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Bohr theorems for slice regular functions over octonions

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In this paper, we mainly investigate two versions of the Bohr theorem for slice regular functions over the largest alternative division algebras of octonions $\mathbb{O}$. To this end, we establish the coefficient estimates for self-maps of the unit ball of $\mathbb{O}$ and the Carathéodory class in this setting. As a further application of the coefficient estimate, the 1/2-covering theorem is also proven for slice regular functions with convex image.

Keywords: function of one hypercomplex variable, division algebras, Bohr theorem, covering theorem

2010 *Mathematics subject classification:* Primary 30G35
Secondary 17A35, 30C25

1. Introduction

In the study of Dirichlet series, Harald Bohr in 1914 showed a remarkable phenomenon saying that

THEOREMAL 1.1 Bohr theorem. *Let* $F(z)=\sum_{n=0}^{+\infty} a_n z^n$ *be a holomorphic function in the open unit disk* $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ *such that* $|F(z)|\leqslant 1$ *for all* $z\in\mathbb{D}$. *Then*

$$\sum_{n=0}^{+\infty}|a_n z^n|\leqslant 1,\quad |z|\leqslant\frac{1}{3}. \tag{1.1}$$

Moreover, the constant $1/3$*, called the Bohr radius, is the best possible.*

This inequality for the best constant $1/3$ was actually established independently by Wiener, Riesz and Schur. This old result was forgotten seemly until 1995 when Dixon applied the Bohr theorem to the characterization of the long-standing problem of Banach algebras satisfying the von Neumann inequality [**14**]. Over the last two decades, the Bohr theorem has obtained great attention. The study of the Bohr radius for holomorphic functions in several complex variables becomes very active, see e.g., [**2**, **5**, **9**, **12**]. See [**26**] for an operator-theoretic proof of Bohr's inequality and more operator-valued generalizations in the single variable case [**27**], in multivariable cases for free holomorphic functions in non-commutative setting [28–30]. The Bohr type inequality has also been formulated for holomorphic functions valued in Banach spaces [8, 23, 25]. The reader may refer to the survey [1] and references therein for various generalizations and variants of the Bohr theorem, such as in the setting of complex polynomials, harmonic (or poly-harmonic) mappings, holomorphic functions in several complex variables and more abstract settings.

Historically, Bohr in [10] discovered a weak version of theorem 1.1 for $|z| \leqslant 1/6$ by the Carathéodory inequality

$$|F(z)| \leqslant |\gamma| + \frac{1+r}{1-r}|\beta| + \frac{2r}{1-r}\sup_{z\in\mathbb{D}} \operatorname{Re} F(z), \quad r = |z| < 1,$$

for the holomorphic function $F(z)$ in $\mathbb{D}$ with $F(0) = \beta + \gamma i$ for $\beta, \gamma \in \mathbb{R}$.

The Carathéodory inequality above reveals that the modulus of a holomorphic function is essentially bounded by its real part. Another related and classical inequality is the well-known Borel-Carathéodory theorem (cf. [24]).

Theoremal 1.2 Borel-Carathéodory theorem. *Let $F(z)$ be a holomorphic function in $\mathbb{D}$ with $A = \sup_{z\in\mathbb{D}} \operatorname{Re} F(z) < +\infty$. Then, we have, for $r = |z| < 1$,*

$$|F(z) - F(0)| \leqslant \frac{2r}{1-r}(A - \operatorname{Re} F(0)),$$

and its corollary

$$|F(z)| \leqslant \frac{1+r}{1-r}|F(0)| + \frac{2r}{1-r}A.$$

As a generalization of the class of holomorphic functions of one complex variable, the theory of slice regular functions of one quaternionic variable was initiated by Gentili and Struppa [16] and further developed for Clifford algebras [11] and octonions [17]. Based on the concept of stem functions, these three function classes were eventually unified and generalized into real alternative algebras [18]. Following the historical path, theorems 1.1 and 1.2 recently have been generalised into the non-commutative (but associative) algebra of quaternions for slice regular functions [13, 31]. The interested readers refer to [21] and [22] for the monogenic versions of theorems 1.1 and 1.2 in the framework of quaternionic analysis, respectively.

In the present paper, we shall establish the Bohr theorem for slice regular functions over the non-commutative and non-associative algebra of octonions, which is the largest (finite-dimensional) alternative division algebras.

Theorem 1.3. *Let $f(x) = \sum_{n=0}^{+\infty} x^n a_n$ with $a_n \in \mathbb{O}$ be a slice regular function in the open unit ball $\mathbb{B}$ of $\mathbb{O}$ such that $|f(x)| \leqslant 1$ for all $x \in \mathbb{B}$. Then*

$$\sum_{n=0}^{+\infty} |x^n a_n| \leqslant 1, \quad |x| \leqslant \frac{1}{3}.$$

Moreover, the constant $1/3$ is sharp.

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REMARK 1.4. Note that Della Rocchetta, Gentili and Sarfatti have established the special case of theorem 1.3 for quaternions in [**13**] where regular quaternionic rational transformations in [**6**, **32**] are heavily used due to the fact that quaternions are a skew but associate field. However, these results are still unknown for the algebra of octonions. To overcome the difficulties caused by the non-commutativity and non-associativity of octonions, our proof turns back to the relationship between slice regularity and complex holomorphy (see lemma 2.3) and requires some more technical results (see lemmas 3.1 and 5.1).

REMARK 1.5. Restricted to each slice, the function f described in theorem 1.3 can be viewed, by lemma 2.3, as the vector-valued holomorphic function from $\mathbb{D}$ into the unit ball of $\mathbb{C}^4$ endowed with the standard Euclidean norm. Blasco introduced the Bohr radius for holomorphic functions from $\mathbb{D}$ into the unit ball of $\mathbb{C}^n (n \geqslant 2)$ and showed that it is zero; see [**8**, theorem 1.2] for more details. From this point of view, the theory of slice regular functions is different from the vector-valued holomorphic functions.

See [**33**] for generalized Bohr radius for slice regular functions over quaternions. What is more, by a recent result in [**34**, proposition 3.6], we formulate a new generalized version of theorem 1.1 for the associative algebra of quaternions $\mathbb{H}$.

THEOREM 1.6. *Let $f(x) = \sum_{n=0}^{+\infty} x^n a_n$ with $a_n \in \mathbb{H}$ and $g(x) = \sum_{n=0}^{+\infty} x^n b_n$ with $b_n \in \mathbb{H}$ be slice regular functions in the open unit ball $\mathbb{B}_{\mathbb{H}}$ of $\mathbb{H}$ such that $|f(x)| \leqslant |g(x)|$ for all $x \in \mathbb{B}_{\mathbb{H}}$. Then*

$$\sum_{n=0}^{+\infty} |x^n a_n| \leqslant \sum_{n=0}^{+\infty} |x^n b_n|, \quad |x| \leqslant \frac{1}{3}.$$

Note that inequality (1.1) can be rewritten as

$$\sum_{n=1}^{+\infty} |a_n z^n| \leqslant 1 - |F(0)| = \operatorname{dist}(F(0), \partial\mathbb{D}), \quad |z| \leqslant \frac{1}{3}.$$

From this viewpoint, we give another version of the Bohr theorem for octonions as follows.

THEOREM 1.7. *Let $f(x) = \sum_{n=0}^{+\infty} x^n a_n$ with $a_n \in \mathbb{O}$ be a slice regular function in the open unit ball $\mathbb{B}$ of $\mathbb{O}$ such that $f(\mathbb{B}) \subset \Pi := \{x \in \mathbb{O} : \operatorname{Re} x \leqslant 1\}$. Then*

$$\sum_{n=1}^{+\infty} |x^n a_n| \leqslant \operatorname{dist}(f(0), \partial\Pi), \quad |x| \leqslant \frac{1}{3}.$$

The remaining part of this paper is organized as follows. Section 2 is devoted to necessary preliminaries for slice regular functions over octonions. To prove theorem 1.3, we establish in § 3 the analogue of Wiener inequality for slice regular functions over octonions making use of the splitting lemma. Thereafter, the proof of theorem 1.6 is also given in § 3. In § 4, we first formulate the coefficient

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estimate for the Carathéodory class in $\mathbb{O}$ which allows to extend theorem 1.2 into the octonionic setting (see theorem 4.2) and then apply it to prove theorem 1.7. Besides, the growth theorem is proven for the Carathéodory class, which has its own independent interest. In § 5, as a further application of theorem 4.2, the 1/2-covering theorem is established for slice regular functions over octonions with convex image.

2. Preliminaries

In this section, we recall necessary definitions and preliminary results used in the sequel for slice regular functions from [**18**].

2.1. The algebra of octonions

Denote by $\mathbb{C}, \mathbb{H}, \mathbb{O}$ the algebras of complex numbers, quaternions and octonions, respectively. Let $\{1, i, j, k\}$ be the standard basis of the non-commutative, associative, real algebra of quaternions with the multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1.$$

The *conjugate* of $a = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$ $(x_0, x_1, x_2, x_3 \in \mathbb{R})$ is defined as $\overline{a} = x_0 - x_1 i - x_2 j - x_3 k$. By the well-known *Cayley-Dickson process*, the real algebra of octonions can be built from $\mathbb{H}$ as $\mathbb{O} = \mathbb{H} + l\mathbb{H}$ with $\overline{a + lb} = \overline{a} - lb$, $(a + lb) + (c + ld) = (a + c) + l(b + d)$ and $(a + lb)(c + ld) = (ac - d\overline{b}) + l(\overline{a}d + cb)$ for all $a, b, c, d \in \mathbb{H}$. As a consequence, $\{1, i, j, k, l, li, lj, lk\}$ forms the canonical real vector basis of $\mathbb{O}$. Every element $x \in \mathbb{O}$ can be composed into the *real* part $\mathrm{Re}\, x = (x + \overline{x})/2$ and the *imaginary* part $\mathrm{Im}\, x = x - \mathrm{Re}\,(x)$. Define the *modulus* of x as $|x| = \sqrt{x\overline{x}}$, which is exactly the usual Euclidean norm in $\mathbb{R}^8$. Furthermore, the modulus is multiplicative, i.e., $|xy| = |x||y|$ for all $x, y \in \mathbb{O}$. Every non-zero element $x \in \mathbb{O}$ has a multiplicative *inverse* given by $x^{-1} = |x|^{-2}\overline{x}$. The construction above shows that $\mathbb{O}$ is a non-commutative, non-associative, normed and division algebra. See for instance [**4**] for more explanation on the octonions.

The set of square roots of -1 in $\mathbb{O}$ is the six-dimensional unit sphere given by

$$\mathbb{S} = \{I \in \mathbb{O} \mid I^2 = -1\}.$$

For each $I \in \mathbb{S}$, denote by $\mathbb{C}_I := \langle 1, I \rangle \cong \mathbb{C}$ the subalgebra of $\mathbb{O}$ generated over $\mathbb{R}$ by 1 and I.

Notice that each $x \in \mathbb{O}$ can be expressed as $x = \alpha + \beta I_x$ with $\alpha \in \mathbb{R}, \beta \in \mathbb{R}^+$ and $I_x \in \mathbb{S}$. This inconspicuous observation allows decomposing $\mathbb{O}$ into ‘complex slices’

$$\mathbb{O} = \bigcup_{I \in \mathbb{S}} \mathbb{C}_I,$$

which derives the remarkable notion of slice regularity over octonions.

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2.2. Slice functions

Given an open set D of $\mathbb{C}$, invariant under the complex conjugation, its *circularization* Ω_D is defined by

$$\Omega_D = \bigcup_{I \in \mathbb{S}} \big\{\alpha + \beta I : \exists\ \alpha, \beta \in \mathbb{R}, \text{s.t. } z = \alpha + i\beta \in D\big\}.$$

A subset Ω in $\mathbb{O}$ is called to be *circular* if $\Omega = \Omega_D$ for some $D \subseteq \mathbb{C}$. The open unit ball $\mathbb{B} = \{x \in \mathbb{O} : |x| < 1\}$ and the right half-space $\{x \in \mathbb{O} : \operatorname{Re} x > 0\}$ are two typical examples of the circular domain.

Definition 2.1. A function $F : D \longrightarrow \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ on an open set $D \subseteq \mathbb{C}$ invariant under the complex conjugation is called a *stem function* if the $\mathbb{O}$-valued components F_1, F_2 of $F = F_1 + iF_2$ satisfies

$$F_1(\bar{z}) = F_1(z), \quad F_2(\bar{z}) = -F_2(z), \quad \forall\ z = \alpha + i\beta \in D.$$

Each stem function F induces a (left) *slice function* $f = \mathcal{I}(F) : \Omega_D \longrightarrow \mathbb{O}$ given by

$$f(x) := F_1(z) + IF_2(z), \quad \forall\ x = \alpha + I\beta \in \Omega_D.$$

We will denote the set of all such induced slice functions on Ω_D by

$$\mathcal{S}(\Omega_D) := \Big\{f = \mathcal{I}(F) :\ F \text{ is a stem function on } D\Big\}.$$

Each slice function f is induced by a unique stem function F since F_1 and F_2 are determined by f. In fact, it holds that

$$F_1(z) = \frac{1}{2}\big(f(x) + f(\overline{x})\big), \quad z \in \Omega_D,$$

and

$$F_2(z) = \begin{cases} \frac{1}{2I_x}\big(f(x) - f(\overline{x})\big) & \text{if } z \in \Omega_D \setminus \mathbb{R}, \\ 0, & \text{if } z \in \Omega_D \cap \mathbb{R}, \end{cases}$$

Recall that a C^1 function $F : D \longrightarrow \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ is holomorphic if and only if its components F_1, F_2 satisfy the Cauchy-Riemann equations

$$\frac{\partial F_1}{\partial \alpha} = \frac{\partial F_2}{\partial \beta}, \quad \frac{\partial F_1}{\partial \beta} = -\frac{\partial F_2}{\partial \alpha}, \quad z = \alpha + i\beta \in D.$$

Definition 2.2. A (left) slice function $f = \mathcal{I}(F)$ on Ω_D is *regular* if its stem function F is holomorphic on D. Denote the class of slice regular functions on Ω_D by

$$\mathcal{SR}(\Omega_D) := \Big\{f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D) : F \text{ is holomorphic on } D\Big\}.$$

For $f \in \mathcal{SR}(\Omega_D)$, the *slice derivative* is defined to be the slice regular function f' on Ω_D obtained as

$$f'(x) := \mathcal{I}\left(\frac{\partial F}{\partial z}(z)\right) = \frac{1}{2}\mathcal{I}\left(\frac{\partial F}{\partial \alpha}(z) - i\frac{\partial F}{\partial \beta}(z)\right).$$

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Recall that $\mathbb{O}$ is non-associative but alternative, i.e., the *associator* $(x, y, z) := (xy)z - x(yz)$ of three elements $x, y, z \in \mathbb{O}$ is an alternating function in its arguments. Meanwhile, the Artin theorem asserts that *the subalgebra generated by two elements of* $\mathbb{O}$ *is associative*. Hence, a class of examples of slice regular functions is given by polynomials of one octonionic variable with coefficients in $\mathbb{O}$ on the right side. Indeed, each slice regular function f defined in $\mathbb{B}$ admits the expansion of convergent power series

$$f(x) = \sum_{n=0}^{\infty} x^n a_n, \quad \{a_n\} \subset \mathbb{O},$$

for all $x \in \mathbb{B}$.

For simplicity, let $\mathbb{B}_I$ the intersection $\mathbb{B} \cap \mathbb{C}_I$ for any $I \in \mathbb{S}$. Then the restriction $f\,|_{\mathbb{B}_I}$ is holomorphic on $\mathbb{B}_I$. Furthermore, the relation between slice regularity and complex holomorphy can be presented as follows.

Lemma 2.3 Splitting lemma. *Let* $\{I_0 = 1, I, I_1, II_1, I_2, II_2, I_3, II_3\}$ *be a splitting basis for* $\mathbb{O}$. *For* $f \in \mathcal{SR}(\Omega_D)$, *there exist holomorphic functions* $f_m : \Omega_D \cap \mathbb{C}_I \longrightarrow \mathbb{C}_I, m \in \{0, 1, 2, 3\}$, *such that*

$$f(z) = \sum_{m=0}^{3} f_m(z) I_m, \quad \forall\, z \in \Omega_D \cap \mathbb{C}_I.$$

Due to that the pointwise product of two slice functions is not a slice function generally, the notion of slice product was introduced.

Definition 2.4. Let $f = \mathcal{I}(F)$ and $g = \mathcal{I}(G)$ be in $\mathcal{S}(\Omega_D)$ with stem functions $F = F_1 + iF_2$ and $G = G_1 + iG_2$. Then $FG = F_1G_1 - F_2G_2 + i(F_1G_2 + F_2G_1)$ is still a stem function. The *slice product* of f and g is the slice function on Ω_D defined by

$$f \cdot g := \mathcal{I}(FG).$$

In general, $f \cdot g \neq fg$. If the components F_1, F_2 of the first stem function F are real-valued, then $f = \mathcal{I}(F)$ is termed as *slice preserving*. For the slice preserving function f and slice function g, the slice product $f \cdot g$ coincides with fg.

Definition 2.5. For $f = \mathcal{I}(F) \in \mathcal{S}(\Omega_D)$ with $F = F_1 + iF_2$, define the *slice conjugate* of f as

$$f^c = \mathcal{I}(\overline{F_1} + i\overline{F_2}),$$

and the *normal function* (or *symmetrization*) of f as

$$N(f) = f \cdot f^c = f^c \cdot f,$$

which is slice preserving on Ω_D.

Let $\mathcal{Z}_f$ denote the zero set of f on Ω_D.

Definition 2.6. Let $f \in \mathcal{S}(\Omega_D)$. If f does not vanish identically, then its *slice reciprocal* is defined as

$$f^{-\bullet}(x) := N(f)(x)^{-1} \cdot f^c(x) = N(f)(x)^{-1} f^c(x)$$

which is a slice function on $\Omega_D \setminus \mathcal{Z}_{N(f)}$.

More recently, Ghiloni, Perotti and Stoppato in [19] found a new and nice relation between the values of reciprocals $f^{-\bullet}(x)$ and $f(x)^{-1}$ for slice functions $f \in \mathcal{S}(\Omega_D)$ as

$$f^{-\bullet}(x) = f(T_f(x))^{-1}, \tag{2.1}$$

where T_f is a bijective self-map of $\Omega_D \setminus E$ with $E = \{\alpha + \beta I : I \in \mathbb{S}, z = \alpha + \beta i \in D \text{ for } F_2(z) = 0\}$ given by

$$T_f(x) = (f^c(x)^{-1}((xf^c(x))F_2(z)))F_2(z)^{-1},$$

which reduces to the known result $T_f(x) = f^c(x)^{-1} x f^c(x)$ for the associative algebra of quaternions.

Formula (2.1) allows to draw the following consequence.

Theorem 2.7. *Let $f : \mathbb{B} \to \mathbb{O}$ be a non-constant slice regular function. Then the image $f(\mathbb{B})$ is open.*

3. Proof of theorems 1.3 and 1.6

To prove theorem 1.3, following the idea of Wiener, we first establish the analogue of Wiener inequality for slice regular functions over octonions. Due to the non-associativity of octonions, we need a technical lemma.

Lemma 3.1. *Let f be a slice function in the open unit ball $\mathbb{B}$ of $\mathbb{O}$ and $a \in \mathbb{O} \setminus \{0\}$. Then*

$$|f(x)a| < 1 \text{ on } \mathbb{B} \Leftrightarrow |f(x) \cdot a| < 1 \text{ on } \mathbb{B}.$$

Proof. Firstly, assume that $f(x) = F_1(z) + IF_2(z)$ for $z = \alpha + \beta i \in \mathbb{D}$ with $|f(x)a| < 1$ for all $x = \alpha + \beta I \in \mathbb{B}$. Note that $f(x) \cdot a = F_1(z)a + I(F_2(z)a)$. If $F_2(z) = 0$, then $f(x) \cdot a = f(x)a$ and so $|f(x) \cdot a| < 1$. Otherwise, choosing $J = ((I(F_2(z)a))a^{-1})F_2(z)^{-1} \in \mathbb{S}$, we have $f(x) \cdot a = f(\alpha + \beta J)a$. Hence, $|f(x) \cdot a| < 1$ for all $x \in \mathbb{B}$.

Conversely, if $F_2(z) = 0$, then $f(x)a = f(x) \cdot a$ and so $|f(x)a| < 1$. Otherwise, choosing $K = ((IF_2(z))a)(F_2(z)a)^{-1} \in \mathbb{S}$, we obtain $f(x)a = f(\alpha + \beta K) \cdot a$. Hence, the condition $|f(x) \cdot a| < 1$ for all $x \in \mathbb{B}$ implies that $|f(x)a| < 1$ for all $x \in \mathbb{B}$. The proof is complete. □

Fortunately, we can now establish the following useful coefficient estimates. Its complex version is known as the Wiener inequality; see e.g., [10].

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LEMMA 3.2. *Let $f(x)=\sum_{n=0}^{+\infty} x^n a_n$ be a slice regular function in the open unit ball $\mathbb{B}$ of $\mathbb{O}$ such that $|f(x)| \leqslant 1$ for all $x \in \mathbb{B}$. Then*

$$|a_n| \leqslant 1-|a_0|^2, \quad n=1,2,3,\ldots \tag{3.1}$$

Proof. Suppose $a_0 \in [0,1]$. Let us show the validity of inequality (3.1) for n. Fix n and note that $a_n \in \mathbb{C}_I$ for some $I \in \mathbb{S}$. According to lemma 2.3, there exist holomorphic functions $f_m : \mathbb{B}_I \longrightarrow \mathbb{C}_I, m \in \{0,1,2,3\}$, such that

$$f\mid_{\mathbb{B}_I} = \sum_{m=0}^{3} f_m I_m,$$

which implies

$$a_0 = f(0) = f_0(0), \quad a_n = \frac{f^{(n)}(0)}{n!} = \frac{f_0^{(n)}(0)}{n!}.$$

Notice that $|f_0(z)| \leqslant |f(z)| < 1$. Then the classical Wiener inequality for the complex-valued holomorphic function f_0 gives that

$$\frac{|f_0^{(n)}(0)|}{n!} \leqslant 1-|f_0(0)|^2,$$

i.e.,

$$|a_n| \leqslant 1-|a_0|^2.$$

For the other cases $a_0 \in \mathbb{B} \setminus [0,1]$, the function $f(x)a$, taking the value in $\mathbb{B}$, is not slice regular generally, where $a=\overline{a_0}/|a_0|$. However, taking into account of lemma 3.1, we obtain the slice regular function $g(x)=f(x)\cdot a$ with its norm $|g(x)|<1$ for all $x \in \mathbb{B}$, $g(0)=|a_0| \in (0,1)$, and $g^{(n)}(0)=f^{(n)}(0)a$. Now the desired result follows when we apply the former conclusion to g. The proof is complete. □

REMARK 3.3. The complex version of (3.1) for $n=1$ can be viewed as a special case of the Schwarz-Pick lemma which says that, for holomorphic self-maps F of $\mathbb{D}$,

$$|F'(z)| \leqslant \frac{1-|F(z)|^2}{1-|z|^2}, \quad \forall\, z \in \mathbb{D}. \tag{3.2}$$

Inequalities (3.1) and (3.2) do not hold generally for vector-valued holomorphic functions defined in $\mathbb{D}$. For instance, the function $F(z)=(z,1)/\sqrt{2}$ from $\mathbb{D}$ to $B^2=\{z \in \mathbb{C}^2 : |z|<1\}$ satisfies

$$|F'(0)| = \frac{1}{\sqrt{2}} > \frac{1}{2} = 1-|F(0)|^2.$$

It is worth pointing out that a version of the Schwarz-Pick lemma was proved in [3, 7] for the self-map f of the quaternionic open unit ball which is slice regular while f, restricted to each slice, can be viewed as a holomorphic mapping from $\mathbb{D}$ into B^2.

Proof of theorem 1.3. Under the condition of theorem 1.3, it follows that, by lemma 3.2,

$$|a_n| \leqslant 2(1-|a_0|), \quad n=1,2,3,\ldots$$

which implies that

$$\sum_{n=1}^{+\infty}|x^n a_n| = \sum_{n=1}^{+\infty}|x|^n|a_n| \leqslant 2(1-|a_0|)\sum_{n=1}^{+\infty}|x|^n = 2(1-|a_0|)\frac{|x|}{1-|x|}, \quad |x|<1.$$

Hence,

$$\sum_{n=0}^{+\infty}|x^n a_n| \leqslant 1, \quad |x| \leqslant \frac{1}{3}.$$

To show the sharpness, given $a \in (0,1)$, we consider as in the complex case slice regular function

$$f(x) = (1-xa)^{-\bullet} \cdot (a-x) = a-(1-a^2)\sum_{n=1}^{+\infty} x^n a^{n-1}, \quad x \in \mathbb{B}.$$

Now the inequality

$$\sum_{n=0}^{+\infty}|x^n a_n| = a + \frac{r(1-a^2)}{1-ar} \leqslant 1$$

is equivalent to $r \leqslant 1/(1+2a)$. Hence, the radius $1/3$ cannot be improved if we set $a \to 1^-$. The proof is complete. □

REMARK 3.4. When we restrict the slice functions f and g into the quaternionic setting, the connection between the slice product and point-wise product of two slice functions is explicitly formulated as following results:

$$f \cdot g(x) = f(x)g(f(x)^{-1}xf(x)), \quad x \notin \mathcal{Z}_f,$$

and

$$f^{-\bullet} \cdot g(x) = f(T_f(x))^{-1}g(T_f(x)), \quad x \notin \mathcal{Z}_{N(f)}. \tag{3.3}$$

However, these two formulas above do not hold for the octonionic case as shown by [**19**, examples 4.15 and 4.16].

Note that the function $g(x)^{-1}f(x)$ is not slice regular generally for slice regular functions f and g. Hence, making use of formula (3.3), the authors in [**34**] formulated a useful result which is vital in the proof of theorem 1.6.

PROPOSITION 3.5. *Let f and g be slice regular functions in the circular domain Ω of $\mathbb{H}$ such that $\mathcal{Z}_g = \varnothing$. Then we have*

$$|f(x)| \leqslant |g(x)| \text{ on } \Omega \Leftrightarrow |g^{-\bullet} \cdot f(x)| \leqslant 1 \text{ on } \Omega.$$

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Proof of theorem 1.6. Based on proposition 3.5, we can define the quaternionic slice regular function

$$\varphi(x) = g^{-\bullet} \cdot f(x) : \mathbb{B}_{\mathbb{H}} \setminus \mathcal{Z}_{N(g)} \to \mathbb{H}$$

whose modulus is bounded by one. Furthermore, the domain of definition of φ can be extended to the whole ball $\mathbb{B}_{\mathbb{H}}$ keeping its slice regularity and norm. In fact, the bounded slice regular function g can be factored as (cf. [**15**, theorem 3.36])

$$g(x) = \psi \cdot \widehat{g}(x),$$

where $\psi, \widehat{g}$ are slice regular in $\mathbb{B}_{\mathbb{H}}$ such that $\mathcal{Z}_\psi = \mathcal{Z}_g$ and $\widehat{g}(x) \neq 0$ for all $x \in \mathbb{B}_{\mathbb{H}}$. The condition of $|f(x)| \leqslant |g(x)|$ implies that $\mathcal{Z}_g \subseteq \mathcal{Z}_f$, which gives also the factorization of f as

$$f(x) = \psi \cdot \widehat{f}(x),$$

for some slice regular function $\widehat{f}$ in $\mathbb{B}_{\mathbb{H}}$.

Hence,

$$\varphi = (\psi \cdot \widehat{g})^{-\bullet} \cdot (\psi \cdot \widehat{f}) = (\widehat{g}^{-\bullet} \cdot \psi^{-\bullet}) \cdot (\psi \cdot \widehat{f}) = \widehat{g}^{-\bullet} \cdot (\psi^{-\bullet} \cdot \psi) \cdot \widehat{f} = \widehat{g}^{-\bullet} \cdot \widehat{f}.$$

The slice regular function $\widehat{g}^{-\bullet} \cdot \widehat{f}(x)$ is well defined in $\mathbb{B}_{\mathbb{H}}$ and then its value can be identified as $\varphi(x)$ for $x \in \mathcal{Z}_{N(g)}$.

Now we write the expansion of power series for $\varphi(x)$ as $\sum_{n=0}^{\infty} x^n c_n$ for $x \in \mathbb{B}_{\mathbb{H}}$. Then the equation $f(x) = g \cdot \varphi(x)$ gives

$$\sum_{n=0}^{\infty} x^n a_n = \sum_{n,m=0}^{\infty} x^{n+m} b_n c_m.$$

Note that $|xy| = |x||y|$ for all $x, y \in \mathbb{H}$. The formula above implies

$$\sum_{n=0}^{\infty} |x^n a_n| = \sum_{n,m=0}^{\infty} |x^{n+m} b_n c_m| = \sum_{n=0}^{\infty} |x^n b_n| \sum_{n=0}^{\infty} |x^n c_n|.$$

From the quaternionic version of theorem 1.3, we have

$$\sum_{n=0}^{\infty} |x^n c_n| \leqslant 1, \quad |x| \leqslant \frac{1}{3}.$$

Consequently, for $|x| \leqslant \frac{1}{3}$,

$$\sum_{n=0}^{\infty} |x^n a_n| \leqslant \sum_{n=0}^{\infty} |x^n b_n|,$$

as desired. □

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4. Proof of theorem 1.7

To prove theorem 1.7, we formulate the coefficient estimates for the Carathéodory class in $\mathbb{O}$. The following result has been obtained for quaternionic slice regular function [**31**] in which the method is not valid for $\mathbb{O}$.

THEOREM 4.1. *Let* $f(x) = 1 + \sum_{n=1}^{\infty} x^n a_n$ *be a slice regular function in the open unit ball* $\mathbb{B}$ *of* $\mathbb{O}$ *with* $a_n \in \mathbb{O}$ *for all* n. *If* $\operatorname{Re} f(x) > 0$ *for all* $x \in \mathbb{B}$, *then*

$$\frac{1-|x|}{1+|x|} \leqslant \operatorname{Re} f(x) \leqslant |f(x)| \leqslant \frac{1+|x|}{1-|x|}, \quad \forall\, x \in \mathbb{B}, \tag{4.1}$$

and

$$|a_n| \leqslant 2, \qquad n = 1, 2, \ldots \tag{4.2}$$

Proof. Let us first consider (4.1) in the the case of $x = \alpha + \beta I \in \mathbb{B}_I$ for a fixed $I \in \mathbb{S}$. According to lemma 2.3 for slice regular functions, there exist holomorphic functions $f_m : \mathbb{B}_I \longrightarrow \mathbb{C}_I, m \in \{0, 1, 2, 3\}$, such that

$$f \mid_{\mathbb{B}_I} = \sum_{m=0}^{3} f_m I_m.$$

Then $f_0(0) = f(0) = 1$, and $\operatorname{Re} f_0(x) = \operatorname{Re} f(x) > 0$ on $\mathbb{B}_I$. Applying the Carathéodory theorem to the complex-valued holomorphic function f_0 (cf. [**20**]), we obtain

$$\frac{1-|x|}{1+|x|} \leqslant \operatorname{Re} f_0(x) = \operatorname{Re} f(x), \quad x \in \mathbb{B}_I.$$

Due to the arbitrariness of $I \in \mathbb{S}$, it follows that

$$\frac{1-|x|}{1+|x|} \leqslant \operatorname{Re} f(x), \quad x \in \mathbb{B}. \tag{4.3}$$

The condition $\operatorname{Re} f(x) > 0$ implies that $f(x) \neq 0$ for all $x \in \mathbb{B}$. Then $f^{-\bullet}(x)$ is slice regular on $\mathbb{B}$ with $f^{-\bullet}(0) = 1$ and $\operatorname{Re} f^{-\bullet}(x) > 0$ by [**19**, Corollary 4.12]. The former result (4.3) gives that

$$\frac{1-|x|}{1+|x|} \leqslant \operatorname{Re} f^{-\bullet}(x), \quad x \in \mathbb{B}.$$

Assume that $f(x) = F_1(z) + JF_2(z)$ with $z = \alpha + \beta i \in \mathbb{D}$ and $x = \alpha + \beta J \in \mathbb{B}$ for $J \in \mathbb{S}$. Denote $E = \{\alpha + \beta I : I \in \mathbb{S}, z = \alpha + \beta i \in \mathbb{D} \text{ for } F_2(z) = 0\}$. For $x \in E$, it holds that $f^{-\bullet}(x) = f(x)^{-1}$. Then

$$\frac{1-|x|}{1+|x|} \leqslant \mathrm{Re} f^{-\bullet}(x) = \mathrm{Re} f(x)^{-1} \leqslant \frac{1}{|f(x)|},$$

i.e.,

$$|f(x)| \leqslant \frac{1+|x|}{1-|x|}, \quad x \in E. \tag{4.4}$$

For $x \in \mathbb{B} \setminus E$, by [**19**, theorem 4.5], it holds that

$$f^{-\bullet}(x) = f(T_f(x))^{-1},$$

where T_f is a bijective self-map of $\Omega \setminus E$ given by

$$T_f(x) = (f^c(x)^{-1}((xf^c(x))F_2(z)))F_2(z)^{-1}.$$

Then

$$\frac{1-|x|}{1+|x|} \leqslant \mathrm{Re} f(T_f(x))^{-1} \leqslant \frac{1}{|f(T_f(x))|},$$

i.e.,

$$|f(T_f(x))| \leqslant \frac{1+|x|}{1-|x|}, \quad x \in \mathbb{B} \setminus E.$$

Furthermore, the fact $T_f(T_{f^{-\bullet}}(x)) = x$ for $x \in \Omega \setminus E$ gives that

$$|f(x)| = |f(T_f(T_{f^{-\bullet}}(x)))| \leqslant \frac{1+|T_{f^{-\bullet}}(x)|}{1-|T_{f^{-\bullet}}(x)|} = \frac{1+|x|}{1-|x|}, \quad x \in \mathbb{B} \setminus E. \tag{4.5}$$

Now inequality (4.1) follows from inequalities (4.3), (4.4), and (4.5).

Let us establish the coefficient estimates (4.2) for any $n \geqslant 1$. Fix n and note that $a_n \in \mathbb{C}_I$ for some $I \in \mathbb{S}$. As before, the restriction of f on $\mathbb{B}_I$ can be written as

$$f\,|_{\mathbb{B}_I} = \sum_{m=0}^{3} f_m I_m,$$

where $f_m : \mathbb{B}_I \longrightarrow \mathbb{C}_I, m \in \{0, 1, 2, 3\}$, are holomorphic functions.

Note that

$$f(0) = f_0(0) = 1, \quad a_n = \frac{f^{(n)}(0)}{n!} = \frac{f_0^{(n)}(0)}{n!}.$$

Taking into account the coefficient estimates for the holomorphic Carathéodory class, we obtain the desired inequality $|a_n| \leqslant 2$. The proof is complete. □

As a direct consequence, the Borel-Carathéodory theorem for octonions is obtained.

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THEOREM 4.2. *Let* $f(x) = \sum_{n=0}^{+\infty} x^n a_n$ *be slice regular in the open unit ball* $\mathbb{B}$ *of* $\mathbb{O}$ *with* $A = \sup_{x\in\mathbb{B}} \mathrm{Re}\, f(x) < +\infty$*. Then*

$$|a_n| \leqslant 2(A - \mathrm{Re}\, f(0)), \quad n = 1, 2, 3, \ldots$$

and

$$|f(x) - f(0)| \leqslant \frac{2r}{1-r}(A - \mathrm{Re}\, f(0)), \quad r = |x| < 1.$$

Proof. Define the slice regular function

$$g(x) = \frac{-f(x) - \overline{f(0)} + \mathrm{Re}\, f(0) + A}{A - \mathrm{Re}\, f(0)} : \mathbb{B} \longrightarrow \mathbb{O}$$

with $g(0) = 1$ and $\mathrm{Re}\, g(x) > 0$ on $\mathbb{B}$.

By theorem 4.1, it follows that

$$|a_n| = \frac{|g^{(n)}(0)|}{n!}(A - \mathrm{Re}\, f(0)) \leqslant 2(A - \mathrm{Re}\, f(0)), \quad n = 1, 2, 3, \ldots$$

Hence,

$$|f(x) - f(0)| \leqslant \sum_{n=1}^{\infty} |x|^n |a_n| \leqslant 2(A - \mathrm{Re}\, f(0)) \sum_{n=1}^{\infty} |x|^n = \frac{2r}{1-r}(A - \mathrm{Re}\, f(0)).$$

The proof is complete. □

Proof of theorem 1.7. Under the condition of theorem 1.7, it follows that, by theorem 4.2,

$$|a_n| \leqslant 2\big(1 - \mathrm{Re}\, f(0)\big), \quad n = 1, 2, 3, \ldots$$

which implies that

$$\sum_{n=1}^{+\infty} |x^n a_n| \leqslant 2\big(1 - \mathrm{Re}\, f(0)\big) \sum_{n=1}^{+\infty} |x|^n \leqslant 1 - \mathrm{Re}\, f(0) = \mathrm{dist}\,(f(0), \partial\Pi), \quad |x| \leqslant \frac{1}{3},$$

as desired. □

5. Covering theorem

In this section, as a further application of theorem 4.2, we establish 1/2-covering theorem for slice regular functions over octonions with convex image. We first give a useful lemma, in a more general version.

LEMMA 5.1. *Let* f, g *be two slice functions in the open unit ball* $\mathbb{B}$ *of* $\mathbb{O}$*. Then*

$$\mathrm{Re}\,\big(f \cdot g(x)\big) = \mathrm{Re}\,\big(f(x) g(T_f(x))\big),$$

where $T_f(x) = f(x)^{-1} x f(x)$ *for* $x \notin \mathcal{Z}_f$*.*

In particular, for all $a \in \mathbb{O}$,

$$\mathrm{Re}\big(f(x)\cdot a\big) = \mathrm{Re}\big(f(x)a\big).$$

Proof. Assume that $f(x) = F_1(z) + IF_2(z), g(x) = G_1(z) + IG_2(z)$ for $z = \alpha + \beta i \in \mathbb{D}$ and $x = \alpha + \beta I \in \mathbb{B}$ with $\alpha, \beta \in \mathbb{R}, I \in \mathbb{S}$. By definition, it follows that

$$f \cdot g(x) = F_1(z)G_1(z) - F_2(z)G_2(z) + I(F_1(z)G_2(z) + F_2(z)G_1(z)).$$

Hence,

$$\begin{aligned}
\mathrm{Re}\big(f \cdot g(x)\big) &= \mathrm{Re}\big(F_1(z)G_1(z) + I(F_2(z)G_1(z))\big) \\
&\quad + \mathrm{Re}\big(-F_2(z)G_2(z) + I(F_1(z)G_2(z))\big) \\
&= \mathrm{Re}\big(F_1(z)G_1(z) + (IF_2(z))G_1(z)\big) \\
&\quad + \mathrm{Re}\big(-F_2(z)G_2(z) + (IF_1(z))G_2(z)\big) \\
&= \mathrm{Re}\big(f(x)G_1(z)\big) + \mathrm{Re}\big((-F_2(z) + (IF_1(z)))G_2(z)\big) \\
&= \mathrm{Re}\big(f(x)G_1(z)\big) + \mathrm{Re}\big((I(IF_2(z) + F_1(z)))G_2(z)\big) \\
&= \mathrm{Re}\big(f(x)G_1(z)\big) + \mathrm{Re}\big((If(x))G_2(z)\big).
\end{aligned}$$

As for $T_f(x) = f(x)^{-1}xf(x) = \alpha + \beta J$ with $J = f(x)^{-1}If(x) \in \mathbb{S}$ for $x \notin \mathcal{Z}_f$, we have

$$f(x)g(T_f(x)) = f(x)G_1(z) + f(x)(JG_2(z)),$$

and then

$$\begin{aligned}
\mathrm{Re}\big(f(x)g(T_f(x))\big) &= \mathrm{Re}\big(f(x)G_1(z) + f(x)(JG_2(z))\big) \\
&= \mathrm{Re}\big(f(x)G_1(z) + (f(x)J)G_2(z)\big) \\
&= \mathrm{Re}\big(f(x)G_1(z) + (If(x))G_2(z)\big) \\
&= \mathrm{Re}\big(f \cdot g(x)\big),
\end{aligned}$$

which completes the proof. □

THEOREM 5.2. *Let f be a slice regular function in the open unit ball $\mathbb{B}$ of $\mathbb{O}$ with convex image and normalized by $f'(0) = 1$. Then $f(\mathbb{B})$ contains an open ball centred at $f(0)$ of radius $1/2$. Moreover, the constant $1/2$ is sharp.*

Proof. Let f be as described in the theorem and assume that $f(0) = 0$ for otherwise we can consider the slice regular function $f - f(0)$. Let $p \in \partial f(\mathbb{B})$ be a point at minimum distance from the origin. By theorem 2.7, we have $|p| > 0$. If $|p| = +\infty$, the theorem holds naturally. Otherwise, $0 < |p| < +\infty$, we obtain that

$$\mathrm{Re}\left(f(x)\frac{\overline{p}}{|p|}\right) < |p|, \quad \forall\, x \in \mathbb{B}, \tag{5.1}$$

since the image $f(\mathbb{B})$ is convex.

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By (5.1) and lemma 5.1, the slice regular function $g(x) = f(x) \cdot (\overline{p}/|p|)$ satisfies

$$\operatorname{Re} g(x) < |p|, \quad \forall\, x \in \mathbb{B}.$$

By theorem 4.2, we have

$$|g'(0)| \leqslant 2(|p| - \operatorname{Re} g(0)),$$

i.e.

$$\left| f'(0)\frac{\overline{p}}{|p|} \right| = 1 \leqslant 2(|p| - \operatorname{Re} g(0)) = 2|p|.$$

Therefore, $f(\mathbb{B})$ contains an open ball centred at $f(0)$ of radius $1/2$.

To see that the constant $1/2$ is optimal, we consider the slice regular function given by

$$f(x) = (1-x)^{-\bullet} \cdot x = x(1-x)^{-1}, \quad \forall\, x \in \mathbb{B}.$$

It is easy to show that $f'(0) = 1$ and the image $f(\mathbb{B}) = \{x \in \mathbb{O} : \operatorname{Re} x > -1/2\}$ is convex and contains the open ball centred at 0 of radius 1/2 while 1/2 is optimal, as desired. □

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