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# HAUSDORFF DIMENSION FOR THE SET OF POINTS CONNECTED WITH THE GENERALIZED JARNÍK–BESICOVITCH SET

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**Abstract**

In this article we aim to investigate the Hausdorff dimension of the set of points $x \in [0, 1)$ such that for any $r \in \mathbb{N}$,

$$a_{n+1}(x)a_{n+2}(x) \cdots a_{n+r}(x) \geq e^{\tau(x)(h(x)+\cdots+h(T^{n-1}(x)))}$$

holds for infinitely many $n \in \mathbb{N}$, where $h$ and $\tau$ are positive continuous functions, $T$ is the Gauss map and $a_n(x)$ denotes the $n$th partial quotient of $x$ in its continued fraction expansion. By appropriate choices of $r, \tau(x)$ and $h(x)$ we obtain various classical results including the famous Jarník–Besicovitch theorem.

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## 1. Introduction

From Dirichlet's theorem (1842) it is well known that for any irrational $x \in [0, 1)$ there exist infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{1-1}$$

Statement (1-1) is important as it provides the rate of approximation that works for all $x \in [0, 1)$. Thus, the area of interest is to investigate the sets of irrational numbers satisfying (1-1) but with functions decreasing faster than 1 over the denominator squared. The primary metric result in this connection is due to Khintchine [10], who investigated that for a decreasing function $\phi$, the inequality

$$\left| x - \frac{p}{q} \right| < \phi(q) \tag{1-2}$$

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has infinitely many solutions for almost all (respectively almost no) $x \in [0, 1)$ if and only if $\sum_{q=1}^{\infty} q\phi(q)$ diverges (respectively converges). He used the tool of continued fractions to show this.

The metrical aspect of the theory of continued fractions has been very well studied due to its close connections with Diophantine approximation. This theory can be viewed as arising from the Gauss map $T : [0, 1) \to [0, 1)$, which is defined as

$$T(0) = 0 \quad \text{and} \quad T(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad \text{for } 0 < x < 1,$$

where $\lfloor \cdot \rfloor$ denotes the integral part of any real number. Thus, $T(x)$ represents the fractional part of $1/x$.

Also, note that for any $x \in [0, 1)$, its unique continued fraction expansion is given as

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}, \tag{1-3}$$

where, for each $n \geq 1$, $a_n(x)$ are known as the partial quotients of $x$ such that $a_1(x) = \lfloor 1/x \rfloor$ and $a_n(x) = \lfloor 1/T^{n-1}(x) \rfloor$ for $n \geq 2$. Therefore, (1-3) can also be represented as

$$x = [a_1(x), a_2(x), a_3(x), \ldots, a_n(x) + T^n(x)] =: [a_1(x), a_2(x), a_3(x), \ldots].$$

Note that Khintchine's result is a Lebesgue measure criterion for the set of points satisfying Equation (1-2) and thus it gives no further information about the sizes of null sets. For this reason the Hausdorff dimension and measure are appropriate tools as they help to distinguish between null sets. The starting point to answer this problem is the Jarník–Besicovitch theorem [4, 9], which gives the Hausdorff dimension of the set

$$J(\tau) = \left\{ x \in [0, 1) : \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ for i.m. } (p, q) \in \mathbb{Z} \times \mathbb{N} \right\}. \tag{1-4}$$

Here and throughout 'i.m.' stands for 'infinitely many'.

Recall that for any irrational $x \in [0, 1)$, the irrationality exponent of $x$ is defined as

$$\vartheta(x) := \sup\{\tau : x \in J(\tau)\}.$$

From Equation (1-1), it is known that $\vartheta(x) \geq 2$ for any irrational $x \in [0, 1)$. Moreover, for any $\tau \geq 2$, the Jarník–Besicovitch theorem [4, 9] states that

$$\dim_{\mathrm{H}}\{x \in [0, 1) : \vartheta(x) \geq \tau\} = \dim_{\mathrm{H}}\{x \in [0, 1) : \vartheta(x) = \tau\} = \frac{2}{\tau}.$$

Observe that the exponent $\tau$ in the above sets is constant. Barral and Seuret [3] generalized the Jarník–Besicovitch theorem by considering the set of points for which the irrationality exponent is not fixed in advance but it may vary with $x$ in a continuous way. More precisely, Barral and Seuret [3] showed that for a continuous function $\tau(x)$,

$$\begin{aligned}\dim_{\mathrm{H}}\{x \in [0,1) : \vartheta(x) \geq \tau(x)\} &= \dim_{\mathrm{H}}\{x \in [0,1) : \vartheta(x) = \tau(x)\} \\ &= \frac{2}{\min\{\tau(x) \ : \ x \in [0,1]\}}.\end{aligned}$$

They called such a set the localized Jarník–Besicovitch set. The result of Barral and Seuret was further generalized to the setting of continued fractions by Wang *et al.* in [21]. To refer to their result, we first restate the Jarník–Besicovitch set (1-4) in terms of growth rate of partial quotients,

$$J(\tau) = \{x \in [0,1) : a_n(x) \geq e^{((\tau-2)/2)S_n(\log|T'(x)|)} \text{ for i.m. } n \in \mathbb{N}\}. \tag{1-5}$$

Note that in terms of entries of continued fractions the Jarník–Besicovitch set, as given in Equation (1-5), contains the approximating function that involves the ergodic sum

$$S_n(\log|T'(x)|) = \log|T'(x)| + \cdots + \log|T'(T^{n-1}(x))|$$

and this sum is growing fast as $n \to \infty$. Therefore, having the approximating function in terms of the ergodic sum and the fact that partial quotients of any real number $x \in [0,1)$ completely determine its Diophantine properties, the Jarník–Besicovitch set (and its related variations, which we see in this article) in terms of the growth rate of partial quotients gives us better approximation results. In fact, Wang *et al.* [21] introduced the generalized version of the set (1-5) as

$$J(\tau, h) = \{x \in [0,1) : a_n(x) \geq e^{\tau(x)S_n h(x)} \text{ for i.m. } n \in \mathbb{N}\},$$

where $h(x)$, $\tau(x)$ are positive continuous functions defined on $[0,1]$ and $S_n h(x)$ represents the ergodic sum '$h(x) + \cdots + h(T^{n-1}(x))$.' They called such points the localized $(\tau, h)$ approximable points. Further, they proved the Hausdorff dimension of $J(\tau, h)$ to be

$$s_{\mathbb{N}}^{(1)} = \inf\{s \geq 0 : \mathsf{P}(T, -s\tau_{\min}h - s\log|T'|) \leq 0\},$$

where $\tau_{\min} = \min\{\tau(x) : x \in [0,1]\}$ and $\mathsf{P}$ denotes the pressure function defined below in Section 3.

In this article we introduce the set of points $x \in [0,1)$ for which the product of an arbitrary block of consecutive partial quotients, in their continued fraction expansion, is growing. In fact, we determine the size of such a set in terms of Hausdorff dimension. Motivation for considering the growth of the product of consecutive partial quotients arose from the work of Kleinbock and Wadleigh [12], where they considered improvements to Dirichlet's theorem. We refer the reader to [1, 2, 8, 11–13] for comprehensive metric theory associated with the set of points improving Dirichlet's theorem.

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We prove the following main result of this article. Note that the tempered distortion property is defined in Section 3.

THEOREM 1.1. *Let $h : [0, 1] \to (0, \infty)$ and $\tau : [0, 1] \to [0, \infty)$ be positive continuous functions with h satisfying the tempered distortion property. For $r \in \mathbb{N}$, define the set*

$$\mathcal{R}_r(\tau) := \{x \in [0, 1) : a_{n+1}(x) \cdots a_{n+r}(x) \geq e^{\tau(x)S_nh(x)} \text{ for i.m. } n \in \mathbb{N}\}.$$

*Then*

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau) = s_{\mathbb{N}}^{(r)} = \inf\{s \geq 0 : \mathsf{P}(T, -g_r(s)\tau_{\min}h - s \log |T'|) \leq 0\},$$

*where $\tau_{\min} = \min\{\tau(x) : x \in [0, 1]\}$ and $g_r(s)$ is given by the following recursive formula*

$$g_1(s) = s, \quad g_r(s) = \frac{sg_{r-1}(s)}{1 - s + g_{r-1}(s)} \quad \text{for } r \geq 2.$$

Theorem 1.1 is more general as for different $\tau(x)$ and $h(x)$ it implies various classical results, as we now see.

- When $r = 1$, $\tau(x) =$ constant and $h(x) = \log |T'|$, then we obtain the classical Jarník–Besicovitch theorem [4, 9].

COROLLARY 1.2. *For any $\tau \geq 2$,*

$$\dim_{\mathrm{H}} J(\tau) = \frac{2}{\tau}.$$

- When $r = 1$, we obtain the result by Wang *et al.* [21].

COROLLARY 1.3.

$$\dim_{\mathrm{H}}\{x \in [0, 1) : a_{n+1}(x) \geq e^{\tau(x)S_nh(x)} \text{ for i.m. } n \in \mathbb{N}\} = s_{\mathbb{N}}^{(1)}.$$

- When $r = 1, \tau(x) = 1$ and $h(x) = \log B$, we obtain the result by Wang and Wu [20].

COROLLARY 1.4. *For any $B > 1$,*

$$\dim_{\mathrm{H}}\{x \in [0, 1) : a_{n+1}(x) \geq B^n \text{ for i.m. } n \in \mathbb{N}\} = s_B^{(1)},$$

*where*

$$s_B^{(1)} = \inf\{s \geq 0 : \mathsf{P}(T, -s \log B - s \log |T'|) \leq 0\}.$$

- When $\tau(x) = 1$ and $h(x) = \log B$, we obtain the result by Huang *et al.* [7].

COROLLARY 1.5. *For any $B > 1$,*

$$\dim_{\mathrm{H}}\{x \in [0, 1) : a_{n+1}(x) \cdots a_{n+r}(x) \geq B^n \text{ for i.m. } n \in \mathbb{N}\} = s_B^{(r)},$$

*where*

$$s_B^{(r)} = \inf\{s \geq 0 : \mathsf{P}(T, -g_r(s) \log B - s \log |T'|) \leq 0\}.$$

- When $r = 2$, $\tau(x) =$ constant and $h(x) = \log|T'|$, we obtain the result by Hussain *et al.* [8].

COROLLARY 1.6.

$$\dim_H\{x \in [0,1) : a_{n+1}(x)a_{n+2}(x) \geq q_{n+1}^{\tau+2}(x) \text{ for i.m. } n \in \mathbb{N}\} = \frac{2}{2+\tau}.$$

## 2. Preliminaries

In this section, we gather some fundamental properties of continued fractions of real numbers and recommend [10, 14] to the reader for further details.

For any vector $(a_1, \ldots, a_n) \in \mathbb{N}^n$ with $n \in \mathbb{N}$, define the *basic cylinder* of order $n$ as

$$I_n = I_n(a_1, \ldots, a_n) := \{x \in [0,1) : a_1(x) = a_1, \ldots, a_n(x) = a_n\}.$$

PROPOSITION 2.1. *Let $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$. Then we have the following results.*

(i)

$$I_n(a_1, a_2, \ldots, a_n) = \begin{cases} \left[\dfrac{p_n}{q_n}, \dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}\right) & \text{if } n \text{ is even,} \\ \left(\dfrac{p_n + p_{n-1}}{q_n + q_{n-1}}, \dfrac{p_n}{q_n}\right] & \text{if } n \text{ is odd,} \end{cases}$$

*where the numerator $p_n = p_n(x)$ and the denominator $q_n = q_n(x)$ of the nth convergent of $x$ are obtained from the following recursive relation:*

$$\begin{aligned} p_{-1} &= 1, \; p_0 = 0, \; p_{n+1} = a_{n+1}p_n + p_{n-1}, \\ q_{-1} &= 0, \; q_0 = 1, \; q_{n+1} = a_{n+1}q_n + q_{n-1}. \end{aligned} \tag{2-6}$$

*Also, $p_{n-1}q_n - p_nq_{n-1} = (-1)^n$ for all $n \geq 1$.*

(ii) *The length of $I_n(a_1, a_2, \ldots, a_n)$ is given by*

$$\frac{1}{2q_n^2} \leq |I_n(a_1, \ldots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \leq \frac{1}{q_n^2}. \tag{2-7}$$

(iii) *$q_n \geq 2^{(n-1)/2}$ and, for any $1 \leq k \leq n$,*

$$\begin{aligned} q_{n+k}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k}) &\geq q_n(a_1, \ldots, a_n) \cdot q_k(a_{n+1}, \ldots, a_{n+k}), \\ q_{n+k}(a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+k}) &\leq 2q_n(a_1, \ldots, a_n) \cdot q_k(a_{n+1}, \ldots, a_{n+k}). \end{aligned} \tag{2-8}$$

(iv)

$$\frac{1}{(a_{n+1}+2)q_n^2} < \left|x - \frac{p_n}{q_n}\right| = \frac{1}{q_n(q_{n+1} + T^{n+1}(x)q_n)} < \frac{1}{a_{n+1}q_n^2}$$

*and the derivative of $T^n$ is given by*

$$(T^n)'(x) = \frac{(-1)^n}{(xq_{n-1} - p_{n-1})^2}.$$

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*Further,*

$$q_n^2(x) \le \prod_{k=0}^{n-1} |T'(T^k(x))| \le 4q_n^2(x). \tag{2-9}$$

From (2-6), note that for any $n \ge 1$, $p_n$ and $q_n$ are determined by $a_1, \dots, a_n$. Thus, we can write $p_n$ as $p_n(a_1, \dots, a_n)$ and $q_n$ as $q_n(a_1, \dots, a_n)$. Just to avoid confusion, we can also use the notation $a_n$ and $q_n$ in place of $a_n(x)$ and $q_n(x)$, respectively.

The next theorem, known as Legendre's theorem, connects one-dimensional Diophantine approximation with continued fractions.

LEGENDRE THEOREM . *Let* $(p, q) \in \mathbb{Z} \times \mathbb{N}$. *If*

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2}, \quad \textit{then} \quad \frac{p}{q} = \frac{p_n(x)}{q_n(x)}$$

*for some* $n \ge 1$.

According to Legendre's theorem, if an irrational $x$ is well approximated by a rational $p/q$, then this rational must be a convergent of $x$. Thus, in order to find good rational approximates to an irrational number, we only need to focus on its convergents. Note that, from (iv) of Proposition 2.1, a real number $x$ is well approximated by its convergent $p_n/q_n$ if its $(n + 1)$th partial quotient is sufficiently large.

## 3. The pressure function and $s_{\mathbb{N}}^{(r)}$

In this section, we recall the definition of pressure function and collect some of its basic properties which are useful for proving the main result of this article. For thorough results on pressure functions in infinite conformal iterated function systems, we refer the reader to [16–19]. However, for our purposes, we use the Mauldin and Urbański [17] form of pressure function applicable to the geometry of continued fractions.

Consider a finite or infinite subset $\mathcal{B}$ of the set of natural numbers and define

$$Z_{\mathcal{B}} = \{x \in [0, 1) : a_n(x) \in \mathcal{B} \text{ for all } n \ge 1\}.$$

Then $(Z_{\mathcal{B}}, T)$ is a subsystem of $([0, 1), T)$, where $T$ represents the Gauss map. Let $\psi : [0, 1) \to \mathbb{R}$ be a function. Then the pressure function $\mathsf{P}_{\mathcal{B}}(T, \psi)$ with respect to the potential $\psi$ and restricted to the system $(Z_{\mathcal{B}}, T)$ is given by

$$\mathsf{P}_{\mathcal{B}}(T, \psi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{a_1,\dots,a_n \in \mathcal{B}} \sup_{x \in Z_{\mathcal{B}}} e^{S_n\psi([a_1,\dots,a_n+x])}, \tag{3-10}$$

where $S_n\psi(x)$ denotes the ergodic sum $\psi(x) + \cdots + \psi(T^{n-1}(x))$. For $\mathcal{B} = \mathbb{N}$, we denote $\mathsf{P}_{\mathbb{N}}(T, \psi)$ by $\mathsf{P}(T, \psi)$.

The $n$th variation of a function $\psi$, denoted by $\mathrm{Var}_n(\psi)$, is defined as

$$\mathrm{Var}_n(\psi) := \sup_{x,z \,:\, I_n(x)=I_n(z)} |\psi(x) - \psi(z)|,$$

where $I_n(x)$ represents the basic cylinder of order $n$ that contains $x$ in the continued fraction expansion, that is, $I_n(x) = I_n(a_1(x), \dots, a_n(x))$ (for the definition of cylinder, see Section 2).

A function $\psi$ is said to satisfy the *tempered distortion property* if

$$\mathrm{Var}_1(\psi) < \infty \text{ and } \lim_{n\to\infty} \mathrm{Var}_n(\psi) = 0. \tag{3-11}$$

Throughout the article, we consider $\psi : [0,1) \to \mathbb{R}$ to be a function satisfying the tempered distortion property. The existence of the limit in the definition of the pressure function is guaranteed by the following proposition.

PROPOSITION 3.1 [15, Proposition 2.4]. *The limit defining $\mathsf{P}_{\mathcal{B}}(T,\psi)$ exists. Furthermore, if $\psi : [0,1) \to \mathbb{R}$ is a function satisfying the tempered distortion property, then the value of $\mathsf{P}_{\mathcal{B}}(T,\psi)$ remains unchanged even without taking the supremum over $x \in Z_{\mathcal{B}}$ in (3-10). The independence of $\mathsf{P}_{\mathcal{B}}(T,\psi)$ on the supremum over $x \in Z_{\mathcal{B}}$ follows from the fact that*

$$|S_n\psi([a_1,\dots,a_n+x_1]) - S_n\psi([a_1,\dots,a_n+x_2])| \le \sum_{j=1}^{n} \mathrm{Var}_j(\psi) = o(n)$$

*for any $x_1, x_2 \in Z_{\mathcal{B}}$.*

Thus, if we want to take any point $z$ from the basic cylinder $I_n(a_1,\dots,a_n)$, we can always take it as $z = p_n/q_n = [a_1,\dots,a_n]$.

The next result by Hanus *et al.* [6] shows that when the Gauss system $([0,1),T)$ is approximated by $(Z_{\mathcal{B}},T)$, then in the system of continued fractions the pressure function has a continuity property.

PROPOSITION 3.2 [6]. *Let $\psi : [0,1) \to \mathbb{R}$ be a function satisfying (3-11). Then*

$$\mathsf{P}(T,\psi) = \sup\{\mathsf{P}_{\mathcal{B}}(T,\psi) : \mathcal{B} \text{ is a finite subset of } \mathbb{N}\}.$$

For every $n \ge 1$ and $s \ge 0$, let

$$f_{n,\mathcal{B}}(s) = \sum_{a_1,\dots,a_n\in\mathcal{B}} \frac{1}{e^{g_r(s)\tau_{\min}S_nh(z)}q_n^{2s}}, \tag{3-12}$$

where $z \in I_n(a_1,\dots,a_n)$ and $g_r(s)$ is defined by the formula

$$g_1(s) = s \text{ and } g_r(s) = \frac{sg_{r-1}(s)}{1-s+g_{r-1}(s)} \quad \text{for all } r \ge 2. \tag{3-13}$$

It can be easily checked that for any $s \in (1/2,1)$, we have $g_{r+1}(s) \le g_r(s)$ for all $r \ge 1$.

From now onwards we consider a particular potential

$$\psi_s(x) := -g_r(s)\tau_{\min}h - s\log|T'(x)|.$$

From the definition of the pressure function (3-10) and from (2-9), (3-12) and (3-13),

$$\mathsf{P}_{\mathcal{B}}(T,\psi_s) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{a_1,\ldots,a_n\in\mathcal{B}}\frac{1}{e^{g_r(s)\tau_{\min}S_nh(z)}q_n^{2s}}.$$

Define

$$s_{n,\mathcal{B}}^{(r)} = \inf\{s\geq 0 : f_{n,\mathcal{B}}(s)\leq 1\}$$

and let

$$s_{\mathcal{B}}^{(r)} = \inf\{s\geq 0 : \mathsf{P}_{\mathcal{B}}(T,-g_r(s)\tau_{\min}h - s\log|T'(x)|)\leq 0\},$$
$$s_{\mathbb{N}}^{(r)} = \inf\{s\geq 0 : \mathsf{P}(T,-g_r(s)\tau_{\min}h - s\log|T'(x)|)\leq 0\}.$$

When $\mathcal{B}$ is a finite subset of $\mathbb{N}$, then $s_{n,\mathcal{B}}^{(r)}$ and $s_{\mathcal{B}}^{(r)}$ are the unique solutions to $f_{n,\mathcal{B}}(s)=1$ and $\mathsf{P}_{\mathcal{B}}(T,-g_r(s)\tau_{\min}h - s\log|T'(x)|)=0$, respectively (for details, see [20]). If $\mathcal{B}=\{1,\ldots,M\}$ for any $M\in\mathbb{N}$, write $s_{n,M}^{(r)}$ for $s_{n,\mathcal{B}}^{(r)}$ and $s_M^{(r)}$ for $s_{\mathcal{B}}^{(r)}$.

From Proposition 3.2 and since the potential $\psi_s$ satisfies the tempered distortion property, we have the following result.

COROLLARY 3.3. *For any integer $r\geq 1$,*

$$s_{\mathbb{N}}^{(r)} = \sup\{s_{\mathcal{B}}^{(r)} : \mathcal{B}\textit{ is a finite subset of }\mathbb{N}\}.$$

*Furthermore, the dimensional term $s_{\mathbb{N}}^{(r)}$ is continuous with respect to $\psi_s$, that is,*

$$\lim_{\epsilon\to 0}\inf\{s\geq 0 : \mathsf{P}_{\mathcal{B}}(T,\psi_s+\epsilon)\leq 0\} = \inf\{s\geq 0 : \mathsf{P}_{\mathcal{B}}(T,\psi_s)\leq 0\}. \tag{3-14}$$

From the definition of the pressure function and by Corollary 3.3, we have the following result.

COROLLARY 3.4. *For any $M\in\mathbb{N}$ and $0<\epsilon<1$,*

$$\lim_{n\to\infty}s_{n,M}^{(r)} = s_M^{(r)},\quad \lim_{n\to\infty}s_{n,\mathbb{N}}^{(r)} = s_{\mathbb{N}}^{(r)},\quad \lim_{M\to\infty}s_M^{(r)} = s_{\mathbb{N}}^{(r)}\quad\text{and}\quad |s_{n,M}^{(r)} - s_M^{(r)}|\leq\epsilon.$$

## 4. Proof of Theorem 1.1

The proof of Theorem 1.1 is divided into two main parts:

(i) the upper estimate for $\dim_{\mathrm{H}}\mathcal{R}_r(\tau)$; and
(ii) the lower estimate for $\dim_{\mathrm{H}}\mathcal{R}_r(\tau)$.

**4.1. Proof of Theorem 1.1: the upper estimate.** Upper estimates for the Hausdorff dimension of any set are usually easy to obtain as they involve a natural covering argument. Thus, for the upper bound, we find a natural covering for the set $\mathcal{R}_r(\tau)$. To do this, recall that $h$ is assumed to be a positive continuous function satisfying the

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tempered distortion property. Consequently,

$$(1/n)\sum_{j=1}^{n}\mathrm{Var}_j(h)\to 0 \quad \text{as } n\to\infty.$$

Thus, for any fixed $\lambda > 0$, there exists $N(\lambda)\in\mathbb{N}$ such that for any $n\geq N(\lambda)$ we have $\sum_{j=1}^{n}\mathrm{Var}_j(h)\leq n\lambda$. Therefore, for any $x, z\in[0,1)$ with $I_n(x) = I_n(z)$,

$$\begin{aligned}|S_nh(x)-S_nh(z)| &= \Big|\sum_{j=0}^{n-1}h(T^jx)-\sum_{j=0}^{n-1}h(T^jz)\Big| \\ &\leq \sum_{j=0}^{n-1}|h(T^jx)-h(T^jz)| \\ &\leq \sum_{j=0}^{n-1}\mathrm{Var}_{n-j}(h)\leq n\lambda.\end{aligned}$$

Then

$$\mathcal{R}_r(\tau)\subset C_r(\tau),$$

where

$$C_r(\tau) := \{x\in[0,1) : a_{n+1}(x)\cdots a_{n+r}(x)\geq e^{\tau_{\min}S_n(h-\lambda)(z)} \text{ i.m. } n\in\mathbb{N}\}$$

and $z\in I_n(a_1,\ldots,a_n)$. Thus, for the upper estimate of $\dim_{\mathrm{H}}\mathcal{R}_r(\tau)$, it is sufficient to calculate the upper estimate for $\dim_{\mathrm{H}} C_r(\tau)$; that is, first, we show that

$$\dim_{\mathrm{H}} C_r(\tau)\leq\inf\{s\geq 0 : \mathsf{P}(T, -g_r(s)\tau_{\min}(h-\lambda)-s\log|T'|)\leq 0\} \tag{4-15}$$

by induction on $r$.

For $r = 1$, the result was proved by Wang *et al.* in [21].

Suppose that (4-15) is true for $r = k$. We need to show that (4-15) holds for $r = k + 1$. Note that

$$\begin{aligned}C_{k+1}(\tau) \subseteq\ & \{x\in[0,1) : a_{n+1}(x)\cdots a_{n+k}(x)\geq e^{\tau_{\min}S_n(h-\lambda)(z)} \text{ i.m. } n\in\mathbb{N}\} \\ & \cup\left\{x\in[0,1) : \begin{array}{l} 1\leq a_{n+1}(x)\cdots a_{n+k}(x)\leq e^{\tau_{\min}S_n(h-\lambda)(z)}, \\ a_{n+k+1}(x)\geq \dfrac{e^{\tau_{\min}S_n(h-\lambda)(z)}}{a_{n+1}(x)\cdots a_{n+k}(x)} \text{ i.m. } n\in\mathbb{N}\end{array}\right\}.\end{aligned}$$

Further, for any $1<\gamma\leq e$,

$$\begin{aligned}C_{k+1}(\tau) \subseteq\ & \{x\in[0,1) : a_{n+1}(x)\cdots a_{n+k}(x)\geq \gamma^{\tau_{\min}S_n(h-\lambda)(z)} \text{ i.m. } n\in\mathbb{N}\} \\ & \cup\left\{x\in[0,1) : \begin{array}{l} 1\leq a_{n+1}(x)\cdots a_{n+k}(x)\leq \gamma^{\tau_{\min}S_n(h-\lambda)(z)}, \\ a_{n+k+1}(x)\geq \dfrac{e^{\tau_{\min}S_n(h-\lambda)(z)}}{a_{n+1}(x)\cdots a_{n+k}(x)} \text{ i.m. } n\in\mathbb{N}\end{array}\right\} \\ =:\ & \mathcal{A}(\tau)\cup\mathcal{B}(\tau).\end{aligned}$$

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Thus,

$$\dim_{\mathrm{H}} C_{k+1}(\tau) \leq \inf_{1<\gamma\leq e} \max\{\dim_{\mathrm{H}} \mathcal{A}(\tau), \dim_{\mathrm{H}} \mathcal{B}(\tau)\}.$$

By using the induction hypothesis and since $\gamma^{\tau_{\min} S_n(h-\lambda)(z)} \leq e^{\tau_{\min} S_n(h-\lambda)(z)}$,

$$\dim_{\mathrm{H}} \mathcal{A}(\tau) \leq t_\gamma^k := \inf\{s \geq 0 : \mathsf{P}(T, -g_k(s)\tau_{\min}(h-\lambda)\log\gamma - s\log|T'|) \leq 0\}.$$

For the upper bound of $\dim_{\mathrm{H}} \mathcal{B}(\tau)$, we proceed by finding a natural covering for this set. In terms of the lim sup nature of the set, $\mathcal{B}(\tau)$ can be rewritten as

$$\mathcal{B}(\tau) = \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\left\{x \in [0,1) : \begin{array}{l} 1 \leq a_{n+1}(x)\cdots a_{n+k}(x) \leq \gamma^{\tau_{\min} S_n(h-\lambda)(z)}, \\ a_{n+k+1}(x) \geq \dfrac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{a_{n+1}(x)\cdots a_{n+k}(x)} \end{array}\right\}$$
$$=: \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty} \mathcal{B}^*(\tau).$$

Thus, for each $n \geq N$, the cover for $\mathcal{B}^*(\tau)$ will serve as a natural cover for $\mathcal{B}(\tau)$. Clearly,

$$\mathcal{B}^*(\tau) \subseteq \bigcup_{a_1,\ldots,a_n\in\mathbb{N}}\ \bigcup_{1\leq a_{n+1}\cdots a_{n+k}\leq\gamma^{\tau_{\min} S_n(h-\lambda)(z)}} J_{n+k}(a_1,\ldots,a_{n+k}),$$

where

$$J_{n+k}(a_1,\ldots,a_{n+k}) = \bigcup_{a_{n+k+1}\geq \frac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{a_{n+1}\cdots a_{n+k}}} I_{n+k+1}(a_1,\ldots,a_{n+k+1}).$$

By using Equation (2-7),

$$\begin{aligned}|J_{n+k}(a_1,\ldots,a_{n+k})| &= \sum_{a_{n+k+1}\geq \frac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{a_{n+1}\cdots a_{n+k}}} |I_{n+k+1}(a_1,\ldots,a_{n+k+1})| \\ &\asymp \frac{1}{\frac{e^{\tau_{\min} S_n(h-\lambda)(z)}}{a_{n+1}\cdots a_{n+k}} q_{n+k}^2(a_1,\ldots,a_{n+k})} \\ &\asymp \frac{1}{e^{\tau_{\min} S_n(h-\lambda)(z)}(a_{n+1}\cdots a_{n+k})q_n^2(a_1,\ldots,a_n)}.\end{aligned}$$

Fixing $\delta > 0$ and taking the $(s+\delta)$-volume of the cover of $B^*(\tau)$,

$$\begin{aligned}&\sum_{a_1,\ldots,a_n\in\mathbb{N}}\ \sum_{1\leq a_{n+1}\cdots a_{n+k}\leq\gamma^{\tau_{\min} S_n(h-\lambda)(z)}} \left(\frac{1}{e^{\tau_{\min} S_n(h-\lambda)(z)}(a_{n+1}\cdots a_{n+k})q_n^2(z)}\right)^{s+\delta} \\ &\leq \sum_{a_1,\ldots,a_n\in\mathbb{N}}\ \sum_{1\leq a_{n+1}\cdots a_{n+k}\leq\gamma^{\tau_{\min} S_n(h-\lambda)(z)}} \left(\frac{1}{e^{\tau_{\min} S_n(h-\lambda)(z)}(a_{n+1}\cdots a_{n+k})q_n^2(z)}\right)^{s} e^{-\delta\tau_{\min} S_n(h-\lambda)(z)}\end{aligned}$$

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$$
\begin{aligned}
&\asymp \sum_{a_1,\ldots,a_n\in\mathbb{N}} \frac{(\log \gamma^{\tau_{\min}S_n(h-\lambda)(z)})^{k-1}}{(k-1)!}\gamma^{(1-s)\tau_{\min}S_n(h-\lambda)(z)} \\
&\quad \cdot \Big(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}q_n^2(z)}\Big)^s e^{-\delta\tau_{\min}S_n(h-\lambda)(z)} \\
&\leq \sum_{a_1,\ldots,a_n\in\mathbb{N}} \frac{(\log e^{\tau_{\min}S_n(h-\lambda)(z)})^{k-1}}{(k-1)!}\gamma^{(1-s)\tau_{\min}S_n(h-\lambda)(z)} \\
&\quad \cdot \Big(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}q_n^2(z)}\Big)^s e^{-\delta\tau_{\min}S_n(h-\lambda)(z)} \\
&\leq \sum_{a_1,\ldots,a_n\in\mathbb{N}} \gamma^{(1-s)\tau_{\min}S_n(h-\lambda)(z)}\Big(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}q_n^2(z)}\Big)^s.
\end{aligned}
$$

Therefore, the $(s+\delta)$-dimensional Hausdorff measure of $\mathcal{B}(\tau)$ is

$$
\begin{aligned}
&\mathcal{H}^{s+\delta}(\mathcal{B}(\tau)) \\
&\leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_n\in\mathbb{N}} \sum_{1\leq a_{n+1}\cdots a_{n+k}\leq \gamma^{\tau_{\min}S_n(h-\lambda)(z)}} \Big(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}(a_{n+1}\cdots a_{n+k})q_n^2(z)}\Big)^{s+\delta} \\
&\leq \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{a_1,\ldots,a_n\in\mathbb{N}} \gamma^{(1-s)\tau_{\min}S_n(h-\lambda)(z)}\Big(\frac{1}{e^{\tau_{\min}S_n(h-\lambda)(z)}q_n^2(z)}\Big)^s.
\end{aligned}
$$

Since $\delta > 0$ is arbitrary,

$$\dim_{\mathrm{H}} \mathcal{B}(\tau) \leq u_\gamma^{k+1},$$

where $u_\gamma^{k+1}$ is defined as

$$\inf\{s \geq 0 : \mathsf{P}(T, (1-s)\tau_{\min}(h-\lambda)\log\gamma - s\tau_{\min}(h-\lambda) - s\log|T'|) \leq 0\}.$$

Hence,

$$\dim_{\mathrm{H}} C_{k+1}(\tau) \leq \inf_{1<\gamma\leq e} \max\{t_\gamma^k, u_\gamma^{k+1}\}.$$

As the pressure function $P(T,\psi_s)$ is increasing with respect to the potential $\psi_s$, we know that $t_\gamma^k$ is increasing and $u_\gamma^{k+1}$ is decreasing with respect to $\gamma$. Therefore, the infimum is obtained at the value $\gamma$, where

$$-g_k(s)\tau_{\min}(h-\lambda)\log\gamma - s\log|T'|$$

is equal to

$$(1-s)\tau_{\min}(h-\lambda)\log\gamma - s\tau_{\min}(h-\lambda) - s\log|T'|.$$

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This implies that

$$\begin{aligned}\gamma^{(1-s)\tau_{\min}(h-\lambda)} e^{-s\tau_{\min}(h-\lambda)} &= \gamma^{-g_k(s)\tau_{\min}(h-\lambda)}\\ \iff -\frac{s}{(1-s)+g_k(s)} &= -\log\gamma\\ \iff -g_{k+1}(s) &= -g_k(s)\log\gamma.\end{aligned}$$

Hence,

$$\dim_{\mathrm{H}} C_{k+1}(\tau) \le \inf\{s \ge 0 : \mathsf{P}(T, -g_{k+1}(s)\tau_{\min}(h-\lambda) - s\log|T'|) \le 0\}.$$

Consequently, for any $r \ge 1$,

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau) \le \inf\{s \ge 0 : \mathsf{P}(T, -g_r(s)\tau_{\min}(h-\lambda) - s\log|T'|) \le 0\}.$$

By Equation (3-14) of Corollary 3.3 and letting $\lambda \to 0$, we obtain the desired result.

**4.2. Proof of Theorem 1.1: the lower estimate.** For the lower estimate of the Hausdorff dimension of $\mathcal{R}_r(\tau)$, we adapt similar method as [7]. We first construct the Cantor subset $\mathcal{E}_\infty$ which sits inside the set $\mathcal{R}_r(\tau)$ and then we distribute the measure $\mu > 0$ on $\mathcal{E}_\infty$ and obtain the Holder exponent. Lastly, we apply the mass distribution principle [5].

PROPOSITION 4.1 (Mass distribution principle). *Let $\mathcal{U}$ be a Borel subset of $\mathbb{R}^d$ and $\mu$ be a Borel measure with $\mu(\mathcal{U}) > 0$. Suppose that, for some $s > 0$, there exists a constant $c > 0$ such that, for any $x \in [0, 1)$,*

$$\mu(B(x, d)) \le cd^s,$$

*where $B(x, d)$ denotes an open ball centred at $x$ and of radius $d$. Then*

$$\dim_{\mathrm{H}} \mathcal{U} \ \ge\ s.$$

**Cantor subset:**

Fix $\frac{1}{2} < s < s_{\mathbb{N}}^{(r)}$ and choose $1 \le \gamma_0 \le \gamma_1 \le \cdots \le \gamma_{r-2} \le e$ in a way such that

$$\log\gamma_i = \frac{g_r(s)(1-s)^i}{s^{i+1}} \quad \text{for all } 0 \le i \le r-2. \tag{4-16}$$

Moreover, we have the following lemma, which we prove by induction on $r$.

LEMMA 4.2. *For any $r \ge 1$,*

$$g_r(s) = \frac{s^r(2s-1)}{s^r-(1-s)^r} \tag{4-17}$$

*satisfies the recursive relation defined in (3-13).*

PROOF. When $r = 1$, it is clear from (3-13) that

$$g_1(s) = s = \frac{s(2s-1)}{s-(1-s)}.$$

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Suppose that (4-17) is true for $r = k$; then, for $r = k + 1$,

$$
\begin{aligned}
g_{k+1}(s) &= \frac{sg_k(s)}{1 - s + g_k(s)} \text{ by (3-13)} \\
&= \frac{s\frac{s^k(2s-1)}{s^k-(1-s)^k}}{1 - s + \frac{s^k(2s-1)}{s^k-(1-s)^k}} \text{ (by induction hypothesis)} \\
&= \frac{s^{k+1}(2s-1)}{s^k - (1-s)^k - s^{k+1} + s(1-s)^k + (2s-1)s^k} \\
&= \frac{s^{k+1}(2s-1)}{s^{k+1} - (1-s)^k(1-s)} = \frac{s^{k+1}(2s-1)}{s^{k+1} - (s-1)^{k+1}}.
\end{aligned}
$$

Therefore, (4-17) is true for $r = k + 1$. □

Thus, by using (4-16) and (4-17), it is easy to check that the following equality holds:

$$
\begin{aligned}
\log \gamma_0^{-s} = \log(\gamma_0^{1-s}(\gamma_0\gamma_1)^{-s}) = \cdots &= \log((\gamma_0 \cdots \gamma_{r-3})^{1-s}(\gamma_0 \cdots \gamma_{r-2})^{-s}) \\
&= \log(\gamma_0 \cdots \gamma_{r-2})^{1-s} - s = -g_r(s).
\end{aligned} \tag{4-18}
$$

Further, let $\epsilon > 0$ and $M \in \mathbb{N}$. Fix an irrational $z_0$ and an integer $t_0$ such that for any $z \in I_n(z_0)$ with $n \geq t_0$,

$$\tau(z) \leq \min\{\tau_{\min}(1 + \epsilon), \tau_{\min} + \epsilon\}.$$

Next define two integer sequences $\{t_j\}_{j\geq1}$ and $\{m_j\}_{j\geq1}$ recursively, where $\{m_j\}_{j\geq1}$ is defined to be a largely sparse integer sequence tending to infinity. For each $j \geq 1$, define $t_j = t_0 + j$ and set $n_j = (n_{j-1} + (r - 1)) + t_j + m_j + 1$.

Now we construct the Cantor subset $\mathcal{E}_\infty$ level by level. We start by defining the zero level.

**Level 0.** Let $n_0 + (r - 1) \geq t_2$. Define

$$\nu^{(0,r-1)} = (a_1(z_0), a_2(z_0), \ldots, a_{n_0+(r-1)}(z_0)).$$

Then the zero level $\mathcal{E}_0$ of the Cantor set $\mathcal{E}_\infty$ is defined as

$$\mathcal{E}_0 := \mathcal{F}_0 = \{I_{n_0+(r-1)}(\nu^{(0,r-1)})\}.$$

**Level 1.** Note that $n_1 = (n_0 + (r - 1)) + t_1 + m_1 + 1$. Let us define the collection of basic cylinders of order $n_1 - 1$:

$$\mathcal{F}_1 = \{I_{n_1-1}(\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, b_1^{(1)}, \ldots, b_{m_1}^{(1)}) : 1 \leq b_1^{(1)}, \ldots, b_{m_1}^{(1)} \leq M\}.$$

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For each $I_{n_1-1}(w^{(1)} = (v^{(0,r-1)}, v^{(0,r-1)}|_{t_1}, b_1^{(1)}, \ldots, b_{m_1}^{(1)})) \in \mathcal{F}_1$, define the collection of subcylinders of order $n_1$:

$$\mathcal{E}_{1,0}(w^{(1)}) := \{I_{n_1}(v^{(1,0)}) = I_{n_1}(w^{(1)}, a_{n_1}) : \\ \gamma_o^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \le a_{n_1} < 2\gamma_0^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)}\}, \tag{4-19}$$

where $z_1 \in I_{n_1-(n_0+(r-1))-1}(v^{(0,r-1)}|_{t_1}, b_1^{(1)}, \ldots, b_{m_1}^{(1)})$.

Let $I_{n_1} = I_{n_1}(v^{(0,r-1)}, v^{(0,r-1)}|_{t_1}, b_1^{(1)}, \ldots, b_{m_1}^{(1)}, a_{n_1}) \in \mathcal{E}_{1,0}(w^{(1)})$. The choice of $z_1$ indicates that for any $x \in I_{n_1}$, the continued fraction representations of $z_1$ and $x$ share prefixes up to $t_1$th partial quotients. Hence, $\tau(z_1)$ is close to $\tau(x)$ by the continuity of $\tau$. Further, it can be easily checked that $S_{n_1-(n_0+(r-1))-1}h(z_1) \sim S_{n_1-1}h(x)$ (here '$\sim$' denotes the asymptotic equality of two functions). Consequently,

$$\begin{aligned} &\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1) \sim \tau(x)S_{n_1-1}h(x) \\ \implies &\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)\log\gamma_0 \sim \tau(x)S_{n_1-1}h(x)\log\gamma_0 \\ \implies &\log\gamma_0^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \sim \log\gamma_0^{\tau(x)S_{n_1-1}h(x)} \\ \implies &\log a_{n_1} \sim \log\gamma_0^{\tau(x)S_{n_1-1}h(x)} \text{ from (4-19).} \end{aligned}$$

Thus, $a_{n_1}(x) \sim \gamma_o^{\tau(x)S_{n_1-1}h(x)}$.

Next, for each $I_{n_1}(v^{(1,0)}) \in \mathcal{E}_{1,0}(w^{(1)})$, define

$$\mathcal{E}_{1,1}(v^{(1,0)}) := \{I_{n_1+1}(v^{(1,1)}) = I_{n_1+1}(v^{(1,0)}, a_{n_1+1}) : \\ \gamma_1^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \le a_{n_1+1} < 2\gamma_1^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)}\}.$$

Continuing in this way for each $I_{n_1+(r-3)}(v^{(1,r-3)}) \in \mathcal{E}_{1,r-3}(v^{(1,r-4)})$, collect a family of subcylinders of order $n_{1+(r-2)}$:

$$\mathcal{E}_{1,r-2}(v^{(1,r-3)}) := \{I_{n_1+(r-2)}(v^{(1,r-2)}) = I_{n_1+(r-2)}(v^{(1,r-3)}, a_{n_1+(r-2)}) : \\ \gamma_{r-2}^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \le a_{n_1+(r-2)} < 2\gamma_{r-2}^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)}\}.$$

Further, for each $I_{n_1+(r-2)}(v^{(1,r-2)}) \in \mathcal{E}_{1,r-2}(v^{(1,r-3)})$, collect a family of subcylinders of order $n_{1+(r-1)}$:

$$\mathcal{E}_{1,r-1}(v^{(1,r-2)}) := \Big\{I_{n_1+(r-1)}(v^{(1,r-1)}) = I_{n_1+(r-1)}(v^{(1,r-2)}, a_{n_1+(r-1)}) : \\ \Big(\frac{e}{\gamma_0\cdots\gamma_{r-2}}\Big)^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)} \le a_{n_1+(r-1)} \\ < 2\Big(\frac{e}{\gamma_0\cdots\gamma_{r-2}}\Big)^{\tau(z_1)S_{n_1-(n_0+(r-1))-1}h(z_1)}\Big\}.$$

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Then the first level of the Cantor set $\mathcal{E}_\infty$ is defined as

$$\mathcal{E}_{1,r-1} = \Big\{ I_{n_1+(r-1)}(v^{(1,r-1)}) \in \mathcal{E}_{1,r-1}(v^{(1,r-2)}) : \\ I_{n_1+i}(v^{(1,i)}) \in \mathcal{E}_{1,i}(v^{(1,i-1)}) \text{ for } 1 \le i \le r-2; \\ I_{n_1}(v^{(1,0)}) \in \mathcal{E}_{1,0}(w^{(1)}); I_{n_1-1}(w^{(1)}) \in \mathcal{F}_1 \Big\}.$$

**Level *j*.** Suppose that the $(j-1)$th level $\mathcal{E}_{j-1,r-1}$ has been constructed. The set $\mathcal{E}_{j-1,r-1}$ consists of cylinders all of which are of order $n_{j-1} + (r-1)$. Recall that $n_j = (n_{j-1} + (r-1)) + t_j + m_j + 1$. For each basic cylinder $I_{n_{j-1}+(r-1)}(v^{(j-1,r-1)}) \in \mathcal{E}_{j-1,r-1}$, define the collections of subcylinders of order $n_j - 1$:

$$\mathcal{F}_j(I_{n_{j-1}+(r-1)}(v^{(j-1,r-1)})) = \{I_{n_j-1}(v^{(j-1,r-1)}, v^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \dots, b_{m_j}^{(j)}) : \\ 1 \le b_1^{(j)}, \dots b_{m_j}^{(j)} \le M\}$$

and let

$$\mathcal{F}_j = \bigcup_{I_{n_{j-1}+(r-1)} \in \mathcal{E}_{j-1,r-1}} \mathcal{F}_j(I_{n_{j-1}+(r-1)}(v^{(j-1,r-1)})).$$

Following the same process as for **Level 1**, for each $I_{n_j-1}(w^{(j)}) \in \mathcal{F}_j$ define the collection of subcylinders:

$$\mathcal{E}_{j,0}(w^{(j)}) := \{I_{n_j}(v^{(j,0)}) = I_{n_j}(w^{(j)}, a_{n_j}) : \\ \gamma_0^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \le a_{n_j} < 2\gamma_0^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}\},$$

where $z_j \in I_{n_j-(n_{j-1}+(r-1))-1}(v^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \dots, b_{m_j}^{(j)})$.

Next, for each $I_{n_j}(v^{(j,0)}) \in \mathcal{E}_{j,0}(w^{(j)})$, define

$$\mathcal{E}_{j,1}(v^{(j,0)}) := \{I_{n_j+1}(v^{(j,1)}) = I_{n_j+1}(v^{(j,0)}, a_{n_j+1}) : \\ \gamma_1^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \le a_{n_j+1} < 2\gamma_1^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}\}.$$

Similarly, for each $I_{n_j+i-1}(v^{(j,i-1)}) \in \mathcal{E}_{j,i-1}(v^{(j,i-2)})$ with $2 \le i \le r-2$, we collect a family of subcylinders of order $n_{j+i}$:

$$\mathcal{E}_{j,i}(v^{(j,i-1)}) := \{I_{n_j+i}(v^{(j,i)}) = I_{n_j+i}(v^{(j,i-1)}, a_{n_j+i}) : \\ \gamma_i^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \le a_{n_j+i} < 2\gamma_i^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}\}.$$

Continuing in this way for each $I_{n_j+r-2}(v^{(j,r-2)}) \in \mathcal{E}_{j,r-2}(v^{(j,r-3)})$, we define

$$\mathcal{E}_{j,r-1}(v^{(j,r-2)}) := \Big\{ I_{n_j+(r-1)}(v^{(j,r-1)}) = I_{n_j+(r-1)}(v^{(j,r-2)}, a_{n_j+(r-1)})) : \\ \Big(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\Big)^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \\ \leq a_{n_j+(r-1)} < 2\Big(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\Big)^{\tau(z_j)S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)} \Big\}.$$

Then the $j$th level of the Cantor set $\mathcal{E}_\infty$ is defined as

$$\mathcal{E}_{j,r-1} = \{ I_{n_j+(r-1)}(v^{(j,r-1)}) \in \mathcal{E}_{j,r-1}(v^{(j,r-2)}) : \\ I_{n_j+i}(v^{(j,i)}) \in \mathcal{E}_{j,i}(v^{(j,i-1)}) \text{ for } 1 \leq i \leq r-2; \\ I_{n_j}(v^{(j,0)}) \in \mathcal{E}_{j,0}(w^{(j)}); I_{n_j-1}(w^{(j)}) \in \mathcal{F}_j \}.$$

Then the Cantor set is defined as

$$\mathcal{E}_\infty = \bigcap_{j=1}^{\infty} \bigcup_{I_{n_j+(r-1)}(v^{(j,r-1)})\in\mathcal{E}_{j,r-1}} I_{n_j+(r-1)}(v^{(j,r-1)}).$$

By the same arguments as discussed earlier in defining Level 1, that is, by the continuity of $\tau$ and since $z_j$ and $x$ share common prefixes up to $t_j$th partial quotients,

$$\lim_{j\to\infty} \tau(z_j) = \tau(x) \quad \text{and} \quad \lim_{j\to\infty} \frac{S_{n_j-(n_{j-1}+(r-1))-1}h(z_j)}{S_{n_j-1}h(x)} = 1. \tag{4-20}$$

Therefore, $\mathcal{E}_\infty$ is contained in $\mathcal{R}_r(\tau)$, as for showing this it is sufficient to show that (4-20) holds.

In order to better understand the structure of $\mathcal{E}_\infty$, we utilize the idea of symbolic space. If the continued fraction expansion of a point $x \in \mathcal{E}_\infty$ is represented by $[v_1, v_2, \ldots, v_n, \ldots]$, then the sequence $(v_1, v_2, \ldots, v_n, \ldots)$ is known as an *admissible sequence* and $v = (v_1, v_2, \ldots, v_n)$ is called an *admissible block* for any $n \geq 1$. If $v$ is an admissible block, then $I_n(v) \cap \mathcal{E}_\infty \neq \emptyset$ and such basic cylinders $I_n(v)$ are known as admissible cylinders.

For any $n \geq 1$, denote by $\mathcal{D}_n$ the set of strings defined as

$$\mathcal{D}_n = \{(v_1, v_2, \ldots, v_n) \in \mathbb{N}^n : v = (v_1, v_2, \ldots, v_n) \text{ is an admissible block}\}.$$

We define $\mathcal{D}_n$ for different cases, according to the limitations on the partial quotients defined in the construction of $\mathcal{E}_\infty$.

Let $l_1 = n_1 - (n_0 + (r-1)) - 1 = t_1 + m_1$.

(1a) When $1 \leq n \leq (n_0 + (r-1))$,

$$\mathcal{D}_n = \{v^{(0,r-1)} = (a_1(z_0), a_2(z_0), \ldots, a_{n_0+(r-1)}(z_0))\}.$$

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(1b) When $(n_0 + (r-1)) < n \le (n_0 + (r-1)) + t_1$,

$$\mathcal{D}_n = \{(\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{n-(n_0+(r-1))})\}.$$

(1c) When $(n_0 + (r-1)) + t_1 < n < n_1$,

$$\mathcal{D}_n = \{\nu = (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1}, \ldots, \nu_n) : 1 \le \nu_u \le M, \\ (n_0 + (r-1)) + t_1 < u \le n\}.$$

(1d) When $n = n_1$,

$$\mathcal{D}_n = \{\nu = (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1}, \ldots, \nu_{n_1-1}, \nu_{n_1}) : \\ \gamma_0^{\tau(z_1)S_{l_1}h(z_1)} \le \nu_{n_1} < 2\gamma_0^{\tau(z_1)S_{l_1}h(z_1)} \\ \text{and } 1 \le \nu_u \le M \text{ for } (n_0 + (r-1)) + t_1 < u < n_1\},$$

where $z_1 \in I_{l_1(\nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1}, \ldots, \nu_{n_1-1})}$.

(1e) When $n = n_1 + i$, where $1 \le i \le r-2$,

$$\mathcal{D}_n = \{\nu = (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1}, \ldots, \nu_{n_1+i}) : \\ \gamma_i^{\tau(z_1)S_{l_1}h(z_1)} \le \nu_{n_1+i} < 2\gamma_i^{\tau(z_1)S_{l_1}h(z_1)}, \text{ where } 1 \le i \le r-2, \\ \gamma_0^{\tau(z_1)S_{l_1}h(z_1)} \le \nu_{n_1} < 2\gamma_0^{\tau(z_1)S_{l_1}h(z_1)} \\ \text{and } 1 \le \nu_u \le M \text{ for } (n_0 + (r-1)) + t_1 < u < n_1\}.$$

(1f) When $n = n_1 + (r-1)$,

$$\mathcal{D}_n = \Big\{\nu = (\nu^{(0,r-1)}, \nu^{(0,r-1)}|_{t_1}, \nu_{(n_0+(r-1))+t_1+1}, \ldots, \nu_{n_1}) : \\ \Big(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\Big)^{\tau(z_1)S_{l_1}h(z_1)} \le \nu_{n_1+(r-1)} \\ < 2\Big(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\Big)^{\tau(z_1)S_{l_1}h(z_1)}, \\ \gamma_i^{\tau(z_1)S_{l_1}h(z_1)} \le \nu_{n_1+i} < 2\gamma_i^{\tau(z_1)S_{l_1}h(z_1)}, \text{ where } 1 \le i \le r-2, \\ \gamma_0^{\tau(z_1)S_{l_1}f(z_1)} \le \nu_{n_1} < 2\gamma_0^{\tau(z_1)S_{l_1}h(z_1)} \\ \text{and } 1 \le \nu_u \le M \text{ for } (n_0 + (r-1)) + t_1 < u < n_1\Big\}.$$

Next we define $\mathcal{D}_n$ inductively. For this we suppose that $\mathcal{D}_{n_{j-1}+(r-1)}$ has been given and, for each $j \ge 1$, write $l_j = n_j - (n_{j-1} + (r-1)) - 1 = t_j + m_j$.

(2a) When $(n_{j-1} + (r-1)) < n \le (n_{j-1} + (r-1)) + t_j$,

$$\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{n-(n_{j-1}+(r-1))})\}.$$

(2b) When $(n_{j-1} + (r-1)) + t_j < n < n_j$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1}, \ldots, \nu_n) :\\ \nu^{(j-1,r-1)} \in \mathcal{D}_{n_{j-1}+(r-1)},\\ 1 \le \nu_u \le M,\ (n_{j-1} + (r-1)) + t_j < u \le n\}.\end{aligned}$$

(2c) When $n = n_j$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_1+1)}, \ldots, \nu_{n_j}) :\\ \gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \le \nu_{n_j} < 2\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}\\ \text{and } 1 \le \nu_u \le M \text{ for } (n_{j-1} + (r-1)) + t_j < u < n_j\},\end{aligned}$$

where $z_j \in I_{l_j}(\nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1}, \ldots, \nu_{n_j-1})$.

(2d) When $n = n_j + i$, where $1 \le i \le r-2$,

$$\begin{aligned}\mathcal{D}_n = \{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1)}, \ldots, \nu_{n_j+i}) :\\ \gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \le \nu_{n_j+i} < 2\gamma_i^{\tau(z_j)S_{l_j}h(z_j)},\ \text{where } 1 \le i \le r-2,\\ \gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \le \nu_{n_j} < 2\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}\\ \text{and } 1 \le \nu_u \le M \text{ for } (n_{j-1} + (r-1)) + t_j < u < n_j\}.\end{aligned}$$

(2e) When $n = n_j + (r-1)$,

$$\begin{aligned}\mathcal{D}_n = \Big\{\nu = (\nu^{(j-1,r-1)}, \nu^{(j-1,r-1)}|_{t_j}, \nu_{(n_{j-1}+(r-1))+t_j+1)}, \ldots, \nu_{n_j}) :\\ \Big(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\Big)^{\tau(z_j)S_{l_j}h(z_j)} \le \nu_{n_j+(r-1)} < 2\Big(\frac{e}{\gamma_0\gamma_1\cdots\gamma_{r-2}}\Big)^{\tau(z_j)S_{l_j}h(z_j)},\\ \gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \le \nu_{n_j+i} < 2\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \text{ where } 1 \le i \le r-2,\\ \gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \le \nu_{n_j} < 2\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}\\ \text{and } 1 \le \nu_u \le M,\ \text{for } (n_{j-1} + (r-1)) + t_j < u < n_j\Big\}.\end{aligned}$$

**Fundamental cylinders.** For each $\nu = (\nu_1, \ldots, \nu_n) \in \mathcal{D}_n$, we define a refinement $J_n$ of $I_n$ as the union of its subcylinders with nonempty intersection with $\mathcal{E}_\infty$.

(3a) For $(n_{j-1} + (r-1)) + t_j < n < n_j + 1$, define

$$J_n(\nu) = \bigcup_{1 \le \nu_{n+1} \le M} I_{n+1}(\nu_1, \ldots, \nu_n, \nu_{n+1}). \tag{4-21}$$

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(3b) For $n = n_j - 1$, define

$$J_{n_j-1}(v) = \bigcup_{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} \le v_{n_j} < 2\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}} I_{n_j}(v_1, \ldots, v_{n_j-1}, v_{n_j}), \tag{4-22}$$

where $z_j \in I_{l_j}(v^{(j-1,r-1)}|_{t_j}, v_{(n_{j-1}+(r-1))+t_j+1}, \ldots, v_{n_j-1})$.

(3c) For $n = n_j + i - 1$ with $1 \le i \le r-2$, define

$$J_{n_j+i-1}(v) \bigcup_{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} \le v_{n_j+i} \le 2\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}} I_{n_j+i}(v_1, \ldots, v_{n_{j+i}}). \tag{4-23}$$

(3d) For $n = n_j + (r-2)$, define

$$J_{n_j+(r-2)}(v) = \bigcup_{\left(\frac{e}{\gamma_0\cdots\gamma_{r-2}}\right)^{\tau(z_j)S_{l_j}h(z_j)} \le v_{n_j+(r-1)} \le 2\left(\frac{e}{\gamma_0\cdots\gamma_{r-2}}\right)^{\tau(z_j)S_{l_j}h(z_j)}} I_{n_j+(r-1)}(w), \tag{4-24}$$

where $w = (v_1, \ldots, v_{n_j+(r-1)})$.

(3e) If $n_j + (r-1) \le n \le (n_j + (r-1)) + t_{j+1}$, then, by construction of $\mathcal{E}_\infty$,

$$J_n(v) = I_{n_j+(r-1)+t_{j+1}}(v^{(j,r-1)}, v^{(j,r-1)}|_{t_{j+1}}). \tag{4-25}$$

Clearly,

$$\mathcal{E}_\infty = \bigcap_{n=1}^{\infty} \bigcup_{v \in D_n} J_n(v).$$

*4.2.1. Lengths of fundamental cylinders.* In this subsubsection, we estimate the lengths of the fundamental cylinders defined above.

Let the continued fraction representation for any $x \in \mathcal{E}_\infty$ be

$$[v^{(j-1,r-1)}, v^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \ldots, b_{m_j}^{(j)}, a_{n_j}, \ldots, a_{n_j+(r-1)}, \ldots].$$

(I) If $n = (n_j + (r-1)) + t_{j+1}$, then, by using (2-8),

$$\begin{aligned} &q_{(n_j+(r-1))+t_{j+1}}(v^{(j-1,r-1)}, v^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \ldots, b_{m_j}^{(j)}, a_{n_j}, \ldots, a_{n_j+(r-1)}, v^{(j,r-1)}|_{t_{j+1}}) \\ &\quad \le 2^{3r+2} q_{(n_{j-1}+(r-1))+t_j}(v^{(j-1,r-1)}, v^{(j-1,r-1)}|_{t_j}) \cdot q_{m_j}(b_1^{(j)}, \ldots, b_{m_j}^{(j)}) \\ &\quad\quad \cdot e^{\tau(z_j)S_{l_j}h(z_j)} \cdot q_{t_{j+1}}(v^{(j,r-1)}|_{t_{j+1}}). \end{aligned}$$

Next, from (iii) of Proposition 2.1 and by the choice of $m_j$,

$$\begin{aligned} q_{(n_j+(r-1))+t_{j+1}}(x) &\le q_{(n_{j-1}+(r-1))+t_j}(x) \cdot (q_{l_j}(z_j) e^{\tau(z_j)S_{l_j}h(z_j)})^{1+\epsilon} \\ &\le \prod_{k=1}^{j} (q_{l_k}(z_k) e^{\tau(z_k)S_{l_k}h(z_k)})^{1+\epsilon}, \end{aligned} \tag{4-26}$$

where $z_k \in I_{l_k}(v^{(k-1,r-1)}|_{t_k}, b_1^{(k)}, \ldots, b_{m_k}^{(k)})$ for all $1 \le k \le j$.

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(II) If $(n_j + (r-1)) \le n < (n_j + (r-1)) + t_{j+1}$, then

$$q_n(x) \le q_{(n_j+(r-1))+t_{j+1}}(x) \le \prod_{k=1}^{j} (q_{l_k}(z_k) \cdot e^{\tau(z_k)S_{l_k}h(z_k)})^{1+\epsilon}.$$

(III) When $(n_{j-1} + (r-1)) + t_j \le n \le n_j - 1$, represent $n - (n_{j-1} + (r-1)) - t_j$ by $l$; then

$$\begin{aligned} q_n(x) &\le 2q_{(n_{j-1}+(r-1))+t_j}(x) \cdot q_l(b_1^{(j)}, \dots, b_l^{(j)}) \\ &\le \prod_{k=1}^{j-1} (q_{l_k}(z_k) e^{\tau(z_k)S_{l_k}h(z_k)})^{1+\epsilon} \cdot q_l(b_1^{(j)}, \dots, b_l^{(j)}). \end{aligned}$$

Now we calculate the lengths of fundamental cylinders for different cases (4-21)–(4-25) defined above.

(I) If $(n_{j-1} + (r-1)) \le n \le (n_{j-1} + (r-1)) + t_j$, then, from (2-7), (4-25) and (4-26),

$$\begin{aligned} |J_n(x)| = |I_{(n_{j-1}+(r-1))+t_j}(x)| &\ge \frac{1}{2q^2_{(n_{j-1}+(r-1))+t_j}(x)} \\ &\ge \frac{1}{2} \prod_{k=1}^{j-1} (q_{l_k}(z_k) \cdot e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)}. \end{aligned}$$

(II) If $(n_{j-1} + (r-1)) + t_j < n < n_j - 1$ and $l = n - (n_{j-1} + (r-1)) - t_j - 1$, then, from (2-7) and (4-21),

$$|J_n(x)| \ge \frac{1}{6q_n^2(x)} \ge \frac{1}{6} \prod_{k=1}^{j} (q_{l_k}(z_k) \cdot e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)} \cdot q_l^{-2}(b_1^{(j)}, \dots, b_l^{(j)}).$$

(III) If $n = n_j - 1$, then, following the similar steps as for I and using (4-22),

$$\begin{aligned} |J_{n_j-1}(x)| &\ge \frac{1}{6\nu_{n_j}(x) q^2_{n_j-1}(x)} \ge \frac{1}{6\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j-1}(x)} \\ &\ge \frac{1}{24\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} q^2_{l_j}(x)} \cdot \prod_{k=1}^{j-1} (q_{l_k}(z_k) \cdot e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)}. \end{aligned}$$

(IV) If $n = n_j + i - 1$ where $1 \le i \le r - 2$, then from (4-23) and following the similar steps as for I,

$$\begin{aligned} |J_{n_j+i-1}(x)| &\ge \frac{1}{6\nu_{n_j+i}(x) q^2_{n_j+i-1}(x)} \ge \frac{1}{6\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)} \\ &\ge \frac{1}{6 \cdot 4^i \gamma_i^{\tau(z_j)S_{l_j}h(z_j)} (\gamma_0 \cdots \gamma_{i-1})^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j-1}(x)} \end{aligned}$$

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$$\geq \frac{1}{6\cdot 4^i \gamma_i^{\tau(z_j)S_{l_j}h(z_j)}(\gamma_0\cdots\gamma_{i-1})^{2\tau(z_j)S_{l_j}h(z_j)} q_{l_j}^2(z_j)} \cdot \prod_{k=1}^{j-1}(q_{l_k}(z_k)e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)}.$$

(V) If $n = n_j + (r-2)$, then, from (4-24),

$$\begin{aligned}|J_{n_j+(r-2)}(x)| &\geq \frac{1}{6v_{n_j+(r-1)}(x)q_{n_j+(r-2)}^2(x)}\\ &\geq \frac{1}{6\cdot 4^{r-1}(e\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)} q_{n_j-1}^2(x)}\\ &\geq \frac{1}{6\cdot 4^{r}(e\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)} q_{l_j}^2(z_j)}\\ &\quad\cdot \prod_{k=1}^{j-1}(q_{l_k}(z_k)e^{\tau(z_k)S_{l_k}h(z_k)})^{-2(1+\epsilon)}.\end{aligned}$$

*4.2.2. Supporting measure.* We define a probability measure supported on the set $\mathcal{E}_\infty$.

Define $s_j := s^{(r)}_{(t_j,m_j),M}$ to be the solution of

$$\sum_{\substack{a_1=v_1^{(j-1,r-1)},\ldots,a_{t_j}=v_{t_j}^{(j-1,r-1)}\\ 1\leq b_1^{(j)},\ldots,b_{m_j}^{(j)}\leq M}} \frac{1}{e^{g_r(s)\tau(z_j)S_{l_j}h(z_j)} q_{l_j}^{2s}(z_j)} = 1,$$

where $z_j \in I_{l_j}(v^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \ldots, b_{m_j}^{(j)})$ and the sequences $\{t_j\}_{j\geq 1}$ and $\{m_j\}_{j\geq 1}$ are defined previously.

Consequently, from (4-18),

$$\sum_{\substack{a_1=v_1^{(j-1,r-1)},\ldots,a_{t_j}=v_{t_j}^{(j-1,r-1)}\\ 1\leq b_1^{(j)},\ldots,b_{m_j}^{(j)}\leq M}} \left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} q_{l_j}^2(z_j)}\right)^s = 1, \tag{4-27}$$

where $z_j \in I_{l_j}(v^{(j-1,r-1)}|_{t_j}, b_1^{(j)}, \ldots, b_{m_j}^{(j)})$.

Equality (4-27) induces a measure $\mu$ on a basic cylinder of order $t_j + m_j$ if we consider

$$\mu(I_{n_j+t_j}(a_1,\ldots,a_{t_j}, b_1^{(j)},\ldots,b_{m_j}^{(j)})) = \left(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} q_{l_j}^2(z_j)}\right)^{s_j}$$

for each $a_1 = v_1^{(j-1,r-1)}, \ldots, a_{t_j} = v_{t_j}^{(j-1,r-1)}, 1 \leq b_1^{(j)}, \ldots, b_{m_j}^{(j)} \leq M$.

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We start by assuming that the measure of $I_{n_{j-1}+(r-1)}(x) \in \mathcal{E}_\infty$ has been defined as

$$\mu(I_{n_{j-1}+(r-1)}(x)) = \prod_{k=1}^{j-1}\Big(\Big(\frac{1}{\gamma_0^{\tau(z_k)S_{l_k}h(z_k)} q_{l_k}^2(z_k)}\Big)^{s_k}\frac{1}{e^{\tau(z_k)S_{l_k}h(z_k)}}\Big),$$

where $z_k \in I_{l_k}(v^{(k-1,r-1)}|_{t_k}, b_1^{(k)}, \dots, b_{m_k}^{(k)})$ for all $1 \le k \le j-1$.

*Case 1.* $n_{j-1}+(r-1) < n \le n_{j-1}+(r-1)+t_j$. As the basic cylinder of order $n_{j-1}+(r-1)$ contains only one subcylinder of order $n$ with a nonempty intersection with $\mathcal{E}_\infty$,

$$\mu(I_n(x)) = \mu(I_{n_{j-1}+(r-1)}(x)).$$

*Case 2.* $n = n_j - 1$. Let

$$\mu(I_{n_j-1}(x)) = \mu(I_{n_{j-1}+(r-1)}(x)) \cdot \Big(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)} q_{l_j}^2(z_j)}\Big)^{s_j}.$$

Next we uniformly distribute the measure of $I_{n_j-1}(x)$ on its subcylinders.

*Case 3.* $n = n_j + i - 1$, where $1 \le i \le r-1$.

$$\begin{aligned}\mu(I_{n_j+i-1}(x)) &= \mu(I_{n_j+i-2}(x)) \cdot \frac{1}{\gamma_{i-1}^{\tau(z_j)S_{l_j}h(z_j)}}\\ &= \mu(I_{n_j-1}(x)) \cdot \frac{1}{(\gamma_0 \dots \gamma_{i-1})^{\tau(z_j)S_{l_j}h(z_j)}}.\end{aligned}$$

*Case 4.* $n = n_j + (r-1)$.

$$\begin{aligned}\mu(I_{n_j+r-1}(x)) &= \Big(\frac{\gamma_0 \dots \gamma_{r-2}}{e}\Big)^{\tau(z_j)S_{l_j}h(z_j)} \mu(I_{n_j+(r-2)}(x))\\ &= \Big(\frac{\gamma_0 \dots \gamma_{r-2}}{e}\Big)^{\tau(z_j)S_{l_j}h(z_j)} \frac{1}{(\gamma_0 \dots \gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}}\\ &\quad \cdot \mu(I_{n_j+(r-2)}(x))\\ &= \frac{1}{e^{\tau(z_j)S_{l_j}h(z_j)}}\mu(I_{n_j-1}(x)).\end{aligned}$$

The measure of other basic cylinders of order less than $n_j - 1$ is followed by the consistency property that a measure should satisfy.

For any $n_{j-1}+(r-1)+t_j < n \le n_j - 1$, let

$$\mu(I_n(x)) = \sum_{I_{n_j-1}(x) \subset I_n(x)} \mu(I_{n_j-1}(x)).$$

*4.2.3. The Hölder exponent of the measure $\mu$.* In this subsubsection, we compare the measure of fundamental cylinders with their lengths.

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*Case 1.* $n = n_j - 1$.

$$\begin{aligned}
\mu(J_{n_j-1}(x)) &= \prod_{k=1}^{j-1}\bigg(\bigg(\frac{1}{\gamma_0^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\bigg)^{s_k}\frac{1}{e^{\tau(z_k)S_{l_k}h(z_k)}}\bigg)\cdot\bigg(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\bigg)^{s_j}\\
&\le \prod_{k=1}^{j-1}\bigg(\frac{1}{\gamma_0^{s_k\tau(z_k)S_{l_k}h(z_k)}e^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^{2s_k}(z_k)}\bigg)\cdot\bigg(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\bigg)^{s_M^{(r)}-3\epsilon}\\
&\le \prod_{k=1}^{j-1}\bigg(\frac{1}{e^{2s_k\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^{2s_k}(z_k)}\bigg)\cdot\bigg(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\bigg)^{s_M^{(r)}-3\epsilon}\\
&\le \bigg(\prod_{k=1}^{j-1}\bigg(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\bigg)^{1+\epsilon}\bigg)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\cdot\bigg(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\bigg)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\le 24|J_{n_j-1}(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}.
\end{aligned}$$

From Corollary 3.4, we have $|s_j - s_M^{(r)}| \le 3\epsilon$, which further implies that $s_M^{(r)} - 3\epsilon \le s_j$. In (4-28), we have used the fact that since $1 \le \gamma_0 \dots \gamma_{r-2} \le e$ and $e/\gamma_0 \dots \gamma_{r-2} \ge (e/\gamma_0 \dots \gamma_{r-2})^s$ for any $0 < s < 1$, we have $e\gamma_0^s \ge e^{2s}$. Therefore, it is also true for $s_k$.

*Case 2.* $n = n_j + i - 1$, where $1 \le i \le r - 2$.

$$\begin{aligned}
\mu(J_{n_j+i-1}(x)) &= \mu(I_{n_j-1}(x))\cdot\frac{1}{(\gamma_0\cdots\gamma_{i-1})^{\tau(z_j)S_{l_j}h(z_j)}}\\
&\le \bigg(\prod_{k=1}^{j-1}\bigg(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\bigg)^{1+\epsilon}\bigg)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\quad\cdot\bigg(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\bigg)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\cdot\frac{1}{(\gamma_0\cdots\gamma_{i-1})^{\tau(z_j)S_{l_j}h(z_j)}}\\
&\le \bigg(\prod_{k=1}^{j-1}\bigg(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\bigg)^{1+\epsilon}\bigg)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} &\text{(4-28)}\\
&\quad\cdot\bigg(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\bigg)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\frac{1}{(\gamma_0(\gamma_1\cdots\gamma_{i-1})^2\gamma_i)^{\tau(z_j)S_{l_j}h(z_j)}} &\text{(4-29)}\\
&\le \bigg(\prod_{k=1}^{j-1}\bigg(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\bigg)^{1+\epsilon}\bigg)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\quad\cdot\bigg(\frac{1}{(\gamma_0\gamma_1\cdots\gamma_{i-1})^{2\tau(z_j)S_{l_j}h(z_j)}\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\bigg)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\le 6.4^{i+1}|J_{n_j+i-1}(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\le 6.4^{r-3}|J_{n_j+i-1}(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}.
\end{aligned}$$

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Equation (4-29) is obtained by using the fact that $1/\gamma_0\gamma_1\cdots\gamma_{i-1}\leq(1/\gamma_0(\gamma_1\cdots\gamma_{i-1})^2\gamma_i)^s$ for any $0<s<1$.

*Case 3.* $n=n_j+r-2$.

$$\begin{aligned}\mu(J_{n_j+r-2}(x)) &= \mu(J_{n_j-1}(x))\cdot\frac{1}{(\gamma_0\ldots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}}\\
&\leq\Big(\prod_{k=1}^{j-1}\Big(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\Big)^{1+\epsilon}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\cdot\Big(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\quad\cdot\Big(\frac{1}{(e\gamma_1\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}}\Big)^{s_j}\\
&\leq\Big(\prod_{k=1}^{j-1}\Big(\frac{1}{e^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\Big)^{1+\epsilon}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\quad\cdot\Big(\frac{1}{(e\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\leq 6.4^r|J_{n_j+r-2}(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}.\end{aligned}$$

*Case 4.* $n_j+(r-1)\leq n\leq n_j+t_{j+1}$.

$$\begin{aligned}\mu(J_n(x)) &= \prod_{k=1}^{j-1}\Big(\Big(\frac{1}{\gamma_0^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\Big)^{s_k}\frac{1}{e^{\tau(z_k)S_{l_k}h(z_k)}}\Big)\\
&\quad\cdot\Big(\frac{1}{\gamma_0^{\tau(z_j)S_{l_j}h(z_j)}q_{l_j}^2(z_j)}\Big)^{s_j}\frac{1}{e^{\tau(z_j)S_{l_j}h(z_j)}}\\
&= \prod_{k=1}^{j}\Big(\Big(\frac{1}{\gamma_0^{\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\Big)^{s_k}\frac{1}{e^{\tau(z_k)S_{l_k}h(z_k)}}\Big)\\
&\leq\prod_{k=1}^{j}\Big(\frac{1}{e_0^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\Big)^{s_k}\\
&\leq\prod_{k=1}^{j}\Big(\frac{1}{e_0^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\Big)^{(s_M^{(r)}-3\epsilon)}\\
&\leq\Big(\prod_{k=1}^{j}\Big(\frac{1}{e_0^{2\tau(z_k)S_{l_k}h(z_k)}q_{l_k}^2(z_k)}\Big)^{1+\epsilon}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\leq 2|J_n(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}.\end{aligned}$$

*4.2.4. Gap estimation.* Let $x\in\mathcal{E}_\infty$. In this subsubsection, we estimate the gap between $J_n(x)$ and its adjacent fundamental cylinder $J_n(x')$ of the same order $n$. Assume

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that $a_i(x) = a_i(x')$ for all $1 \le i < n$. These gaps are helpful for estimating the measure on general balls. Also, as $J_n(x)$ and $J_n(x')$ are adjacent, we have $|a_n(x) - a_n(x')| = 1$.

Let the left and the right gaps between $J_n(x)$ and its adjacent fundamental cylinder at each side be represented by $g_n^L(x)$ and $g_n^R(x)$, respectively.

Denote by $g_n^{L,R}(x)$ the minimum distance between $J_n(x)$ and its adjacent cylinder of the same order $n$, that is,

$$g_n^{L,R}(x) = \min\{g_n^L(x), g_n^R(x)\}.$$

Without loss of generality, we assume that $n$ is even and estimate $g_n^R(x)$ only, since if $n$ is odd then for $g_n^L(x)$ we can carry out the estimation in almost the same way.

*Gap I.* When $(n_{j-1} + (r-1)) + t_j < n < n_j - 1$ for all $j \ge 1$,

$$\begin{aligned} g_n^R(x) &\ge \sum_{a_{n+1}>M} |I_{n+1}(a_1, a_2, \ldots, a_{n-1}, a_n + 1, a_{n+1})| \\ &= \frac{(M+1)(p_n + p_{n-1}) + p_{n-1}}{(M+1)(q_n + q_{n-1}) + q_{n-1}} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \\ &= \frac{1}{((M+1)(q_n + q_{n-1}) + q_{n-1})(q_n + q_{n-1})} \\ &\ge \frac{1}{3Mq_n^2} \ge \frac{1}{3M}|I_n(x)|. \end{aligned}$$

*Gap II.* When $n = n_j + i - 1$, where $0 \le i \le r-2$,

$$\begin{aligned} g_n^R(x) &\ge \frac{p_n + p_{n-1}}{q_n + q_{n-1}} - \frac{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} p_n + p_{n-1}}{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q_n + q_{n-1}} \\ &= \frac{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} - 1}{(\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q_n + q_{n-1})(q_n + q_{n-1})} \ge \frac{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} - 1}{4\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q_n^2} \\ &\ge \frac{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}}{8\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q_n^2} \ge \frac{1}{8}|I_n(x)|. \end{aligned}$$

*Gap III.* When $n = n_j + r - 2$,

$$\begin{aligned} g_n^R(x) &\ge \frac{\left(\frac{e}{\gamma_0 \cdots \gamma_{r-2}}\right)^{\tau(z_j)S_{l_j}h(z_j)} - 1}{\left(\left(\frac{e}{\gamma_0 \cdots \gamma_{r-2}}\right)^{\tau(z_j)S_{l_j}h(z_j)} q_n + q_{n-1}\right)(q_n + q_{n-1})} \\ &\ge \frac{1}{8q_n^2} \ge \frac{1}{8}|I_n(x)|. \end{aligned}$$

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*Gap IV.* If $(n_j + (r-1)) \leq n \leq (n_j + (r-1)) + t_{j+1}$, then note that $J_n(x)$ is a small part of $I_{n_j+(r-1)}(x)$ as $I_{(n_j+(r-1))+t_{j+1}}(x) \subset I_{(n_j+(r-1))+2}(x)$. Therefore, the right gap is larger than the distance between the right end point of $J_n(x)$ and that of $I_{n_j+(r-1)}(x)$.

$$\begin{aligned} g_n^R(x) \geq |I_{(n_j+(r-1))+2}(x)| &\geq \frac{1}{2q^2_{(n_j+(r-1))+2}(x)} \geq \frac{1}{32a_1^2a_2^2q^2_{n_j+(r-1)}(x)} \\ &\geq \frac{1}{32a_1^2a_2^2q^2_{(n_j+(r-1))+t_j}(x)} \geq \frac{1}{32a_1^2a_2^2}|I_{(n_j+(r-1))+t_j}(x)| \\ &= \frac{1}{32a_1^2a_2^2}|J_{(n_j+(r-1))+t_j}(x)|, \end{aligned}$$

where $a_1$ represents $a_{(n_j+(r-1))+1}(x)$ and $a_2$ represents $a_{(n_j+(r-1))+2}(x)$.

*4.2.5. The measure $\mu$ on the general ball $B(x,d)$.* We now estimate the measure $\mu$ on any ball $B(x,d)$ with radius $d$ and centred at $x$. Fix $x \in \mathcal{E}_\infty$. There exists a unique sequence $(v_1, v_2, \ldots, v_n, \ldots)$ such that for each $n \geq 1$, $x \in J_n(v_1, \ldots, v_n)$, where $(v_1, \ldots, v_n) \in \mathcal{D}_n$ and $g_{n+1}^R(x) \leq d < g_n^R(x)$. Clearly, $B(x,d)$ can intersect only one fundamental cylinder of order $n$, that is, $J_n(v_1, \ldots, v_n)$.

*Case I.* $(n_{j-1} + (r-1)) + t_j < n < n_j - 1$ for all $j \geq 1$ or $n = n_j + t_{j+1}$. Since in this case $1 \leq a_n(x) \leq M$ and $|J_n(x)| \leq 1/q_n^2$,

$$\begin{aligned} \mu(B(x,d)) \leq \mu(J_n(x)) &\leq c|J_n(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} \\ &\leq c\Big(\frac{1}{q_n^2}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} \\ &\leq c4M^2\Big(\frac{1}{q_{n+1}^2}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} \\ &\leq c8M^2|I_{n+1}(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} \\ &\leq c48M^3(g_{n+1}^R(x))^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} \\ &\leq cc_0^3 d^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}. \end{aligned}$$

*Case II.* $n = n_j + i - 1$, where $0 \leq i \leq r-2$. Since

$$\frac{1}{8\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)}q^2_{n_j+i-1}(x)} \leq |I_{n_j+i}(x)| \leq 8g^R_{n_j+i}(x) \leq 8d$$

implies that

$$1 \leq 64d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)}q^2_{n_j+i-1}(x),$$

the number of fundamental cylinders of order $n_j + i$ contained in $J_{n_j+i-1}(x)$ that the ball $B(x,d)$ intersects is at most

$$\frac{2d}{|I_{n_j+i}(x)|} + 2 \le 16d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x) + 2^7 d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)} = c_0 d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x).$$

Therefore,

$$\begin{aligned}
&\mu(B(x,d))\\
&\le \min\{\mu(J_{n_j+i-1}(x)), c_0 d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)\mu(J_{n_j+i}(x))\}\\
&\le \mu(J_{n_j+i-1}(x))\min\Big\{1, c_0 d\gamma_i^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)\frac{1}{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)}}\Big\}\\
&\le 6\cdot 4^{i+1}|J_{n_j+i-1}(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\min\{1, c_0 d\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)\}\\
&\le c\Big(\frac{1}{\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x)}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} (c_0 d\gamma_i^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+i-1}(x))^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\le cc_0 d^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}.
\end{aligned}$$

Here we have used the fact that $\min\{a,b\} \le a^{1-s}b^s$ for any $a,b > 0$ and $0 \le s \le 1$.

*Case III.* $n = n_j + r - 2$. As

$$\frac{(\gamma_0\cdots\gamma_{r-2})^{2\tau(z_j)S_{l_j}h(z_j)}}{8e^{2\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+r-2}(x)} \le |I_{n_j+r-1}(x)| \le 8g^R_{n_j+r-1}(x) \le 8d,$$

the number of fundamental cylinders of order $n_j + r - 1$ contained in cylinder $J_{n_j+r-2}(x)$ that the ball $B(x,d)$ intersects is at most

$$\frac{2d}{|I_{n_j+r-1}(x)|} + 2 \le c_0 d \frac{e^{2\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{2\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-1}(x).$$

Therefore,

$$\begin{aligned}
&\mu(B(x,d))\\
&\le \min\Big\{\mu(J_{n_j+r-2}(x)), c_0 d\frac{e^{2\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{2\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-2}(x)\mu(J_{n_j+r-1}(x))\Big\}\\
&\le \mu(J_{n_j+r-2}(x))\\
&\quad\cdot\min\Big\{1, c_0 d\frac{e^{2\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{2\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-2}(x)\frac{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}}{e^{\tau(z_j)S_{l_j}h(z_j)}}\Big\}
\end{aligned}$$

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$$\begin{aligned}
&\leq 6\cdot 4^r |J_{n_j+r-2}(x)|^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} \min\Big\{1, c_0 d \frac{e^{\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-2}(x)\Big\}\\
&\leq c\Big(\frac{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}}{e^{\tau(z_j)S_{l_j}h(z_j)} q^2_{n_j+r-2}}\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)} \cdot \Big(c_0 d \frac{e^{\tau(z_j)S_{l_j}h(z_j)}}{(\gamma_0\cdots\gamma_{r-2})^{\tau(z_j)S_{l_j}h(z_j)}} q^2_{n_j+r-2}(x)\Big)^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}\\
&\leq c c_0 d^{(s_M^{(r)}-3\epsilon)/(1+\epsilon)}.
\end{aligned}$$

By combining all the above cases and using the mass distribution principle, we conclude that

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau) \geq \dim_{\mathrm{H}} \mathcal{E}_\infty \geq s_0 = \frac{s_M^{(r)} - 3\epsilon}{1+\epsilon}.$$

Since $\epsilon > 0$ is arbitrary, as $\epsilon \to 0$ we have $s_0 \to s_M^{(r)}$. Further, letting $M \to \infty$,

$$\dim_{\mathrm{H}} \mathcal{R}_r(\tau) \geq s_{\mathbb{N}}^{(r)}.$$

This completes the proof for the lower bound of Theorem 1.1.

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