STOCHASTIC DIFFERENTIAL EQUATIONS

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1. *Introduction.* The work of which this paper is an account began as a study of differential equations for functions whose values are random variables of finite variance. It was intended that all questions of convergence should be treated from the standpoint of strong convergence in Hilbert space—familiar to probabilists from the writings of Karhunen (11) and Loève (13) as *mean-square* convergence. The more general Banach-space approach now adopted was made possible by the discovery of a theorem (Theorem 1 of this paper) which Mr D. G. Kendall, its apparent author, kindly communicated to us.

Our results are stated as abstract Banach-space theorems, but the reader who prefers a probabilistic interpretation may think of the Banach space in the following way. If Ω is a space for which $\mathscr{B}$ (a Borel field of subsets) has been defined, and if μ is a probability measure on $\mathscr{B}$, then each Baire function $x(\cdot)$ on Ω to the set of real (or complex) numbers is a random variable. We shall denote by K_p ($p \geqslant 1$) the set of random variables $x(\cdot)$ such that

$$\int_\Omega |x|^p \, d\mu < \infty.$$

K_p can be partitioned into equivalence classes in the usual way by 'identifying' random variables which are almost certainly equal, i.e. equal almost everywhere (μ) in Ω. The set L_p of these equivalence classes is a Banach space when normed by

$$\| [x] \| = \left(\int_\Omega |x|^p \, d\mu \right)^{1/p},$$

where $[x]$ denotes the equivalence class of which x ($\in K_p$) is a member. In the discussion of particular integrals we prove our theorems only for weakly complete Banach spaces. We remark here that when $p > 1$ the space L_p is reflexive and therefore weakly complete, by a theorem of Pettis (17). For a discussion of weak completeness in the case $p = 1$ the paper of Kakutani (10) may be consulted.

Equations of the type we consider have been used in the study of certain stochastic processes in physics. Reviews of their applications in this field have been given by Chandrasekhar (1) and Moyal (14). One of the simplest important examples is the Langevin equation

$$d\dot{x}(t) + \alpha\dot{x}(t)\, dt = dz(t) \tag{1}$$

for the linear Brownian motion of a particle of unit mass. This equation describes the linear motion of a particle subjected to a series of random molecular impulses. The term $\alpha\dot{x}(t)\,dt$, in which $\alpha > 0$ is constant, represents the viscous drag of the fluid, i.e. the

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mean effect of the impulses. $z(\cdot)$ is a given function of the time t whose values are random variables, and is called the *impulse process*. Together with the term $\alpha\dot{x}(t)\,dt$, it describes the bombardment of the particle by molecules of the surrounding fluid. The inclusion of the term $\alpha\dot{x}(t)\,dt$ therefore allows us to assume that the expectation of $z(t)$ is zero for all t. We shall show that when the function z satisfies certain conditions it is possible to give a precise meaning to, and a rigorous solution of, equation (1). It is convenient at this point to explain why this equation is written in differential form.

In the usual model of Brownian motion, based on the classical kinetic theory of fluids, the impulse process z is a function whose values are Gaussian random variables with zero mean. If we identify random variables which are almost certainly equal, then the range of z spans a subspace of a Hilbert space L_2 and, moreover, z is a function with *orthogonal increments*, and is such that

$$\| z(t+h) - z(t) \|^2 = \sigma^2 \, | h |$$

for all real t and h, where σ is a real constant. It follows (Moyal (14)) that z is nowhere strongly differentiable. Hence if one writes down, as most of the early authors did, the 'natural' equation of motion

$$\ddot{x}(t) + \alpha\dot{x}(t) = \dot{z}(t), \tag{2}$$

one is embarrassed by the fact that the derivatives $\ddot{x}$ and $\dot{z}$ do not exist. In order to avoid this difficulty Doob (2) has suggested that, instead of (2), equation (1) (with a special convention about its differential notation) should be considered. A precise definition of its meaning will be given in §4. Our use differs slightly from that of Doob (2, 3), because his discussion is from the standpoint of almost certain convergence.

Apart from the generalization, already mentioned, to Banach space, it is also possible to consider, instead of (1), an nth-order linear equation, with coefficients which are given real or complex-valued continuous functions of t. For convenience we discuss only the second-order case, but our results may be extended to the case where $n \neq 2$ without difficulty. Simultaneous sets of linear equations could also be considered. There is no possibility of a further generalization, from the present standpoint, to non-linear equations unless X is a Banach algebra. We do not consider this situation.

In §2 we discuss the definition of bounded variation adopted in this paper. §3 is a brief account of the theory of the complementary function. The main theorems of the paper are to be found in §4. Two examples to illustrate the theory are given in the final section.

2. *Bounded variation in Banach space.* Before we consider differential equations, it is convenient to give a short account of some properties of vector-valued functions which are of bounded variation in the sense introduced by Dunford (4). The results of this section will be used later in our discussion of particular integrals.

Let X be a real or complex Banach space, and let X^* be its first adjoint. A function $z(\cdot)$ from the real line R^1 to X is said to be of *Dunford bounded variation* on the interval $[a, b]$ when there exists a finite real constant $K[a, b]$ such that, if $k \geqslant 1$ and

$$a \leqslant t_1 < t_2 < \ldots < t_{2k} \leqslant b,$$

then

$$\left\| \sum_{r=1}^{k} (z(t_{2r}) - z(t_{2r-1})) \right\| \leqslant K[a, b].$$

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The least such K will be denoted by $V[a,b]$ and will be called the *variation* of $z(\cdot)$ on $[a,b]$.

By an application of the uniform boundedness principle Dunford has shown that $z(\cdot)$ is of Dunford bounded variation on $[a,b]$ if and only if $x^*z(\cdot)$ is of bounded variation on $[a,b]$, in the ordinary sense, for each $x^* \in X^*$. By using a theorem of Gelfand (6) we have shown that, in the special case where X is a Hilbert space, $z(\cdot)$ is of Dunford bounded variation on $[a,b]$ if and only if the *covariance function*

$$\rho(t,s) \equiv (z(t), z(s))$$

is of bounded variation on $[a,b]^2$ in the sense of Fréchet (5). An account of this and some related theorems will be given in a forthcoming paper by one of us.

When X is a weakly complete Banach space the function $V[a,b]$ has an important property, which we shall now prove with the help of the theory of unconditional convergence. We recall that a series $\sum_{r=1}^{\infty} x_r$ of elements of X is said to be unconditionally convergent whenever all its subseries, with the order of terms undisturbed, converge strongly. If $\{n_r\}$ is any sequence of integers such that $n_{r+1} > n_r > n_0 = 0$ for $r = 1, 2, 3, \ldots$, and if $y_r = \sum_{i=n_{r-1}+1}^{n_r} x_i$, then the series $\sum_{r=1}^{\infty} y_r$ will be called a *bracketing* of $\sum_{r=1}^{\infty} x_r$. Pettis (16) has shown that if $\sum_{r=1}^{\infty} x_r$ is unconditionally convergent, and if $\sum_{r=1}^{\infty} y_r$ is any bracketing of $\sum_{r=1}^{\infty} x_r$, then $\| y_r \| \to 0$ as $r \to \infty$. It is also known from the work of Orlicz (15) that, when X is weakly complete, a sufficient condition for the unconditional convergence of $\sum_{r=1}^{\infty} x_r$ is that $\sum_{r=1}^{\infty} | x^*(x_r) | < \infty$ for each $x^* \in X^*$.

We need a lemma, first proved by Dunford (4) for real Banach spaces.

Lemma 1. *If $x_r \in X$ for $r = 1, 2, \ldots, N$, then*

$$\sum_{r=1}^{N} | x^*(x_r) | \leqslant 4K \| x^* \| \quad \textit{for each} \quad x^* \in X^*,$$

where $K = \sup_\sigma \| \sum_\sigma x_r \|$, the supremum being taken over all subsets σ of the set of integers $1, 2, \ldots, N$.

For each $x^* \in X^*$

$$\sum_{r=1}^{N} \Re x^*(x_r) = \Sigma^{(+)} \Re x^*(x_r) + \Sigma^{(-)} \Re x^*(x_r),$$

where the terms under $\Sigma^{(+)}$ are positive and those under $\Sigma^{(-)}$ non-positive. Then

$$\begin{aligned}\sum_{r=1}^{N} | \Re x^*(x_r) | &= \Sigma^{(+)} \Re x^*(x_r) - \Sigma^{(-)} \Re x^*(x_r) \\ &= \Re x^*(\Sigma^{(+)} x_r - \Sigma^{(-)} x_r) \leqslant 2K \| x^* \|.\end{aligned}$$

This, together with the corresponding result for the imaginary part, completes the proof.

THEOREM 1. *If $z(\cdot)$ assumes its values in a weakly complete Banach space X and is of Dunford bounded variation on $[t_0, t_1]$, where $t_0 < t_1$, and is strongly continuous-to-the-right at $t = t_0$, then $V[t_0, t_0+h]$ tends to zero as $h \to 0+$.*

Since $V[t_0, t]$ is a non-decreasing non-negative function of t $(t_0 < t \leqslant t_1)$, it follows that $\lim_{h\to 0+} V[t_0, t_0+h] = \delta$ exists and is non-negative. Suppose, if possible, that $\delta > 0$. Then we can choose a finite $n_1 \geqslant 1$, τ_{2n_1} and a set of t_j $(j = 2, 3, \ldots, (2n_1 - 1))$ satisfying

$$t_0 \leqslant \tau_{2n_1} < t_{2n_1-1} < \ldots < t_2 < t_1$$

and such that

$$\left\| \sum_{r=1}^{n_1-1} (z(t_{2r-1}) - z(t_{2r})) + (z(t_{2n_1-1}) - z(\tau_{2n_1})) \right\| > \tfrac{1}{2}\delta.$$

If $\tau_{2n_1} > t_0$ let $t_{2n_1} = \tau_{2n_1}$. If $\tau_{2n_1} = t_0$, choose t_{2n_1} in such a way that $t_0 < t_{2n_1} < t_{2n_1-1}$ and

$$\| z(t_{2n_1}) - z(\tau_{2n_1}) \| < \tfrac{1}{4}\delta.$$

The possibility of such a choice is a consequence of the strong right-hand continuity of $z(\cdot)$ at $t = t_0$. In either case we have

$$\left\| \sum_{r=1}^{n_1} (z(t_{2r-1}) - z(t_{2r})) \right\| > \tfrac{1}{4}\delta.$$

Now repeat this procedure with the interval $[t_0, t_{2n_1}]$ to define $t_{2n_1+1}, t_{2n_1+2}, \ldots, t_{2n_2}$ satisfying

$$t_0 < t_{2n_2} < \ldots < t_{2n_1+1} < t_{2n_1}$$

and such that

$$\left\| \sum_{r=n_1+1}^{n_2} (z(t_{2r-1}) - z(t_{2r})) \right\| > \tfrac{1}{4}\delta,$$

and so on. We obtain in this way a monotonic decreasing sequence $\{t_n\}$ bounded below by t_0. If we define

$$x_r = z(t_{2r-1}) - z(t_{2r}), \quad y_s = \sum_{i=n_{s-1}+1}^{n_s} x_i,$$

we see that $\sum_{s=1}^{\infty} y_s$, a bracketing of $\sum_{r=1}^{\infty} x_r$, has the property that $\| y_s \| > \frac{1}{4}\delta$ for all s. Hence, by the theorem of Pettis (16), $\sum_{r=1}^{\infty} x_r$ cannot converge unconditionally. On the other hand, using Lemma 1 we have

$$\sum_{r=1}^{N} | x^*(x_r) | \leqslant 4 \| x^* \| V[t_0, t_1],$$

for $N = 1, 2, \ldots$ and for each $x^* \in X^*$. It follows that $\sum_{r=1}^{\infty} | x^*(x_r) | < \infty$ for every $x^* \in X^*$, and so (from weak completeness) that $\sum_{r=1}^{\infty} x_r$ converges unconditionally. We therefore have a contradiction and conclude that $\delta = 0$. This completes the proof.

This important result appears to be due to Mr D. G. Kendall. It is not known whether the condition of weak completeness is a necessary one.

We shall use this theorem in conjunction with the following elementary property of the Riemann–Stieltjes integral. For an account of this integral Dunford (4) and Hille (8) may be consulted.

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THEOREM 2. *If $z(\cdot)$ assumes its values in a Banach space X and is of Dunford bounded variation on $[a,b]$, and if $\phi(\cdot)$ is a real or complex-valued function defined and continuous on $[a,b]$, then the strong Riemann–Stieltjes integral $\int_a^b \phi(t)\,dz(t)$ exists and satisfies*

$$\left\|\int_a^b \phi(t)\,dz(t)\right\| \leqslant 4 \sup_{a\leqslant t\leqslant b} |\phi(t)|\; V[a,b].$$

The existence of the integral has been proved by Dunford (4). Its value is the strong limit of a sequence of sums of the type $\sum_{i=1}^{n} \phi(\tau_i)\,[z(t_{i+1})-z(t_i)]$, where

$$a = t_1 < t_2 < \ldots < t_{n+1} = b$$

and $$t_i \leqslant \tau_i \leqslant t_{i+1} \quad \text{for} \quad i = 1,2,3,\ldots,n.$$

Applying Lemma 1 we have

$$\begin{aligned}\left\|\sum_{i=1}^{n} \phi(\tau_i)\,[z(t_{i+1})-z(t_i)]\right\| &= \sup_{\|x^*\|=1}\left|\sum_{i=1}^{n} \phi(\tau_i)\,x^*(z(t_{i+1})-z(t_i))\right| \\ &\leqslant \sup_{1\leqslant i\leqslant n} |\phi(\tau_i)| \sup_{\|x^*\|=1}\sum_{i=1}^{n} |x^*(z(t_{i+1})-z(t_i))| \\ &\leqslant 4 \sup_{a\leqslant t\leqslant b} |\phi(t)|\; V[a,b].\end{aligned}$$

Since this is true for each such sum we have

$$\left\|\int_a^b \phi(t)\,dz(t)\right\| \leqslant 4 \sup_{a\leqslant t\leqslant b} |\phi(t)|\; V[a,b].$$

3. *The complementary function.* It is proved in Hille (8) that the Banach-algebra analogue of the classical Cauchy–Lipschitz theorem is true. We require this theorem only for the special case of linear equations, and in this case the existence of a complementary function may be proved by a direct appeal to the classical theory, and the theorem obtained is valid for Banach *spaces*. We state the results of this section mainly for the sake of completeness.

THEOREM 3. *If X is a Banach space over the complex field and if $p(\cdot)$ and $q(\cdot)$ are complex-valued functions defined and continuous on $[0,T]$, where $0 < T < \infty$, then there exists exactly one function $x(\cdot)$ which assumes its values in X and is such that*

(i) *$\dot{x}$ exists as a strong derivative (one-sidedly at $t = 0$ and $t = T$) throughout $[0,T]$;*

(ii) *$x(0) = y_1$, $\dot{x}(0) = y_2$, where $y_1, y_2 \in X$;*

(iii) *$\ddot{x}$ exists as the strong derivative of $\dot{x}$ (one-sidedly at $t = 0$ and $t = T$) and satisfies $\ddot{x}(t) + p(t)\,\dot{x}(t) + q(t)\,x(t) = \theta$, whenever $0 \leqslant t \leqslant T$, where θ is the zero element of X.*

We know from the classical theorem, in which X is the set of complex numbers, that there exists a unique pair of twice differentiable complex-valued functions $u_1(\cdot)$ and $u_2(\cdot)$ which satisfy

$$\ddot{u}_i(t) + p(t)\,\dot{u}_i(t) + q(t)\,u_i(t) = 0 \quad (i = 1,2), \tag{3}$$

whenever $0 \leqslant t \leqslant T$, and are such that

$$u_1(0) = 1, \quad \dot{u}_1(0) = 0, \quad u_2(0) = 0, \quad \dot{u}_2(0) = 1. \tag{4}$$

It follows that $$x(t) = y_1 u_1(t) + y_2 u_2(t)$$

defines a function which satisfies the conditions of the theorem. The proof of the uniqueness of this solution is the strong convergence analogue of the classical proof. Our proof yields the immediate

COROLLARY. *The function $x(\cdot)$ defined in Theorem 3 assumes its values in the linear manifold spanned by y_1 and y_2.*

Except where special care seems necessary we shall omit from now on the remark about derivatives being one-sided at $t = 0$ and $t = T$.

The classical theory of the complementary function for the two-point boundary problem (see, for example, Ince (9)) admits of a similar generalization. As before, we suppose that $p(\cdot)$ and $q(\cdot)$ are complex-valued functions continuous on $[0, T]$, and we also suppose that $0 \leqslant a < b \leqslant T < \infty$. If $\alpha_1, \alpha'_1, \ldots, \beta_2, \beta'_2$ are complex constants such that the set of equations

$$\left.\begin{aligned} L(\xi) &\equiv \ddot{\xi}(t) + p(t)\,\dot{\xi}(t) + q(t)\,\xi(t) = 0, \\ U_1(\xi) &\equiv \alpha_1 \xi(a) + \alpha'_1 \dot{\xi}(a) + \beta_1 \xi(b) + \beta'_1 \dot{\xi}(b) = 0, \\ U_2(\xi) &\equiv \alpha_2 \xi(a) + \alpha'_2 \dot{\xi}(a) + \beta_2 \xi(b) + \beta'_2 \dot{\xi}(b) = 0, \end{aligned}\right\} \tag{5}$$

has no complex-valued solution $\xi(\cdot)$ on $[0, T]$, other than $\xi(t) \equiv 0$, then we say that these equations are *incompatible*. It is well known (Ince (9)) that, if $\{v_1, v_2\}$ is any fundamental system of complex-valued solutions on $[0, T]$ of $L(\xi) = 0$, then equations (5) are incompatible when and only when

$$\begin{vmatrix} U_1(v_1) & U_1(v_2) \\ U_2(v_1) & U_2(v_2) \end{vmatrix} \neq 0.$$

In particular, this will be true of $\Delta = \det[U_i(u_j)]$. Using this condition it is easy to prove

THEOREM 4. *If X is a Banach space over the complex field, and if equations (5) are incompatible then there exists a unique function $x(\cdot)$, assuming its values in X, and such that*

(i) *$\dot{x}$ exists as a strong derivative throughout $[0, T]$;*

(ii) *$U_1(x) = y_1$, $U_2(x) = y_2$, where $y_1, y_2 \in X$;* (6)

(iii) *$\ddot{x}$ exists as the strong derivative of $\dot{x}$ and satisfies $\ddot{x}(t) + p(t)\,\dot{x}(t) + q(t)\,x(t) = \theta$, whenever $0 \leqslant t \leqslant T$.*

If $x(\cdot)$ is any function from $[0, T]$ to X which satisfies conditions (i) and (iii) of this theorem then, by Theorem 3, it has a unique expression in the form

$$x(t) = x_1 u_1(t) + x_2 u_2(t),$$

where $x_1 = x(0)$ and $x_2 = \dot{x}(0)$. In order that (ii) should also be satisfied we must have

$$x_1 U_1(u_1) + x_2 U_1(u_2) = y_1, \quad x_1 U_2(u_1) + x_2 U_2(u_2) = y_2.$$

Since $\Delta \neq 0$ (because of incompatibility), these equations have a unique solution. It is easy to see that the function thus defined is the only one satisfying the conditions of the theorem.

If $\Delta = 0$ there may still be a solution, but it will no longer be unique. We shall not consider this case.

4. *The particular integral and the general solution.* We now give a precise meaning to the differential notation used in §1.

If $z(\cdot)$ is a given function defined on $[0, T]$, where $0 < T < \infty$, and assuming its values in a complex Banach space X, and if $p(\cdot)$ and $q(\cdot)$ are complex-valued functions defined and continuous on $[0, T]$, then the equation

$$d\dot{x}(t) + p(t)\,\dot{x}(t)\,dt + q(t)\,x(t)\,dt = dz(t) \tag{7}$$

will be said to possess a solution on $[0, T]$ satisfying the boundary conditions

$$x(0) = y_1, \quad \dot{x}(0) = y_2, \tag{8}$$

where $y_1, y_2 \in X$, whenever there exists a function $x(\cdot)$ defined on $[0, T]$, assuming its values in the closed linear manifold spanned by y_1 and y_2 together with the range of $z(\cdot)$, and such that

(i) $\dot{x}$ exists as a strong derivative and is strongly continuous on $[0, T]$;

(ii) x satisfies the boundary conditions (8);

(iii) $$[\dot{x}(t)]_{t_1}^{t_2} + \int_{t_1}^{t_2} p(t)\,\dot{x}(t)\,dt + \int_{t_1}^{t_2} q(t)\,x(t)\,dt = [z(t)]_{t_1}^{t_2} \tag{9}$$
whenever $0 \leqslant t_1 < t_2 \leqslant T$, where the integrals are taken in the Riemann-Graves sense (see Graves (7)).

Since the integrands in (9) are all strongly continuous, it is clear that condition (iii) can be replaced by

(iii*a*) The strong derivative $\dfrac{d}{dt}[\dot{x}(t) - z(t)]$ exists and satisfies

$$\frac{d}{dt}[\dot{x}(t) - z(t)] + p(t)\,\dot{x}(t) + q(t)\,x(t) = \theta$$

throughout $[0, T]$.

It follows from (iii*a*) and (i) that the strong continuity of z on $[0, T]$ is a necessary condition for equation (7) to have a solution. In our next theorem we state a set of conditions sufficient for equation (7) to have a solution in the sense of this definition, and we show that the solution is unique. We first prove

Lemma 2. *If $y(\cdot)$ is a strongly continuous function from $[0, T]$ to X and if*

$$\text{strong} \lim_{h \to 0+} \frac{w(t+h) - w(t)}{h} = y(t) \quad \textit{when } 0 \leqslant t < T,$$

then the strong derivative $\dot{w}$ exists (one-sidedly at $t = 0$ and $t = T$) and satisfies

$$\dot{w}(t) = y(t) \tag{10}$$

throughout $[0, T]$.

Let

$$\left.\begin{aligned} w_0(t) &= w(t) - \int_0^t y(\tau)\,d\tau \quad \text{when } 0 \leqslant t < T, \\ &= w(T) - \int_0^T y(\tau)\,d\tau \quad \text{when } t \geqslant T. \end{aligned}\right\}$$

Then

$$\text{strong} \lim_{h \to 0+} \frac{w_0(t+h) - w_0(t)}{h} = \theta \quad \text{for all } t \geqslant 0.$$

Hence

$$\lim_{h \to 0+} \frac{x^* w_0(t+h) - x^* w_0(t)}{h} = 0$$

for each $x^* \in X^*$ and all $t \geqslant 0$. Consequently, by a well-known theorem of Dini (see Saks (18)),

$$x^* w_0(t) = x^* w_0(0)$$

for all $t \geqslant 0$ and each $x^* \in X^*$, so that

$$w_0(t) = w_0(0) \quad \text{for all } t \geqslant 0.$$

It follows that when $0 < t < T$ the two-sided strong derivative $\dot{w}$ exists, and that $\dot{w}$ also exists as a right-hand strong derivative at $t = 0$ and as a left-hand strong derivative at $t = T$, and that (10) is satisfied throughout $[0, T]$.

The use of this lemma was suggested to us by Mr D. G. Kendall. One immediate consequence is that condition (iii) in the definition at the beginning of this section can be replaced by

(iii b) For each $t \in [0, T)$

$$\| \Delta_h \dot{x}(t) + hp(t)\dot{x}(t) + hq(t)x(t) - \Delta_h z(t) \| = o(h) \qquad \text{as } h \to 0+,$$

where $\Delta_h f(t) \equiv f(t+h) - f(t)$.

For it follows from the lemma, on putting

$$w(t) \equiv \dot{x}(t) - z(t), \quad y(t) \equiv -p(t)\dot{x}(t) - q(t)x(t),$$

that (iii a) and (iii b) are equivalent.

We are now able to prove the fundamental existence and uniqueness theorem for the one-point boundary problem.

Theorem 5. *If $z(\cdot)$ is a function defined on $[0, T]$, where $0 < T < \infty$, which assumes its values in a weakly complete Banach space X and which is strongly continuous and of Dunford bounded variation on $[0, T]$, then equation (7) has a unique solution on $[0, T]$ satisfying the boundary conditions (8).*

Let $g(t, \tau)$ be the Green function, so that

(i) $g(t, \tau)$ is continuous, together with $\partial g/\partial t$ and $\partial^2 g/\partial t^2$ on $[0, T]^2$;

(ii) $g(t, t) = 0$, $[\partial g/\partial t]_{\tau = t} = 1$ whenever $0 \leqslant t \leqslant T$;

(iii) $\dfrac{\partial^2 g}{\partial t^2} + p(t)\dfrac{\partial g}{\partial t} + q(t)\, g = 0$ throughout $[0, T]^2$.

It is well known (see, for example, Ince (9)) that the Green function has a unique expression in the form

$$g(t, \tau) = u_1(t)\, v_1(\tau) + u_2(t)\, v_2(\tau), \tag{11}$$

where $\{u_1, u_2\}$ is the canonical fundamental system defined by (3) and (4) and where $\{v_1, v_2\}$ is a pair of complex-valued functions, each continuous on $[0, T]$.

The following strong Riemann–Stieltjes integrals exist:

$$\left.\begin{aligned} x_k(t) &= \int_0^t g^{(k)}(t,\tau)\,dz(\tau), \\ \text{where} \qquad g^{(k)}(t,\tau) &= \frac{\partial^k}{\partial t^k} g(t,\tau). \end{aligned}\right\} \quad (k = 0,1,2). \tag{12}$$

We propose to show that $x_0(\cdot)$ is a particular integral; that is, a solution of (7) with boundary conditions

$$x_0(0) = \dot{x}_0(0) = \theta. \tag{13}$$

We first show that $x_1(t)$ is the strong derivative of $x_\theta(t)$ whenever $0 \leqslant t \leqslant T$. Using (11) we have, when $0 \leqslant t < t+h < T$,

$$\Delta_h x_0(t) - hx_1(t) = (\Delta_h u_1(t) - h\dot{u}_1(t)) \int_0^t v_1(\tau)\,dz(\tau) + (\Delta_h u_2(t) - h\dot{u}_2(t)) \int_0^t v_2(\tau)\,dz(\tau) + \int_t^{t+h} g(t+h,\tau)\,dz(\tau).$$

Consequently

$$\|\Delta_h x_0(t) - hx_1(t)\| \leqslant |\Delta_h u_1(t) - h\dot{u}_1(t)| \left\| \int_0^t v_1(\tau)\,dz(\tau) \right\| + |\Delta_h u_2(t) - h\dot{u}_2(t)| \left\| \int_0^t v_2(\tau)\,dz(\tau) \right\| + \left\| \int_t^{t+h} g(t+h,\tau)\,dz(\tau) \right\|.$$

For each fixed $t \in [0, T)$ the first two terms on the right are $o(h)$ as $h \to 0+$. Next, by Theorem 2,

$$\left\| \int_t^{t+h} g(t+h,\tau)\,dz(\tau) \right\| \leqslant 4 \sup_{t \leqslant \tau \leqslant t+h} |g(t+h,\tau)|\, V[t,t+h].$$

But, from Theorem 1,

$$\lim_{h \to 0+} V[t,t+h] = 0.$$

Moreover, since $g^{(1)}(t,\tau)$ is continuous in t for each fixed τ, we have, using the relation

$$g(\tau,\tau) = 0 \quad \text{for all } \tau,$$

$$|g(t+h,\tau)| = |g(t+h,\tau) - g(\tau,\tau)| = |h|\,|g^{(1)}(t+\theta h,\tau)|,$$

where $0 < \theta < 1$, whenever $t \leqslant \tau \leqslant t+h$. But $g^{(1)}$ is continuous, and hence bounded, on $[0,T]^2$, and so, combining these results,

$$\left\| \int_t^{t+h} g(t+h,\tau)\,dz(\tau) \right\| = o(h)$$

as $h \to 0+$, for each $t \epsilon [0,T)$.

In order to apply Lemma 2 we now prove that x_1 is strongly continuous on $[0,T]$. If $0 \leqslant s < t \leqslant T$, then

$$\|x_1(t) - x_1(s)\| \leqslant |\dot{u}_1(t) - \dot{u}_1(s)| \left\| \int_0^s v_1(\tau)\,dz(\tau) \right\| + |\dot{u}_2(t) - \dot{u}_2(s)| \left\| \int_0^s v_2(\tau)\,dz(\tau) \right\| + \left\| \int_s^t g^{(1)}(t,\tau)\,dz(\tau) \right\|.$$

So, by Theorem 2,

$$\| x_1(t) - x_1(s) \| \leqslant 4 \, | \dot{u}_1(t) - \dot{u}_1(s) | \sup_{[0,T]} | v_1(\tau) | \, V[0,T]$$
$$+ 4 \, | \dot{u}_2(t) - \dot{u}_2(s) | \sup_{[0,T]} | v_2(\tau) | \, V[0,T] + 4 \Big(\sup_{[0,T]^2} | g^{(1)}(t,\tau) | \Big) \, V[s,t].$$

It now follows from Theorem 1 and the continuity of the $\dot{u}_i$ and v_i on $[0,T]$ that $\| x_1(t) - x_1(s) \| \to 0$ as $t \to s$ with s fixed and as $s \to t$ with t fixed.

The function x_1 is therefore strongly continuous on $[0,T]$ and so, by Lemma 2, $\dot{x}_0$ exists as a strong derivative (one-sidedly at $t = 0$ and $t = T$) and satisfies

$$\dot{x}_0(t) = x_1(t)$$

throughout $[0,T]$.

Next, when $0 \leqslant t < t + h < T$,

$$\| \Delta_h x_1(t) - h x_2(t) - \Delta_h z(t) \|$$
$$\leqslant \left\| \int_0^t (g^{(1)}(t+h,\tau) - g^{(1)}(t,\tau) - h g^{(2)}(t,\tau)) \, dz(\tau) \right\|$$
$$+ \left\| \int_t^{t+h} (g^{(1)}(t+h,\tau) - 1) \, dz(\tau) \right\|.$$

As before, the first term on the right is $o(h)$, as $h \to 0+$. And, by another application of the mean-value theorem, using the fact that $g^{(1)}(\tau,\tau) = 1$ when $0 \leqslant \tau \leqslant T$, we can show that the second term is also $o(h)$ as $h \to 0+$. Thus

$$\| \Delta_h x_1(t) - h x_2(t) - \Delta_h z(t) \| = o(h),$$

as $h \to 0+$, for each $t \in [0,T)$. Using the relations

$$x_2(t) = \int_0^t g^{(2)}(t,\tau) \, dz(\tau) = \int_0^t (-p(t) \, g^{(1)}(t,\tau) - q(t) \, g(t,\tau)) \, dz(\tau)$$
$$= -p(t) \, x_1(t) - q(t) \, x_0(t)$$

and

$$x_1(t) = \dot{x}_0(t),$$

we obtain

$$\| \Delta_h \dot{x}_0(t) + h p(t) \, \dot{x}_0(t) + h q(t) \, x_0(t) - \Delta_h z(t) \| = o(h)$$

as $h \to O+$ for each $t \in [0,T)$.

Thus x_0 satisfies condition (iii$b$). It also satisfies condition (i) and the boundary conditions (13), and is therefore a particular integral. It is, in fact, the only particular integral. For suppose $x(\cdot)$ and $y(\cdot)$ are both particular integrals, and let

$$w(t) \equiv x(t) - y(t)$$

on $[0,T]$. Then $w(\cdot)$ is strongly differentiable and

$$\left\| \frac{1}{h} \Delta_h \dot{w}(t) + p(t) \dot{w}(t) + q(t) \, w(t) \right\| = o(1)$$

as $h \to 0+$, for each $t \in [0,T)$. But $\dot{w}$ is strongly continuous and so, by Lemma 2, $\ddot{w}$ exists as the strong derivative of $\dot{w}$ (one sidedly at $t = 0$ and $t = T$) and satisfies

$$\ddot{w}(t) + p(t) \dot{w}(t) + q(t) \, w(t) = \theta$$

for all $t \in [0, T]$. Moreover, $\dot{w}(0) = w(0) = \theta.$

Hence, by Theorem 3, the trivial solution $w(t) \equiv \theta$ on $[0, T]$ is also the only solution of these equations. The particular integral is therefore unique.

It follows at once, as in the classical theory, that the general solution of (7) with boundary conditions (8) is

$$x(t) = \int_0^t g(t,\tau)\, dz(\tau) + y_1 u_1(t) + y_2 u_2(t),$$

and that this solution is also unique. It is evident from the form of this solution that the range of the function x does lie in the closed linear manifold spanned by the range of z together with the vectors y_1 and y_2. The proof of Theorem 5 is thus complete.

In the discussion preceding Theorem 5 we may replace the boundary conditions (8) by the two-point boundary conditions (6). By a slightly more difficult argument we shall prove

THEOREM 6. *If $z(\cdot)$ is a function defined on $[0, T]$, where $0 < T < \infty$, which assumes its values in a weakly complete Banach space X and which is strongly continuous and of Dunford bounded variation on $[0, T]$, and if equations (5) are incompatible, then equation (7) possesses a unique solution satisfying the boundary conditions (6).*

We shall consider only the case $a \leq t \leq b$. The proof for the cases omitted

$$(0 \leq t < a, b < t \leq T)$$

is an easy modification of that given here.

The Green function for this problem is a complex-valued function $g(t, \tau)$ defined on $[a, b]^2$ and having the following properties:

(i) $\partial^2 g/\partial t^2$ exists and satisfies $\dfrac{\partial^2 g}{\partial t^2} + p(t)\dfrac{\partial g}{\partial t} + q(t)\, g = 0$ throughout $[a, b]^2$, except on the line $t = \tau$

(ii) $U_1(g) = U_2(g) = 0$ when $a < \tau < b$;

(iii) g is continuous on $[a, b]^2$ and $g^{(1)}$ is continuous on $[a, b]^2$ except on the line $t = \tau$, where it satisfies

$$g^{(1)}(\tau + 0, \tau) - g^{(1)}(\tau - 0, \tau) = 1$$

for all $\tau \in (a, b)$. (14)

Ince (9) gives an account of the construction of the Green function. Briefly, the procedure is as follows. Let

$$g_+(t,\tau) = u_1(t)\, v_1(\tau) + u_2(t)\, v_2(\tau) \quad g_-(t,\tau) = u_1(t)\, w_1(\tau) + u_2(t)\, w_2(\tau),$$

where v_1, v_2, w_1, w_2 are complex-valued and continuous on $[a, b]$. We show that these functions can be chosen in such a way that the function g defined by

$$\left.\begin{aligned} g(t,\tau) &= g_+(t,\tau) \quad \text{when } t \geq \tau \\ &= g_-(t,\tau) \quad \text{when } t < \tau \end{aligned}\right\}$$

has the desired properties. We first observe that g_+ and g_- both satisfy

$$\frac{\partial^2 h}{\partial t^2} + p(t)\frac{\partial h}{\partial t} + q(t)\, h = 0$$

throughout $[a,b]^2$. Condition (14) (iii) on the line $t=\tau$ yields the equations

$$\left.\begin{aligned}[][v_1(\tau)-w_1(\tau)]\,u_1(\tau)+[v_2(\tau)-w_2(\tau)]\,u_2(\tau) &= 0,\\ [v_1(\tau)-w_1(\tau)]\,\dot{u}_1(\tau)+[v_2(\tau)-\dot{w}_2(\tau)]\,\dot{u}_2(\tau) &= 1,\end{aligned}\right\} \tag{15}$$

for all $\tau\in(a,b)$. But, since $\{u_1,u_2\}$ is a fundamental system, its Wronskian W has the property that

$$\inf_{a\leqslant t\leqslant b}|\,W(t)\,| = \delta>0.$$

Thus equations (15) can be solved uniquely for the differences $[v_1(\tau)-w_1(\tau)]$ and $[v_2(\tau)-w_2(\tau)]$ for each $\tau\in[a,b]$. Next, from the conditions (14) (ii) we have, when $a<\tau<b$ (and hence, by continuity of the v_i and w_i, when $a\leqslant\tau\leqslant b$),

$$\sum_{j=1}^{2}[A_i(u_j)\,w_j(\tau)+B_i(u_j)\,v_j(\tau)] = 0 \quad (i=1,2)$$

or

$$\sum_{j=1}^{2}U_i(u_j)\,w_j(\tau) = -\sum_{j=1}^{2}B_i(u_j)\,[v_j(\tau)-w_j(\tau)] \quad (i=1,2), \tag{16}$$

where $A_i(\xi)=\alpha_i\xi(a)+\alpha_i'\dot{\xi}(a)$ and $B_i(\xi)=\beta_i\xi(b)+\beta_i'\dot{\xi}(b)$.

From §3 we know that $\det[U_i(u_j)]\neq 0$; and so, since we have already found the differences $[v_j(\tau)-w_j(\tau)]$, it follows that equations (16) may be solved uniquely for the $w_j(\tau)$ in terms of known functions which are continuous on $[a,b]$. Hence we can also find the $v_j(\tau)$. The function $g(t,\tau)$ thus obtained satisfies the conditions (14). We remark that these conditions imply that $g^{(2)}$ is continuous on $[a,b]^2$ except on the line $t=\tau$.

As in the one-point boundary problem, we can now define the strong Riemann–Stieltjes integrals*

$$x_k(t)=\int_a^b g^{(k)}(t,\tau)\,dz(\tau) \quad (k=0,1,2).$$

Then, if $a\leqslant t<t+h<b$

$$\begin{aligned}\|\,\Delta_h x_0(t)-hx_1(t)\,\| \leqslant{}& \left\|\int_a^t (g_+(t+h,\tau)-g_+(t,\tau)-hg_+^{(1)}(t,\tau))\,dz(\tau)\right\|\\ &+\left\|\int_t^b (g_-(t+h,\tau)-g_-(t,\tau)-hg_-^{(1)}(t,\tau))\,dz(\tau)\right\|\\ &+\left\|\int_t^{t+h}(g_+(t+h,\tau)-g_-(t+h,\tau))\,dz(\tau)\right\|.\end{aligned}$$

It can now be shown, as in the proof of Theorem 5, that the first two terms on the right are $o(h)$ as $h\to 0+$. Also, from the continuity of g on $[a,b]^2$

$$g_+(\tau,\tau)=g_-(\tau,\tau)$$

* For, by Theorem 2, $\int_a^t g_+^{(k)}(t,\tau)\,dz(\tau)$ exists. And (if $t<b$) by Theorems 1 and 2, $\int_{t+0}^b g_-^{(k)}(t,\tau)\,dz(\tau)$ exists and is equal to $\int_t^b g_-^{(k)}(t,\tau)\,dz(\tau)$.

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whenever $a \leqslant \tau \leqslant b$. Hence, by an application of Theorems 1 and 2 and the mean-value theorem,

$$\left\| \int_t^{t+h} (g_+(t+h,\tau) - g_-(t+h,\tau))\, dz(\tau) \right\|$$
$$\leqslant 4 \sup_{t \leqslant \tau \leqslant t+h} | g_+(t+h,\tau) - g_+(\tau,\tau) - g_-(t+h,\tau) + g_-(\tau,\tau) |\, V[t,t+h]$$
$$= o(h),$$

as $h \to 0+$ for each $t \in [a,b)$. As before, x_1 is strongly continuous on $[a,b]$; and so, using Lemma 2, we see that $\dot{x}_0$ exists as a strong derivative on $[a,b]$ (one-sidedly at $t = a$ and $t = b$) and that

$$\dot{x}_0(t) = x_1(t)$$

throughout $[a,b]$.

Next,
$$\| \Delta_h x_1(t) - hx_2(t) - \Delta_h z(t) \|$$
$$\leqslant \left\| \int_a^t (g_+^{(1)}(t+h,\tau) - g_+^{(1)}(t,\tau) - hg_+^{(2)}(t,\tau))\, dz(\tau) \right\|$$
$$+ \left\| \int_t^b (g_-^{(1)}(t+h,\tau) - g_-^{(1)}(t,\tau) - hg_-^{(2)}(t,\tau))\, dz(\tau) \right\|$$
$$+ \left\| \int_t^{t+h} (g_+^{(1)}(t+h,\tau) - g_-^{(1)}(t+h,\tau) - 1)\, dz(\tau) \right\|.$$

By the now familiar argument, the first two terms on the right are $o(h)$ as $h \to 0+$ for each $t \in [a,b)$. From condition (14) (iii) we have

$$g_+^{(1)}(\tau,\tau) - g_-^{(1)}(\tau,\tau) = 1 \quad \text{when } a \leqslant \tau \leqslant b,$$

so that, as before,

$$\left\| \int_t^{t+h} (g_+^{(1)}(t+h,\tau) - g_-^{(1)}(t+h,\tau) - 1)\, dz(\tau) \right\|$$
$$\leqslant 4 \sup_{t \leqslant \tau \leqslant t+h} | g_+^{(1)}(t+h,\tau) - g_-^{(1)}(t+h,\tau) - g_+^{(1)}(\tau,\tau) + g_-^{(1)}(\tau,\tau) |\, V[t,t+h]$$
$$= o(h),$$

as $h \to 0+$, for each $t \in [a,b)$. We have thus shown that

$$\| \Delta_h x_1(t) - hx_2(t) - \Delta_h z(t) \| = o(h)$$

as $h \to 0+$, for each $t \in [a,b)$. The argument may now be concluded, as in the proof of Theorem 5, to show that $x_0(\cdot)$ is a particular integral on $[a,b]$ (i.e. a solution of (7) with boundary conditions $U_1(x_0) = U_2(x_0) = \theta$). The general solution on $[a,b]$ is

$$x(t) = \int_a^b g(t,\tau)\, dz(\tau) + y_1 \phi_1(t) + y_2 \phi_2(t), \tag{17}$$

where $\{\phi_1, \phi_2\}$ is the unique pair of complex-valued solutions of

$$\ddot{\phi} + p\dot{\phi} + q\phi = 0$$

such that $U_i(\phi_j) = \delta_{ij}$. The uniqueness of this solution follows, using the same method as before, by an application of Theorem 4. It is evident that the function x of (17) assumes its values in the closed linear manifold spanned by the vectors y_1 and y_2 together with the range of z.

This, together with a similar argument for the cases $0 \leqslant t < a$ and $b < t \leqslant T$, completes the proof of Theorem 6.

5. *Examples.* A collection of formal solutions of well-known equations with one-point boundary conditions has been published by Moyal(14). The theorems of the present paper show these solutions to be valid, in the sense explained in §4, under the conditions stated by Moyal. In this section we therefore confine ourselves to the two-point boundary problem.

Example 1. *Simple Brownian motion.* The Langevin equation,

$$d\dot{x}(t) + \alpha\dot{x}(t)\,dt = dz(t) \quad (\alpha > 0), \tag{18}$$

has already been discussed. It is convenient to depart slightly from the notation of Theorem 6, *via* a change of origin and time scale, and to seek a solution of (18) on the interval $[-T, T]$ which satisfies the boundary conditions

$$x(t_0) = y_+, \quad x(-t_0) = y_-, \tag{19}$$

where $0 < t_0 \leqslant T$ and $y_+, y_- \in X$.

If $z(\cdot)$ assumes its values in a weakly complete Banach space X and is strongly continuous and of Dunford bounded variation on $[-T, T]$, then there is a unique solution given by

$$x(t) = \int_{-t_0}^{t_0} \frac{(1-e^{\alpha(t_0+\tau)})\,(e^{-\alpha t}-e^{-\alpha t_0})}{2\alpha \sinh \alpha t_0}\,dz(\tau) + \frac{1}{\alpha}\int_{t_0}^{t} (1-e^{-\alpha(t-\tau)})\,dz(\tau)$$
$$+ \frac{1}{2\sinh \alpha t_0}(y_+(e^{\alpha t_0}-e^{-\alpha t}) - y_-(e^{-\alpha t_0}-e^{-\alpha t})) \quad (-T \leqslant t \leqslant T). \tag{20}$$

In the usual discussions of (18), X is taken to be a Hilbert space of random variables of finite variance and z a random function whose expectation is everywhere zero, whose increments over disjoint intervals are orthogonal, and which satisfies

$$\| z(s) - z(t) \|^2 = \sigma^2 \,| s - t |,$$

where σ is a real constant, for all real s and t. z is thus everywhere strongly continuous, and it can also be shown that z is of Dunford bounded variation on every bounded interval. Hence T can be chosen arbitrarily large, and it follows in this case that the solution (20) is valid for all finite values of t. If y_+ and y_- are taken to be random variables of zero variance (i.e. random variables which are almost certainly constant), we find that, when $s = t + \tau$ with $\tau\,(>0)$ constant,

$$\mathscr{E}[\dot{x}(t)] \to 0 \quad \text{as } t \to \infty,$$

where $\mathscr{E}[\cdot]$ denotes the mathematical expectation, and

$$(\dot{x}(s), \dot{x}(t)) = e^{-\alpha(s-t)}(\dot{x}(t), \dot{x}(t)) \to \frac{\sigma^2}{2\alpha} e^{-\alpha\tau} \quad \text{as } t \to \infty.$$

Thus $\dot{x}(t)$ tends towards a process stationary to the second order with zero mean and covariance

$$\mu(s, t) = \frac{\sigma^2}{2\alpha} e^{-\alpha|s-t|}.$$

Example 2. *The Brownian oscillator* has the following equation:

$$d\dot{x}(t) + 2\alpha\dot{x}(t)\,dt + \omega_0^2 x(t)\,dt = dz(t) \quad (0 < \alpha < \omega_0). \tag{21}$$

This represents the linear motion of a particle of unit mass subject to the same random impulse process as in Example 1 and also to the viscous drag $2\alpha\dot{x}(t)$ and the linear restoring force $\omega_0^2 x(t)$.

When $z(\cdot)$ satisfies the conditions stated in Example 1, equation (21) with boundary conditions (19) has the solution (when $-T \leqslant t \leqslant T$)

$$x(t) = \frac{1}{\omega}\int_{-t_0}^{t_0} e^{-\alpha(t-\tau)} \frac{\sin\omega(t-t_0)\sin\omega(t_0+\tau)}{\sin 2\omega t_0}\, dz(\tau) + \frac{1}{\omega}\int_{t_0}^{t} e^{-\alpha(t-\tau)} \sin\omega(t-\tau)\, dz(\tau) + \frac{e^{-\alpha t}}{\sin 2\omega t_0}(y_+ e^{\alpha t_0}\sin\omega(t+t_0) - y_- e^{-\alpha t_0}\sin(t-t_0)),$$

where $\omega = (\omega_0^2 - \alpha^2)^{\frac{1}{2}}$.

In the special case of Hilbert space it can be shown, on making the same hypotheses as in Example 1, that

$$\mathscr{E}[\dot{x}(t)] \to 0 \quad \text{as } t \to \infty,$$

and

$$(\dot{x}(s), \dot{x}(t)) \to \frac{\sigma^2}{4\alpha} e^{-\alpha\tau}\left(\cos\omega\tau - \frac{\alpha}{\omega}\sin\omega\tau\right) \quad \text{as } t \to \infty.$$

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