10	0.079056265312896	0.066845181259216	0.087022659575664	0.215751935782790

TABLE 2

Using $\cos^2 x = 1 - \sin^2 x$, results for greater values of b can be found from

$$I_p(a, 2n + \lambda) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} I_p(a + 2i, \lambda), \text{ where } \lambda = 0 \text{ or } 2,$$

but it is easier to calculate them directly on the spreadsheet.

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95.34 On evaluating the probability integral

In a previous issue of the *Gazette* [1], Nick Lord showed how the identity

$$\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$$

could be derived from Wallis's formula for π in an elementary way. While his argument was later simplified [2], the following proof is even quicker. It is certainly not new – it appears tucked away as an extended exercise in Spivak's famous text [3, p. 371] – and undoubtedly it has been discovered and rediscovered many times. Still it deserves to be far better known than it is.

We begin with the elementary inequality

$$e^x \geqslant 1 + x \qquad \text{for every } x.$$

Indeed, the graph of the exponential function is concave up and so never lies

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