COMPOSITIO MATHEMATICA

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Abstract

Rapoport–Zink spaces are deformation spaces for p-divisible groups with additional structure. At infinite level, they become preperfectoid spaces. Let $\mathscr{M}_\infty$ be an infinite-level Rapoport–Zink space of EL type, and let $\mathscr{M}_\infty^\circ$ be one connected component of its geometric fiber. We show that $\mathscr{M}_\infty^\circ$ contains a dense open subset which is cohomologically smooth in the sense of Scholze. This is the locus of p-divisible groups which do not have any extra endomorphisms. As a corollary, we find that the cohomologically smooth locus in the infinite-level modular curve $X(p^\infty)^\circ$ is exactly the locus of elliptic curves E with supersingular reduction, such that the formal group of E has no extra endomorphisms.

1. Main theorem

Let p be a prime number. Rapoport–Zink spaces [RZ96] are deformation spaces of p-divisible groups equipped with some extra structure. This article concerns the geometry of Rapoport–Zink spaces of EL type (endomorphisms + level structure). In particular we consider the infinite-level spaces $\mathscr{M}_{\mathcal{D},\infty}$, which are preperfectoid spaces [SW13]. An example is the space $\mathscr{M}_{H,\infty}$, where $H/\overline{\mathbf{F}}_p$ is a p-divisible group of height n. The points of $\mathscr{M}_{H,\infty}$ over a nonarchimedean field K containing $W(\overline{\mathbf{F}}_p)$ are in correspondence with isogeny classes of p-divisible groups $G/\mathcal{O}_K$ equipped with a quasi-isogeny $G \otimes_{\mathcal{O}_K} \mathcal{O}_K/p \to H \otimes_{\overline{\mathbf{F}}_p} \mathcal{O}_K/p$ and an isomorphism $\mathbf{Q}_p^n \cong VG$ (where VG is the rational Tate module).

The infinite-level space $\mathscr{M}_{\mathcal{D},\infty}$ appears as the limit of finite-level spaces, each of which is a smooth rigid-analytic space. We would like to investigate the question of smoothness for the space $\mathscr{M}_{\mathcal{D},\infty}$ itself, which is quite a different matter. We need the notion of cohomological smoothness [Sch17], which makes sense for general morphisms of analytic adic spaces, and which is reviewed in § 4. Roughly speaking, an adic space is cohomologically smooth over C (where $C/\mathbf{Q}_p$ is complete and algebraically closed) if it satisfies local Verdier duality. In particular, if U is a quasi-compact adic space which is cohomologically smooth over $\mathrm{Spa}(C, \mathcal{O}_C)$, then the cohomology group $H^i(U, \mathbf{F}_\ell)$ is finite for all i and all primes $\ell \neq p$.

Our main theorem shows that each connected component of the geometric fiber of $\mathscr{M}_{\mathcal{D},\infty}$ has a dense open subset which is cohomologically smooth.

THEOREM 1.0.1. *Let $\mathcal{D}$ be a basic EL datum (cf. § 2). Let C be a complete algebraically closed extension of the field of scalars of $\mathscr{M}_{\mathcal{D},\infty}$, and let $\mathscr{M}_{\mathcal{D},\infty}^{\circ}$ be a connected component of the base change $\mathscr{M}_{\mathcal{D},\infty,C}$. Let $\mathscr{M}_{\mathcal{D},\infty}^{\circ,\mathrm{non\text{-}sp}} \subset \mathscr{M}_{\mathcal{D},\infty}^{\circ}$ be the nonspecial locus (cf. § 3.5), corresponding to p-divisible groups without extra endomorphisms. Then $\mathscr{M}_{\mathcal{D},\infty}^{\circ,\mathrm{non\text{-}sp}}$ is cohomologically smooth over C.*

We remark that outside of trivial cases, $\pi_0(\mathscr{M}_{\mathcal{D},\infty,C})$ has no isolated points, which implies that no open subset of $\mathscr{M}_{\mathcal{D},\infty,C}$ can be cohomologically smooth. (Indeed, the H^0 of any quasi-compact open fails to be finitely generated.) Therefore it really is necessary to work with individual connected components of the geometric fiber of $\mathscr{M}_{\mathcal{D},\infty}$.

Theorem 1.0.1 is an application of the perfectoid version of the Jacobian criterion for smoothness, due to Fargues and Scholze [FS]; cf. Theorem 4.2.1. The latter theorem involves the Fargues–Fontaine curve X_C (reviewed in § 3). It asserts that a functor $\mathscr{M}$ on perfectoid spaces over $\mathrm{Spa}(C, \mathcal{O}_C)$ is cohomologically smooth, when $\mathscr{M}$ can be interpreted as global sections of a smooth morphism $Z \to X_C$, subject to a certain condition on the tangent bundle Tan_{Z/X_C}.

In our application to Rapoport–Zink spaces, we construct a smooth morphism $Z \to X_C$, whose moduli space of global sections is isomorphic to $\mathscr{M}_{\mathcal{D},\infty}^{\circ}$ (Lemma 5.2.1). Next, we show that a geometric point $x \in \mathscr{M}_{\mathcal{D},\infty}^{\circ}(C)$ lies in $\mathscr{M}_{\mathcal{D},\infty}^{\circ,\mathrm{non\text{-}sp}}(C)$ if and only if the corresponding section $s\colon X_C \to Z$ satisfies the condition that all slopes of the vector bundle $s^* \mathrm{Tan}_{Z/X_C}$ on X_C are positive (Theorem 5.5.1). This is exactly the condition on Tan_{Z/X_C} required by Theorem 4.2.1, so we can conclude that $\mathscr{M}_{\mathcal{D},\infty}^{\circ}$ is cohomologically smooth.

The geometry of Rapoport–Zink spaces is related to the geometry of Shimura varieties. As an example, consider the tower of classical modular curves $X(p^{\infty})$, considered as rigid spaces over C. There is a perfectoid space $X(p^{\infty})$ over C for which $X(p^{\infty}) \sim \varprojlim_n X(p^n)$, and a Hodge–Tate period map $\pi_{HT}\colon X(p^{\infty}) \to \mathbf{P}_C^1$ [Sch15], which is $\mathrm{GL}_2(\mathbf{Q}_p)$-equivariant. Let $X(p^{\infty})^{\circ} \subset X(p^{\infty})$ be a connected component.

COROLLARY 1.0.2. *The following are equivalent for a C-point x of $X(p^{\infty})^{\circ}$.*

 (i) *The point x corresponds to an elliptic curve E, such that the p-divisible group $E[p^{\infty}]$ has $\mathrm{End}\, E[p^{\infty}] = \mathbf{Z}_p$.*
 (ii) *The stabilizer of $\pi_{HT}(x)$ in $\mathrm{PGL}_2(\mathbf{Q}_p)$ is trivial.*
 (iii) *There is a neighborhood of x in $X(p^{\infty})^{\circ}$ which is cohomologically smooth over C.*

2. Review of Rapoport–Zink spaces at infinite level

2.1 The infinite-level Rapoport–Zink space $\mathscr{M}_{H,\infty}$

Let k be a perfect field of characteristic p, and let H be a p-divisible group of height n and dimension d over k. We review here the definition of the infinite-level Rapoport–Zink space associated with H.

First there is the formal scheme $\mathscr{M}_H$ over $\mathrm{Spf}\, W(k)$ parametrizing deformations of H up to isogeny, as in [RZ96]. For a $W(k)$-algebra R in which p is nilpotent, $\mathscr{M}_H(R)$ is the set of isomorphism classes of pairs (G, ρ), where G/R is a p-divisible group and $\rho\colon H \otimes_k R/p \to G \otimes_R R/p$ is a quasi-isogeny.

The formal scheme $\mathscr{M}_H$ locally admits a finitely generated ideal of definition. Therefore it makes sense to pass to its adic space $\mathscr{M}_H^{\mathrm{ad}}$, which has generic fiber $(\mathscr{M}_H^{\mathrm{ad}})_\eta$, a rigid-analytic space over $\mathrm{Spa}(W(k)[1/p], W(k))$. Then $(\mathscr{M}_H^{\mathrm{ad}})_\eta$ has the following moduli interpretation: it is the

sheafification of the functor assigning to a complete affinoid $(W(k)[1/p], W(k))$-algebra (R, R^+) the set of pairs (G, ρ), where G is a p-divisible group defined over an open and bounded subring $R_0 \subset R^+$, and $\rho \colon H \otimes_k R_0/p \to G \otimes_{R_0} R_0/p$ is a quasi-isogeny. There is an action of $\operatorname{Aut} H$ on $\mathscr{M}_H^{\mathrm{ad}}$ obtained by composition with ρ.

Given such a pair (G, ρ), Grothendieck–Messing theory produces a surjection $M(H) \otimes_{W(k)} R \to \operatorname{Lie} G[1/p]$ of locally free R-modules, where $M(H)$ is the covariant Dieudonné module. There is a Grothendieck–Messing period map $\pi_{GM} \colon (\mathscr{M}_H^{\mathrm{ad}})_\eta \to \mathcal{F}\ell$, where $\mathcal{F}\ell$ is the rigid-analytic space parametrizing rank d locally free quotients of $M(H)[1/p]$. The morphism π_{GM} is equivariant for the action of $\operatorname{Aut} H$. It has open image $\mathcal{F}\ell^a$ (the admissible locus).

We obtain a tower of rigid-analytic spaces over $(\mathscr{M}_H^{\mathrm{ad}})_\eta$ by adding level structures. For a complete affinoid $(W(k)[1/p], W(k))$-algebra (R, R^+), and an element of $(\mathscr{M}_H^{\mathrm{ad}})_\eta(R, R^+)$ represented locally on $\operatorname{Spa}(R, R^+)$ by a pair (G, ρ) as above, we have the Tate module $TG = \varprojlim_m G[p^m]$, considered as an adic space over $\operatorname{Spa}(R, R^+)$ with the structure of a $\mathbf{Z}_p$-module [SW13, (3.3)]. Finite-level spaces $\mathscr{M}_{H,m}$ are obtained by trivializing the $G[p^m]$; these are finite étale covers of $(\mathscr{M}_H^{\mathrm{ad}})_\eta$. The infinite-level space is obtained by trivializing all of TG at once, as in the following definition.

DEFINITION 2.1.1 [SW13, Definition 6.3.3]. Let $\mathscr{M}_{H,\infty}$ be the functor which sends a complete affinoid $(W(k)[1/p], W(k))$-algebra (R, R^+) to the set of triples (G, ρ, α), where (G, ρ) is an element of $(\mathscr{M}_H)_\eta^{\mathrm{ad}}(R, R^+)$, and $\alpha \colon \mathbf{Z}_p^n \to TG$ is a $\mathbf{Z}_p$-linear map which is an isomorphism pointwise on $\operatorname{Spa}(R, R^+)$.

There is an equivalent definition in terms of *isogeny* classes of triples (G, ρ, α), where this time $\alpha \colon \mathbf{Q}_p^n \to VG$ is a trivialization of the rational Tate module. Using this definition, it becomes clear that $\mathscr{M}_{H,\infty}$ admits an action of the product $\operatorname{GL}_n(\mathbf{Q}_p) \times \operatorname{Aut}^0 H$, where Aut^0 means automorphisms in the isogeny category. Then the period map $\pi_{GM} \colon \mathscr{M}_{H,\infty} \to \mathcal{F}\ell$ is equivariant for $\operatorname{GL}_n(\mathbf{Q}_p) \times \operatorname{Aut}^0 H$, where $\operatorname{GL}_n(\mathbf{Q}_p)$ acts trivially on $\mathcal{F}\ell$.

We remark that $\mathscr{M}_{H,\infty} \sim \varprojlim_m \mathscr{M}_{H,m}$ in the sense of [SW13, Definition 2.4.1].

One of the main theorems of [SW13] is the following.

THEOREM 2.1.2. *The adic space $\mathscr{M}_{H,\infty}$ is a preperfectoid space.*

This means that for any perfectoid field K containing $W(k)$, the base change $\mathscr{M}_{H,\infty} \times_{\operatorname{Spa}(W(k)[1/p], W(k))} \operatorname{Spa}(K, \mathcal{O}_K)$ becomes perfectoid after p-adically completing.

We sketch here the proof of Theorem 2.1.2. Consider the 'universal cover' $\tilde{H} = \varprojlim_p H$ as a sheaf of $\mathbf{Q}_p$-vector spaces on the category of k-algebras. This has a canonical lift to the category of $W(k)$-algebras [SW13, Proposition 3.1.3(ii)], which we continue to call $\tilde{H}$. The adic generic fiber $\tilde{H}_\eta^{\mathrm{ad}}$ is a preperfectoid space, as can be checked 'by hand': it is a product of the d-dimensional preperfectoid open ball $(\operatorname{Spa} W(k)[\![T_1^{1/p^\infty}, \ldots, T_d^{1/p^\infty}]\!])_\eta$ by the constant adic space $VH^{\text{ét}}$, where $H^{\text{ét}}$ is the étale part of H. Given a triple (G, ρ, α) representing an element of $\mathscr{M}_{H,\infty}(R, R^+)$, the quasi-isogeny ρ induces an isomorphism $\tilde{H}_\eta^{\mathrm{ad}} \times_{\operatorname{Spa}(W(k)[1/p], W(k))} \operatorname{Spa}(R, R^+) \to \tilde{G}_\eta^{\mathrm{ad}}$; composing this with α gives a morphism $\mathbf{Q}_p^n \to \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)$. We have therefore described a morphism $\mathscr{M}_{H,\infty} \to (\tilde{H}_\eta^{\mathrm{ad}})^n$.

Theorem 2.1.2 follows from the fact that the morphism $\mathscr{M}_{H,\infty} \to (\tilde{H}^{\mathrm{ad}})_\eta^n$ presents $\mathscr{M}_{H,\infty}$ as an open subset of a Zariski closed subset of $(\tilde{H}^{\mathrm{ad}})_\eta^n$. We conclude this subsection by spelling

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out how this is done. We have a *quasi-logarithm* map $\mathrm{qlog}_H \colon \tilde{H}_\eta^{\mathrm{ad}} \to M(H)[1/p] \otimes_{W(k)[1/p]} \mathbf{G}_a$ [SW13, Definition 3.2.3], a $\mathbf{Q}_p$-linear morphism of adic spaces over $\mathrm{Spa}(W(k)[1/p], W(k))$.

Now suppose (G, ρ) is a deformation of H to (R, R^+). The logarithm map on G fits into an exact sequence of $\mathbf{Z}_p$-modules:

$$0 \to G_\eta^{\mathrm{ad}}[p^\infty](R, R^+) \to G_\eta^{\mathrm{ad}}(R, R^+) \to \mathrm{Lie}\, G[1/p].$$

After taking projective limits along multiplication-by-p, this turns into an exact sequence of $\mathbf{Q}_p$-vector spaces,

$$0 \to VG(R, R^+) \to \tilde{G}_\eta^{\mathrm{ad}}(R, R^+) \to \mathrm{Lie}\, G[1/p].$$

On the other hand, we have a commutative diagram.

$$
\begin{array}{ccc}
\tilde{H}_\eta(R, R^+) & \xrightarrow{\;\cong\;} & \tilde{G}_\eta(R, R^+) \\
{\scriptstyle \mathrm{qlog}_H}\Big\downarrow & & \Big\downarrow{\scriptstyle \log_G} \\
M(H) \otimes_{W(k)} R & \longrightarrow & \mathrm{Lie}\, G[1/p]
\end{array}
$$

The lower horizontal map $M(H) \otimes_{W(k)} R \to \mathrm{Lie}\, G[1/p]$ is the quotient by the R-submodule of $M(H) \otimes_{W(k)} R$ generated by the image of $VG(R, R^+) \to \tilde{G}_\eta^{\mathrm{ad}}(R, R^+) \cong \tilde{H}_\eta^{\mathrm{ad}}(R, R^+) \to M(H) \otimes_{W(k)} R$.

Now suppose we have a point of $\mathscr{M}_{H,\infty}(R, R^+)$ represented by a triple (G, ρ, α). Then we have a $\mathbf{Q}_p$-linear map $\mathbf{Q}_p^n \to \tilde{H}_\eta^{\mathrm{ad}}(R, R^+) \to M(H) \otimes_{W(k)} R$. The cokernel of its R-extension $R^n \to M(H) \otimes_{W(k)} R$ is a projective R-module of rank d, namely $\mathrm{Lie}\, G[1/p]$. This condition on the cokernel allows us to formulate an alternate description of $\mathscr{M}_{H,\infty}$ which is independent of deformations.

PROPOSITION 2.1.3. *The adic space $\mathscr{M}_{H,\infty}$ is isomorphic to the functor which assigns to a complete affinoid $(W(k)[1/p], W(k))$-algebra (R, R^+) the set of n-tuples $(s_1, \ldots, s_n) \in \tilde{H}_\eta^{\mathrm{ad}}(R, R^+)^n$ such that the following conditions are satisfied.*

(i) *The quotient of $M(H) \otimes_{W(k)} R$ by the R-span of the $\mathrm{qlog}(s_i)$ is a projective R-module W of rank d.*
(ii) *For all geometric points $\mathrm{Spa}(C, \mathcal{O}_C) \to \mathrm{Spa}(R, R^+)$, the sequence*

$$0 \to \mathbf{Q}_p^n \xrightarrow{(s_1, \ldots, s_n)} \tilde{H}_\eta^{\mathrm{ad}}(C, \mathcal{O}_C) \to W \otimes_R C \to 0$$

is exact.

2.2 Infinite-level Rapoport–Zink spaces of EL type

This article treats the more general class of Rapoport–Zink spaces of EL type. We review these here.

DEFINITION 2.2.1. Let k be an algebraically closed field of characteristic p. A *rational EL datum* is a quadruple $\mathcal{D} = (B, V, H, \mu)$, where:

– B is a semisimple $\mathbf{Q}_p$-algebra;
– V is a finite B-module;

- H is an object of the isogeny category of p-divisible groups over k, equipped with an action $B \to \operatorname{End} H$;
- μ is a conjugacy class of $\overline{\mathbf{Q}}_p$-rational cocharacters $\mathbf{G}_m \to \mathbf{G}$, where $\mathbf{G}/\mathbf{Q}_p$ is the algebraic group $\operatorname{GL}_B(V)$.

These are subject to the following conditions.

- If $M(H)$ is the (rational) Dieudonné module of H, then there exists an isomorphism $M(H) \cong V \otimes_{\mathbf{Q}_p} W(k)[1/p]$ of $B \otimes_{\mathbf{Q}_p} W(k)[1/p]$-modules. In particular $\dim V = \operatorname{ht} H$.
- In the weight decomposition of $V \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p \cong \bigoplus_{i \in \mathbf{Z}} V_i$ determined by μ, only weights 0 and 1 appear, and $\dim V_0 = \dim H$.

The *reflex field* E of $\mathcal{D}$ is the field of definition of the conjugacy class μ. We remark that the weight filtration (but not necessarily the weight decomposition) of $V \otimes_{\mathbf{Q}_p} \overline{\mathbf{Q}}_p$ may be descended to E, and so we will be viewing V_0 and V_1 as $B \otimes_{\mathbf{Q}_p} E$-modules.

The infinite-level Rapoport–Zink space $\mathscr{M}_{\mathcal{D},\infty}$ is defined in [SW13] in terms of moduli of deformations of the p-divisible group H along with its B-action. It admits an alternate description along the lines of Proposition 2.1.3.

PROPOSITION 2.2.2 [SW13, Theorem 6.5.4]. *Let* $\mathcal{D} = (B, V, H, \mu)$ *be a rational EL datum. Let* $\check{E} = E \cdot W(k)$. *Then* $\mathscr{M}_{\mathcal{D},\infty}$ *is isomorphic to the functor which inputs a complete affinoid* $(\check{E}, \mathcal{O}_{\check{E}})$-*algebra* (R, R^+) *and outputs the set of B-linear maps*

$$s \colon V \to \tilde{H}_\eta^{\mathrm{ad}}(R, R^+),$$

subject to the following conditions.

- *Let W be the quotient*

$$V \otimes_{\mathbf{Q}_p} R \xrightarrow{\operatorname{qlog}_H \circ s} M(H) \otimes_{W(k)} R \to W \to 0.$$

  Then W is a finite projective R-module, which locally on R is isomorphic to $V_0 \otimes_E R$ as a $B \otimes_{\mathbf{Q}_p} R$-module.
- *For any geometric point $x = \operatorname{Spa}(C, \mathcal{O}_C) \to \operatorname{Spa}(R, R^+)$, the sequence of B-modules*

$$0 \to V \to \tilde{H}(\mathcal{O}_C) \to W \otimes_R C \to 0$$

  is exact.

If $\mathcal{D} = (\mathbf{Q}_p, \mathbf{Q}_p^n, H, \mu)$, where H has height n and dimension d and $\mu(t) = (t^{\oplus d}, 1^{\oplus(n-d)})$, then $E = \mathbf{Q}_p$ and $\mathscr{M}_{\mathcal{D},\infty} = \mathscr{M}_{H,\infty}$.

In general, we call $\check{E}$ the field of scalars of $\mathscr{M}_{\mathcal{D},\infty}$, and for a complete algebraically closed extension C of $\check{E}$, we write $\mathscr{M}_{\mathcal{D},\infty,C} = \mathscr{M}_{\mathcal{D},\infty} \times_{\operatorname{Spa}(\check{E}, \mathcal{O}_{\check{E}})} \operatorname{Spa}(C, \mathcal{O}_C)$ for the corresponding geometric fiber of $\mathscr{M}_{\mathcal{D},\infty}$.

The space $\mathscr{M}_{\mathcal{D},\infty}$ admits an action by the product group $\mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where $J/\mathbf{Q}_p$ is the algebraic group $\operatorname{Aut}_B^\circ(H)$. A pair $(\alpha, \alpha') \in \mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$ sends s to $\alpha' \circ s \circ \alpha^{-1}$.

There is once again a Grothendieck–Messing period map $\pi_{GM} \colon \mathscr{M}_{\mathcal{D},\infty} \to \mathcal{F}\ell_\mu$ onto the rigid-analytic variety whose (R, R^+)-points parametrize $B \otimes_{\mathbf{Q}_p} R$-module quotients of $M(H) \otimes_{W(k)} R$ which are projective over R, and which are of type μ in the sense that they are (locally on R) isomorphic to $V_0 \otimes_E R$. The morphism π_{GM} sends an (R, R^+)-point of $\mathscr{M}_{\mathcal{D},\infty}$ to the quotient

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W of $M(H) \otimes_{W(k)} R$ as above. It is equivariant for the action of $\mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where $\mathbf{G}(\mathbf{Q}_p)$ acts trivially on $\mathcal{F}\ell_\mu$. In terms of deformations of the p-divisible group H, the period map π_{GM} sends a deformation G to $\mathrm{Lie}\, G$.

There is also a Hodge–Tate period map $\pi_{HT} \colon \mathscr{M}_{\mathcal{D},\infty} \to \mathcal{F}\ell'_\mu$, where $\mathcal{F}\ell'_\mu(R, R^+)$ parametrizes $B \otimes_{\mathbf{Q}_p} R$-module quotients of $V \otimes_{\mathbf{Q}_p} R$ which are projective over R, and which are (locally on R) isomorphic to $V_1 \otimes_E R$. The morphism π_{HT} sends an (R, R^+)-point of $\mathscr{M}_{\mathcal{D},\infty}$ to the image of $V \otimes_{\mathbf{Q}_p} R \to M(H) \otimes_{W(k)} R$. It is equivariant for the action of $\mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$, where this time $J(\mathbf{Q}_p)$ acts trivially on $\mathcal{F}\ell'_\mu(R, R^+)$. In terms of deformations of the p-divisible group H, the period map π_{HT} sends a deformation G to $(\mathrm{Lie}\, G^\vee)^\vee$.

3. The Fargues–Fontaine curve

3.1 Review of the curve

We briefly review here some constructions and results from [FF18]. First we review the absolute curve, and then we cover the version of the curve which works in families.

Fix a perfectoid field F of characteristic p, with $F^\circ \subset F$ its ring of integral elements. Let $\varpi \in F^\circ$ be a pseudo-uniformizer for F, and let k be the residue field of F. Let $W(F^\circ)$ be the ring of Witt vectors, which we equip with the $(p, [\varpi])$-adic topology. Let $\mathcal{Y}_F = \mathrm{Spa}(W(F^\circ), W(F^\circ)) \setminus \{|p[\varpi]| = 0\}$. Then $\mathcal{Y}_F$ is an analytic adic space over $\mathbf{Q}_p$. The Frobenius automorphism of F induces an automorphism ϕ of $\mathcal{Y}_F$. Let $B_F = H^0(\mathcal{Y}_F, \mathcal{O}_{\mathcal{Y}_F})$, a $\mathbf{Q}_p$-algebra endowed with an action of ϕ. Let P_F be the graded ring $P_F = \bigoplus_{n \geq 0} B_F^{\phi = p^n}$. Finally, the Fargues–Fontaine curve is $X_F = \mathrm{Proj}\, P_F$. It is shown in [FF18] that X_F is the union of spectra of Dedekind rings, which justifies the use of the word 'curve' to describe X_F. Note however that there is no 'structure morphism' $X_F \to \mathrm{Spec}\, F$.

If $x \in X_F$ is a closed point, then the residue field of x is a perfectoid field F_x containing $\mathbf{Q}_p$ which comes equipped with an inclusion $i \colon F \hookrightarrow F_x^\flat$, which presents $F_x^\flat$ as a finite extension of F. Such a pair (F_x, i) is called an untilt of F. Then $x \mapsto (F_x, i)$ is a bijection between closed points of X_F and isomorphism classes of untilts of F, modulo the action of Frobenius on i. Thus if $F = E^\flat$ is the tilt of a given perfectoid field $E/\mathbf{Q}_p$, then $X_{E^\flat}$ has a canonical closed point ∞, corresponding to the untilt E of $E^\flat$.

An important result in [FF18] is the classification of vector bundles on X_F. (By a vector bundle on X_F we are referring to a locally free $\mathcal{O}_{X_F}$-module $\mathcal{E}$ of finite rank. We will use the notation $V(\mathcal{E})$ to mean the corresponding geometric vector bundle over X_F, whose sections correspond to sections of $\mathcal{E}$.) Recall that an *isocrystal* over k is a finite-dimensional vector space N over $W(k)[1/p]$ together with a Frobenius semilinear automorphism ϕ of N. Given N, we have the graded P_F-module $\bigoplus_{n \geq 0} (N \otimes_{W(k)[1/p]} B_F)^{\phi = p^n}$, which corresponds to a vector bundle $\mathcal{E}_F(N)$ on X_F. Then the Harder–Narasimhan slopes of $\mathcal{E}_F(N)$ are negative to those of N. If F is algebraically closed, then every vector bundle on X_F is isomorphic to $\mathcal{E}_F(N)$ for some N.

It is straightforward to 'relativize' the above constructions. If $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over k, one can construct the adic space $\mathcal{Y}_S$, the ring B_S, the scheme X_S, and the vector bundles $\mathcal{E}_S(N)$ as above. Frobenius-equivalences classes of untilts of S correspond to effective Cartier divisors of X_S of degree 1.

In our applications, we will start with an affinoid perfectoid space S over $\mathbf{Q}_p$. We will write $X_S = X_{S^\flat}$, and we will use ∞ to refer to the canonical Cartier divisor of X_S corresponding to the untilt S of $S^\flat$. Thus if N is an isocrystal over k, and $S = \mathrm{Spa}(R, R^+)$ is an affinoid perfectoid space over $W(k)[1/p]$, then the fiber of $\mathcal{E}_S(N)$ over ∞ is $N \otimes_{W(k)[1/p]} R$.

Let $S = \mathrm{Spa}(R, R^+)$ be as above and let ∞ be the corresponding Cartier divisor. We denote the completion of the ring of functions on $\mathcal{Y}_S$ along ∞ by $B_{\mathrm{dR}}^+(R)$. It comes equipped with a surjective homomorphism $\theta \colon B_{\mathrm{dR}}^+(R) \to R$, whose kernel is a principal ideal $\ker(\theta) = (\xi)$.

3.2 Relation to p-divisible groups

Here we recall the relationships between p-divisible groups and global sections of vector bundles on the Fargues–Fontaine curve. Let us fix a perfect field k of characteristic p, and write $\mathrm{Perf}_{W(k)[1/p]}$ for the category of perfectoid spaces over $W(k)[1/p]$. Given a p-divisible group H over k with covariant isocrystal N, if H has slopes $s_1, \ldots, s_k \in \mathbb{Q}$, then N has the slopes $1 - s_1, \ldots, 1 - s_k$. For an object S in $\mathrm{Perf}_{W(k)[1/p]}$ we define the vector bundle $\mathcal{E}_S(H)$ on X_S by

$$\mathcal{E}_S(H) = \mathcal{E}_S(N) \otimes_{\mathcal{O}_{X_S}} \mathcal{O}_{X_S}(1).$$

Under this normalization, the Harder–Narasimhan slopes of $\mathcal{E}_S(H)$ are (pointwise on S) the same as the slopes of H.

Let us write $H^0(\mathcal{E}(H))$ for the sheafification of the functor on $\mathrm{Perf}_{W(k)[1/p]}$, which sends S to $H^0(X_S, \mathcal{E}_S(H))$.

PROPOSITION 3.2.1. *Let H be a p-divisible group over a perfect field k of characteristic p, with isocrystal N. There is an isomorphism $\tilde{H}_\eta^{\mathrm{ad}} \cong H^0(\mathcal{E}(H))$ of sheaves on $\mathrm{Perf}_{W(k)[1/p]}$ making the following diagram commute,*

where the morphism $H^0(\mathcal{E}(H)) \to N \otimes_{W(k)[1/p]} \mathbf{G}_a$ sends a global section of $\mathcal{E}(H)$ to its fiber at ∞.

Proof. Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over $W(k)[1/p]$. Then $\tilde{H}_\eta^{\mathrm{ad}}(R, R^+) \cong \tilde{H}(R^\circ) \cong \tilde{H}(R^\circ/p)$. Observe that $\tilde{H}(R^\circ/p) = \mathrm{Hom}_{R^\circ/p}(\mathbf{Q}_p/\mathbf{Z}_p, H)[1/p]$, where the Hom is taken in the category of p-divisible groups over R°/p. Recall the crystalline Dieudonné functor $G \mapsto M(G)$ from p-divisible groups to Dieudonné crystals [Mes72]. Since the base ring R°/p is semiperfect, the latter category is equivalent to the category of finite projective modules over Fontaine's period ring $A_{\mathrm{cris}}(R^\circ/p) = A_{\mathrm{cris}}(R^\circ)$, equipped with Frobenius and Verschiebung.

Now we apply [SW13, Theorem A]: since R°/p is f-semiperfect, the crystalline Dieudonné functor is fully faithful up to isogeny. Thus

$$\mathrm{Hom}_{R^\circ/p}(\mathbf{Q}_p/\mathbf{Z}_p, H)[1/p] \cong \mathrm{Hom}_{A_{\mathrm{cris}}(R^\circ), \phi}(M(\mathbf{Q}_p/\mathbf{Z}_p), M(H))[1/p],$$

where the latter Hom is in the category of modules over $A_{\mathrm{cris}}(R^\circ)$ equipped with Frobenius. Recall that $B_{\mathrm{cris}}^+(R^\circ) = A_{\mathrm{cris}}(R^\circ)[1/p]$. Since H arises via base change from k, we have $M(H)[1/p] = B_{\mathrm{cris}}^+(R^\circ) \otimes_{W(k)[1/p]} N$. For its part, $M(\mathbf{Q}_p/\mathbf{Z}_p)[1/p] = B_{\mathrm{cris}}^+(R^\circ)e$, for a basis element e on which Frobenius acts as p. Therefore

$$\tilde{H}(R^\circ) \cong (B_{\mathrm{cris}}^+(R^\circ) \otimes_{W(k)[1/p]} N)^{\phi = p}.$$

On the Fargues–Fontaine curve side, we have by definition $H^0(X_S, \mathcal{E}_S(H)) = (B_S \otimes_{W(k)[1/p]} N)^{\phi=p}$. The isomorphism between $(B_S \otimes_{W(k)[1/p]} N)^{\phi=p}$ and $(B_{\mathrm{cris}}^+(R^\circ) \otimes_{W(k)[1/p]} N)^{\phi=p}$ is discussed in [LB18, Remarque 6.6].

The commutativity of the diagram in the proposition is [SW13, Proposition 5.1.6(ii)], at least in the case that S is a geometric point, but this suffices to prove the general case. $\qquad\square$

With Proposition 3.2.1 we can reinterpret the infinite-level Rapoport Zink spaces as moduli spaces of *modifications* of vector bundles on the Fargues–Fontaine curve. First we do this for $\mathcal{M}_{H,\infty}$. In the following, we consider $\mathcal{M}_{H,\infty}$ as a sheaf on the category of perfectoid spaces over $W(k)[1/p]$.

PROPOSITION 3.2.2. *Let H be a p-divisible group of height n and dimension d over a perfect field k. Let N be the associated isocrystal over k. Then $\mathcal{M}_{H,\infty}$ is isomorphic to the functor which inputs an affinoid perfectoid space $S = \mathrm{Spa}(R, R^+)$ over $W(k)[1/p]$ and outputs the set of exact sequences*

$$0 \to \mathcal{O}_{X_S}^n \xrightarrow{s} \mathcal{E}_S(H) \to i_{\infty*}W \to 0, \qquad (3.2.1)$$

where $i_\infty \colon \mathrm{Spec}\, R \to X_S$ is the inclusion, and W is a projective $\mathcal{O}_S$-module quotient of $N \otimes_{W(k)[1/p]} \mathcal{O}_S$ of rank d.

Proof. We briefly describe this isomorphism on the level of points over $S = \mathrm{Spa}(R, R^+)$. Suppose that we are given a point of $\mathcal{M}_{H,\infty}(S)$, corresponding to a p-divisible group G over R°, together with a quasi-isogeny $\iota \colon H \otimes_k R^\circ/p \to G \otimes_{R^\circ} R^\circ/p$ and an isomorphism $\alpha \colon \mathbf{Q}_p^n \to VG$ of sheaves of $\mathbf{Q}_p$-vector spaces on S. The logarithm map on G fits into an exact sequence of sheaves of $\mathbf{Z}_p$-modules on S,

$$0 \to G_\eta^{\mathrm{ad}}[p^\infty] \to G_\eta^{\mathrm{ad}} \to \mathrm{Lie}\, G[1/p] \to 0.$$

After taking projective limits along multiplication-by-p, this turns into an exact sequence of sheaves of $\mathbf{Q}_p$-vector spaces on S,

$$0 \to VG \to \tilde{G}_\eta^{\mathrm{ad}} \to \mathrm{Lie}\, G[1/p] \to 0.$$

The quasi-isogeny induces an isomorphism $\tilde{H}_\eta^{\mathrm{ad}} \times_{\mathrm{Spa}\, W(k)[1/p]} S \cong \tilde{G}_\eta^{\mathrm{ad}}$; composing this with the level structure gives an injective map $\mathbf{Q}_p^n \to \tilde{H}_\eta^{\mathrm{ad}}(S)$, whose cokernel W is isomorphic to the projective R-module $\mathrm{Lie}\, G$ of rank d. In light of Theorem 3.2.1, the map $\mathbf{Q}_p^n \to \tilde{H}_\eta^{\mathrm{ad}}(S)$ corresponds to an $\mathcal{O}_{X_S}$-linear map $s \colon \mathcal{O}_{X_S}^n \to \mathcal{E}_S(H)$, which fits into the exact sequence in (3.2.1). $\qquad\square$

Similarly, we have a description of $\mathcal{M}_{\mathcal{D},\infty}$ in terms of modifications.

PROPOSITION 3.2.3. *Let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum. Then $\mathcal{M}_{\mathcal{D},\infty}$ is isomorphic to the functor which inputs an affinoid perfectoid space S over $\breve{E}$ and outputs the set of exact sequences of $B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S}$-modules*

$$0 \to V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \xrightarrow{s} \mathcal{E}_S(H) \to i_{\infty*}W \to 0,$$

where W is a finite projective $\mathcal{O}_S$-module, which is locally isomorphic to $V_0 \otimes_{\mathbf{Q}_p} \mathcal{O}_S$ as a $B \otimes_{\mathbf{Q}_p} \mathcal{O}_S$-module (using notation from Definition 2.2.1).

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3.3 The determinant morphism, and connected components

If we are given a rational EL datum $\mathcal{D}$, there is a *determinant morphism* $\det\colon \mathscr{M}_{\mathcal{D},\infty} \to \mathscr{M}_{\det \mathcal{D},\infty}$, which we review below. For an algebraically closed perfectoid field C containing $W(k)[1/p]$, the base change $\mathscr{M}_{\det \mathcal{D},\infty,C}$ is a locally profinite set of copies of $\operatorname{Spa} C$. For a point $\tau \in \mathscr{M}_{\det \mathcal{D},\infty}(C)$, let $\mathscr{M}_{\mathcal{D},\infty}^{\tau}$ be the fiber of $\mathscr{M}_{\mathcal{D},\infty} \to \mathscr{M}_{\det \mathcal{D},\infty}$ over τ. We will prove in §5 that each $\mathscr{M}_{\mathcal{D},\infty}^{\tau,\text{non-sp}}$ is cohomologically smooth if $\mathcal{D}$ is basic. This implies that $\pi_0(\mathscr{M}_{\mathcal{D},\infty}^{\tau,\text{non-sp}})$ is discrete, so that cohomogical smoothness of $\mathscr{M}_{\mathcal{D},\infty}^{\tau,\text{non-sp}}$ is inherited by each of its connected components. This is Theorem 1.0.1. In certain cases (for example Lubin–Tate space) it is known that $\mathscr{M}_{\mathcal{D},\infty}^{\tau}$ is already connected [Che14].

We first review the determinant morphism for the space $\mathscr{M}_{H,\infty}$, where H is a p-divisible group of height n and dimension d over a perfect field k of characteristic p. Let $\breve{E} = W(k)[1/p]$. For a perfectoid space $S = \operatorname{Spa}(R, R^+)$ over $\breve{E}$, we have the vector bundle $\mathcal{E}_S(H)$ and its determinant $\det \mathcal{E}_S(H)$, a line bundle of degree d. (This does not correspond to a p-divisible group 'det H' unless $d \leq 1$.) We define $\mathscr{M}_{\det H,\infty}(S)$ to be the set of morphisms $s\colon \mathcal{O}_{X_S} \to \det \mathcal{E}_S(H)$, such that the cokernel of s is a projective $B_{\mathrm{dR}}^+(R)/(\xi)^d$-module of rank 1, where (ξ) is the kernel of $B_{\mathrm{dR}}^+(R) \to R$. The morphism $\det\colon \mathscr{M}_{H,\infty} \to \mathscr{M}_{\det H,\infty}$ is simply $s \mapsto \det s$.

Regarding the structure of $\mathscr{M}_{\det H,\infty}$: we claim that for an algebraically closed perfectoid field $C/\breve{E}$, the set $\mathscr{M}_{\det H,\infty}(C)$ is a $\mathbf{Q}_p^\times$-torsor. Indeed, since the vector bundle $\mathcal{E}_C(H)$ has degree d, so does the line bundle $\det \mathcal{E}_C(H)$, so that $\det \mathcal{E}_C(H) \cong \mathcal{O}_{X_C}(d)$. A C-point of $\mathscr{M}_{\det H,\infty}$ is therefore a global section of $\mathcal{O}_{X_C}(d)$ with a zero of order d at ∞. In other words, it is a nonzero element of $\operatorname{Fil}^0 B_C^{\phi=p^d} \cong \mathbf{Q}_p(d)$.

For the general case, let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum. Let $F = Z(B)$ be the center of B. Then F is a semisimple commutative $\mathbf{Q}_p$-algebra; i.e., it is a product of fields. The idea is now to construct the determinant datum $\det \mathcal{D} = (F, \det_F V, \det_F H, \det_F \circ \mu)$, noting once again that there may not be a p-divisible group 'det$_F H$'. The determinant $\det_F V$ is a free F-module of rank 1. For a perfectoid space $S = \operatorname{Spa}(R, R^+)$ over $\breve{E}$, we have the $F \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S}$-module $\mathcal{E}_S(H)$ and its determinant $\det_F \mathcal{E}_S(H)$; the latter is a locally free $F \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S}$-module of rank 1. Let d be the degree of $\det_F \mathcal{E}_S(H)$, considered as a function on $\operatorname{Spec} F$. We define $\mathscr{M}_{\det \mathcal{D},\infty}(S)$ to be the set of F-linear morphisms $s\colon \det_F V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \to \det_F \mathcal{E}_S(H)$, such that the cokernel of s is (locally on $\operatorname{Spec} F$) a projective $B_{\mathrm{dR}}^+(R)/(\xi)^d$-module of rank 1. (We remark here that the $\det_F$ in $\det_F \circ \mu$ means the morphism from $\mathbf{G} = \operatorname{Aut}_B(V)$ to $\mathbf{G}^{\mathrm{ab}} = \operatorname{Aut}_F(\det_F V) = \operatorname{Res}_{F/\mathbf{Q}_p} \mathbf{G}_m$. If $\det_F \mu$ is a minuscule cocharacter, meaning that it is a vector of only 0s and 1s in the character group $X_*(\mathbf{G}^{\mathrm{ab}}) \cong \mathbf{Z}^{[F:\mathbf{Q}_p]}$), then $\det \mathcal{D}$ is an honest rational EL datum.) The morphism $\mathscr{M}_{\mathcal{D},\infty} \to \mathscr{M}_{\det \mathcal{D},\infty}$ sends a $B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S}$-linear map $s\colon V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \to \mathcal{E}_S(H)$ to the $F \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S}$-linear map $\det s\colon \det_F V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \to \det_F \mathcal{E}_S(H)$.

An argument similar to the above shows that for an algebraically closed perfectoid field $C/\breve{E}$, the set $\mathscr{M}_{\det \mathcal{D},\infty}(C)$ is an $F^\times$-torsor, equal to the set of F-bases for $F(d)$. Here the Tate twist is interpreted (locally on $\operatorname{Spec} F$) as the dth tensor power of the rational Tate module of the Lubin–Tate module for F.

3.4 Basic Rapoport–Zink spaces

The main theorem of this article concerns basic Rapoport–Zink spaces, so we recall some facts about these here.

Let H be a p-divisible group over a perfect field k of characteristic p. The space $\mathcal{M}_{H,\infty}$ is said to be basic when the p-divisible group H (or rather, its Dieudonné module $M(H)$) is isoclinic.

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This is equivalent to saying that the natural map

$$\operatorname{End}^{\circ} H \otimes_{\mathbf{Q}_p} W(k)[1/p] \to \operatorname{End}_{W(k)[1/p]} M(H)[1/p]$$

is an isomorphism, where on the right the endomorphisms are not required to commute with Frobenius.

More generally we have a notion of basicness for a rational EL datum (B, H, V, μ), referring to the following equivalent conditions.

– The $\mathbf{G}$-isocrystal ($\mathbf{G} = \operatorname{Aut}_B V$) associated to H is basic in the sense of Kottwitz [Kot85].
– The natural map

$$\operatorname{End}_B^{\circ}(H) \otimes_{\mathbf{Q}_p} W(k)[1/p] \to \operatorname{End}_{B \otimes_{\mathbf{Q}_p} W(k)[1/p]} M(H)[1/p]$$

is an isomorphism.
– Considered as an algebraic group over $\mathbf{Q}_p$, the automorphism group $J = \operatorname{Aut}_B^{\circ} H$ is an inner form of $\mathbf{G}$.
– Let $D' = \operatorname{End}_B^{\circ} H$. For any algebraically closed perfectoid field C containing $W(k)$, the map

$$D' \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \to \mathscr{E}\!nd_{(B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C})} \mathcal{E}_C(H)$$

is an isomorphism.

In brief, the duality theorem from [SW13] says the following. Given a basic EL datum $\mathcal{D}$, there is a dual datum $\check{\mathcal{D}}$, for which the roles of the groups $\mathbf{G}$ and J are reversed. There is a $\mathbf{G}(\mathbf{Q}_p) \times J(\mathbf{Q}_p)$-equivariant isomorphism $\mathscr{M}_{\mathcal{D},\infty} \cong \mathscr{M}_{\check{\mathcal{D}},\infty}$ which exchanges the roles of π_{GM} and π_{HT}.

3.5 The special locus

Let $\mathcal{D} = (B, V, H, \mu)$ be a basic rational EL datum relative to a perfect field k of characteristic p, with reflex field E. Let F be the center of B. Define F-algebras D and D' by

$$D = \operatorname{End}_B V,$$
$$D' = \operatorname{End}_B H.$$

Finally, let $\mathbf{G} = \operatorname{Aut}_B V$ and $J = \operatorname{Aut}_B H$, considered as algebraic groups over $\mathbf{Q}_p$. Then $\mathbf{G}$ and J both contain $\operatorname{Res}_{F/\mathbf{Q}_p} \mathbf{G}_m$.

Let C be an algebraically closed perfectoid field containing $\check{E}$, and let $x \in \mathscr{M}_{\mathcal{D},\infty}(C)$. Then x corresponds to a p-divisible group G over $\mathcal{O}_C$ with endomorphisms by B, and also it corresponds to a $B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}$-linear map $s \colon V \otimes_{\mathbf{Q}_p} \mathcal{O}_X \to \mathcal{E}_C(N)$ as in Proposition 3.2.3. Define $A_x = \operatorname{End}_B G$ (endomorphisms in the isogeny category). Then A_x is a semisimple F-algebra. In light of Proposition 3.2.3, an element of A_x is a pair (α, α'), where $\alpha \in \operatorname{End}_{B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}} V \otimes \mathcal{O}_{X_C} = \operatorname{End}_B V = D$ and $\alpha' \in \operatorname{End}_{B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}} \mathcal{E}_C(H) = D'$ (the last equality is due to basicness), such that $s \circ \alpha = \alpha' \circ s$. Thus we have

$$A_x \cong \{(\alpha, \alpha') \in D \times D' \mid s \circ \alpha = \alpha' \circ s\}.$$

LEMMA 3.5.1. *The following are equivalent.*

(i) *The F-algebra A_x strictly contains F.*
(ii) *The stabilizer of $\pi_{GM}(x) \in \mathcal{F}\ell_\mu(C)$ in $J(\mathbf{Q}_p)$ strictly contains $F^\times$.*
(iii) *The stabilizer of $\pi_{HT}(x) \in \mathcal{F}\ell'_\mu(C)$ in $\mathbf{G}(\mathbf{Q}_p)$ strictly contains $F^\times$.*

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Proof. As in Proposition 3.2.3, let $s\colon V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \xrightarrow{s} \mathcal{E}_S(H)$ be the modification corresponding to x.

Note that the condition (1) is equivalent to the existence of an invertible element $(\alpha, \alpha') \in A_x$ not contained in (the diagonally embedded) F. Also note that if one of α, α' lies in F, then so does the other, in which case they are equal.

Suppose $(\alpha, \alpha') \in A_x$ is invertible. The point $\pi_{GM}(x) \in \mathcal{F}\ell_\mu$ corresponds to the cokernel of the fiber of s at ∞. Since $\alpha' \circ s = s \circ \alpha$, the cokernels of $\alpha' \circ s$ and s are the same, which means exactly that $\alpha' \in J(\mathbf{Q}_p)$ stabilizes $\pi_{GM}(x)$. Thus (1) implies (2). Conversely, if there exists $\alpha' \in J(\mathbf{Q}_p)\backslash F^\times$ which stabilizes $\pi_{GM}(x)$, it means that the $B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}$-linear maps s and $\alpha' \circ s$ have the same cokernel, and therefore there exists $\alpha \in \mathrm{End}_{B \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}} V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} = D$ such that $s \circ \alpha = \alpha' \circ s$, and then $(\alpha, \alpha') \in A_x \backslash F^\times$. This shows that (2) implies (1).

The equivalence between (1) and (3) is proved similarly. $\square$

DEFINITION 3.5.2. The *special locus* in $\mathscr{M}_{\mathcal{D}, \infty}$ is the subset $\mathscr{M}_{\mathcal{D}, \infty}^{\mathrm{sp}}$ defined by the condition $A_x \neq F$. The *nonspecial locus* $\mathscr{M}_{\mathcal{D}, \infty}^{\mathrm{non-sp}}$ is the complement of the special locus.

The special locus is built out of 'smaller' Rapoport–Zink spaces, in the following sense. Let A be a semisimple F-algebra, equipped with two F-embeddings $A \to D$ and $A \to D'$, so that $A \otimes_F B$ acts on V and H. Also assume that a cocharacter in the conjugacy class μ factors through a cocharacter $\mu_0\colon \mathbf{G}_m \to \mathrm{Aut}_{A \otimes_F B} V$. Let $\mathcal{D}_0 = (A \otimes_F B, V, H, \mu_0)$. Then there is an evident morphism $\mathscr{M}_{\mathcal{D}_0, \infty} \to \mathscr{M}_{\mathcal{D}, \infty}$. The special locus $\mathscr{M}_{\mathcal{D}, \infty}^{\mathrm{sp}}$ is the union of the images of all the $\mathscr{M}_{\mathcal{D}_0, \infty}$, as A ranges through all semisimple F-subalgebras of $D \times D'$ strictly containing F.

4. Cohomological smoothness

Let Perf be the category of perfectoid spaces in characteristic p, with its pro-étale topology [Sch17, Definition 8.1]. For a prime $\ell \neq p$, there is a notion of ℓ-cohomological smoothness [Sch17, Definition 23.8]. We only need the notion for morphisms $f\colon Y' \to Y$ between sheaves on Perf which are separated and representable in locally spatial diamonds. If such an f is ℓ-cohomologically smooth, and Λ is an ℓ-power torsion ring, then the relative dualizing complex $Rf^!\Lambda$ is an invertible object in $D_{\text{ét}}(Y', \Lambda)$ (thus, it is v-locally isomorphic to $\Lambda[n]$ for some $n \in \mathbf{Z}$), and the natural transformation $Rf^!\Lambda \otimes f^* \to Rf^!$ of functors $D_{\text{ét}}(Y, \Lambda) \to D_{\text{ét}}(Y', \Lambda)$ is an equivalence [Sch17, Proposition 23.12]. In particular, if f is projection onto a point, and $Rf^!\Lambda \cong \Lambda[n]$, one derives a statement of Poincaré duality for Y':

$$R\mathrm{Hom}(R\Gamma_c(Y', \Lambda), \Lambda) \cong R\Gamma(Y', \Lambda)[n].$$

We will say that f is cohomologically smooth if it is ℓ-cohomologically smooth for all $\ell \neq p$. As an example, if $f\colon Y' \to Y$ is a separated smooth morphism of rigid-analytic spaces over $\mathbf{Q}_p$, then the associated morphism of diamonds $f^\diamond\colon (Y')^\diamond \to Y^\diamond$ is cohomologically smooth [Sch17, Proposition 24.3]. There are other examples where f does not arise from a finite-type map of adic spaces. For instance, if $\tilde{B}_C = \mathrm{Spa}\, C\langle T^{1/p^\infty}\rangle$ is the perfectoid closed ball over an algebraically closed perfectoid field C, then $\tilde{B}_C$ is cohomologically smooth over C.

If Y is a perfectoid space over an algebraically closed perfectoid field C, it seems quite difficult to detect whether Y is cohomologically smooth over C. We will review in § 4.2 a 'Jacobian criterion' from [FS] which applies to certain kinds of Y. But first we give a classical analogue of this criterion in the context of schemes.

4.1 The Jacobian criterion: classical setting

PROPOSITION 4.1.1. *Let X be a smooth projective curve over an algebraically closed field k. Let $Z \to X$ be a smooth morphism. Define $\mathscr{M}_Z$ to be the functor which inputs a k-scheme T and outputs the set of sections of $Z \to X$ over X_T, that is, the set of morphisms s making*

$$
\begin{array}{ccc}
 & & Z \\
 & \nearrow\!\!\!^{s} & \downarrow \\
X \times_k T & \longrightarrow & X
\end{array}
$$

commute, subject to the condition that, fiberwise on T, the vector bundle $s^* \mathrm{Tan}_{Z/X}$ has vanishing H^1. Then $\mathscr{M}_Z \to \mathrm{Spec}\, k$ is formally smooth.

Here $\mathrm{Tan}_{Z/X}$ is the tangent bundle, equal to the $\mathcal{O}_Z$-linear dual of the sheaf of differentials $\Omega_{Z/X}$, which is locally free of finite rank. Let $\pi\colon X \times_k T \to T$ be the projection. For $t \in T$, let X_t be the fiber of π over t, and let $s_t\colon X_t \to Z$ be the restriction of s to X_t. By proper base change, the fiber of $R^1\pi_* s^* \mathrm{Tan}_{Z/X}$ at $t \in T$ is $H^1(X_t, s_t^* \mathrm{Tan}_{Z/X})$. The condition about the vanishing of H^1 in the proposition is equivalent to $H^1(X_t, s_t^* \mathrm{Tan}_{Z/X}) = 0$ for each $t \in T$. By Nakayama's lemma, this condition is equivalent to $R^1\pi_* s^* \mathrm{Tan}_{Z/X} = 0$.

Proof. Suppose we are given a commutative diagram,

$$
\begin{array}{ccc}
T_0 & \longrightarrow & \mathscr{M}_Z \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathrm{Spec}\, k
\end{array}
\tag{4.1.1}
$$

where $T_0 \to T$ is a first-order thickening of affine schemes; thus T_0 is the vanishing locus of a square-zero ideal sheaf $I \subset \mathcal{O}_T$. Note that I becomes an $\mathcal{O}_{T_0}$-module.

The morphism $T_0 \to \mathscr{M}_Z$ in (4.1.1) corresponds to a section of $Z \to X$ over T_0. Thus there is a solid diagram.

$$
\begin{array}{ccc}
X \times_k T_0 & \xrightarrow{\;s_0\;} & Z \\
\downarrow & \nearrow^{s} & \downarrow \\
X \times_k T & \longrightarrow & X
\end{array}
\tag{4.1.2}
$$

We claim that there exists a dotted arrow making the diagram commute. Since $Z \to X$ is smooth, it is formally smooth, and therefore this arrow exists Zariski-locally on X. Let $\pi\colon X \times_k T \to T$ and $\pi_0\colon X \times_k T_0 \to T_0$ be the projections. Then $X \times_k T_0$ is the vanishing locus of the ideal sheaf $\pi^* I \subset \mathcal{O}_{X \times_k T}$. Note that sheaves of sets on $X \times_k T$ are equivalent to sheaves of sets on $X \times_k T_0$; under this equivalence, $\pi^* I$ and $\pi_0^* I$ correspond. By [Sta14, Remark 36.9.6], the set of such morphisms form a (Zariski) sheaf of sets on $X \times_k T$, which when viewed as a sheaf on $X \times_k T_0$ is a torsor for

$$
\mathscr{H}\!om_{\mathcal{O}_{X \times_k T_0}}(s_0^* \Omega_{Z/X}, \pi_0^* I) \cong s_0^* \mathrm{Tan}_{Z/X} \otimes \pi_0^* I.
$$

This torsor corresponds to class in

$$H^1(X \times_k T_0, s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I).$$

This H^1 is the limit of a spectral sequence with terms

$$H^p(T_0, R^q \pi_{0*}(s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I)).$$

But since T_0 is affine and $R^q \pi_{0*}(s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I)$ is quasi-coherent, the above terms vanish for all $p > 0$, and therefore

$$H^1(X \times_k T_0, s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I) \cong H^0(T_0, R^1 \pi_{0*}(s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I)).$$

Since $s_0^* \operatorname{Tan}_{Z/X}$ is locally free, we have $s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I \cong s_0^* \operatorname{Tan}_{Z/X} \otimes^{\mathbf{L}} \pi_{0*} I$, and we may apply the projection formula [Sta14, Lemma 35.21.1] to obtain

$$R\pi_{0*}(s_0^* \operatorname{Tan}_{Z/X} \otimes \pi_0^* I) \cong R\pi_{0*} s_0^* \operatorname{Tan}_{Z/X} \otimes^{\mathbf{L}} I.$$

Now we apply the hypothesis about vanishing of H^1, which implies that $R\pi_{0*} s_0^* \operatorname{Tan}_{Z/X}$ is quasi-isomorphic to the locally free sheaf $\pi_{0*} s_0^* \operatorname{Tan}_{Z/X}$ in degree 0. Therefore the complex displayed above has $H^1 = 0$.

Thus our torsor is trivial, and so a morphism $s \colon X \times_k T \to Z$ exists filling in (4.1.2). The final thing to check is that s corresponds to a morphism $T \to \mathscr{M}_Z$, i.e., that it satisfies the fiberwise $H^1 = 0$ condition. But this is automatic, since T_0 and T have the same schematic points. $\qquad\square$

In the setup of Proposition 4.1.1, let $s \colon X \times_k \mathscr{M}_Z \to Z$ be the universal section. That is, the pull-back of s along a morphism $T \to \mathscr{M}_Z$ is the section $X \times_k T \to Z$ to which this morphism corresponds. Let $\pi \colon X \times_k \mathscr{M}_Z \to \mathscr{M}_Z$ be the projection. By Proposition 4.1.1 $\mathscr{M}_Z \to \operatorname{Spec} k$ is formally smooth. There is an isomorphism

$$\pi_* s^* \operatorname{Tan}_{Z/X} \cong \operatorname{Tan}_{\mathscr{M}_Z / \operatorname{Spec} k}.$$

Indeed, the proof of Proposition 4.1.1 shows that $\pi_* s^* \operatorname{Tan}_{Z/X}$ has the same universal property with respect to first order deformations as $\operatorname{Tan}_{\mathscr{M}_Z / \operatorname{Spec} k}$.

The following example is of similar spirit as our main application of the perfectoid Jacobian criterion below.

Example 4.1.2. Let $X = \mathbf{P}^1$ over the algebraically closed field k. For $d \in \mathbf{Z}$, let

$$V_d = \underline{\operatorname{Spec}}_X \operatorname{Sym}_{\mathcal{O}_X}(\mathcal{O}(-d))$$

be the geometric vector bundle over X whose global sections are $\Gamma(X, \mathcal{O}(d))$. Fix integers n, d, $\delta > 0$ and let P be a homogeneous polynomial over k of degree δ in n variables. Then P defines a morphism $P \colon \prod_{i=1}^n V_d \to V_{d\delta}$, by sending sections $(s_i)_{i=1}^n$ of V_d to the section $P(s_1, \ldots, s_n)$ of $V_{d\delta}$. Fix a global section $f \colon X \to V_{d\delta}$ to the projection morphism and consider the pull-back of P along f.

$$
\begin{array}{ccccc}
Z & \hookrightarrow & P^{-1}(f) & \longrightarrow & X \\
& & \downarrow & & \downarrow f \quad \searrow^{\operatorname{id}_X} \\
& & \prod_{i=1}^n V_d & \xrightarrow{\ P\ } & V_{d\delta} \ \longrightarrow\ X
\end{array}
$$

Moreover, let Z be the smooth locus of $P^{-1}(f)$ over X. It is an open subset. The derivatives $\partial P/\partial x_i$ of P are homogeneous polynomials of degree $\delta - 1$ in n variables, hence can be regarded as functions $\prod_{i=1}^n V_d \to V_{d(\delta-1)}$. A point $y \in P^{-1}(f)$ lies in Z if and only if $(\partial P/\partial x_i)(y)$, $i = 1, \ldots, n$ are not all zero. We wish to apply Proposition 4.1.1 to Z/X. Let $\mathscr{M}'_Z$ denote the space of global sections of Z over X, that is for a k-scheme T, $\mathscr{M}'_Z(T)$ is the set of morphisms $s\colon X \times_k T \to Z$ as in the proposition (without any further conditions). A k-point $g \in \mathscr{M}'_Z(k)$ corresponds to a section $g\colon X \to \prod_{i=1}^n V_d$, satisfying $P \circ g = f$. In general, for a (geometric) vector bundle V on X with corresponding locally free $\mathcal{O}_X$-module $\mathscr{E}$, the pull-back of the tangent space $\mathrm{Tan}_{V/X}$ along a section $s\colon X \to V$ is canonically isomorphic to $\mathscr{E}$. Hence in our situation (using that $Z \subseteq P^{-1}(f)$ is open) the tangent space $g^*\mathrm{Tan}_{Z/X}$ can be computed from the short exact sequence,

$$0 \to g^*\mathrm{Tan}_{Z/X} \to \bigoplus_{i=1}^n \mathcal{O}(d) \xrightarrow{D_g P} \mathcal{O}(d\delta) \to 0,$$

where $D_g P$ is the derivative of P at g. It is the $\mathcal{O}_X$-linear map given by $(t_i)_{i=1}^n \mapsto \sum_{i=1}^n (\partial P/\partial x_i)(g)t_i$ (note that $(\partial P/\partial x_i)(g)$ are global sections of $\mathcal{O}(d(\delta - 1))$). Note that $D_g P$ is surjective: by Nakayama, it suffices to check this fiberwise, where it is true by the condition defining Z.

The space $\mathscr{M}_Z$ is the subfunctor of $\mathscr{M}'_Z$ consisting of all g such that (fiberwise) $g^*\mathrm{Tan}_{Z/X} = \ker(D_g P)$ has vanishing H^1. Writing $\ker(D_g P) = \bigoplus_{i=1}^r \mathcal{O}(m_i)$ $(m_i \in \mathbf{Z})$, this is equivalent to $m_i \geq -1$. By the Proposition 4.1.1 we conclude that $\mathscr{M}_Z$ is formally smooth over k.

Consider now a numerical example. Let $n = 3$, $d = 1$ and $\delta = 4$ and let $g \in \mathscr{M}'_Z(k)$. Then $D_g P \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}(1)^{\oplus 3}, \mathcal{O}(4)) = \Gamma(X, \mathcal{O}(3)^{\oplus 3})$, a 12-dimensional k-vector space, and moreover, $D_g P$ lies in the open subspace of surjective maps. We have the short exact sequence of $\mathcal{O}_X$-modules

$$0 \to g^*\mathrm{Tan}_{Z/X} \to \mathcal{O}(1)^{\oplus 3} \xrightarrow{D_g P} \mathcal{O}(4) \to 0. \tag{4.1.3}$$

This shows that $g^*\mathrm{Tan}_{Z/X}$ has rank 2 and degree -1. Moreover, being a subbundle of $\mathcal{O}(1)^{\oplus 3}$ it only can have slopes less than or equal to 1. There are only two options, either $g^*\mathrm{Tan}_{Z/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}$ or $g^*\mathrm{Tan}_{Z/X} \cong \mathcal{O}(-2) \oplus \mathcal{O}(1)$. The point g lies in $\mathscr{M}_Z$ if and only if the first option occurs for g. Which option occurs can be seen from the long exact cohomology sequence associated to (4.1.3):

$$0 \to \Gamma(X, g^*\mathrm{Tan}_{Z/X}) \to \underbrace{\Gamma(X, \mathcal{O}(1))^{\oplus 3}}_{\text{6-dim'l}} \xrightarrow{\Gamma(D_g P)} \underbrace{\Gamma(X, \mathcal{O}(4))}_{\text{5-dim'l}} \to \mathrm{H}^1(X, g^*\mathrm{Tan}_{Z/X}) \to 0.$$

It is clear that $\Gamma(X, g^*\mathrm{Tan}_{Z/X})$ is 1-dimensional if and only if $g^*\mathrm{Tan}_{Z/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}$ and 2-dimensional otherwise. The first option is generic, i.e., $\mathscr{M}_Z$ is an open subscheme of $\mathscr{M}'_Z$.

4.2 The Jacobian criterion: perfectoid setting

We present here the perfectoid version of Proposition 4.1.1

THEOREM 4.2.1 (Fargues and Scholze [FS]). *Let $S = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space in characteristic p. Let $Z \to X_S$ be a smooth morphism of schemes. Let $\mathscr{M}_Z^{\geq 0}$ be the functor which inputs a perfectoid space $T \to S$ and outputs the set of sections of $Z \to X_S$ over T,*

that is, the set of morphisms s making

commute, subject to the condition that, fiberwise on T, all Harder–Narasimhan slopes of the vector bundle $s^* \operatorname{Tan}_{Z/X_S}$ are positive. Then $\mathscr{M}_Z^{>0} \to S$ is a cohomologically smooth morphism of locally spatial diamonds.

Example 4.2.2. Let $S = \eta = \operatorname{Spa}(C, \mathcal{O}_C)$, where C is an algebraically closed perfectoid field of characteristic 0, and let $Z = \mathbf{V}(\mathcal{E}_S(H)) \to X_S$ be the geometric vector bundle attached to $\mathcal{E}_S(H)$, where H is a p-divisible group over the residue field of C. Then $\mathscr{M}_Z = H^0(\mathcal{E}_S(H))$ is isomorphic to $\tilde{H}_\eta^{\mathrm{ad}}$ by Proposition 3.2.1. Let $s\colon X_{\mathscr{M}_Z} \to Z$ be the universal morphism; then $s^* \operatorname{Tan}_{Z/X_S}$ is the constant Banach–Colmez space associated to H (i.e., the pull-back of $\mathcal{E}_S(H)$ along $X_{\mathscr{M}_Z} \to X_S$). This has vanishing H^1 if and only if H has no étale part. This is true if and only if $\mathscr{M}_Z^{>0}$ is isomorphic to a perfectoid open ball. The perfectoid open ball is cohomologically smooth, in accord with Theorem 4.2.1. In contrast, if the étale quotient $H^{\mathrm{ét}}$ has height $d > 0$, then $\pi_0(\tilde{H}_\eta^{\mathrm{ad}}) \cong \mathbf{Q}_p^d$ implies that $\tilde{H}_\eta^{\mathrm{ad}}$ is not cohomologically smooth.

In the setup of Theorem 4.2.1, suppose that $x = \operatorname{Spa}(C, \mathcal{O}_C) \to S$ is a geometric point, and that $x \to \mathscr{M}_Z^{>0}$ is an S-morphism, corresponding to a section $s\colon X_C \to Z$. Then $s^* \operatorname{Tan}_{Z/X_S}$ is a vector bundle on X_C. In light of the discussion in the previous section, we are tempted to interpret $H^0(X_C, s^* \operatorname{Tan}_{Z/X_S})$ as the 'tangent space of $\mathscr{M}_Z^{>0} \to S$ at x'. At points x where $s^* \operatorname{Tan}_{Z/X_S}$ has only positive Harder–Narasimhan slopes, this tangent space is a perfectoid open ball.

5. Proof of the main theorem

5.1 Dilatations and modifications

As preparation for the proof of Theorem 1.0.1, we review the notion of a dilatation of a scheme at a locally closed subscheme [BLR90, § 3.2].

Throughout this subsection, we fix some data. Let X be a curve, meaning that X is a scheme which is locally the spectrum of a Dedekind ring. Let $\infty \in X$ be a closed point with residue field C. Let $i_\infty \colon \operatorname{Spec} C \to X$ be the embedding, and let $\xi \in \mathcal{O}_{X,\infty}$ be a local uniformizer at ∞.

PROPOSITION 5.1.1. *Let $V \to X$ be a morphism of finite type, and let $Y \subset V_\infty$ be a locally closed subscheme of the fiber of V at ∞.*

There exists a morphism of X-schemes $V' \to V$ which is universal for the following property: $V' \to X$ is flat at ∞, and $V'_\infty \to V_\infty$ factors through $Y \subset V_\infty$.

The X-scheme V' is the *dilatation* of V at Y. We review here its construction.

First suppose that $Y \subset V_\infty$ is closed. Let $\mathscr{I} \subset \mathcal{O}_V$ be the ideal sheaf which cuts out Y. Let $B \to V$ be the blow-up of V along Y. Then $\mathscr{I} \cdot \mathcal{O}_B$ is a locally principal ideal sheaf. The dilatation V' of V at Y is the open subscheme of B obtained by imposing the condition that the ideal $(\mathscr{I} \cdot \mathcal{O}_B)_x \subset \mathcal{O}_{B,x}$ is generated by ξ at all $x \in B$ lying over ∞.

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We give here an explicit local description of the dilatation V'. Let $\operatorname{Spec} A$ be an affine neighborhood of ∞, such that $\xi \in A$, and let $\operatorname{Spec} R \subset V$ be an open subset lying over $\operatorname{Spec} A$. Let $I = (f_1, \ldots, f_n)$ be the restriction of $\mathscr{I}$ to $\operatorname{Spec} R$, so that I cuts out $Y \cap \operatorname{Spec} A$. Then the restriction of $V' \to V$ to $\operatorname{Spec} R$ is $\operatorname{Spec} R'$, where

$$R' = R\left[\frac{f_1}{\xi}, \ldots, \frac{f_n}{\xi}\right]\bigg/(\xi\text{-torsion}).$$

Now suppose $Y \subset V_\infty$ is only locally closed, so that Y is open in its closure $\overline{Y}$. Then the dilatation of V at Y is the dilatation of $V \backslash (\overline{Y} \backslash Y)$ at Y.

Note that a dilatation $V' \to V$ is an isomorphism away from ∞, and that it is affine.

Example 5.1.2. Let

$$0 \to \mathcal{E}' \to \mathcal{E} \to i_{\infty*}W \to 0$$

be an exact sequence of $\mathcal{O}_X$-modules, where $\mathcal{E}$ (and thus $\mathcal{E}'$) is locally free, and W is a C-vector space. (This is an elementary modification of the vector bundle $\mathcal{E}$.) Let $K = \ker(\mathcal{E}_\infty \to W)$.

Let $\mathbf{V}(\mathcal{E}) \to X$ be the geometric vector bundle corresponding to $\mathcal{E}$. Similarly, we have $\mathbf{V}(\mathcal{E}') \to X$, and an X-morphism $\mathbf{V}(\mathcal{E}') \to \mathbf{V}(\mathcal{E})$. Let $\mathbf{V}(K) \subset \mathbf{V}(\mathcal{E})_\infty$ be the affine space associated to $K \subset \mathcal{E}_\infty$. We claim that $\mathbf{V}(\mathcal{E}')$ is isomorphic to the dilatation $\mathbf{V}(\mathcal{E})'$ of $\mathbf{V}(\mathcal{E})$ at $\mathbf{V}(K)$. Indeed, by the universal property of dilatations, there is a morphism $\mathbf{V}(\mathcal{E}') \to \mathbf{V}(\mathcal{E})'$, which is an isomorphism away from ∞.

To see that $\mathbf{V}(\mathcal{E}') \to \mathbf{V}(\mathcal{E})'$ is an isomorphism, it suffices to work over $\mathcal{O}_{X,\infty}$. Over this base, we may give a basis $f_1, \ldots, f_n$ of global sections of $\mathcal{E}$, with $f_1, \ldots, f_k$ lifting a basis for $K \subset \mathcal{E}_\infty$. Then the localization of $\mathbf{V}(\mathcal{E})' \to \mathbf{V}(\mathcal{E})$ at ∞ is isomorphic to

$$\operatorname{Spec} \mathcal{O}_{X,\infty}\left[\frac{f_1}{\xi}, \ldots, \frac{f_k}{\xi}, f_{k+1}, \ldots, f_n\right] \to \operatorname{Spec} \mathcal{O}_{X,\infty}[f_1, \ldots, f_n].$$

This agrees with the localization of $\mathbf{V}(\mathcal{E}') \to \mathbf{V}(\mathcal{E})$ at ∞.

LEMMA 5.1.3. *Let $V \to X$ be a smooth morphism, let $Y \subset V_\infty$ be a smooth locally closed subscheme, and let $\pi \colon V' \to V$ be the dilatation of V at Y. Then $V' \to X$ is smooth, and $\operatorname{Tan}_{V'/X}$ lies in an exact sequence of $\mathcal{O}_{V'}$-modules*

$$0 \to \operatorname{Tan}_{V'/X} \to \pi^* \operatorname{Tan}_{V/X} \to \pi^* j_* N_{Y/V_\infty} \to 0, \tag{5.1.1}$$

where N_{Y/V_∞} is the normal bundle of $Y \subset V_\infty$, and $j \colon Y \to V$ is the inclusion.

Finally, let $T \to X$ be a morphism which is flat at ∞, and let $s \colon T \to V$ be a morphism of X-schemes, such that s_∞ factors through Y. By the universal property of dilatations, s factors through a morphism $s' \colon T \to V'$. Then we have an exact sequence of $\mathcal{O}_V$-modules

$$0 \to (s')^* \operatorname{Tan}_{V'/X} \to s^* \operatorname{Tan}_{V/X} \to i_{T_\infty*} s_\infty^* N_{Y/V_\infty} \to 0. \tag{5.1.2}$$

Proof. One reduces to the case that Y is closed in V_∞. The smoothness of $V' \to X$ is [BLR90, § 3.2, Proposition 3]. We turn to the exact sequence (5.1.1). The morphism $\operatorname{Tan}_{V'/X} \to \pi^* \operatorname{Tan}_{V/X}$ comes from functoriality of the tangent bundle. To construct the morphism $\pi^* \operatorname{Tan}_{V/X} \to$

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$\pi^* j_* N_{Y/V_\infty}$, we consider the diagram
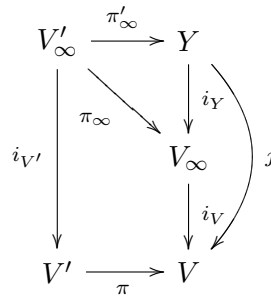
in which the outer rectangle is cartesian. For its part, the normal bundle N_{Y/V_∞} sits in an exact sequence of $\mathcal{O}_Y$-modules

$$0 \to \mathrm{Tan}_{Y/C} \to i_Y^* \, \mathrm{Tan}_{V_\infty/C} \to N_{Y/V_\infty} \to 0.$$

The composite

$$
\begin{aligned}
i_{V'}^* \pi^* \, \mathrm{Tan}_{V/X} &= \pi_\infty^* i_V^* \, \mathrm{Tan}_{V/X} \\
&\cong \pi_\infty^* \, \mathrm{Tan}_{V_\infty/C} \\
&= (\pi_\infty')^* i_Y^* \, \mathrm{Tan}_{V/C} \\
&\to (\pi_\infty')^* N_{Y/V_\infty}
\end{aligned}
$$

induces by adjunction a morphism

$$\pi^* \, \mathrm{Tan}_{V/X} \to i_{V'*}(\pi_\infty')^* N_{Y/V_\infty} \cong \pi^* j_* N_{Y/V_\infty},$$

where the last step is justified because j is a closed immersion.

We check that (5.1.1) is exact using our explicit description of V'. The sequence is clearly exact away from the preimage of Y in V', since on the complement of this locus, the morphism π is an isomorphism, and $\pi^* j_* = 0$. Therefore we let $y \in Y$ and check exactness after localization at y. Let $\mathcal{I} \subset \mathcal{O}_V$ be the ideal sheaf which cuts out Y, and let $I \subset \mathcal{O}_{V,y}$ be the localization of $\mathcal{I}$ at y. Then $\mathcal{O}_{V_\infty,y} = \mathcal{O}_{V,y}/\xi$. Since $Y \subset V_\infty$ are both smooth at y, we can find a system of local coordinates $\overline{f}_1, \dots, \overline{f}_n \in \mathcal{O}_{V_\infty,y}$ (meaning that the differentials $d\overline{f}_i$ form a basis for $\Omega^1_{V_\infty/C,y}$), such that $\overline{f}_{k+1}, \dots, \overline{f}_n$ generate I/ξ. If $\partial/\partial \overline{f}_i$ are the dual basis, then the stalk of N_{Y/V_∞} at y is the free $\mathcal{O}_{Y,y}$-module with basis $\partial/\partial \overline{f}_{k+1}, \dots, \partial/\partial \overline{f}_n$.

Choose lifts $f_i \in \mathcal{O}_{V,y}$ of the $\overline{f}_i$. Then I is generated by $\xi, f_k, \dots, f_n$. The localization of $V' \to V$ over y is $\mathrm{Spec}\, \mathcal{O}_{V',y}$, where $\mathcal{O}_{V',y} = \mathcal{O}_{V,y}[g_{k+1}, \dots, g_n]/(\xi\text{-torsion})$, where $\xi g_i = f_i$ for $i = k+1, \dots, n$. Then the stalk of $\mathrm{Tan}_{V'/X}$ at y is the free $\mathcal{O}_{V',y}$-module with basis $\partial/\partial f_1, \dots, \partial/\partial f_k, \partial/\partial g_{k+1}, \dots, \partial/\partial g_n$, whereas the stalk of $\pi^* \mathrm{Tan}_{V/X}$ at y is the free $\mathcal{O}_{V',y}$-module with basis $\partial/\partial f_1, \dots, \partial/\partial f_n$. The quotient between these stalks is evidently the free module over $\mathcal{O}_{V',y}/\xi$ with basis $\partial/\partial f_{k+1}, \dots, \partial/\partial f_n$, and this agrees with the stalk of $\pi^* j_* N_{Y/V_\infty}$.

Given a morphism of X-schemes $s \colon T \to V$ as in the lemma, we apply $(s')^*$ to (5.1.1); this is exact because s' is flat. The term on the right is $s^* j_* N_{Y/V_\infty} \cong i_{T_\infty*} s_\infty^* N_{Y/V_\infty}$ (once again, this is valid because j is a closed immersion). $\qquad\square$

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5.2 The space $\mathscr{M}_{H,\infty}$ as global sections of a scheme over X_C

We will prove Theorem 1.0.1 for the Rapoport–Zink spaces of the form $\mathscr{M}_{H,\infty}$ before proceeding to the general case. Let H be a p-divisible group of height n and dimension d over a perfect field k. In this context, $\breve{E} = W(k)[1/p]$. Let $\mathcal{E} = \mathcal{E}_C(H)$. Throughout, we will be interpreting $\mathscr{M}_{H,\infty}$ as a functor on $\mathrm{Perf}_{\breve{E}}$ as in Proposition 3.2.2.

We have a determinant morphism $\det: \mathscr{M}_{H,\infty} \to \mathscr{M}_{\det H,\infty}$. Let $\tau \in \mathscr{M}_{\det H,\infty}(C)$ be a geometric point of $\mathscr{M}_{\det H,\infty}$. This point corresponds to a section τ of $\mathbf{V}(\det \mathcal{E}) \to X_C$, which we also call τ. Let $\mathscr{M}_{H,\infty}^\tau$ be the fiber of $\det$ over τ.

Our first order of business is to express $\mathscr{M}_{H,\infty}^\tau$ as the space of global sections of a smooth morphism $Z \to X_C$, defined as follows. We have the geometric vector bundle $\mathbf{V}(\mathcal{E}^n) \to X$, whose global sections parametrize morphisms $s: \mathcal{O}_{X_C}^n \to \mathcal{E}$. Let U_{n-d} be the locally closed subscheme of the fiber of $\mathbf{V}(\mathcal{E}^n)$ over ∞, which parametrizes all morphisms of rank $n-d$. We consider the dilatation $\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty=n-d} \to \mathbf{V}(\mathcal{E}^n)$ of $\mathbf{V}(\mathcal{E}^n)$ along U_{n-d}. For any flat X_C-scheme T, $\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty=n-d}(T)$ is the set of all $s: \mathcal{O}_T^n \to \mathcal{E}_T$ such that $\mathrm{cok}(s) \otimes C$ is projective $\mathcal{O}_T \otimes C$-module of rank d. Define Z as the following cartesian product.

$$
\begin{array}{ccc}
Z & \longrightarrow & \mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty=n-d} \\
\downarrow & & \downarrow{\scriptstyle \det} \\
X_C & \xrightarrow{\ \tau\ } & \mathbf{V}(\det \mathcal{E})
\end{array}
\tag{5.2.1}
$$

LEMMA 5.2.1. *Let $\mathscr{M}_Z$ be the functor which inputs a perfectoid space T/C and outputs the set of sections of $Z \to X_C$ over X_T. Then $\mathscr{M}_Z$ is isomorphic to $\mathscr{M}_{H,\infty}^\tau$.*

Proof. Let $T = \mathrm{Spa}(R, R^+)$ be an affinoid perfectoid space over C. The morphism $X_T \to X_C$ is flat. (This can be checked locally: $B_{\mathrm{dR}}^+(R)$ is torsion-free over the discrete valuation ring $B_{\mathrm{dR}}^+(C)$, and so it is flat.) By the description in (5.2.1), an X_T-point of $\mathscr{M}_Z$ corresponds to a morphism $\sigma: \mathcal{O}_{X_T}^n \to \mathcal{E}_T(H)$ which has the following properties.

(1) The cokernel of σ_∞ is a projective R-module quotient of $\mathcal{E}_T(H)_\infty$ of rank d.
(2) The determinant of σ equals τ.

On the other hand, by Proposition 3.2.2, $\mathscr{M}_{H,\infty}(T)$ is the set of morphisms $\sigma: \mathcal{O}_{X_T}^n \to \mathcal{E}_T(H)$ satisfying the following.

(1′) The cokernel of σ is $i_{\infty*}W$, for a projective R-module quotient W of $\mathcal{E}_T(H)_\infty$ of rank d.
(2) The determinant of σ equals τ.

We claim the two sets of conditions are equivalent for a morphism $\sigma: \mathcal{O}_{X_T}^n \to \mathcal{E}_T(H)$. Clearly (1′) implies (1), so that (1′) and (2) together imply (1) and (2) together. Conversely, suppose (1) and (2) hold. Since τ represents a point of $\mathscr{M}_{\det H,\infty}$, it is an isomorphism outside of ∞, and therefore so is σ. This means that $\mathrm{cok}\,\sigma$ is supported at ∞. Thus $\mathrm{cok}\,\sigma$ is a $B_{\mathrm{dR}}^+(R)$-module. For degree reasons, the length of $(\mathrm{cok}\,\sigma) \otimes_{B_{\mathrm{dR}}^+(R)} B_{\mathrm{dR}}^+(C')$ has length d for every geometric point $\mathrm{Spa}(C', (C')^+) \to T$. Whereas condition (1) says that $(\mathrm{cok}\,\sigma) \otimes_{B_{\mathrm{dR}}^+(R)} R$ is a projective R-module of rank d. This shows that $(\mathrm{cok}\,\sigma)$ is already a projective R-module of rank d, which is condition (1′). $\qquad\square$

LEMMA 5.2.2. *The morphism $Z \to X_C$ is smooth.*

Proof. Let $\infty' \in X_C$ be a closed point, with residue field C'. It suffices to show that the stalk of Z at ∞' is smooth over $\operatorname{Spec} B_{\mathrm{dR}}^+(C')$.

If $\infty' \neq \infty$, then this stalk is isomorphic to the variety $(\mathbf{A}^{n^2})^{\det=\tau}$ consisting of $n \times n$ matrices with fixed determinant τ. Since τ is invertible in $B_{\mathrm{dR}}^+(C')$, this variety is smooth.

Now suppose $\infty' = \infty$. Let ξ be a generator for the kernel of $B_{\mathrm{dR}}^+(C) \to C$. Then the stalk of Z at ∞ is isomorphic to the flat $B_{\mathrm{dR}}^+(C)$-scheme Y, whose T-points for a flat $B_{\mathrm{dR}}^+(C)$-scheme T are $n \times n$ matrices with coefficients in $\Gamma(T, \mathcal{O}_T)$, which are rank $n-d$ modulo ξ, and which have fixed determinant τ (which must equal $u\xi^d$ for a unit $u \in B_{\mathrm{dR}}^+(C)$). Consider the open subset $Y_0 \subset Y$ consisting of matrices M where the first $(n-d)$ columns have rank $(n-d)$. Then the final d columns of M are congruent modulo ξ to a linear combination of the first $(n-d)$ columns. After row reduction operations only depending on those first $(n-d)$ columns, M becomes

$$\left(\begin{array}{c|c} I_{n-d} & P \\ \hline 0 & \xi Q \end{array} \right),$$

with $\det Q = w$ for a unit $w \in B_{\mathrm{dR}}^+(C)$ which only depends on the first $(n-d)$ columns of M. We therefore have a fibration $Y_0 \to \mathbf{A}^{n(n-d)}$, namely projection onto the first $(n-d)$ columns, whose fibers are $\mathbf{A}^{d(n-d)} \times (\mathbf{A}^{d^2})^{\det=w}$, which is smooth. Therefore Y_0 is smooth. The variety Y is covered by opens isomorphic to Y_0, and so it is smooth. $\square$

We intend to apply Theorem 4.2.1 to the morphism $Z \to X$, and so we need some preparations regarding the relative tangent space of $\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}_\infty = n-d} \to X_C$.

5.3 A linear algebra lemma

Let $f \colon V' \to V$ be a rank r linear map between n-dimensional vector spaces over a field C. Thus there is an exact sequence

$$0 \to W' \to V' \xrightarrow{f} V \xrightarrow{q} W \to 0,$$

with $\dim W = \dim W' = n - r$.

Consider the minor map $\Lambda \colon \operatorname{Hom}(V', V) \to \operatorname{Hom}(\bigwedge^{r+1} V', \bigwedge^{r+1} V)$ given by $\sigma \mapsto \bigwedge^{r+1} \sigma$. This is a polynomial map, whose derivative at f is a linear map

$$D_f \Lambda \colon \operatorname{Hom}(V', V) \to \operatorname{Hom}\left(\bigwedge^{r+1} V', \bigwedge^{r+1} V \right).$$

Explicitly, this map is

$$D_f \Lambda(\sigma)(v_1 \wedge \cdots \wedge v_{r+1}) = \sum_{i=1}^{r+1} f(v_1) \wedge f(v_2) \wedge \cdots \wedge \sigma(v_i) \wedge \cdots \wedge f(v_{r+1}). \tag{5.3.1}$$

LEMMA 5.3.1. *Let*

$$K = \ker(\operatorname{Hom}(V', V) \to \operatorname{Hom}(W', W))$$

be the kernel of the map $\sigma \mapsto q \circ (\sigma|_{W'})$. Then $\ker D_f \Lambda = K$.

Proof. Suppose $\sigma \in K$. Since f has rank r, the exterior power $\bigwedge^{r+1} V'$ is spanned over C by elements of the form $v_1 \wedge \cdots \wedge v_{r+1}$, where $v_{r+1} \in \ker f = W'$. Since $f(v_{r+1}) = 0$, the sum in

(5.3.1) reduces to

$$D_f\Lambda(\sigma)(v_1 \wedge \cdots \wedge v_{r+1}) = f(v_1) \wedge \cdots \wedge f(v_r) \wedge \sigma(v_{r+1}).$$

Since $\sigma \in K$ and $v_{r+1} \in W'$ we have $\sigma(v_{r+1}) \in \ker q = f(V')$, which means that

$$D_f\Lambda(\sigma)(v_1, \ldots, v_{r+1}) \in \bigwedge^{r+1} f(V') = 0.$$

Thus $\sigma \in \ker D_f\Lambda$.

Now suppose $\sigma \in \ker D_f\Lambda$. Let $w \in W'$. We wish to show that $\sigma(w) \in f(V')$. Let $v_1, \ldots, v_r \in V'$ be vectors for which $f(v_1), \ldots, f(v_r)$ is a basis for $f(V')$. Since $\sigma \in \ker D_f\Lambda$, we have $D_f\Lambda(\sigma)(v_1 \wedge \cdots \wedge v_r \wedge w) = 0$. On the other hand,

$$D_f\Lambda(\sigma)(v_1 \wedge \cdots \wedge v_r \wedge w) = f(v_1) \wedge \cdots \wedge f(v_r) \wedge \sigma(w),$$

because all other terms in the sum in (5.3.1) are 0, owing to $f(w) = 0$. Since the wedge product above is 0, and the $f(v_i)$ are a basis for $f(V')$, we must have $\sigma(w) \in f(V')$. Thus $\sigma \in K$. $\qquad\square$

We interpret Lemma 5.3.1 as the calculation of a certain normal bundle. Let $Y = \mathbf{V}(\mathrm{Hom}(V', V))$ be the affine space over C representing morphisms $V' \to V$ over a C-scheme, and let $j \colon Y^{\mathrm{rk}=r} \to Y$ be the locally closed subscheme representing morphisms which are everywhere of rank r. Thus, $Y^{\mathrm{rk}=r}$ is an open subset of the fiber over 0 of (the geometric version of) the minor map Λ. It is well known that $Y^{\mathrm{rk}=r}/C$ is smooth of codimension $(n-r)^2$ in Y/C, and so the normal bundle $N_{Y^{\mathrm{rk}=r}/Y}$ is locally free of this rank.

We have a universal morphism of $\mathcal{O}_{Y^{\mathrm{rk}=r}}$-modules $\sigma \colon \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C V' \to \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C V$. Let $\mathcal{W}' = \ker\sigma$ and $\mathcal{W} = \mathrm{cok}\,\sigma$, so that $\mathcal{W}'$ and $\mathcal{W}$ are locally free $\mathcal{O}_{Y^{\mathrm{rk}=r}}$-modules of rank $n - r$. We also have the $\mathcal{O}_{Y^{\mathrm{rk}=r}}$-linear morphism $D\Lambda \colon \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C \mathrm{Hom}(V', V) \to \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C \mathrm{Hom}(\Lambda^{r+1}V', \Lambda^{r+1}V)$, whose kernel is precisely $\mathrm{Tan}_{Y^{\mathrm{rk}=r}/C}$. The geometric interpretation of Lemma 5.3.1 is a commutative diagram with short exact rows.

$$\begin{array}{ccccc}
\ker D\Lambda & \longrightarrow & \mathcal{O}_{Y^{\mathrm{rk}=r}} \otimes_C \mathrm{Hom}(V', V) & \longrightarrow & \mathscr{H}\!om(\mathcal{W}', \mathcal{W}) \\
\cong \big\downarrow & & \cong \big\downarrow & & \big\downarrow \cong \\
\mathrm{Tan}_{Y^{\mathrm{rk}=r}/C} & \longrightarrow & j^*\mathrm{Tan}_{Y/C} & \longrightarrow & N_{Y^{\mathrm{rk}=r}/Y}
\end{array} \qquad (5.3.2)$$

5.4 Moduli of morphisms of vector bundles with fixed rank at ∞

We return to the setup of §5.1. We have a curve X and a closed point $\infty \in X$, with inclusion map i_∞ and residue field C.

Let $\mathcal{E}$ and $\mathcal{E}'$ be rank n vector bundles over X, with fibers $V = \mathcal{E}_\infty$ and $V' = \mathcal{E}'_\infty$. We have the geometric vector bundle $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E})) \to X$. If $f \colon T \to X$ is a morphism, then T-points of $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))$ classify $\mathcal{O}_T$-linear maps $f^*\mathcal{E}' \to f^*\mathcal{E}$.

Let $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))^{\mathrm{rk}_\infty=r}$ be the dilatation of $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))$ at the locally closed subscheme $\mathbf{V}(\mathrm{Hom}(V', V))^{\mathrm{rk}=r}$ of the fiber $\mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))_\infty = \mathbf{V}(\mathrm{Hom}(V', V))$. This has the following property, for a flat morphism $f \colon T \to X$: the X-morphisms $s \colon T \to \mathbf{V}(\mathscr{H}\!om(\mathcal{E}', \mathcal{E}))^{\mathrm{rk}_\infty=r}$ parametrize those $\mathcal{O}_T$-linear maps $\sigma \colon f^*\mathcal{E}' \to f^*\mathcal{E}$, for which the fiber $\sigma_\infty \colon f_\infty^*V' \to f_\infty^*V$ has rank r everywhere on T_∞.

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Given a morphism s as above, corresponding to a morphism $\sigma\colon f^*\mathcal{E}' \to f^*\mathcal{E}$, we let $\mathcal{W}'$ and $\mathcal{W}$ denote the kernel and cokernel of σ_∞. Then $\mathcal{W}'$ and $\mathcal{W}$ are locally free $\mathcal{O}_{T_\infty}$-modules of rank r. Let $i_{T_\infty}\colon T_\infty \to T$ denote the pull-back of i_∞ through f.

We intend to use Lemma 5.1.3 to compute $s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}\!om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X}$. The tangent bundle $\mathrm{Tan}_{\mathbf{V}(\mathscr{H}\!om(\mathcal{E}',\mathcal{E}))/X}$ is isomorphic to the pull-back $f^*\mathscr{H}\!om(\mathcal{E}',\mathcal{E})$. Also, we have identified the normal bundle $N_{\mathbf{V}(\mathrm{Hom}(V',V))^{\mathrm{rk}=r}/\mathbf{V}(\mathrm{Hom}(V',V))}$ in (5.3.2). So when we apply the lemma to this situation, we obtain an exact sequence of $\mathcal{O}_T$-modules

$$0 \to s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}\!om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X} \to f^*\mathscr{H}\!om(\mathcal{E}',\mathcal{E}) \to i_{T_\infty*}\mathscr{H}\!om(\mathcal{W}',\mathcal{W}) \to 0, \qquad (5.4.1)$$

where the third arrow is adjoint to the map

$$i_{T_\infty}^* f^*\mathscr{H}\!om(\mathcal{E}',\mathcal{E}) = \mathrm{Hom}(f_\infty^*V', f_\infty^*V) \to \mathscr{H}\!om(\mathcal{W}',\mathcal{W}),$$

which sends $\sigma \in \mathscr{H}\!om(f_\infty^*V', f_\infty^*V)$ to the composite

$$\mathcal{W}' \to f_\infty^*V' \overset{\sigma_\infty}{\to} f_\infty^*V \to \mathcal{W}.$$

The short exact sequence in (5.4.1) identifies the $\mathcal{O}_T$-module $s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}\!om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X}$ as a modification of $f^*\mathscr{H}\!om(\mathcal{E}',\mathcal{E})$ at the divisor T_∞. We can say a little more in the case that σ itself is a modification. Let us assume that σ fits into an exact sequence

$$0 \to f^*\mathcal{E}' \overset{\sigma}{\to} f^*\mathcal{E} \overset{\alpha}{\to} i_{T_\infty*}\mathcal{W} \to 0.$$

Dualizing gives another exact sequence

$$0 \to f^*(\mathcal{E}^\vee) \overset{\sigma^\vee}{\to} f^*(\mathcal{E}')^\vee \overset{\alpha'}{\to} i_{T_\infty*}(\mathcal{W}')^\vee \to 0.$$

Then

$$s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}\!om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X} = \ker[f^*\mathscr{H}\!om(\mathcal{E}',\mathcal{E}) \to i_{T_\infty*}\mathscr{H}\!om(\mathcal{W}',\mathcal{W})]$$
$$\cong \ker(\alpha \otimes \alpha').$$

The kernel of $\alpha \otimes \alpha'$ can be computed in terms of $\ker\alpha = f^*\mathcal{E}'$ and $\ker\alpha' = f^*(\mathcal{E}^\vee)$, see Lemma 5.4.1 below. It sits in a diagram.

$$(5.4.2)$$

$$
\begin{array}{ccccccccc}
& & 0 & & & & & & \\
& & \downarrow & & & & & & \\
& & f^*\mathscr{H}\!om(\mathcal{E},\mathcal{E}') & & & & & & \\
& & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{F} & \longrightarrow & f^*\mathscr{H}\!om(\mathcal{E},\mathcal{E}) \oplus f^*\mathscr{H}\!om(\mathcal{E}',\mathcal{E}') & \longrightarrow & s^*\mathrm{Tan}_{\mathbf{V}(\mathscr{H}\!om(\mathcal{E}',\mathcal{E}))^{\mathrm{rk}_\infty=r}/X} & \longrightarrow & 0 \\
& & \downarrow & & & & & & \\
& & \mathrm{Tor}_1(i_{\infty*}\mathcal{W}', i_{\infty*}\mathcal{W}) & & & & & & \\
& & \downarrow & & & & & & \\
& & 0 & & & & & &
\end{array}
$$

Lemma 5.4.1. *Let $\mathcal{A}$ be an abelian $\otimes$-category. Let*

$$0 \to K \xrightarrow{i} A \xrightarrow{f} B \to 0,$$

$$0 \to K' \xrightarrow{i'} A' \xrightarrow{f'} B' \to 0$$

be two exact sequences in $\mathcal{A}$, with A, A', K, K' projective. The homology of the complex

$$K \otimes K' \xrightarrow{(i \otimes 1_{K'}, 1_K \otimes i')} (A \otimes K') \oplus (K \otimes A) \xrightarrow{1_A \otimes i' - i \otimes 1_{A'}} A \otimes A'$$

is given by $H_2 = 0$, $H_1 \cong \mathrm{Tor}_1(B, B')$, and $H_0 \cong B \otimes B'$. Thus, $K'' = \ker(f \otimes f' \colon A \otimes A' \to B \otimes B')$ appears in a diagram

$$
\begin{array}{ccccccc}
& & 0 & & & & \\
& & \downarrow & & & & \\
& & K \otimes K' & & & & \\
& & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & (A \otimes K') \oplus (K \otimes A) & \longrightarrow & K' & \longrightarrow & 0 \\
& & \downarrow & & & & \\
& & \mathrm{Tor}_1(B, B') & & & & \\
& & \downarrow & & & & \\
& & 0 & & & &
\end{array}
$$

where both sequences are exact.

Proof. Let $C_\bullet$ be the complex $K \to A$, and let $C'_\bullet$ be the complex $K' \to A'$. Since $C'_\bullet$ is a projective resolution of B', we have a Tor spectral sequence [Sta14, Tag 061Z]

$$E^2_{i,j} \colon \ \mathrm{Tor}_j(H_i(C_\bullet), B') \implies H_{i+j}(C_\bullet \otimes C'_\bullet).$$

We have $E^2_{0,0} = B \otimes B'$ and $E^2_{0,1} = \mathrm{Tor}_1(B, B')$, and $E^2_{i,j} = 0$ for all other (i, j). Therefore $H_0(C_\bullet \otimes C'_\bullet) \cong B \otimes B'$ and $H_1(C_\bullet \otimes C'_\bullet) \cong \mathrm{Tor}_1(B, B')$, which is the lemma. $\qquad\square$

5.5 A tangent space calculation

We return to the setup of § 5.2. Thus we have fixed a p-divisible group H over a perfect field k, and an algebraically closed perfectoid field C containing $W(k)[1/p]$. But now we specialize to the case that H is isoclinic. Therefore $D = \mathrm{End}\, H$ (up to isogeny) is a central simple $\mathbf{Q}_p$-algebra. Let $\mathcal{E} = \mathcal{E}_C(H)$; we have $\mathscr{H}\!om(\mathcal{E}, \mathcal{E}) \cong D \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}$.

Recall the scheme $Z \to X_C$, defined as a fiber product in (5.2.1). Let $s \colon X_C \to Z$ be a section. This corresponds to a morphism $\sigma \colon \mathcal{O}^n_{X_C} \to \mathcal{E}$. Let W' and W be the cokernel of σ_∞; these are C-vector spaces.

We are interested in the vector bundle $s^*\mathrm{Tan}_{Z/X_C}$. This is the kernel of the derivative of the determinant map:

$$s^*\mathrm{Tan}_{Z/X_C} = \ker(D_s \det \colon s^*\mathrm{Tan}_{\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}\infty = n-d}/X_C} \to \det \mathcal{E}).$$

We apply (5.4.2) to give a description of $s^*\mathrm{Tan}_{\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}\infty = n-d}/X_C}$. We get a diagram of $\mathcal{O}_{X_C}$-modules.

$$
\begin{array}{c}
0 \\
\downarrow \\
(\mathcal{E}^\vee)^n \\
\downarrow \\
0 \longrightarrow \mathcal{F} \longrightarrow (M_n(\mathbf{Q}_p) \times D) \otimes \mathcal{O}_{X_C} \longrightarrow s^*\mathrm{Tan}_{\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}\infty = n-d}/X_C} \longrightarrow 0. \quad (5.5.1) \\
\downarrow \\
\mathrm{Tor}_1(i_{\infty*}W', i_{\infty*}W) \\
\downarrow \\
0
\end{array}
$$

On the other hand, the horizontal exact sequence fits into a diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F} & \longrightarrow & (M_n(\mathbf{Q}_p) \times D) \otimes \mathcal{O}_{X_C} & \longrightarrow & s^*\mathrm{Tan}_{\mathbf{V}(\mathcal{E}^n)^{\mathrm{rk}\infty \leqslant n-d}/X_C} & \longrightarrow & 0 \\
& & & & \downarrow{\scriptstyle \mathrm{tr}} & & \downarrow{\scriptstyle D_s \det} & & \\
& & & & \mathcal{O}_{X_C} & \xrightarrow{\quad \tau \quad} & \det \mathcal{E} & &
\end{array} \quad (5.5.2)
$$

The arrow labeled tr is induced from the $\mathbf{Q}_p$-linear map $M_n(\mathbf{Q}_p) \times D \to \mathbf{Q}_p$ carrying (α', α) to $\mathrm{tr}(\alpha') - \mathrm{tr}(\alpha)$ (reduced trace on D). The commutativity of the lower right square boils down to the identity, valid for sections $s_1, \ldots, s_n \in H^0(X_C, \mathcal{E})$ and $\alpha \in D$:

$$((\alpha s_1) \wedge s_2 \wedge \cdots \wedge s_n) + \cdots + (s_1 \wedge \cdots \wedge (\alpha s_n)) = (\mathrm{tr}\,\alpha)(s_1 \wedge \cdots \wedge s_n).$$

(There is a similar identity for $\alpha' \in M_n(\mathbf{Q}_p)$.) Because the arrow labeled τ is injective, we can combine (5.5.1) and (5.5.2) to arrive at a description of $s^*\mathrm{Tan}_{Z/X_C}$.

$$
\begin{array}{c}
0 \\
\downarrow \\
(\mathcal{E}^\vee)^n \\
\downarrow \\
0 \longrightarrow \mathcal{F} \longrightarrow (M_n(\mathbf{Q}_p) \times D)^{\mathrm{tr}=0} \otimes \mathcal{O}_X \longrightarrow s^*\mathrm{Tan}_{Z/X_C} \longrightarrow 0 \quad (5.5.3) \\
\downarrow \\
\mathrm{Tor}_1(i_{\infty*}W', i_{\infty*}W) \\
\downarrow \\
0
\end{array}
$$

We pass to duals to obtain the following.

$$
\begin{array}{ccccccccc}
& & & & & & 0 & & \\
& & & & & & \downarrow & & \\
0 \longrightarrow (s^*\mathrm{Tan}_{Z/X_C})^\vee \longrightarrow ((M_n(\mathbf{Q}_p) \times D)/\mathbf{Q}_p) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} & \longrightarrow & \mathcal{F}^\vee & \longrightarrow & 0 \\
& & & & & & \downarrow & & \\
& & & & & & \mathcal{E}^n & & \\
& & & & & & \downarrow & & \\
& & & & \mathrm{Tor}_1(i_{\infty*}((W')^\vee, i_{\infty*}W^\vee) & & & & \\
& & & & & & \downarrow & & \\
& & & & & & 0 & &
\end{array}
\tag{5.5.4}
$$

The dotted arrow is induced from the map $(M_n(\mathbf{Q}_p) \times D) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \to \mathcal{E}^n$ sending $(\alpha', \alpha) \otimes 1$ to $\alpha \circ \sigma - \sigma \circ \alpha'$.

THEOREM 5.5.1. *If s is a section to $Z \to X_C$ corresponding, under the isomorphism of Lemma 5.2.1, to a point $x \in \mathcal{M}_{H,\infty}^\tau(C)$, then the following are equivalent.*

(i) *The vector bundle $s^*\mathrm{Tan}_{Z/X_C}$ has a Harder–Narasimhan slope which is less than or equal to 0.*

(ii) *The point x lies in the special locus $\mathscr{M}_{H,\infty}^{\tau,\mathrm{sp}}$.*

Proof. Let $\sigma\colon \mathcal{O}_{X_C}^n \to \mathcal{E}$ denote the homomorphism corresponding to x. Condition (1) is true if and only if $H^0(X_C, s^*\mathrm{Tan}_{Z/X_C}^\vee) \neq 0$. We now take H^0 of (5.5.4), noting that $H^0(X_C, \mathcal{F}^\vee) \to H^0(X_C, \mathcal{E}^n)$ is injective. We find that

$$
H^0(X_C, s^*\mathrm{Tan}_{Z/X_C}^\vee) \cong \{(\alpha', \alpha) \in M_n(\mathbf{Q}_p) \times D \mid \alpha \circ \sigma = \sigma \circ \alpha'\}/\mathbf{Q}_p
$$
$$
= A_x/\mathbf{Q}_p.
$$

This is nonzero exactly when x lies in the special locus. $\qquad\square$

Combining Theorem 5.5.1 with the criterion for cohomological smoothness in Theorem 4.2.1 proves Theorem 1.0.1 for the space $\mathscr{M}_{H,\infty}$.

Naturally we wonder whether it is possible to give a complete description of $s^*\mathrm{Tan}_{Z/X_C}$, as this is the 'tangent space' of $\mathscr{M}_{H,\infty}^\tau$ at the point x. Note that $s^*\mathrm{Tan}_{Z/X_C}$ can only have nonnegative slopes, since it is a quotient of a trivial bundle. Therefore Theorem 5.5.1 says that 0 appears as a slope of $s^*\mathrm{Tan}_{Z/X_C}$ if and only if s corresponds to a special point of $\mathscr{M}_{H,\infty}^\tau$.

Example 5.5.2. Consider the case that H has dimension 1 and height n, so that $\mathscr{M}_{H,\infty}$ is an infinite-level Lubin–Tate space. Suppose that $x \in \mathscr{M}_{H,\infty}(C)$ corresponds to a section $s\colon X_C \to Z$. Then $s^*\mathrm{Tan}_{Z/X_C}$ is a vector bundle of rank $n^2 - 1$ and degree $n - 1$, with slopes lying in $[0, 1/n]$; this already limits the possibilities for the slopes to a finite list.

If $n = 2$ there are only two possibilities for the slopes appearing in $s^*\mathrm{Tan}_{Z/X_C}$: $\{1/3\}$ and $\{0, 1/2\}$. These correspond exactly to the nonspecial and special loci, respectively.

If $n = 3$, there are a priori five possibilities for the slopes appearing in $s^*\mathrm{Tan}_{Z/X_C}$: $\{1/4, 1/4\}$, $\{1/3, 1/5\}$, $\{1/3, 1/3, 0, 0\}$, $\{2/7, 0\}$, and $\{1/3, 1/4, 0\}$. But in fact the final two cases cannot occur: if 0 appears as a slope, then x lies in the special locus, so that $A_x \neq \mathbf{Q}_p$. But as A_x is

isomorphic to a subalgebra of $\mathrm{End}^{\circ} H$, the division algebra of invariant $1/3$, it must be the case that $\dim_{\mathbf{Q}_p} A_x = 3$, which forces 0 to appear as a slope with multiplicity $\dim_{\mathbf{Q}_p} A_x/\mathbf{Q}_p = 2$. On the nonspecial locus, we suspect that the generic (semistable) case $\{1/4, 1/4\}$ always occurs, as otherwise there would be some unexpected stratification of $\mathscr{M}_{H,\infty}^{\circ,\mathrm{non\text{-}sp}}$. But currently we do not know how to rule out the case $\{1/3, 1/5\}$.

5.6 The general case

Let $\mathcal{D} = (B, V, H, \mu)$ be a rational EL datum over k, with reflex field E. Let F be the center of B. As in §3.5, let $D = \mathrm{End}_B V$ and $D' = \mathrm{End}_B H$, so that D and D' are both F-algebras.

Let C be a perfectoid field containing $\breve{E}$, and let $\tau \in \mathscr{M}_{\det \mathcal{D}, \infty}(C)$. Let $\mathscr{M}_{\mathcal{D},\infty}^{\tau}$ be the fiber of the determinant map over τ. We will sketch the proof that $\mathscr{M}_{\mathcal{D},\infty}^{\tau} \to \mathrm{Spa}\, C$ is cohomologically smooth. It is along the same lines as the proof for $\mathscr{M}_{H,\infty}$, but with some extra linear algebra added.

The space $\mathscr{M}_{\mathcal{D},\infty}^{\tau}$ may be expressed as the space of global sections of a smooth morphism $Z \to X_C$, defined as follows. We have the geometric vector bundle $\mathbf{V}(\mathscr{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_X, \mathcal{E}_C(H)))$. In its fiber over ∞, we have the locally closed subscheme whose R-points for a C-algebra R are morphisms, whose cokernel is as a $B \otimes_{\mathbf{Q}_p} R$-module isomorphic to $V_0 \otimes_{\breve{E}} R$, where V_0 is the weight 0 subspace of $V \otimes_{\mathbf{Q}_p} \breve{E}$ determined by μ. We then have the dilatation $\mathbf{V}(\mathscr{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)))^{\mu}$ of $\mathbf{V}(\mathscr{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_X, \mathcal{E}_C(H)))$ at this locally closed subscheme. Its points over $S = \mathrm{Spa}(R, R^+)$ parametrize B-linear morphisms $s\colon V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_S} \to \mathcal{E}_S(H)$, such that (locally on S) the cokernel of the fiber s_∞ is isomorphic as a $(B \otimes_{\mathbf{Q}_p} R)$-module to $V_0 \otimes_{\breve{E}} R$. Finally, the morphism $Z \to X_C$ is defined by the following cartesian diagram.

$$
\begin{array}{ccc}
Z & \longrightarrow & \mathbf{V}(\mathscr{H}om_B(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)))^{\mu} \\
\downarrow & & \downarrow{\scriptstyle \det} \\
X_C & \xrightarrow{\ \tau\ } & \mathbf{V}(\mathscr{H}om_F(\det_F V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \det_F \mathcal{E}_C(H)))
\end{array}
$$

Let $x \in \mathscr{M}_{\mathcal{D},\infty}(C)$ correspond to a B-linear morphism $s\colon V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \to \mathcal{E}_C(H)$ and a section of $Z \to X_C$ which we also call s. Define $B \otimes_{\mathbf{Q}_p} C$-modules W' and W by

$$
0 \to W' \to V \otimes_{\mathbf{Q}_p} C \xrightarrow{s_\infty} \mathcal{E}_C(H)_\infty \to W \to 0.
$$

The analogue of (5.5.4) is a diagram which computes the dual of $s^* \mathrm{Tan}_{Z/X_C}$.

$$
\begin{array}{ccccccccc}
& & & & & & 0 & & \\
& & & & & & \downarrow & & \\
0 & \longrightarrow & (s^*\mathrm{Tan}_{Z/X_C})^{\vee} & \longrightarrow & ((D' \times D)/F) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} & \longrightarrow & \mathcal{F}^{\vee} & \longrightarrow & 0 \\
& & & & & & \downarrow & & \\
& & & & & & \mathscr{H}om(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H)) & & \\
& & & & & & \downarrow & & \\
& & & & & & \mathrm{Tor}_1^F(i_{\infty *}((W')^{\vee}), i_{\infty *} W^{\vee}) & & \\
& & & & & & \downarrow & & \\
& & & & & & 0 & &
\end{array}
\qquad (5.6.1)
$$

This time, the dotted arrow is induced from the map $(D' \times D) \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C} \to \mathcal{H}om(V \otimes_{\mathbf{Q}_p} \mathcal{O}_{X_C}, \mathcal{E}_C(H))$ sending $(\alpha', \alpha) \otimes 1$ to $\alpha \circ s - s \circ \alpha'$. Taking H^0 in (5.6.1) shows that $H^0(X_C, s^* \mathrm{Tan}^{\vee}_{Z/X_C}) = A_x/F$, and this is nonzero exactly when x lies in the special locus.

5.7 Proof of Corollary 1.0.2

We conclude with a discussion of the infinite-level modular curve $X(p^\infty)$. Recall from [Sch15] the following facts about the Hodge–Tate period map $\pi_{HT} \colon X(p^\infty) \to \mathbf{P}^1$. The ordinary locus in $X(p^\infty)$ is sent to $\mathbf{P}^1(\mathbf{Q}_p)$. The supersingular locus is isomorphic to finitely many copies of $\mathscr{M}_{H,\infty,C}$, where H is a connected p-divisible group of height 2 and dimension 1 over the residue field of C; the restriction of π_{HT} to this locus agrees with the π_{HT} we had already defined on each $\mathscr{M}_{H,\infty,C}$.

We claim that the following are equivalent for a C-point x of $X(p^\infty)^\circ$.

(i) The point x corresponds to an elliptic curve $E/\mathcal{O}_C$, such that the p-divisible group $E[p^\infty]$ has $\mathrm{End}\, E[p^\infty] = \mathbf{Z}_p$.

(ii) The stabilizer of $\pi_{HT}(x)$ in $\mathrm{PGL}_2(\mathbf{Q}_p)$ is trivial.

(iii) There is a neighborhood of x in $X(p^\infty)^\circ$ which is cohomologically smooth over C.

First we discuss the equivalence of (1) and (2). If E is ordinary, then $E[p^\infty] \cong \mathbf{Q}_p/\mathbf{Z}_p \times \mu_{p^\infty}$ certainly has endomorphism ring larger than $\mathbf{Z}_p$, so that (1) is false. Meanwhile, the stabilizer of $\pi_{HT}(x)$ in $\mathrm{PGL}_2(\mathbf{Q}_p)$ is a Borel subgroup, so that (2) is false as well. The equivalence between (1) and (2) in the supersingular case is a special case of the equivalence discussed in § 3.5.

Theorem 1.0.1 tells us that $\mathscr{M}_{H,\infty}^{\circ,\text{non-sp}}$ is cohomologically smooth, which implies that shows that (2) implies (3). We therefore are left with showing that if (2) is false for a point $x \in X(p^\infty)^\circ$, then no neighborhood of x is cohomologically smooth.

First suppose that x lies in the ordinary locus. This locus is fibered over $\mathbf{P}^1(\mathbf{Q}_p)$. Suppose U is a sufficiently small neighborhood of x. Then U is contained in the ordinary locus, and so $\pi_0(U)$ is nondiscrete. This implies that $H^0(U, \mathbf{F}_\ell)$ is infinite, and so U cannot be cohomologically smooth.

Now suppose that x lies in the supersingular locus, and that $\pi_{HT}(x)$ has nontrivial stabilizer in $\mathrm{PGL}_2(\mathbf{Q}_p)$. We can identify x with a point in $\mathscr{M}_{H,\infty}^{\circ,\text{sp}}(C)$. We intend to show that every neighborhood of x in $\mathscr{M}_{H,\infty}^\circ$ fails to be cohomologically smooth.

Not knowing a direct method, we appeal to the calculations in [Wei16], which constructed semistable formal models for each $\mathscr{M}_{H,m}^\circ$. The main result we need is Theorem 5.1.2, which uses the term 'CM points' for what we have called special points. There exists a decreasing basis of neighborhoods $Z_{x,0} \supset Z_{x,1} \supset \cdots$ of x in $\mathscr{M}_{H,\infty}^\circ$. For each affinoid $Z = \mathrm{Spa}(R, R^+)$, let $\overline{Z} = \mathrm{Spec}\, R^+ \otimes_{\mathcal{O}_C} \kappa$, where κ is the residue field of C. For each $m \geq 0$, there exists a nonconstant morphism $\overline{Z}_{x,m} \to C_{x,m}$, where $C_{x,m}$ is an explicit nonsingular affine curve over κ. This morphism is equivariant for the action of the stabilizer of $Z_{x,m}$ in $\mathrm{SL}_2(\mathbf{Q}_p)$. For infinitely many m, the completion $C_{x,m}^{\text{cl}}$ of $C_{x,m}$ is a projective curve with positive genus.

Let $U \subset \mathscr{M}_{H,\infty}^\circ$ be an affinoid neighborhood of x. Then there exists $N \geq 0$ such that $Z_{x,m} \subset U$ for all $m \geq N$. Let $K \subset \mathrm{SL}_2(\mathbf{Q}_p)$ be a compact open subgroup which stabilizes U, so that U/K is an affinoid subset of the rigid-analytic curve $\mathscr{M}_{H,\infty}^\circ/K$. For each $m \geq N$, let $K_m \subset K$ be the stabilizer of $Z_{x,m}$, so that K_m acts on $C_{x,m}$.

There exists an integral model of U/K whose special fiber contains as a component the completion of each $\overline{Z}_{x,m}/K_m$ which has positive genus. Since there is a nonconstant morphism

$\overline{Z}_{x,m}/K_m \to C_{x,m}/K_m$, we must have

$$\dim_{\mathbf{F}_\ell} H^1(U/K, \mathbf{F}_\ell) \geq \sum_{m \geq N} \dim_{\mathbf{F}_\ell} H^1(C_{x,m}^{\mathrm{cl}}/K_m, \mathbf{F}_\ell).$$

Now we take a limit as K shrinks. Since $U \sim \varprojlim U/K$, we have $H^1(U, \mathbf{F}_\ell) \cong \varinjlim H^1(U/K, \mathbf{F}_\ell)$. Also, for each m, the action of K_m on $C_{x,m}$ is trivial for all sufficiently small K. Therefore

$$\dim_{\mathbf{F}_\ell} H^1(U, \mathbf{F}_\ell) \geq \sum_{m \geq N} \dim_{\mathbf{F}_\ell} H^1(C_{x,m}^{\mathrm{cl}}, \mathbf{F}_\ell) = \infty.$$

This shows that U is not cohomologically smooth.

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