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SET FORCING AND STRONG CONDENSATION FOR $H(\omega_2)$

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Abstract. The Axiom of Strong Condensation, first introduced by Woodin in [14], is an abstract version of the Condensation Lemma of L. In this paper, we construct a set-sized forcing to obtain Strong Condensation for $H(\omega_2)$. As an application, we show that "ZFC + Axiom of Strong Condensation + $\neg\square_{\omega_1}$" is consistent, which answers a question in [14]. As another application, we give a partial answer to a question of Jech by proving that "ZFC + there is a supercompact cardinal + any ideal on ω_1 which is definable over $H(\omega_2)$ is not precipitous" is consistent under sufficient large cardinal assumptions.

§1. Introduction. In this paper we investigate the Axiom of Strong Condensation, which is an abstract version of Gödel's Condensation Lemma proposed by Woodin. As a fundamental feature of the Constructible Universe L, the Condensation Lemma has immense consequences on the consistency of mathematical statements inside and outside of set theory.

The purpose of our study is to discover to what extent the Axiom of Strong Condensation and its localized versions capture the power of the Condensation Lemma. There are two main objectives. The first objective is to explore the consequences of the Axiom of Strong Condensation. This is almost achieved in the work of Law ([10]) and Woodin ([14]). Their results suggest that the appearance of the Condensation Lemma can be replaced by the Axiom of Strong Condensation in most arguments and constructions. The second objective, which is also the main focus of this paper, is to seek for theorems of "ZFC+$V = L$" which cannot be derived from Strong Condensation alone.

For this purpose, we need to examine various models of Strong Condensation derived from different approaches. L is such a model, which is certainly of no interest here. On the other hand, there are also several known constructions of nontrivial models of Strong Condensation, including the models from Beller–Jensen–Welch ([2]) and Woodin ([14]). However, their models, satisfying forms of "fine-structural" properties, are very similar to L. As the main result of this paper, we construct a not so "fine-structural" model of Strong Condensation for $H(\omega_2)$ using set-sized forcing[1]:

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[1]It was independently shown by Sy Friedman that Strong Condensation for $H(\omega_2)$ holds in the forcing extension in [7].

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THEOREM 1.1. *Assume CH and $2^{\omega_1} = \omega_2$. Then there is a cardinal-preserving set forcing $\mathbb{P}$ which forces Strong Condensation for $H(\omega_2)$.*

As the main application of Theorem 1.1, we study the relationship between the Axiom of Strong Condensation and $\square_{\omega_1}$. $\square_{\omega_1}$ is certainly a natural candidate for the second objective, since it seems impossible to prove $\square_{\omega_1}$ using the Condensation Lemma alone. The first proof that $\square_{\omega_1}$ holds in L is to be found in Jensen's ground breaking paper ([9]) which also gives birth to both the square principle and fine structure theory. It was then followed by a proof of Silver using Silver machines ([4]). Finally, Friedman–Koepke ([6]) gives a proof using hyperfine structure. All of these proofs involve some forms of fine structure. Woodin asked whether the converse is also true, i.e., whether the Axiom of Strong Condensation is consistent with the failure of $\square_{\omega_1}$. We answer this question affirmatively:

THEOREM 1.2. *Assume there exists κ such that the set*

$$S = \{\eta < \kappa \mid \eta \text{ is a measurable cardinal}\}$$

is stationary below κ, then ZFC+ Axiom of Strong Condensation $+\neg\square_{\omega_1}$ is consistent.

Theorem 1.1 also has some effect on results related to large cardinals. Jech asked whether certain large cardinals entail the existence of a precipitous ideal on ω_1 in the same model.[2] As the second application of Theorem 1.1, we construct a model in which a supercompact cardinal exists and no ideal definable over $H(\omega_2)$ is precipitous. This generalizes previous results of Foreman–Magidor-Shelah ([5]), Schimmerling–Velickovic ([11]), Woodin ([14]).

The article is organized as follows. In Section 2, we present the basic definitions and a summary of the background. We also prove several lemmas which will be used in later sections. In Section 3, we provide the forcing construction for the main theorem. In Section 4, we construct a model in which the Axiom of Strong Condensation holds and $\square_{\omega_1}$ fails. In Section 5, we study the application on precipitous ideals. We prove that supercompact cardinals do not entail any precipitous ideal on ω_1 definable over $H(\omega_2)$ and show that this result is somewhat optimal. In Section 6, we give some final remarks.

Most of the notations in this paper are standard. For a set S, we use $P_{\omega_1}(S)$ to denote the set of countable subsets of S and $tc(S)$ to denote the transitive closure of S. For any $X \prec M$, $a \in X$, $P \subset M$, let $\bar{X}$ stand for the transitive collapse of X, $\pi_X : X \to \bar{X}$ be the collapsing map, a_X be the image of a under the collapsing map and P_X the pointwise image of P under the collapsing map. For any structure $\langle X, P\rangle$, if it is clear from the context, we always identify the structure with its underlying set X. For any function F, we write $\mathrm{dom}(F)$ for the domain of F, $\mathrm{ran}(F)$ for the range of F and $\mathrm{Field}(F)$ for $\mathrm{dom}(F)\cup\mathrm{ran}(F)$. If $X \subset \mathrm{dom}(F)$, we let $F[X]$ denote the set $\{a \in \mathrm{ran}(F) \mid (\exists b \in X) a = F(b)\}$. We use $\mathrm{Add}(\omega_1, 1)$ to denote the forcing to add one ω_1-Cohen set. Our treatment of iterated forcing is based on [3].

§2. Preliminaries. The study of abstract condensation properties was initialized by Woodin in [14]. Most of the content in this section is due to Woodin. However,

[2]Note that Jech's question is not a question about large cardinal strength. He proved ([8]) that the existence of a precipitous ideal is equiconsistent to the existence of one measurable cardinal.

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some of these facts never appeared in the literature, let alone their proofs. In this section, we try to systematically summarize the known facts about the Strong Condensation property and present their proofs, which were not yet available. We believe that the proofs in this section are mostly identical to the original unpublished proofs by Woodin.

2.1. Strong condensation. In [14], Woodin defines the Strong Condensation property.

DEFINITION 2.1 ([14]). Suppose that M is a transitive set closed under the Gödel operations and that

$$F : Ord \cap M \to M$$

is a bijection.

We say that the function F witnesses *Strong Condensation* for M if for any $X \prec \langle M, F\rangle$,

$$F_X = F \restriction (Ord \cap \bar{X}).$$

We say that *Strong Condensation* holds for M if such an F exists.

He also defines the following global version.

DEFINITION 2.2 ([14]). The *Axiom of Strong Condensation* is the statement that for each regular cardinal κ, Strong Condensation holds for $H(\kappa)$.

From now on, we will abbreviate Strong Condensation by SC, SC for $H(\kappa)$ by SC_κ and the Axiom of Strong Condensation by ASC. L is the canonical model of ASC. At first glance, ASC does not capture the full strength of the condensation property in L, i.e., it does not assert the existence of a global bijection $F : \mathrm{Ord} \to V$ such that $F \restriction \kappa$ witnesses SC_κ for all uncountable regular κ, while in L, $<_L$ induces such a bijection. Nevertheless, it turns out that such a bijection exists.

FACT 2.3.

(1) *If $\kappa < \eta$ are two uncountable regular cardinals, then SC_η implies SC_κ. In particular, if F witnesses SC_η, then $F \restriction \kappa$ witnesses SC_κ.*
(2) *If F_1 and F_2 both witness SC_κ for some regular cardinal κ, then*

$$F_1 = F_2 \leftrightarrow F_1 \restriction \omega_1 = F_2 \restriction \omega_1.$$

Fact 2.3(1) is a direct corollary of Lemma 2.6 below. Fact 2.3(2) is clear.

COROLLARY 2.4. *Assume ASC. Then there is an $A \subseteq \omega_1$, such that $V = HOD[A]$. In particular, there is a $\Delta_1(A)$-definable global well-ordering.*

PROOF. Define a class function $f : Reg \setminus \{\omega\} \to V$ by letting $f(\kappa)$ be the set of all functions witnessing SC_κ. Let

$$K = \{F \in P(H(\omega_1)) \mid \forall\alpha\exists\kappa\exists F'(\kappa > \alpha \wedge F' \in f(\kappa) \wedge F' \restriction \omega_1 = F)\}.$$

Since for any κ, $f(\kappa)$ is nonempty, K is also nonempty. By Fact 2.3(1), for each $F \in K$ and all regular κ, there is an $F' \in f(\kappa)$ such that $F' \restriction \omega_1 = F$. By Fact 2.3(2), for any $F \in K$ and any κ, there is a unique $F' \in f(\kappa)$ such that $F' \restriction \omega_1 = F$. Fix some $F \in K$. Now let F'' be a class function with domain Ord such that for all regular κ, $F'' \restriction \kappa$ is the unique $F' \in f(\kappa)$ such that $F' \restriction \omega_1 = F$. It follows that $F'' : \mathrm{Ord} \to V$ is a global bijection witnessing ASC.

It is straightforward to verify that $V = \mathrm{HOD}[F]$. Since ASC trivially implies GCH, the corollary follows by coding F into a subset A of ω_1. $\dashv$

Like L, SC is absolute between models.

THEOREM 2.5 ([14]). *Suppose that M is a transitive set closed under the Gödel operations and $F : Ord \cap M \to M$ is a bijection. Suppose that N is a transitive inner model such that*

(1) *$N \models ZC + \Sigma_1$-Replacement,*
(2) *$\{M, F\} \subset N$,*
(3) *F witnesses SC for M in N.*

Then F witnesses SC for M.

ASC is arguably the strongest abstract condensation property that can be extracted from L, as all truths in L whose known proofs only involve Gödel's Condensation Lemma remain true in models of ASC.[3] We list some of them in Table 1.[4]

The proof of Fact 2.3 relies heavily on the following characterization of SC. This characterization will be used throughout this paper. In particular, in the proof of Theorem 1.1, we will force an F witnessing this characterization.

LEMMA 2.6. *Assume GCH. For any regular cardinal $\kappa > \omega_1$, the following are equivalent:*

(1) *SC_κ.*
(2) *there is a bijection F from κ to $H(\kappa)$ and a club C of $P_{\omega_1}(H(\kappa))$ such that for every $X \in C$, X is a countable elementary submodel of $\langle H(\kappa), F\rangle$, and $F_X \subset F$.*

L	$M \models$ ASC
L_α-hierarchies	$F[\alpha]$-hierarchies
Acceptability(GCH)	F-Acceptability(GCH)
$\diamondsuit_\kappa$	$\diamondsuit_\kappa$ ([14])
no ω_1-Erdős cardinal	no ω_1-Erdős cardinal ([7], [14])
no precipitous ideal	no precipitous ideal ([10])
Δ_1 global well-ordering	$\Delta_1(A)$ global well-ordering for some $A \subset \omega_1$ (§2)
$0^\sharp$ exists iff $\exists j : L \prec L$ nontrivial	$M^\sharp$ exists iff $\exists j : M \prec M$ nontrivial

TABLE 1. Comparison between L and models of ASC.

[3] See [14] and [7] for the definition of various weaker forms of the condensation principle and their relationship with ASC.

[4] *Acceptability* is the following statement: If there is a subset of δ in $L_{\gamma+1} \setminus L_\gamma$, then there is a surjection of δ onto L_γ in $L_{\gamma+1}$. *F-Acceptability* is similarly obtained by replacing the L_α-hierarchies by the $F[\alpha]$-hierarchies. The $F[\alpha]$-hierarchies consist of all $F[\alpha]$ such that $F[\alpha]$ is transitive, closed under Gödel operations and $F \upharpoonright \alpha$ witnesses SC for $F[\alpha]$.

$M^\sharp$ can be defined as the set of true sentences of $M = L[A]$ with ω many order indiscernibles and all ordinals $\alpha < \omega_1$ as constants. Here although A is a class predicate, we only care about the information from its restriction $A \cap \omega_1$. This is because under ASC, $A \cap \omega_1$ captures the information of A up to arbitrary height. For the exact definition see [13]. Also see Section 2.2 for some basic facts about sharps.

Woodin proves that in any model M of the Axiom of Condensation (a weaker abstract condensation property, see [14] for the definition), there is no precipitous ideal (a proof can be found in [10]). It follows that any model of ASC contains no precipitous ideal.

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Proof. Clearly, (1) implies (2). We will prove that (2) also implies (1). We first reduce the requirement in the definition of SC_κ:

Claim 2.7. *If F is a bijection from κ to $H(\kappa)$ such that for every countable elementary submodel X of $\langle H(\kappa), F\rangle$, $F_X \subset F$ holds, then F witnesses SC_κ.*

Proof. Assume the claim fails and let $F : \kappa \to H(\kappa)$ be a witness. By the definition of SC_κ, there is an uncountable elementary submodel X of $\langle H(\kappa), F\rangle$ such that $F_X \not\subset F$. We can find $x, y \in X$ such that $(x <_F y \wedge \pi_X(y) <_F \pi_X(x))$, where $<_F$ is the well-ordering on $H(\kappa)$ derived from F.

Fix such a pair $\{x, y\}$ and let δ be a sufficiently large regular cardinal. Choose a countable $K \prec H(\delta)$ such that $\{\kappa, F, x, y, X\} \subset K$. By elementarity, $K \models x <_F y \wedge \pi_X(y) <_F \pi_X(x)$. As $F, X \in K$, $\langle K \cap X, F \cap K \cap X\rangle \prec \langle H(\kappa), F\rangle$. We write X^K for $\langle K \cap X, F \cap K \cap X\rangle$. Since $x <_F y$ and X^K is a countable elementary submodel of $\langle H(\kappa), F\rangle$, by our requirement on F, $\pi_{X^K}(x) <_F \pi_{X^K}(y)$. On the other hand, since $K \models \pi_X(y) <_F \pi_X(x)$, $\langle K \cap H(\kappa), F \cap K\rangle \models \pi_X(y) <_F \pi_X(x)$. By the Tarski criterion, we have $\langle K \cap H(\kappa), F \cap K\rangle \prec \langle H(\kappa), F\rangle$. Using the assumption and the countability of $K \cap H(\kappa)$ again, $\pi_{K\cap H(\kappa)}(\pi_X(y)) <_F \pi_{K\cap H(\kappa)}(\pi_X(x))$.

Via an induction on the rank of elements of X^K, we show that for all $z \in X^K$, $\pi_{X^K}(z) = \pi_K(\pi_X(z))$ as follows: If $a \in \pi_{X^K}(z)$, then there is an $a' \in X^K$ such that $\pi_{X^K}(a') = a$ and $a' \in z$. By the induction hypothesis, $a = \pi_{X^K}(a') = \pi_K(\pi_X(a'))$. However, as $a', z \in X^K$, $a' \in z \to \pi_X(a') \in \pi_X(z) \to \pi_K(\pi_X(a')) \in \pi_K(\pi_X(z))$. Hence $a \in \pi_K(\pi_X(z))$.

Now assume $a \in \pi_K(\pi_X(z))$. Then there is an $a' \in K$ such that $a = \pi_K(a')$ and $a' \in \pi_X(z)$. Since $\pi_X(z) \in H(\kappa)_X$ and $H(\kappa)_X$ is transitive, $a' \in H(\kappa)_X$. Let $a'' \in X$ be such that $\pi_X(a'') = a'$. Since $X \in K$, by elementarity $a'' \in K$ and $a'' \in z$. By the induction hypothesis, $a = \pi_K(\pi_X(a'')) = \pi_{X^K}(a'') \in \pi_{X^K}(z)$.

In conclusion, $\pi_{X^K}(y) = \pi_K(\pi_X(y)) = \pi_{K\cap H(\kappa)}(\pi_X(y)) <_F \pi_{K\cap H(\kappa)}(\pi_X(x)) = \pi_K(\pi_X(x)) = \pi_{X^K}(x)$. This leads to a contradiction. ⊣

Returning to the proof of the lemma, let C and F be as stated in the lemma. By our claim, we need to prove that $F_X \subset F$ for all countable $X \prec \langle H(\kappa), F\rangle$. Since $\kappa > \omega_1$, it suffices to show that for all $\alpha \in X \cap \mathrm{Ord}$, if $F[\alpha]$ is transitive and of uncountable size, then $(F \upharpoonright \alpha)_X = F_{F[\alpha]\cap X} \subset F$.

Fix one such $\alpha \in X$ and pick $D \subset C \upharpoonright F[\alpha]$ to be a club of $P_{\omega_1}(F[\alpha])$. This is possible since $|F[\alpha]| > \omega$ and C is a club. Without loss of generality, we can assume that there is a function $d : (F[\alpha])^{<\omega} \to F[\alpha]$ such that $D = \{A \subset F[\alpha] \mid d[A^{<\omega}] \subset A\}$. Now for each K closed under d, there is a countable $Y \prec \langle H(\kappa), F\rangle$ such that $K = Y \cap F[\alpha]$ and $F_Y \subset F$. Moreover, since $F[\alpha]$ is transitive, F_K is an initial segment of F_Y. Thus $F_K \subset F \upharpoonright \alpha$. As $|\alpha| < \kappa$, $d \in H(\kappa)$, so $\langle H(\kappa), F\rangle \models$ there is a function $d : (F[\alpha])^{<\omega} \to F[\alpha]$ such that whenever $K \in P_{\omega_1}(F[\alpha])$ is closed under d then $F_K \subset F \upharpoonright \alpha$. By elementarity, the last statement is also true in X. Fix a witnessing function $d_X \in X$. Then by elementarity, $F[\alpha] \cap X$ is closed under d_X, which means $F_{F[\alpha]\cap X} \subset F \upharpoonright \alpha \subset F$. ⊣

By examining the above proof, it can be seen that a version of Lemma 2.6 remains true when replacing $H(\kappa)$ by any M which is an uncountable transitive model such that $H(\omega_1) \in M$ and $M \prec H(\kappa)$. In this situation, SC for M holds iff there is a bijection F from Ord^M to M and a club C of $P_{\omega_1}(M)$ such that for every $X \in C$, X is a countable elementary submodel of $\langle M, F\rangle$, and $F_X \subset F$. This implies that if

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SC_κ holds, then there are unboundedly many $\alpha < \kappa$ such that SC holds for $F[\alpha]$. This justifies the definition of $F[\alpha]$-hierarchies.

2.2. Models of SC. We summarize some approaches to construct models of SC. Clearly, for any real r, $L[r]$ is a model of ASC. The following propositions indicate that for any $A \subset \omega_1$, $L[A]$ is a model of ASC if CH holds and $r^\sharp$ exists for all reals r.

PROPOSITION 2.8. $CH \leftrightarrow SC_{\omega_1}$.

PROOF. Note that any elementary submodel of $H(\omega_1)$ is transitive. Hence any bijection $F : \omega_1 \to H(\omega_1)$ witnesses SC_{ω_1}. ⊣

To state the next proposition, we recall some standard background of sharps. The reader is referred to [12] for further details. For any transitive a, consider the structure

$$\langle L_\delta(a), \in, a, x_k, b\rangle_{k\in\omega, b\in a},$$

where δ is a limit ordinal and $\{x_k \mid k \in \omega\}$ is a set of ordinal indiscernibles for $\langle L_\delta(a), \in, a, b\rangle_{b\in a}$ indexed in increasing order. We denote the language of the above structure by $\mathfrak{L}$. In general, due to the lack of a definable global well-order, this structure does not have a built-in Skolem function. However, we can still define partial Skolem functions which moreover suffice for the general theory of sharps. For any finite subset B of a, we can choose an $\langle L_\delta(a), \in\rangle$ definable well-ordering $<_B$ on $\{d \in L_\delta(a) \mid d$ is $\langle L_\delta(a), \in\rangle$-definable from $B \cup \mathrm{Ord}\}$ in a uniform way.[5] Note $\{x_k \mid k \in \omega\}$ remains an indiscernible sequence of the structure $\langle L_\delta(a), \in, a, b, <_B\rangle_{b\in a, B\in[a]^{<\omega}}$. Expand $\mathfrak{L}$ to $\mathfrak{L}'$, the language of $\langle L_\delta(a), \in, a, x_k, b, <_B\rangle_{k\in\omega, b\in a, B\in[a]^{<\omega}}$. We further extend $\mathfrak{L}'$ to a language $\mathfrak{L}_a$ by inductively introducing the following partial Skolem terms. Suppose that $t_0, \dots, t_k$ are terms which have been defined. Suppose that $\phi(c_0, \dots, c_n, d_0, \dots, d_k)$ is a formula of set theory with free variables c_i, d_j. Then for any $b_0, \dots, b_m \in a$, define

$$t^{\phi}_{\{b_0,\dots,b_m\}}(t_0, \dots, t_k) = \begin{cases} <_{\{b_0,\dots,b_m\}} \text{-least } \vec{y} \text{ such that } L_\delta(a) \models \phi(\vec{y}, t_0, \dots, t_k) & \text{if such } y \text{ exists,} \\ \emptyset & \text{otherwise.} \end{cases}$$

Since any element of $L_\delta(a)$ is definable from $a \cup \mathrm{Ord}$ in $\langle L_\delta(a), \in, a, b\rangle_{b\in a}$, any $x \in L_\delta(a)$ is definable from a finite set $B_x \subset a$ and a finite set O_x of ordinals using a formula $\varphi_x(s, t)$. Now if $L_\delta(a) \models \exists x\phi(x, \vec{t})$ holds for some $\mathfrak{L}_a$ formula ϕ, then there is such an x definable from finite sets $B_x \subset a$ and $O_x \subset \mathrm{Ord}$. Thus the term $t^{\phi}_{B_x}(\vec{t})$ witnesses that $L_\delta(a) \models \exists x\phi(x, \vec{t})$. Hence in our setting, it suffices to use the partial Skolem terms, which will be called "Skolem terms" subsequently, to define the sharps. A useful remark is that by alternating the order of free variables, for any permutation $\vec{t'}$ of $\vec{t}$, and any formula ϕ, there is a formula ϕ' such that

[5]The exact definition of $<_B$ is irrelevant. We present one possible definition as following. Note that d is definable in $\langle L_\delta(a), \in\rangle$ using parameters from $B \cup \mathrm{Ord}$ iff there is an ordinal $\gamma < \delta$ and a formula ψ such that d is definable over $\langle L_\gamma, \in\rangle$ using the formula ψ with parameters $\{B\} \cup O$, where $O \in \gamma^{<\omega}$. Let *Form* be the set of formulas equipped with the usual Gödel's well-ordering. Let $<'$ be the lexicographical well-ordering on $\delta \times Form \times \mathrm{Ord}^{<\omega}$. Now for any such d, let $\langle \gamma_d, \psi_d, O_d\rangle$ be the $<'$-least triple witnessing the definition. Now define $d <_B e$ if $\langle \gamma_d, \psi_d, O_d\rangle <' \langle \gamma_e, \psi_e, O_e\rangle$.

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$t^\phi_B(\vec{t}) = t^{\phi'}_B(\vec{t}')$. Thus when we write $t^\phi_{B_x}(\vec{t})$, we can always assume that $\vec{t}$ is of some fixed style. This simplifies the following definition.

Definition 2.9. An EM blueprint for a is a complete theory in $\mathfrak{L}_a$ with underlying structure $\langle L_\delta(a), \in, a, x_k, b, <_B\rangle_{k\in\omega, b\in a, B\in[a]^{<\omega}}$.

For such a, we say $a^\sharp$ exists if there is an EM blueprint T for a such that

(1) (Unboundedness) For any n-ary Skolem term $t^\phi_B(\cdot)$, T contains the sentence: $t^\phi_B(x_0, \dots, x_{n-1}) \in \mathrm{Ord} \to t^\phi_B(x_0, \dots, x_{n-1}) < x_n$.

(2) (Remarkability) For any $(m+n+1)$-ary Skolem term $t^\phi_B(\cdot)$, and a finite sequence $\vec{c}$ from b, T contains the sentence:

$$t^\phi_B(\vec{c}, x_0, \dots, x_{m+n}) < x_m \to$$
$$t^\phi_B(\vec{c}, x_0, \dots, x_{m+n}) = t^\phi_B(\vec{c}, x_0, \dots, x_{m-1}, x_{m+n+1}, \dots, x_{m+2n+1}).$$

(3) (Well-foundedness) For every $\alpha < \omega_1$, $(\mathcal{M}, \alpha)$ is wellfounded, where $(\mathcal{M}, \alpha)$ is the unique (up to isomorphism) model satisfying T that is generated from α-many indiscernibles, which means that $(\mathcal{M}, \alpha)$ is equal to the Skolem Hull of a and α-many indiscernibles using the built-in Skolem terms in $(\mathcal{M}, \alpha)$.

(4) (Witness condition) Whenever $\exists x\phi(x) \in T$, then for some term t involving no indiscernibles not appearing in $\phi(x)$, $\phi(t) \in T$.

By the general analysis of sharps, unboundedness implies that the set of indiscernibles in any $(\mathcal{M}, \alpha)$ is closed. Remarkability implies that in any $(\mathcal{M}, \alpha)$, if x is generated from a finite set X of indiscernibles and a term t, then the rank of x is below the least indiscernible greater than $\sup X$. It is known that the EM blueprint T witnessing that $a^\sharp$ exists is unique. To simplify the presentation, we thus define $a^\sharp$ to be the unique transitive (M, ω) model satisfying T. In particular, $a^\sharp = \langle L_\delta(a), \in, x_k, a, <_B\rangle_{k\in\omega, b\in a, B\in[a]^{<\omega}}$ such that $L_\delta(a)$ is equal to the Skolem Hull of $a \cup \{x_k \mid k < \omega\}$ using the Skolem terms from $\mathfrak{L}_a$. More generally, for any set y, we define $y^\sharp$ to be $(tc(y))^\sharp$. We will not use any specific property of $a^\sharp$ other than the following fact: $L_\delta(a) \prec L(a)$. Another remark is that if $|a| \leq \alpha$ and $a^\sharp$ exists,then $a^\sharp \in H(\alpha^+)$.

Proposition 2.10. *Suppose $\kappa > \omega$ is regular. Suppose F witnesses SC_κ and for any $x \in H(\kappa)$, $x^\sharp$ exists. Suppose that in $L[F]$, there is no inaccessible cardinal above κ. Then $L[F]$ is a model of ZFC+ASC and $H(\kappa) \subset L[F]$.*

Proof. Fix $\alpha < \kappa$ and consider $F \restriction \alpha \in H(\alpha^+)$. By assumption, $(F \restriction \alpha)^\sharp$ exists. Let x_0 be the first indiscernible of $(F \restriction \alpha)^\sharp$. Then by the usual analysis of indiscernibles, x_0 must be an inaccessible cardinal in $L(tc(F \restriction \alpha))$ and $x_0 > \alpha$. It follows that for any $\alpha < \kappa$, there is $\eta < \alpha^+$ such that

$$L(tc(F \restriction \alpha)) \models \eta \text{ is an inaccessible cardinal greater than } \alpha.$$

Without loss of generality, we can assume that $F(0) = \emptyset$. In what follows, we define a function $F' : \mathrm{Ord} \to H(\kappa)$ by inductively assigning value to $F'(\alpha)$. Set $F'(0) = F(0) = \emptyset$. Inductively on $\alpha < \kappa$, we define as follows:

Case 1) $\alpha = \beta + 1$ is a successor ordinal. As F is a bijection from κ to $H(\kappa)$ and $F'(\beta) \in H(\kappa)$ is defined, there is a unique ordinal $\bar{\beta}$ such that $F'(\beta) = F(\bar{\beta})$.

Subcase 1a) If $\beta = 0$ or $\bar{\beta} \neq 0$, then set $F'(\alpha) = F(\bar{\beta} + 1)$.

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SUBCASE 1b) If $\bar{\beta} = 0$ and $\beta \neq 0$, then we set $F'(\alpha) = \emptyset$.

CASE 2) α is a limit ordinal. Let $\xi_\alpha = \sup\{\beta < \alpha \mid F'(\beta) \neq \emptyset\}$.

SUBCASE 2a) If $\xi_\alpha = \alpha$ or $L(tc(F \restriction \xi_\alpha)) \models$ there is no inaccessible cardinal η s.t $\xi_\alpha \leq \eta \leq \alpha$, then let $F'(\alpha) = \emptyset$.

SUBCASE 2b) $\xi_\alpha < \alpha$ and $L(tc(F \restriction \xi_\alpha)) \models \alpha$ is the least inaccessible above ξ_α.[6] Denote $\sup\{\beta < \kappa \mid (\exists \gamma < \alpha)(F'(\gamma) = F(\beta))\}$ by δ_α.

SUBSUBCASE 2ba) $F(\delta_\alpha)$ is undefined. Set $F'(\alpha) = \emptyset$.[7]

SUBSUBCASE 2bb) $F(\delta_\alpha)$ is defined. Set $F'(\alpha) = F(\delta_\alpha)$.

Basically, F' can be viewed as a function stretched from F. Let $<_{L[F']}$ be the canonical Σ_1-well ordering of $L[F']$. Let $\bar{F} : \mathrm{Ord} \to L[F']$ be the bijection derived from $<_{L[F']}$. We claim that for any uncountable regular cardinal η, $\bar{F} \restriction \eta$ witnesses SC_η in $L[F']$ and $H(\kappa) \subset L[F']$.

CLAIM 2.11. $H(\kappa) \subset L[F']$.

PROOF. We will show that $H(\lambda) \subset F'[\lambda]$ for any uncountable regular cardinal $\lambda \leq \kappa$.

We first assume $\lambda > \omega$ is a successor cardinal. A simple observation from the construction is that $F'[\lambda] = F[\gamma]$ for some $\gamma \leq \kappa$ and $F' \restriction (\lambda \setminus F'^{-1}[\{\emptyset\}])$ is an injection. Thus it suffices to verify that $\gamma = \lambda$. As $|F'[\alpha]| < \lambda$ for any $\alpha < \lambda$ and λ is the first ordinal β such that $|F[\beta]| \geq \lambda$, $\gamma \leq \lambda$. It remains to show that $\gamma \geq \lambda$. Assume otherwise, $\gamma < \lambda$. It is clear that γ must be a limit ordinal. As $F' \restriction (\lambda \setminus F'^{-1}[\{\emptyset\}])$ is an injection, there is a least $\bar{\gamma} < \lambda$ such that $F[\gamma] = F'[\bar{\gamma}]$. $\bar{\gamma}$ must be a limit ordinal by construction. It follows that for any $\alpha \in [\bar{\gamma}, \lambda)$, $\xi_\alpha = \bar{\gamma}$. By our assumption, there is $\eta \in [\bar{\gamma}, \bar{\gamma}^+)$ such that $L(tc(F \restriction \bar{\gamma})) \models \eta$ is the least inaccessible cardinal greater than $\bar{\gamma}$. But as λ is a cardinal, $[\bar{\gamma}, \bar{\gamma}^+) \subset [\bar{\gamma}, \lambda)$ and hence $\eta < \lambda$. Now on stage η, $\xi_\eta = \bar{\gamma}$ and $\delta_\eta = \gamma$. Hence by our construction $F'(\eta) = F(\gamma)$. Contradiction.

Now when λ is a limit cardinal, the statement follows easily. ⊣

This proof also implies $F'(\alpha) = \emptyset$ whenever $\alpha \geq \kappa$. Hence for any cardinal η, $L_\eta[F] \subset L_\eta[F']$. On the other hand, by induction on rank, we can check that for any $X \in L_\eta[F']$, the transitive closure $tc(X)$ of X is in $H(\eta)^{L[F]}$. However, note that $H(\eta)^{L[F]} = L_\eta[F]$.[8] Hence $X \in L_\eta[F]$. It follows that $L_\eta[F] = L_\eta[F']$ for all η.

Next we prove that $\bar{F} \restriction \eta$ witnesses SC_η in $L[F']$ for every uncountable regular η. We first treat the case when $\eta \leq \kappa$. For $\eta \leq \omega_1$, there is nothing to show. Assume $\eta > \omega_1$, we will apply Lemma 2.6. Note that $\bar{F} \restriction \eta$ is a bijection from η to $H(\eta)^{L[F']} = L_\eta[F]$. Consider the club consisting of all countable $X \prec \langle L_\eta[F], F \rangle$. We will show that for any X in the club, $(\bar{F} \restriction \eta)_X = \bar{F} \restriction (\mathrm{ot}(X \cap \eta))$. By the definition of $\bar{F}$ and elementarity between X and $L_\eta[F]$, this amounts to showing that $(F' \restriction \eta)_X = F' \restriction (\mathrm{ot}(X \cap \eta))$. As F witnesses SC, $F_X \subset F$.

[6] Equivalently, Subcase 2a) 2b) can be defined as:

2a) $L(tc(F \restriction \xi_\alpha)) \models \alpha = \xi_\alpha$ or α is not inaccessible.
2b) $L(tc(F \restriction \xi_\alpha)) \models \alpha > \xi_\alpha$ and α is inaccessible.

Although the current definition is more complicated, it does clarify the later presentation.

[7] Note this happens when $\delta_\alpha = \kappa$.

[8] For $\eta \leq \kappa$, $F \restriction \eta$ is a surjection onto $H(\eta)^{L[F]}$. For $\eta > \kappa$, as $tc(F) \in L_\eta[F]$, $H(\eta)^{L[F]} = L_\eta[F]$ by the condensation lemma for $L[F]$.

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By induction on $\gamma \in X \cap \mathrm{Ord}$, we need to verify that $F'(\gamma_X) = (F'(\gamma))_X$. The successor and 0 cases are trivial. Assume γ is limit. Note that by induction hypothesis, $(\xi_\gamma)_X = \sup\{\beta < \gamma_X \mid F'_X(\beta) \neq \emptyset\} = \sup\{\beta < \gamma_X \mid F'(\beta) \neq \emptyset\} = \xi_{\gamma_X}$ and thus $\xi_\gamma = \gamma \leftrightarrow \xi_{\gamma_X} = \gamma_X$. We need the following claim to transfer the sharps along the collapsing maps.

Claim 2.12. *Suppose X is a countable elementary submodel of $H(\gamma^+)$ and $a \in X$ is transitive. Suppose that $a^\sharp$ exists in X. Then $(a_X)^\sharp$ exists and is equal to $(a^\sharp)_X$.*

Proof. Write $a^\sharp$ as $\langle L_\delta(a), \in, x_k, a, b, <_B\rangle_{k\in\omega, b\in a, B\in[a]^{<\omega}}$. Then $(a^\sharp)_X = \langle L_{\delta_X}(a_X), \in, (x_k)_X, a_X, b, <_B\rangle_{k\in\omega, b\in a_X, B\in[a_X]^{<\omega}}$.

As $(a^\sharp)_X \cong a^\sharp \restriction X$, $\{(x_k)_X \mid k < \omega\}$ is an increasing sequence of indiscernibles for the structure $\langle L_{\delta_X}(a_X), \in, a_X, b\rangle_{b\in a_X}$. Hence the theory T of $(a^\sharp)_X$ is an EM blueprint for a_X.

We will verify that the requirements (1)–(4) of Definition 2.9 hold for T. (1), (2), and (4) follow routinely from the elementarity between X and $H(\gamma^+)$. We only need to verify (3).

For $\alpha < \omega_1$, we need to construct a well-founded $(\mathcal{M}, \alpha)$ model for T. Consider a well-founded $(\mathcal{M}, \alpha)$ model $\hat{M}$ for the theory of $a^\sharp$. Let $\{x_\beta \mid \beta < \alpha\}$ be the corresponding indiscernible sequence. Note that $a^\sharp$ can be canonically embedded into $\hat{M}$ by mapping all constants accordingly and all indiscernibles to the first ω many x_β. Thus we can identify $(L_{x_\omega}(a))^{\hat{M}}$ with $L_\delta(a)$. Let K be the Skolem Hull of $(L_\delta(a) \cap X) \cup \{x_\beta \mid \beta < \alpha\}$ using the built-in Skolem terms t^ϕ_B of $a^\sharp$ in $\hat{M}$ for $B \subset X \cap b$.

We claim that K is an $(\mathcal{M}, \alpha)$ model for T. We first show that $K \cap (L_{x_\omega}(a))^{\hat{M}} = L_\delta(a) \cap X$. Clearly $K \cap (L_{x_\omega}(a))^{\hat{M}} \supset L_\delta(a) \cap X$ by the definition of K. On the other hand, suppose that $c \in K \cap (L_{x_\omega}(a))^{\hat{M}}$. Now in $\hat{M}$, $c = t^\phi_B(A)$ for some $B \subset a \cap X$ and A is a finite sequence of $(L_\delta(a) \cap X) \cup \{x_\beta \mid \beta < \alpha\}$. By remarkability, $c = t^\phi_B(A')$, A' is a finite sequence of $(L_\delta(a) \cap X) \cup \{x_\beta \mid \beta < \omega\}$. Note that as $L_\delta \prec \hat{M}$ and $<^{L_\delta}_B = <^{\hat{M}}_B \cap L_\delta$, $c = (t^\phi_B(A'))^{L_\delta}$. By elementarity between X and $H(\gamma^+)$, $c \in L_\delta(a) \cap X$.

Hence $K \cap (L_{x_\omega}(a))^{\hat{M}}$ is isomorphic to $(a^\sharp)_X$. Now for any $c \in L_\delta(a) \cap X$, c is generated by some Skolem term in $a^\sharp$ using parameters from $a \cap X$ and a finite set of x_k. Hence c is generated by the same Skolem term in $K \cap (L_{x_\omega}(a))^{\hat{M}}$ using parameters from $a \cap X$ and a finite set of x_k. Hence $L_\delta(a) \cap X$ is contained in the Skolem Hull of $(a \cap X) \cup \langle x_\beta \mid \beta < \alpha\rangle$ in K. It then follows that K is the Skolem Hull of $(a \cap X) \cup \{x_\beta \mid \beta < \alpha\}$ in K. But as $\{x_\beta \mid \beta < \alpha\}$ is an indiscernible sequence for K, K is clearly an $(\mathcal{M}, \alpha)$ model for T, the theory of $(a^\sharp)_X$. ⊣

Using the above claim, we have

$(*)\qquad L(\mathrm{tc}(F \restriction \xi_{\gamma_X})) \models \gamma_X$ is the least inaccessible cardinal greater than ξ_{γ_X}

if and only if

$\gamma_X \in (F \restriction \xi_{\gamma_X})^\sharp \wedge (F \restriction \xi_{\gamma_X})^\sharp \models \gamma_X$ is the least inaccessible cardinal greater than ξ_{γ_X}

if and only if

$$\gamma_X \in ((F \restriction \xi_\gamma)^\sharp)_X \wedge ((F \restriction \xi_\gamma)^\sharp)_X \models \gamma_X \text{ is the least inaccessible cardinal greater than } (\xi_\gamma)_X$$

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if and only if

$$\gamma \in (F \restriction \xi_\gamma)^\sharp \wedge (F \restriction \xi_\gamma)^\sharp \models \gamma \text{ is the least inaccessible cardinal greater than } \xi_\gamma$$

if and only if

$$(**) \qquad L((\text{tc}(F \restriction \xi_\gamma)) \models \gamma \text{ is the least inaccessible cardinal greater than } \xi_\gamma.$$

If both $(*)$ and $(**)$ hold and $\xi_\gamma < \gamma$, then $F'(\gamma_X) = F(\xi_{\gamma_X}) = (F(\xi_\gamma))_X = (F'(\gamma))_X$. Otherwise, $F'(\gamma_X) = \emptyset = (F'(\gamma))_X$. This ends the induction and the case $\eta \leq \kappa$.

Now we deal with the case $\eta > \kappa$. Now $\bar{F} \restriction \eta$ is a bijection from η to $H(\eta)^{L[F']} = L_\eta[F]$ and definable over $\langle H(\eta)^{L[F']}, F\rangle$. By the condensation lemma for the relativized constructible universe, for any countable $X \prec L_\eta[F]$ such that $\kappa \in X$, there is $\beta < \omega_1$ such that $X \cong L_\beta[F_X]$. Thus $\bar{F}_X$ is derived from the canonical well-ordering of $L_\beta[F'_X]$. As in the last case, it remains to show that $F'_X \subset F'$. Since $\kappa \in X$, $L_\kappa[F] = H(\kappa) \in X$. Hence, $X \cap H(\kappa) \prec H(\kappa)$. By the proof for the last case, $F' \restriction \kappa_X = F'_X \restriction \kappa_X$. On the other hand, we will inductively show that for all $\gamma_X \in [\kappa_X, \beta)$, $F'(\gamma_X)$ is trivial and $\xi_{\gamma_X} = \kappa_X$. When $\gamma_X = \kappa_X$, by our construction $\xi_{\gamma_X} = \kappa_X$. Hence we are in Subcase 2a) and thus $F'(\gamma_X)$ is trivial. Suppose the induction arrives at a $\gamma_X > \kappa_X$, Then by induction hypothesis, it is routine to check that $\xi_{\gamma_X} = \kappa_X$. We also know that

$$L[F] \models \gamma \text{ is not an inaccessible cardinal.}$$

As ran $F = tc(F) = H(\kappa)$, $L[F] = L[tc(F)]$. Moreover as F codes a well-ordering of $tc(F)$, $L(tc(F)) = L[tc(F)] = L[F]$. Thus

$$L(tc(F)) \models \gamma \text{ is not an inaccessible cardinal.}$$

Hence

$$L_\eta(tc(F)) \models \gamma \text{ is not an inaccessible cardinal.}$$

Therefore we are in Subcase 2a) and $F'(\gamma_X)$ is trivial. Now note that $F'_X \restriction [\kappa_X, \beta)$ is also trivial by the definition of F' above κ. Therefore $F' \restriction [\kappa_X, \beta) = F'_X \restriction [\kappa_X, \beta)$. Hence $F'_X \subset F'$. This ends the case $\eta > \kappa$. ⊣

We remark here that the inaccessibility can be replaced by any lightface Π_1 property of ordinals. In particular, we could require the final model compatible with several large cardinal properties below ω_1-Erdős. For example, if we need a inaccessible cardinal above κ, then we only need to require that $L[F] \models$ "there is a unique inaccessible cardinal above κ" and modify the proof accordingly.

The following remarkable theorem of Beller–Jensen–Welch [2] provides a class forcing extension of V that satisfies ASC:

Theorem 2.13 (Jensen's Coding Theorem). *There is a class forcing $\mathbb{P}$ such that if G is $\mathbb{P}$-generic over V then $V[G] \models ZFC + V = L[R]$, $R \subset \omega$. If $V \models GCH$ then $\mathbb{P}$ preserves cardinals.*

In [14], Woodin describes a different approach to construct models of SC. In contrast to L, these models carry a rather complicated structure and serve as the base for $\mathbb{P}_{max}$ variants for several club guessing properties.

Theorem 2.14. *Assume AD holds in $L(\mathbb{R})$ and $x \in \mathbb{R}$. Let*

$$N = HOD^{L(\mathbb{R})}[x].$$

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Suppose that γ is an uncountable cardinal of N which is below the least weakly compact cardinal of N. Then SC holds for $(H(\gamma))^N$ in N.[9]

Woodin then asked how to obtain models of SC via a set forcing notion. A solution to this question, together with Proposition 2.10 provide an approach to prove consistency results related to ASC. By Proposition 2.8, the first nontrivial target is to force SC_{ω_2}.[10]

§3. **Forcing SC_{ω_2}.** In this section, we present the forcing construction for Theorem 1.1. As mentioned, we will force the existence of a function F witnessing (2) of Lemma 2.6. For technical reasons, instead of F, we construct a bijection $H : \omega_2 \to P(\omega_1)$ satisfying a variant of the condensation property, and show that the desired function F is induced by H. We describe the general framework as follows. Our forcing $\mathbb{P}$ is constructed as an ω_2-length countable support iteration $\mathbb{P}_{\omega_2} = \langle \mathbb{P}_\alpha, \dot{Q}_\alpha \mid \alpha \in [\omega_1, \omega_2)\rangle$. This nonstandard index set is designed to simplify the presentation. The iteration starts with a single forcing $\mathbb{P}_{\omega_1}$, which we will also denote by $\dot{Q}_{\omega_1^-}$. $\dot{Q}_{\omega_1^-}$ and $\mathbb{P}_{\omega_1}$ should be compared with Q_0 and $\mathbb{P}_1$ in the usual definition of iterated forcing. We also set $\omega_1^- + 1 = \omega_1$. Now for any $\alpha \in [\omega_1, \omega_2)$ and $q \in \mathbb{P}_\alpha$, $q(\omega_1^-)$ is a $\mathbb{P}_{\omega_1}$-condition. After forcing with $\mathbb{P}_{\omega_1}$ we obtain $H \restriction \omega_1$. Then inductively on $\alpha \in [\omega_1, \omega_2)$, we assign values to $H(\alpha)$ and define Q_α simultaneously. Finally we construct the desired bijection F using H. We will then verify that F witnesses SC_{ω_2} and thus complete the proof of the theorem.[11] For any condition $p \in \mathbb{P}_\alpha$, set $\mathrm{spt}(p) = \{\beta \in \alpha \cup \{\omega_1^-\} \mid p \restriction \beta \Vdash p(\beta)$ is not trivial$\}$.

Along with the definition of $\mathbb{P}_{\omega_2}$, we define the following objects:

- A sequence $\langle \dot{S}_\alpha \mid \alpha \in [\omega_1, \omega_2)\rangle$ such that each $\dot{S}_\alpha$ is a $\mathbb{P}_\alpha$-name of a subset of $(\omega_1)^{V^{\mathbb{P}_\alpha}}$ in $V^{\mathbb{P}_\alpha}$ for $\alpha \in [\omega_1, \omega_2)$;
- A sequence $\langle \dot{S}''_\alpha \mid \alpha \in [\omega_1, \omega_2)\rangle$ such that each $\dot{S}''_\alpha$ is a $\mathbb{P}_\alpha$-name of a stationary, co-stationary subset of $(\omega_1)^{V^{\mathbb{P}_\alpha}}$ in $V^{\mathbb{P}_\alpha}$ for $\alpha \in [\omega_1, \omega_2)$;
- A sequence $\langle \dot{H}_\alpha \mid \alpha \in [\omega_1, \omega_2)\rangle$ such that each $\dot{H}_\alpha$ is a $\mathbb{P}_\alpha$-name of an injection from α to $P(\omega_1)^{V^{\mathbb{P}_\alpha}}$ and for $\beta < \alpha$, $\Vdash_{\mathbb{P}_\alpha} \dot{H}_\beta \subset \dot{H}_\alpha$.

We also fix a bookkeeping bijection $h : [\omega_1, \omega_2) \to [\omega_1, \omega_2) \times \omega_2$ such that $(h(\alpha))_0 \le \alpha$ for all $\alpha \in [\omega_1, \omega_2)$. This h will be used to enumerate $P(\omega_1)^{V^{\mathbb{P}_{\omega_2}}}$. During the construction, we often identify a $\mathbb{P}_\alpha$-name as a $\mathbb{P}_\beta$-name for $\omega_1 \le \alpha < \beta < \omega_2$. We will repeatedly use the forcing maximality principle to construct names, i.e. to define a name $\dot{a}$, we only need to describe how to evaluate this name in any fixed

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[9] In fact, via HOD analysis for $L(\mathbb{R})$ in the context of AD, the initial segment of $\mathrm{HOD}^{L(\mathbb{R})}$ is a extender model and has fine structure. On the other hand, this model contains reals like M_n, the minimal mouse containing n Woodin cardinals, and is not of the form $L_\gamma[A]$ for any bounded subset A of γ.

[10] One may ask whether it is possible to force over some V_κ using Jensen's Coding Theorem to get models of ASC, where κ is inaccessible. The answer is no. Let G be $\mathbb{P}$-generic over V, where $\mathbb{P}$ is the forcing for Jensen's Coding Theorem defined as a class in V_κ. It can be verified that $V_\kappa[G] \models V = L[R]$ for some $R \subset \omega$. Nevertheless, $V[G]_\kappa \neq V_\kappa[G]$ as some real is not in $V_\kappa[G]$. For example, the theory of $L_\kappa[R]$.

[11] It will be clear that $\mathbb{P}_{\omega_2}$ is a totally proper forcing notion, i.e., a proper forcing that does not add reals. However, the iterants of $\mathbb{P}_{\omega_2}$ are not proper, and thus we will directly deal with $\mathbb{P}_{\omega_2}$ rather than adopting the general framework of proper forcing.

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generic extension. We shall always keep and verify the following inductive hypothesis (*) for all $\alpha \in [\omega_1, \omega_2)$ during the construction:

(1) $\mathbb{P}_\alpha$ is ω_1-distributive and has an ω_1-sized dense subset.
(2) If G is $\mathbb{P}_\alpha$-generic and $\omega_1 \leq \beta < \alpha$, then in $V[G]$,

$$C_\beta = \{\eta \in \omega_1 \mid (\exists p \in G \upharpoonright \beta) p \Vdash_{\mathbb{P}_\beta} (\exists q \in G \upharpoonright \beta + 1)\eta \in q(\beta)\}$$

is a club subset of ω_1 and $C_\beta \notin V[G \upharpoonright \mathbb{P}_\gamma]$ for any $\gamma \in [\omega_1, \beta)$.

Now we start the induction on $\alpha \in [\omega_1, \omega_2)$. $\mathbb{P}_{\omega_1}$ is defined as follows:

- $p \in \mathbb{P}_{\omega_1}$ if p is an injection from α to $P_{\omega_1}(\omega_1)$, where $\alpha < \omega_1$;
- $p <_{\mathbb{P}_{\omega_1}} q$ if $q \subset p$.

$\mathbb{P}_{\omega_1}$ is essentially the forcing $\mathrm{Add}(\omega_1, 1)$. Hence $\mathbb{P}_{\omega_1}$ is cardinal preserving and does not add new countable sets of ordinals. Let G_{ω_1} be a $\mathbb{P}_{\omega_1}$-generic filter over V. In $V[G_{\omega_1}]$, we define H_{ω_1} to be $\bigcup G_{\omega_1}$. Clearly, H_{ω_1} is a bijection from ω_1 to $P_{\omega_1}(\omega_1)$. Let A_{ω_1} be the structure $\langle \mathrm{Field}(H_{\omega_1}), H_{\omega_1}, \in \rangle$. Let

$$S'_{\omega_1} = \{X \subset A_{\omega_1} \mid |X| = \omega \wedge \langle X, H_\alpha \upharpoonright X, \in \rangle \prec A_{\omega_1} \wedge (H_{\omega_1})_X \subset H_{\omega_1})\}.$$

Fix a bijection e_{ω_1} from ω_1 to A_{ω_1}. Let S''_{ω_1} be the set $\{X \in \omega_1 \mid e_{\omega_1}[X] \in S'_{\omega_1}\}$. We will verify that S''_{ω_1} is stationary co-stationary later in the section. It is also clear that (*) holds for $\mathbb{P}_{\omega_1}$.

Now assume $\alpha = \beta + 1$ is a successor. Assume that $\mathbb{P}_\beta$, $\dot{H}_\beta$ and $\dot{S}''_\beta$ have been constructed and satisfy the desired properties. We need to define $\dot{Q}_\beta$, $\dot{H}_\alpha$, $\dot{S}_\beta$, and $\dot{S}''_\alpha$. Let G_β be any $\mathbb{P}_\beta$-generic over V. In what follows, we define Q_β, H_α, S_α, and S''_α in $V[G_\beta]$. Q_β is the forcing which shoots a club through S''_β, i.e. the conditions are the countable closed subsets of S''_β, ordered by end-extension. As (*) holds for $\mathbb{P}_\beta$, it is routine to verify (*) for $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{Q}_\beta$ using the properties of club-shooting forcing. In particular, if G_α is $\mathbb{P}_\alpha$-generic, then the following hold in $V[G_\alpha]$:

- $\aleph_\alpha = (\aleph_\alpha)^V$, for $\alpha \in \mathrm{Ord}$;
- $\mathrm{Ord}^\omega = (\mathrm{Ord}^\omega)^V$;
- CH and $2^{\omega_1} = \omega_2$;
- If $C_\beta = \bigcup_{p \in G_\alpha} p(\beta)$, then $C_\beta \subset S''_\beta$ is a club in ω_1.

If $\beta = \omega_1$, then we let W_{ω_1} be a bijection from ω_2 to $P(\omega_1)^{V[G_{\omega_1}]} \backslash P_{\omega_1}(\omega_1)$. Otherwise $\beta > \omega_1$, then fix a bijection W_β from ω_2 to $P(\omega_1)^{V[G_\beta]} \setminus \bigcup_{\gamma < \beta} V[G_\gamma]$.[12] In any case, let S_β be $W_{h(\beta)_0}(h(\beta)_1)$. Since $h(\beta)_0 \leq \beta$, S_β is well-defined and unbounded in ω_1. The unboundedness follows from the fact that (*)(1) implies any bounded subset of ω_1 is in the ground model. Let H_α be $H_\beta \cup \{\langle \beta, S_\beta \rangle\}$. Let A_α be the structure $\langle \mathrm{Field}(H_\alpha), H_\alpha, \in \rangle$. Let

$$S'_\alpha = \{X \subset A_\alpha \mid |X| = \omega \wedge \langle X, H_\alpha \upharpoonright X, \in \rangle \prec A_\alpha \wedge (H_\alpha)_X \subset H_\alpha)\}.$$

[12] Note that $P(\omega_1)^{V[G_\beta]} \backslash \bigcup_{\gamma<\beta} V[G_\gamma]$ is definable in the forcing language and thus in $V[G_\beta]$. By (*)(2), when $\beta > \omega_1$ is a successor ordinal, then C_β is in $P(\omega_1)^{V[G_\beta]} \setminus \bigcup_{\gamma<\beta} V[G_\gamma]$. When β is a limit ordinal, by (*)(2), there is a subset A of ω_1 coding all previous C_γ and thus A is in $P(\omega_1)^{V[G_\beta]} \setminus \bigcup_{\gamma<\beta} V[G_\gamma]$. In both cases, using (*)(1), we know that $P(\omega_1)^{V[G_\beta]} \setminus \bigcup_{\gamma<\beta} V[G_\gamma]$ is of size ω_2 and thus W_β can be defined.

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Fix a bijection e_α from ω_1 to A_α. Let S''_α be the set $\{X \in \omega_1 \mid e_\alpha[X] \in S'_\alpha\}$. We will show that both S''_α and S'_α are stationary and co-stationary in their corresponding structures. It is also not difficult to observe that for any $\gamma < \alpha$, S''_α is contained in S''_γ modulo the nonstationary ideal.

If $\alpha > \omega_1$ is limit, $\mathbb{P}_\alpha$ is defined following the rule of countable support iterated forcing. We will verify (*) for such α later. Let $\dot{H}_\alpha$ be a $\mathbb{P}_\alpha$-name of the function $\bigcup_{\omega_1<\beta<\alpha} \dot{H}_\beta$. We then define A_α, S'_α, and S''_α using exactly the same definition as in the successor case. It will also be shown that S''_α and S'_α are stationary and co-stationary sets and that for any $\gamma < \alpha$, S''_α is almost contained in S''_γ.

To finish the definition of $\mathbb{P}_{\omega_2}$, we need to prove that $\dot{S}''_\alpha$ is stationary and co-stationary for all α and that (*) holds for limit α. This will be shown by verifying that $\mathbb{P}_\alpha$ has a dense set of "complete and flat" conditions.

Definition 3.1. A $\mathbb{P}_\alpha$-condition p is *flat* if there is a unique $\gamma < \omega_1$ and a sequence $\langle p_i \mid i \in \operatorname{spt}(p)\rangle \subset V$ such that

$$(\forall \lambda \in \operatorname{spt}(p) \setminus \{\omega_1^-\})(p \restriction \lambda \Vdash p(\lambda) = p_\lambda \wedge \sup(p_\lambda) = \gamma^{13}).$$

For any sufficiently large θ and any countable $M \prec H(\theta)$ containing $\mathbb{P}_\alpha$[14], a condition q is *$(M, \mathbb{P}_\alpha)$-complete* if the set $\{p \in M \cap \mathbb{P}_\alpha \mid q < p\}$ is $\mathbb{P}_\alpha$-generic over M.

Lemma 3.2. *Suppose (*) holds for all $\beta < \alpha$. For sufficiently large θ, any countable $M \prec H(\theta)$ containing $\mathbb{P}_\alpha$ and $p \in M$, there is a flat $(M, \mathbb{P}_\alpha)$-complete condition q extending p.*

Proof. Fix an enumeration $\langle D_n \mid n < \omega\rangle$ of the open dense subsets of $\mathbb{P}_\alpha$ in M. Fix a bijection $\pi : \omega_1 \leftrightarrow P_{\omega_1}(\omega_1)$ in M. Let $\langle \alpha_n \mid n < \omega\rangle$ be an increasing sequence of ordinals with the supremum $M \cap \omega_1$. Thus $\pi[M \cap \omega_1] = (P_{\omega_1}(\omega_1))^M = M \cap P_{\omega_1}(\omega_1)$. Fix an enumeration $\langle \rho_n \mid n < \omega\rangle$ of $M \cap \alpha$. We also assume that M inherits a well-ordering $<_\theta$ from $H(\theta)$. We inductively construct a sequence of $\mathbb{P}_\alpha$-conditions $\langle p_n \mid n < \omega\rangle$ hitting some appropriate dense sets and obtain a lower bound in the end. We will ensure $p_0 = p$, $p_n \in M$, and $p_{n+1} < p_n$ for all $n < \omega$ during the construction.

Suppose now p_n has been constructed. We will choose $p_{n+1} < p_n$ such that[15]

i) $\operatorname{dom}((p_{n+1})(\omega_1^-)) > \alpha_n$, $\operatorname{ran}((p_{n+1})(\omega_1^-)) \supset \pi[\alpha_n]$ and $p_{n+1} \in D_n$.
ii) There are objects $\langle B^n_{\rho_m} \mid m \leq n\rangle$ and $\langle \gamma^n_{\rho_m,i} \mid i \leq n+1, m \leq n\rangle$ such that for all $i \leq n+1$ and $m \leq n$, $B^n_{\rho_m}$ is a subset of α_n in M, $\gamma^n_{\rho_m,i}$ is in the interval $(\alpha_n, M \cap \omega_1)$ and[16]

[13]Note this implies p_λ is forced to be a closed subset of $\gamma + 1$ with maximum γ.

[14]From now on when we say $\mathbb{P}_\alpha \in M$, we implicitly assume that the definition of $\mathbb{P}_\alpha$ can be carried out in M, i.e., M contains all necessary parameters used in the definition of $\mathbb{P}_\alpha$.

[15]An alternative argument is to show that when the p_n's hit all D_n's, some p_n must satisfy the requirement below. This can be done by showing the corresponding set of conditions is dense, which is essentially the same as the current argument.

[16]It is clear that if p_{n+1} forces this sentence, then $p_{n+1} \restriction \rho_m$ already forces this in $\mathbb{P}_{\rho_m}$. The same also holds for the next two items.

$\gamma^n_{\rho_m,n+1}$ is designed to show that S_{ρ_m} is unbounded in ω_1. $\gamma^n_{\rho_m,k}$ is designed to show $S_{\rho_m} \neq S_{\rho_k}$.

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$$p_{n+1} \Vdash_{\mathbb{P}_\alpha} \quad \dot{S}_{\rho_m} \cap \alpha_n = B^n_{\rho_m} \wedge \gamma^n_{\rho_m,n+1} \in \dot{S}_{\rho_m} \wedge (\forall k \leq n) (\rho_k < \rho_m \rightarrow (\gamma^n_{\rho_m,k} \in \dot{S}_{\rho_k} \leftrightarrow \gamma^n_{\rho_m,k} \notin \dot{S}_{\rho_m})).$$

iii) There is a system $\langle \bar{B}^n_\gamma \mid n < \omega \wedge \gamma \in M \cap \alpha\rangle$ such that if $\gamma \in \operatorname{spt}(p_n)$, then

$$p_{n+1} \Vdash_{\mathbb{P}_\alpha} \sup(p_{n+1}(\gamma)) > \alpha_n \wedge p_{n+1}(\gamma) \cap \alpha_n = \bar{B}^n_\gamma.$$

iv) There is a sequence $\langle \eta^n_\gamma \mid \gamma \in \operatorname{spt}(p_n)\rangle$ such that if $\gamma \in \operatorname{spt}(p_n)$, then

$$\eta^n_\gamma \in (\alpha_n, M \cap \omega_1) \wedge p_{n+1} \Vdash_{\mathbb{P}_\alpha} \dot{e}_\gamma[\eta^{n-1}_\gamma] \prec \dot{e}_\gamma[\eta^n_\gamma] \prec \dot{A}_{\gamma+1}.$$

Since α_n is countable and $\pi \in M$, we can choose $p^0_n < p_n$ in $D_n \cap M$ such that $\operatorname{dom}((p^0_n)(\omega_1^-)) > \alpha_n$ and $\operatorname{ran}((p^0_n)(\omega_1^-)) \supset \pi[\alpha_n]$. Now

$$p^0_n \Vdash_{\mathbb{P}_\alpha} \quad (\forall \beta \in \operatorname{spt}(p_n))(\{X \in P_{\omega_1}(\dot{A}_{\beta+1}) \mid \dot{e}_\beta[\eta^{n-1}_\beta] \prec X \prec \dot{A}_{\beta+1}\} \text{ contains a club in } P_{\omega_1}(\dot{A}_{\beta+1}))) \wedge \dot{e}_\beta[\omega_1] = \dot{A}_{\beta+1})$$

It follows that $p^0_n \Vdash_{\mathbb{P}_\alpha} (\forall \beta \in \operatorname{spt}(p_n))(\exists \eta^n_\beta > \alpha_n)\dot{e}_\beta[\eta^{n-1}_\beta] \prec \dot{e}_\beta[\eta^n_\beta] \prec \dot{A}_{\beta+1}$. So

$$p^0_n \Vdash_{\mathbb{P}_\alpha} \quad (\exists \langle \eta^n_\beta \mid \beta \in \operatorname{spt}(p_n)\rangle)(\forall \beta \in \operatorname{spt}(p_n))\eta^n_\beta > \alpha_n \wedge \dot{e}_\beta[\eta^{n-1}_\beta] \prec \dot{e}_\beta[\eta^n_\beta] \prec \dot{A}_{\beta+1}.$$

Again we can pick $\langle \eta^n_\beta \mid \beta \in \operatorname{spt}(p_n)\rangle$ and p^1_n in M such that for every $\beta \in \operatorname{spt}(p_n)$, $p^1_n \Vdash_{\mathbb{P}_\alpha} \eta^n_\beta > \alpha_n \wedge \dot{e}_\beta[\eta^{n-1}_\beta] \prec \dot{e}_\beta[\eta^n_\beta] \prec \dot{A}_{\beta+1}$.

By our construction and the hypothesis (*)(1), for all $\beta_1 < \beta_2 < \alpha$, S_{β_1} and S_{β_2} are unbounded in ω_1 with all initial segments in V and $S_{\beta_1} \neq S_{\beta_2}$. Hence

$$p^1_n \Vdash_{\mathbb{P}_\alpha} \quad (\forall m < n)(\exists B^n_{\rho_m} \in P_{\omega_1}(\omega_1)^V)(\exists \langle \gamma^n_{\rho_m,i} \mid i \leq n+1\rangle)(\dot{S}_{\rho_m} \cap \alpha_n = B^n_{\rho_m} \wedge \gamma^n_{\rho_m,n+1} \in \dot{S}_{\rho_m} \wedge (\forall k \leq n)(\rho_k < \rho_m \rightarrow (\gamma^n_{\rho_m,k} \in \dot{S}_{\rho_k} \leftrightarrow \gamma^n_{\rho_m,k} \notin \dot{S}_{\rho_m}))).$$

Now we can pick $\langle B^n_{\rho_m} \mid m \leq n\rangle$, $\langle \gamma^n_{\rho_m,i} \mid i \leq n+1, m \leq n\rangle$ and p^2_n in M such that whenever $m < n$,

$$p^2_n \Vdash_{\mathbb{P}_\alpha} \quad \dot{S}_{\rho_m} \cap \alpha_n = B^n_{\rho_m} \wedge \gamma^n_{\rho_m,n+1} \in \dot{S}_{\rho_m} \wedge (\forall k \leq n)(\rho_k < \rho_m \rightarrow (\gamma^n_{\rho_m,k} \in \dot{S}_{\rho_k} \leftrightarrow \gamma^n_{\rho_m,k} \notin \dot{S}_{\rho_m})).$$

Note that each $\gamma^n_{\rho_m,i}$ is bounded by $M \cap \omega_1$.

We will inductively construct a sequence of conditions $\langle p_{n,\beta} \mid \beta \in \operatorname{spt}(p^2_n)\rangle$ in M which will have the properties that for all $\beta \in \operatorname{spt}(p^2_n)$, $\operatorname{spt}(p_{n,\beta}) = \operatorname{spt}(p^2_n) \cap (\beta+1)$ and $p_{n,\beta} <_{\mathbb{P}_{\beta+1}} p^2_n \upharpoonright (\beta+1)$.

When $\beta = \omega_1^-$, let $p_{n,\omega_1} = p^2_n \upharpoonright \omega_1$. Suppose $\beta \in \operatorname{spt}(p^2_n)$ and $p_{n,\gamma}$ is constructed for every $\gamma < \beta$ and $\gamma \in \operatorname{spt}(p^2_n)$, then we define $p_{n,\beta}$ as follows. For $\gamma < \beta$, if $\gamma \in \operatorname{spt}(p^2_n)$, then let $p_{n,\beta}(\gamma) = p_{n,\gamma}(\gamma)$, otherwise let $p_{n,\beta}(\gamma) = \dot{1}_{Q_\gamma}$. $p_{n,\beta} \upharpoonright \beta$ is well-defined and is a $\mathbb{P}_\beta$-condition. Since $\operatorname{spt}(p^2_n) \cap \beta \in M$ and the previous construction can be carried out in M, $p_{n,\beta} \upharpoonright \beta \in M$. Then we choose $p_{n,\beta}(\beta) \in M$ to be the $<_\theta$-least $\mathbb{P}_\beta$-name $\dot{t}$ of a $\dot{Q}_\beta$-condition such that $p^2_n \upharpoonright \beta \Vdash_{\mathbb{P}_\beta} \dot{t} <_{\dot{Q}_\beta} p^2_n(\beta) \wedge \sup(\dot{t}) > \alpha_n$. Note that $p_{n,\beta}$ is in M and satisfies our requirement.

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Now define p_n^3 as

$$p_n^3(\gamma) = \begin{cases} 1_{Q_\gamma} & \gamma \notin \mathrm{spt}(p_n^2), \\ p_{n,\gamma}(\gamma) & \gamma \in \mathrm{spt}(p_n^2). \end{cases}$$

By our construction and the hypothesis $(*)(1)$, for all $\gamma \in \mathrm{spt}(p_n)$, $p_n^3(\gamma)$ is forced to be in V and to have supremum larger than α_n. Since $p_n^3 \in M$ and its support is countable, there are $p_{n+1} < p_n^3$ and $\langle \bar{B}_\gamma^n \mid \gamma \in \mathrm{spt}(p_n)\rangle$ in M such that for all $\gamma \in \mathrm{spt}(p_n)$,

$$p_{n+1} \restriction \gamma \Vdash p_{n+1}(\gamma) \cap \alpha_n = \bar{B}_\gamma^n.$$

It is now routine to verify p_{n+1} satisfies (i)–(iv).

Now we define a flat condition q such that $q < p_n$ for all $n < \omega$. Let

$$q(\omega_1^-) = \bigcup_{n<\omega} (p_n(\omega_1^-))^\frown \langle \langle \mathrm{ot}(M \cap \beta), \bigcup_{n<\omega} B_\beta^n \rangle \mid \beta \in [\omega_1, \alpha) \cap M\rangle.$$

We first verify that $q(\omega_1^-)$ is a $\mathbb{P}_{\omega_1}$-condition. It is clear that $q(\omega_1^-)$ is a function from $\mathrm{ot}(M \cap \alpha)$ to $P_{\omega_1}(\omega_1)$. We only need to verify that for any $\beta_1, \beta_2 \in M \cap (\alpha \setminus \omega_1)$ and any $\beta \in M \cap \omega_1$,

$$\bigcup_{n<\omega} B_{\beta_1}^n \neq q(\omega_1^-)(\beta) \text{ and } \bigcup_{n<\omega} B_{\beta_1}^n \neq \bigcup_{n<\omega} B_{\beta_2}^n.$$

For the first inequality, notice that for any $\beta \in M \cap \omega_1$, there is a $k < \omega$ such that $q(\omega_1^-)(\beta) = p_k(\omega_1^-)(\beta) \in P_{\omega_1}(\omega_1) \cap M$, and hence $q(\omega_1^-)(\beta)$ is bounded in $M \cap \omega_1$. On the other hand, let m be such that $\rho_m = \beta_1$, then for all $m < n < \omega$, $\gamma_{\rho_m,n+1}^n \in \bigcup_{n<\omega} B_{\rho_m}^n$. Notice that the supremum of $\langle \gamma_{\rho_m,n+1}^n \mid n < \omega\rangle$ is $M \cap \omega_1$, thus $\bigcup_{n<\omega} B_{\beta_1}^n = \bigcup_{n<\omega} B_{\rho_m}^n$ is unbounded in $M \cap \omega_1$, hence $\bigcup_{n<\omega} B_{\beta_1}^n \neq q(\omega_1^-)(\beta)$. For the second inequality, let i be such that $\beta_2 = \rho_i$, then there is a $k < \omega$ such that $\gamma_{\rho_m,i}^k \in B_{\rho_m}^k \leftrightarrow \gamma_{\rho_m,i}^k \notin B_{\rho_i}^k$. Note also that by the properties of B_β^k, for any $j < k < \omega$ and $\beta \in M \cap (\alpha \setminus \omega_1)$, $B_\beta^j = B_\beta^k \cap (\max(B_\beta^j) + 1)$. Hence $\gamma_{\rho_m,i}^k \in \bigcup_{n<\omega} B_{\beta_1}^n = \bigcup_{n<\omega} B_{\rho_m}^n$ if and only if $\gamma_{\rho_m,i}^k \notin \bigcup_{n<\omega} B_{\beta_2}^n = \bigcup_{n<\omega} B_{\rho_i}^n$.

For each $\beta \in \bigcup_{n<\omega} \mathrm{spt}(p_n)$, we define $q(\beta)$ to be a $\mathbb{P}_\beta$-name such that

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} q(\beta) = \bigcup_{n<\omega} p_n(\beta) \cup \{M \cap \omega_1\}.$$

For each $\beta \notin \bigcup_{n<\omega} \mathrm{spt}(p_n)$, we let $q(\beta)$ be the trivial condition. By induction on $\beta \in [\omega_1, \alpha)$, we show $q \restriction \beta$ is a $\mathbb{P}_\beta$-condition and for all $n < \omega$, $q \restriction \beta <_\beta p_n \restriction \beta$. This justifies the definition of q and implies that q is a $\mathbb{P}_\alpha$-condition.

We have already shown that $q(\omega_1^-)$ is a $\mathbb{P}_{\omega_1}$-condition stronger than $p_n(\omega_1^-)$ for all $n < \omega$. Now suppose $\beta \in [\omega_1, \alpha)$ and $q(\gamma)$ has been constructed for all $\gamma \in [\omega_1, \beta) \cup \{\omega_1^-\}$. By the induction hypothesis, $q \restriction \beta$ is a $\mathbb{P}_\beta$-condition, and for all $n < \omega$, $q \restriction \beta <_\beta p_n \restriction \beta$. There are two cases:

Case 1. $\beta \notin \bigcup_{i<\omega} \mathrm{spt}(p_i)$. $q(\beta)$ is the trivial condition and for each $n < \omega$, $p_n(\beta)$ is also trivial. Hence the induction hypothesis is true at $\beta + 1$.

Case 2. $\beta \in \bigcup_{i<\omega} \mathrm{spt}(p_i)$, then there is an $i < \omega$ such that $(\forall n > i)\beta \in \mathrm{spt}(p_n)$. We firstly verify that $q \restriction \beta \Vdash_{\mathbb{P}_\beta} M \cap \omega_1 \in \dot{S}_\beta''$.

Note that if $i < n$, $q \restriction \beta < p_{n+1} \restriction \beta \Vdash_{\mathbb{P}_\beta} \sup(p_{n+1}(\beta)) > \alpha_n$ and $p_{n+1} \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{e}_\beta[\eta_\beta^{n-1}] \prec \dot{e}_\beta[\eta_\beta^n] \prec \dot{A}_{\beta+1}$. So for all $i < m < n < \omega$, $p_{n+1} \restriction \beta \Vdash_{\mathbb{P}_\beta}$

$\dot{e}_\beta[\eta^m_\beta] \prec \dot{e}_\beta[\eta^n_\beta] \prec \dot{A}_{\beta+1}$. Since for all $n > i$, $\eta^n_\beta < M \cap \omega_1$ and $\bigcup_{n<\omega} \eta^n_\beta = \omega_1 \cap M$, $\langle \dot{e}_\beta[\eta^n_\beta] \mid n < \omega \rangle$ is forced to be an elementary chain with limit $\dot{e}_\beta[M \cap \omega_1]$. Since $q \restriction \beta$ is stronger than all $p_n \restriction \beta$,

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{e}_\beta[M \cap \omega_1] \prec \dot{A}_{\beta+1}.$$

Now we can verify $q \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{e}_\beta[M \cap \omega_1] \in \dot{S}'_\beta$. Note $q \restriction \beta \Vdash_{\mathbb{P}_\beta} (\omega_1)_{\dot{e}_\beta[M\cap\omega_1]} = M \cap \omega_1$. Let π_β be the $\mathbb{P}_\beta$-name for the collapsing map for $\dot{e}_\beta[M \cap \omega_1]$. So $q \restriction \beta \Vdash_{\mathbb{P}_\beta} \pi_\beta(\dot{S}_\beta) = \dot{S}_\beta \cap (M \cap \omega_1) = \bigcup_{n<\omega} B^n_\beta$. Thus $q \restriction \beta \Vdash_{\mathbb{P}_\beta} (H_{\beta+1})_{\dot{e}_\beta(M\cap\omega_1)}(\mathrm{ot}(M \cap \beta)) = H_{\beta+1}(\beta) \cap M = q(\omega_1^-)(\mathrm{ot}(M \cap \beta))$. On the other hand, by the induction hypothesis, for any $\gamma \in M \cap [\omega_1, \beta)$, $q \restriction \gamma$ is a $\mathbb{P}_\gamma$-condition. Hence $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma} (H_{\gamma+1})_{\dot{e}_\gamma[M\cap\omega_1]} = q(\omega_1^-) \restriction \mathrm{ot}(M \cap (\gamma+1))$. Moreover as for any $\gamma \in M \cap [\omega_1, \beta)$, $\Vdash_{\mathbb{P}_\beta} (H_{\gamma+1})_{\dot{e}_\gamma[M\cap\omega_1]} \subset (H_{\beta+1})_{\dot{e}_\beta[M\cap\omega_1]}$, $q \restriction \beta \Vdash_{\mathbb{P}_\beta} (H_{\beta+1})_{\dot{e}_\beta[M\cap\omega_1]} = q(\omega_1^-) \restriction \mathrm{ot}(M \cap (\beta+1)) \subset \dot{H}_{\beta+1}$. So by the definition of $\dot{S}'_\beta$ and $\dot{S}''_\beta$, $q \restriction \beta \Vdash_{\mathbb{P}_\beta} \dot{e}_\beta[M \cap \omega_1] \in S'_\beta$ and $M \cap \omega_1 \in \dot{S}''_\beta$.

By our assumption, $p_{n+1} \restriction \beta \Vdash_{\mathbb{P}_\beta} \sup(p_{n+1}(\beta)) > \alpha_n$. As $q \restriction \beta <_{\mathbb{P}_\beta} p_{n+1} \restriction \beta$, $q \restriction \beta \Vdash_{\mathbb{P}_\beta} \sup(p_{n+1}(\beta)) > \alpha_n$. Therefore

$$q \restriction \beta \Vdash_{\mathbb{P}_\beta} M \cap \omega_1 = \sup_{n<\omega} \alpha_n \le \bigcup_{n<\omega} \sup(p_n(\beta)) = \sup\left(\bigcup_{n<\omega} p_n(\beta)\right) \le M \cap \omega_1.$$

Since $q \restriction \beta \Vdash_{\mathbb{P}_\beta} M \cap \omega_1 \in \dot{S}''_\beta$, it follows from the definition of $\dot{Q}_\beta$ that $q \restriction \beta \Vdash_{\mathbb{P}_\beta} q(\beta)$ is a $\dot{Q}_\beta$-condition and $q(\beta) < p_n(\beta)$ for all $n < \omega$. It then follows from the induction hypothesis that $q \restriction \beta+1$ is a $\mathbb{P}_\beta$-condition such that $q \restriction \beta+1 < p_n \restriction \beta+1$ for all $n < \omega$.

Now $q \in \bigcap_{n<\omega} D_n$ and is thus an $(M, \mathbb{P}_\alpha)$-complete condition. Moreover, for each $\gamma \in \mathrm{spt}(q) \setminus \omega_1$, $q \restriction \gamma \Vdash q(\gamma) = \bigcup_{n<\omega} \bar{B}^n_\gamma \cup \{M \cap \omega_1\}$. Hence, q is a flat condition. ⊣

Definition 3.3. We say that q is an $(M, \mathbb{P}_\alpha)$-complete flat condition if q is defined as in the proof of Lemma 3.2 for M and $\mathbb{P}_\alpha$.

Note that if q is an $(M, \mathbb{P}_\alpha)$-complete flat condition, then the following facts hold.

Fact 3.4.

(1) *$dom(q(\omega_1^-)) = ot(M \cap \alpha)$ and $\sup(q(\beta)) = M \cap \omega_1$ for all $\beta \in spt(q) \setminus \omega_1$.*
(2) *For all $\beta \in spt(q) \setminus \{\omega_1^-\}$, $q \Vdash M \cap \omega_1 \in S''_\beta$.*
(3) *$q \Vdash (H_\alpha)_{\dot{e}_\alpha[M\cap\omega_1]} = q(\omega_1^-) \restriction ot(M \cap \alpha)$.*

As a corollary, we have

Proposition 3.5. *(*) holds at α.*

Proof. We only need to check (1). By Lemma 3.2, $\mathbb{P}_\alpha$ is ω_1-distributive. Moreover, the set of all $(M, \mathbb{P}_\alpha)$-complete flat conditions is of size ω_1 and dense in $\mathbb{P}_\alpha$. ⊣

We finish the definition of $\mathbb{P}_{\omega_2}$ by proving:

Lemma 3.6. *For any $\alpha \in [\omega_1, \omega_2)$, $\dot{S}''_\alpha$ is a stationary and co-stationary subset of ω_1.*

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Proof. We first show that S''_α is a stationary subset of ω_1 via a density argument. This amounts to showing the following: For any $p \in \mathbb{P}_\alpha$ and any $\mathbb{P}_\alpha$-name $\dot{C}$ such that $p \Vdash \dot{C}$ is a club subset of ω_1, there is a $q <_{\mathbb{P}_\alpha} p$ such that $q \Vdash_{\mathbb{P}_\alpha} \dot{C} \cap \dot{S}''_\alpha \neq \emptyset$.

Let θ be a sufficiently large regular cardinal. Pick a countable $M \prec H(\theta)$ such that $\{\mathbb{P}_\alpha, \dot{C}, p\} \subset M$ and M contains all parameters involved in the definition of $\mathbb{P}_\alpha$. Hence $\dot{H}_\alpha \in M$. Applying Lemma 3.2, choose an $(M, \mathbb{P}_\alpha)$-complete condition $q' <_{\mathbb{P}_\alpha} p$. The following can be verified using a density argument as in the proof of Lemma 3.2.

(1) $q' \Vdash$ "$\dot{C} \cap M$ is unbounded in $M \cap \omega_1$", and thus $q' \Vdash M \cap \omega_1 \in \dot{C}$.
(2) There is an unbounded subset B_α of $M \cap \omega_1$ such that $q' \Vdash$ "$\dot{S}_\alpha \cap M = B_\alpha$ and for all $\beta \in \operatorname{spt}(q')$, $B_\alpha \neq \dot{S}_\beta \cap M$".
(3) $q' \Vdash \dot{e}_\alpha[M \cap \omega_1] \prec \dot{A}_{\alpha+1}$.

Let q be defined as follows:

$$q(\gamma) = \begin{cases} q'(\gamma) \cup \{\langle \operatorname{ot}(M \cap (\alpha+1)), B_\alpha \rangle\}, & \text{if } \gamma = \omega_1, \\ q'(\gamma), & \text{otherwise.} \end{cases}$$

By (2) and Fact 3.4 (1), $q(\omega_1^-)$ is an injection from $\operatorname{ot}(M \cap (\alpha+1))$ to $P_{\omega_1}(\omega_1)$. It follows that q is a $\mathbb{P}_\alpha$-condition. By (1), $q \Vdash M \cap \omega_1 \in \dot{C}$.

We finish the proof by verifying $q \Vdash_{\mathbb{P}_\alpha} M \cap \omega_1 \in \dot{S}''_\alpha$. Note $q \Vdash_{\mathbb{P}_\alpha} (\omega_1)_{\dot{e}_\alpha[M\cap\omega_1]} = M \cap \omega_1$. Let π_α be the $\mathbb{P}_\alpha$-name of the collapsing map for $\dot{e}_\alpha[M \cap \omega_1]$. By Fact 3.4(3), $q \Vdash (H_\alpha)_{\dot{e}_\alpha[M\cap\omega_1]} = q(\omega_1^-) \restriction \operatorname{ot}(M \cap \alpha)$. Since $q \Vdash_{\mathbb{P}_\alpha} \pi_\alpha(\dot{S}_\alpha) = B_\alpha$, $q \Vdash (H_{\alpha+1})_{\dot{e}_\alpha[M\cap\omega_1]} = (H_\alpha)_{\dot{e}_\alpha[M\cap\omega_1]} \cup \{\langle \operatorname{ot}(M \cap \alpha+1), B_\alpha\rangle\} = q(\omega_1^-) \subset \dot{H}_{\alpha+1}$. So by (3) and the definition of $\dot{S}'_\alpha$, $q \Vdash \dot{e}_\alpha[M \cap \omega_1] \in \dot{S}'_\alpha$. By the definition of $\dot{S}''_\alpha$, $q \Vdash M \cap \omega_1 \in \dot{S}''_\alpha$. This completes the proof of the stationarity of $\dot{S}''_\alpha$.

To show that $\dot{S}''_\alpha$ is co-stationary, we only need to repeat the same argument with one exception. Let $q' <_{\mathbb{P}_\alpha} p$ be defined as in the above proof. Now define q as follows:

$$q(\gamma) = \begin{cases} q'(\gamma) \cup \{\langle \operatorname{ot}(M \cap (\alpha+1)), B \rangle\}, & \text{if } \gamma = \omega_1, \\ q'(\gamma), & \text{otherwise.} \end{cases}$$

It is then routine to verify that $q(\omega_1^-)$ is a injection from $\operatorname{ot}(M \cap (\alpha+1))$ to $P_{\omega_1}(\omega_1)$ and thus q is a condition. It is also easy to check, via the same argument, that $q \Vdash_{\mathbb{P}_\alpha} M \cap \omega_1 \notin \dot{S}''_\alpha$. ⊣

Finally, we let $\mathbb{P}_{\omega_2}$ be the direct limit of all $\mathbb{P}_\alpha$ for $\alpha \in [\omega_1, \omega_2)$.

Proposition 3.7. *$\mathbb{P}_{\omega_2}$ is ω_1-distributive and ω_2-c.c..*

Proof. The proof of Lemma 3.2 implies that for any $p \in \mathbb{P}_{\omega_2}$, any sufficiently large θ and any countable $M \prec H(\theta)$ containing $\mathbb{P}_\alpha$, there is an $(M, \mathbb{P}_{\omega_2})$-complete condition $q < p$. Thus $\mathbb{P}_{\omega_2}$ is ω_1-distributive. Since $\mathbb{P}_{\omega_2}$ is a ω_2-length countable support iteration and all $\mathbb{P}_\alpha$ are ω_2-c.c., by Proposition 7.8 of [3], it follows that $\mathbb{P}_{\omega_2}$ is ω_2-c.c. ⊣

Suppose G_{ω_2} is a $\mathbb{P}_{\omega_2}$-generic over V. Then in $V[G_{\omega_2}]$ the following hold:

- $\aleph_\alpha = (\aleph_\alpha)^V$, for $\alpha \in \mathrm{Ord}$;
- $\mathrm{Ord}^\omega = (\mathrm{Ord}^\omega)^V$;
- $CH + 2^{\omega_1} = \omega_2$.

Let H be $\bigcup_{\beta<\omega_2} H_\beta$, and let A_{ω_2} be the structure $\langle \text{Field}(H), H, \in\rangle$. By construction H is an injection from ω_2 to $\bigcup_{\alpha\in[\omega_1,\omega_2)}(P(\omega_1))^{V[G_\alpha]}$. Since $\mathbb{P}_{\omega_2}$ is ω_2-c.c. and is an ω_2-length countable support iteration, every subset of ω_1 in $V[G_{\omega_2}]$ appears in some $V[G_\alpha]$, where $\alpha \in [\omega_1, \omega_2)$. Hence $(P(\omega_1))^{V[G_{\omega_2}]} = \bigcup_{\alpha\in(\omega_1,\omega_2)}(P(\omega_1))^{V[G_\alpha]}$. It follows that H is a bijection from ω_2 to $(P(\omega_1))^{V[G_{\omega_2}]}$.

We will now verify that the structure A_{ω_2} has the desired SC property.

Lemma 3.8. *In $V[G_{\omega_2}]$, there is a club C in $P_{\omega_1}(A_{\omega_2})$ such that whenever Y is in C, $\langle Y, H \cap Y, \in\rangle \prec A_{\omega_2} \wedge H_Y \subset H$.*

Proof. Fix κ sufficiently large and let

$$D = \{M \prec H(\kappa) \mid |M| = \omega \wedge \{H, \mathbb{P}_{\omega_2}, G_{\omega_2}\} \subset M.$$

Let C be the restriction of D onto A_{ω_2}. Then C is a club subset of $\mathbb{P}_{\omega_1}(A_{\omega_2})$. Fix an arbitrary $Y \in C$ and let $X \in D$ be such that $X \cap A_{\omega_2} = Y$. By elementarity, for any β, if $\beta \in \omega_2 \cap X = \omega_2 \cap Y$, then C_β and e_β are both in X. Since C_β is a club, $X \cap \omega_1 \in C_\beta \subset S''_\beta$. Applying e_β it follows that $X \cap A_\alpha = e_\beta[X \cap \omega_1] \in S'_\beta$. By the definition of S'_α, $H_{X\cap A_\alpha} \subset H_\alpha \subset H$. Since $X \cap \omega_2$ is unbounded in $\sup(X \cap \omega_2)$, $H_Y = H_{X\cap A_{\omega_2}} = \bigcup_{\alpha\in X\cap\omega_2} H_{X\cap A_\alpha} \subset H$. The inclusion holds because $X \cap A_\alpha$ is transitive in X. ⊣

We can now construct F from H. Recall that every element X in $H(\omega_2)$ can be uniformly decoded from some subset of ω_1 as follows: For any $X \in H(\omega_2)$, let $H_0(X) = \text{tc}(X) \cup \{X\}$. Note that H_0 is an injection from $H(\omega_2)$ to $H(\omega_2)$. Now for each subset E' of $\omega_1 \times \omega_1$ coding a well-founded ω_1-sized binary relation $<_{E'}$, there is a unique $Y \in H(\omega_2)$ such that $(Y, \in) \cong (E', <_{E'})$. Let E be the inverse image of E' under the Gödel pairing function. Hence, for all such $E \subset \omega_1$ we can uniformly decode a unique element $H_0^{-1}(Y)$ of $H(\omega_2)$. We inductively define $F(\alpha) \in H(\omega_2)$ to be decoded from the $<_H$-least $E \in P(\omega_1)$ such that E can be decoded and no $X \in F[\alpha]$ can be decoded from E. Note that F is a bijection from ω_2 to $H(\omega_2)$ such that $F \restriction \omega_1$ is a bijection from ω_1 to $H(\omega_1)$ and F is Δ_1-definable over $\langle H(\omega_2), H\rangle$, thus we have:

Proposition 3.9. *If $M \prec \langle H(\omega_2), H, \in\rangle$, then $F \restriction M$ is a bijection from Ord^M to M.*

Proof. By definability and elementarity, $F \restriction M$ is definable over M, and is a bijection. ⊣

Lemma 3.10. *In $V[G_{\omega_2}]$, there is a club C in $P_{\omega_1}(H(\omega_2))$ such that $\langle X, F \cap X, \in\rangle \prec \langle H(\omega_2), F, \in\rangle \wedge F_X \subset F$ whenever $X \in C$.*

Proof. Consider the club C given by Lemma 3.8. We show this C works. Fix $X \in C$, then $\langle X, H, \in\rangle \prec \langle H(\omega_2), H, \in\rangle$ and $H_X \subset H$. By induction on $\text{Ord} \cap X$, we show $F(\alpha)_X = F(\alpha_X)$. Suppose $\alpha < \omega_1$. Then $F(\alpha) \in H(\omega_1)$ and thus $F(\alpha)_X = F(\alpha) = F(\alpha_X)$.

Assume that $\alpha \in \text{Ord} \cap X$ and $F(\beta)_X = F(\beta_X)$ for all $\beta \in \alpha \cap X$. Let $A = F(\alpha)$. By elementarity, let $\alpha' \in X$ be such that $H(\alpha')$ witnesses the definition of $F(\alpha)$, i.e., $H(\alpha')$ codes A and for all $\beta' < \alpha'$, either $H(\beta')$ is not a code or $H(\beta')$ codes some $F(\beta)$ with $\beta < \alpha$. Hence

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$$\begin{aligned} &X \models H(\alpha') \text{ codes } A \\ \text{iff} \quad &\bar{X} \models H(\alpha')_X \text{ codes } A_X \\ \text{iff} \quad &\bar{X} \models H(\alpha'_X) \text{ codes } A_X. \end{aligned}$$

Similarly, for all $\beta' \in \alpha' \cap X$, there is a $\beta \in \alpha \cap X$ such that

$$\bar{X} \models \text{ either } H(\beta'_X) \text{ is not a code or } H(\beta'_X) \text{ codes } F(\beta)_X.$$

Since for all $\beta \in \alpha \cap X$, $F(\beta)_X = F(\beta_X)$ and the coding is absolute between any transitive models, we have that for all $\beta' < \alpha'_X$, $H(\beta')$ is not a code or $H(\beta')$ codes $F(\beta_X)$ for some $\beta \in \alpha \cap X$ and $H(\alpha'_X)$ codes A_X. Note that A_X is not equal to $F(\beta_X)$ for $\beta \in \alpha \cap X$, hence $H(\alpha'_X)$ witnesses the definition of $F(\alpha_X)$. It follows that $F(\alpha_X) = A_X = F(\alpha)_X$. ⊣

Remark 3.11. For any predicate $A \subset \omega_2$ in V, we can modify $\mathbb{P}_{\omega_2}$ so that if $M \prec H(\omega_2)^{V[G_{\omega_2}]}$ is closed under F, then $\langle M, M \cap A\rangle \prec \langle H(\omega_2), A\rangle$.

We end this section by imposing a notation for later usage. The forcing defined in this section will be denoted by $\mathbb{P}_e$, where e is a parameter coding the construction of $\mathbb{P}_{\omega_2}$. In particular, the information coded by e includes the bookkeeping function h, the bijections $\{\dot{e}_\alpha \mid \alpha < \omega_2\}$, and the names of the well-ordering $\{\dot{W}_\alpha \mid \alpha < \omega_2\}$. We can moreover require e to be a subset of ω_2. Note also that if $M \subset V$ are two transitive class models of ZFC such that $H(\omega_2)^M = H(\omega_2)^V$, then the forcing poset $\mathbb{P}_e$ defined in M is also a poset for SC_{ω_2} in V.

§4. Joint consistency of ASC and $\neg\Box_{\omega_1}$. In this section, we show the joint consistency of ASC and $\neg\Box_{\omega_1}$. Recall that a sequence $\langle C_\xi \mid \xi \in \lim(\omega_2)\rangle$ is a $\Box_{\omega_1}$- *sequence* if for any $\xi \in \lim(\omega_2)$, the following hold:

(1) C_ξ is a club subset of ξ,
(2) $\mathrm{ot}(C_\xi) \leq \omega_1$,
(3) if $\beta \in \lim(C_\xi)$, then $C_\beta = C_\xi \cap \beta$.

We say $\neg\Box_{\omega_1}$ holds if there is no $\Box_{\omega_1}$- sequence.

Theorem 4.1. *Assume there exists a κ such that the set*

$$S = \{\alpha < \kappa \mid \alpha \textit{ is a measurable cardinal}\}$$

is stationary below κ, then it is consistent that $ZFC + ASC + \neg\Box_{\omega_1}$.

We basically follow the framework for constructing models of the failure of square on the successor of regular cardinals, due to Solovay. We outline the framework as follows. First we collapse a large cardinal κ to ω_2. If there is a square sequence $\vec{C}$ in the generic extension, then we show the forcing can be canonically factorized at some limit level α such that $\vec{C} \restriction \alpha$ is decided by the generic up to the same level. Then we show the quotient forcing is not able to thread $\vec{C} \restriction \alpha$. But as $\vec{C}$ is a square sequence, C_α must thread $\vec{C} \restriction \alpha$. This gives a contradiction. The large cardinal required for this argument is one Mahlo cardinal. In our current setting, we need to first collapse κ to ω_2 and then force SC for ω_2. We may then factorize the forcing as above. However, we meet one obstacle during the attempt to show that the quotient forcing cannot thread the sequence up to that stage, which is caused by the fact that the quotient forcing is not ω-closed. This is the reason we need a stronger

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large cardinal assumption for the construction. We want to remark that the large cardinal assumption seems to be far from optimal. We conjecture that stationary many Ramsey cardinals suffice.

The following lemma is the only place where measurability is required.

LEMMA 4.2. *Let α be a measurable cardinal and let $\theta > \alpha$ be a regular cardinal. Suppose $c \in H(\theta)$. Then there is a pair of models M_0 and M_1 such that*:

(1) $M_0 \prec \langle H(\theta), \alpha, c\rangle$, $M_1 \prec \langle H(\theta), \alpha, c\rangle$.
(2) *If* $\gamma = \sup(\alpha \cap M_0 \cap M_1)$, *then* $P_{\omega_1}(M_0 \cap V_\gamma) \subset M_1$.
(3) $\sup(M_0 \cap \alpha) = \sup(M_1 \cap \alpha) > \gamma$.
(4) $|M_0| = \omega$, $|M_1| = \omega_1$ *and* $H(\omega_1) \subset M_1$.

PROOF. We construct two sequences of models $\langle M_n^0 \mid n < \omega\rangle$ and $\langle M_n^1 \mid n < \omega\rangle$ such that for $i = 0, 1$ and $n < \omega$:

(1) $|M_n^0| = \omega$ and $|M_n^1| = \omega_1$.
(2) $M_n^i \prec M_{n+1}^i \prec \langle H(\theta), \alpha, c\rangle$.
(3) If $\gamma_n^i = \sup(M_n^i \cap \alpha)$, then $M_{n+1}^i \cap \gamma_n^i = M_n^i \cap \gamma_n^i$ and $\gamma_{n+1}^i > \gamma_n^i$.
(4) $(\alpha \cap M_{n+1}^i \cap M_{n+1}^{1-i}) \setminus \min\{\gamma_n^i, \gamma_n^{1-i}\} = \emptyset$.
(5) $P_{\omega_1}(M_0^0 \cap V_\alpha) \subset M_0^1$ and $H(\omega_1) \subset M_0^1$.

Letting $M_0 = \bigcup_{n<\omega} M_n^0$ and $M_1 = \bigcup_{n<\omega} M_n^1$, it is routine to verify that

- $\gamma = \sup(M_0 \cap M_1 \cap \alpha) = \sup(M_0^0 \cap \alpha)$.
- $P_{\omega_1}(M_0^0 \cap V_\alpha) = P_{\omega_1}(M_0 \cap V_\gamma) \subset M_1$.
- $\sup(M_0 \cap \alpha) = \bigcup_{n<\omega} \gamma_n^0 = \bigcup_{n<\omega} \gamma_n^1 = \sup(M_1 \cap \alpha)$.

Hence M_0 and M_1 are as required.

The construction is based on the following standard fact. We include a proof for the reader's convenience.

FACT 4.3 (folklore). *Suppose $\theta > \alpha$ is a regular cardinal, $M \prec \langle H(\theta), \alpha, <\rangle$ and $\xi \in \bigcap_{A\in\mu\cap M} A$, where $\mu \in M$ is a normal measure over α. If $M(\xi) = Sk^{\langle H(\theta),\alpha,<\rangle}(M \cup \{\xi\})$, then $M(\xi) \cap \xi = M \cap \xi = M \cap \alpha$.*

PROOF. For the first equality, fix a Skolem term t and an $a \in M$ such that $t(a, \xi) < \xi$. Now $A = \{\gamma < \alpha \mid t(a, \gamma) < \gamma\} \in M$ must be in μ. By the normality of μ, there is a $B \subset A$ and a δ in M such that $B \in \mu$ and $t(a, \gamma) = \delta$ for all $\gamma \in B$. Hence $t(a, \xi) = \delta \in M$.

For the second equality, note that ξ is greater than any ordinal $\gamma \in M \cap \alpha$ as ξ is in the measure one set (γ, α). ⊣

We can now inductively construct $\langle M_n^0 \mid n < \omega\rangle$ and $\langle M_n^1 \mid n < \omega\rangle$. Choose arbitrary M_0^0 and M_0^1 such that $P_{\omega_1}(M_0^0 \cap V_\alpha) \subset M_0^1$ and $H(\omega_1) \subset M_0^1$. Assume M_n^0 and M_n^1 are constructed. Fix a $\xi_0 \in \bigcap_{A\in\mu\cap M_n^0} A$. Let $M_{n+1}^0 = M_n^0(\xi_0)$. Fix $\xi_1 \in \bigcap_{A\in\mu\cap M_n^1} A \setminus M_{n+1}^0$. Let $M_{n+1}^1 = M_n^1(\xi_1)$. Clearly (1) and (2) hold. By Fact 4.3, it is routine to check that (3) and (4) hold. ⊣

Without loss of generality, we start from a model where GCH holds and there is a κ witnessing the assumption. Moreover, we also assume that there is no inaccessible cardinal above κ. Now force with $\mathrm{Col}(\omega_1, <\kappa) * \mathbb{P}_e$, where $\mathbb{P}_e$ is the forcing defined in Section 3. Let G be $\mathrm{Col}(\omega_1, <\kappa) * \mathbb{P}_e$-generic over V. By Theorem 1.1, SC_{ω_2} holds in $V[G]$. We will show later in this section that there is no $\square_{\omega_1}$ sequence in $V[G]$.

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Now in $V[G]$, we can verify the assumption of Proposition 2.10 as follows: Let A be a bounded subset of ω_2. By the κ-c.c., A is in some intermediate model $V[\bar{G}]$ generated by some complete subforcing of $Col(\omega_1, <\kappa) \times P_e$ of size less than κ. Thus In $V[\bar{G}]$, there remain some measurable cardinals above $|tc(A)|$. Hence $A^\sharp$ exists in $V[\bar{G}]$ and thus exists in $V[G]$ by the absoluteness of sharps.

Now applying Proposition 2.10, $L[F]$ is a model of ASC and ZFC which agrees on ω_1 and ω_2 with $V[G]$. As $L[F]$ is an inner model of $V[G]$, if there is no $\Box_{\omega_1}$-sequence in $V[G]$, then there is no $\Box_{\omega_1}$-sequence in $L[F]$.

The rest of this section is devoted to the proof of the failure of $\Box_{\omega_1}$ in $V[G]$.

Lemma 4.4. *There is no $\Box_{\omega_1}$ sequence in $V[G]$.*

For any $\alpha < \beta \leq \kappa$, denote the forcing $\mathrm{Col}(\omega_1, <\beta)$ by $\mathbb{C}_\beta$ and $\mathrm{Col}(\omega_1, [\alpha, \beta))$ by $\mathbb{C}_{\alpha,\beta}$. We rely on the following fact revealing the analogy between $\mathbb{C}_\kappa * \mathbb{P}_e$ and $\mathbb{P}_e$. The proof is an easy modification of the proof of Lemma 3.2 and is omitted here.

Definition 4.5. A $\mathbb{C}_\kappa * \mathbb{P}_e$-condition $p = p_0 * p_1$ is flat if there is a unique ordinal $\gamma_p < \omega_1$ and sequences $\langle p_i^0 \mid i \in \mathrm{spt}(p_0)\rangle \in V$ and $\langle p_i^1 \mid i \in \mathrm{spt}(p_1)\rangle \in V$ such that

$$(\forall i \in \mathrm{spt}(p_0))(p_0(i) = p_i^0 \wedge \mathrm{dom}(p_i^0) = \gamma_p),$$

$$(\forall i \in \mathrm{spt}(p_1))(p \upharpoonright i \Vdash p_1(i) = p_i^1 \wedge (i \neq \omega_1 \rightarrow \sup(p_i^1) = \gamma_p)).$$

Lemma 4.6. *For all sufficiently large θ, any countable $M \prec H(\theta)$ and $p \in M$, there is a flat $(M, \mathbb{C}_\kappa * \mathbb{P}_e)$-complete condition $q = q_0 * q_1$ extending p such that*

(1) $dom(q_1(\omega_1^-)) = ot(M \cap \kappa)$, $spt(q_0) = spt(q_1) = M \cap \kappa$.
(2) $\gamma_q = M \cap \omega_1$.

We give some simple analysis of $\mathbb{C}_\kappa * \mathbb{P}_e$. By the analysis of $\mathbb{P}_e$ in Section 3, we may assume $\mathbb{P}_e$ consists only of countable conditions which are elements of $V^{\mathbb{C}_\kappa}$ and for all $\alpha < \kappa$, $\mathbb{P}_\alpha$ is of size ω_1 in $V^{\mathbb{C}_\kappa}$. Since $\mathbb{C}_\kappa$ is countably closed, we may further assume $\mathbb{P}_e \subset V$. For any $\alpha < \kappa$, we factor $\mathbb{C}_\kappa$ as $\mathbb{C}_\alpha \times \mathbb{C}_{\alpha,\kappa}$. In $V^{\mathbb{C}_\kappa}$, we factor $\mathbb{P}_e$ as $\mathbb{P}_\alpha * \mathbb{P}_{[\alpha,\kappa)}$. For any $\alpha_1, \alpha_2 < \kappa$, if $\mathbb{P}_{\alpha_2} \in V^{\mathbb{C}_{\alpha_1}}$, then $\mathbb{C}_\kappa * \mathbb{P}_e = \mathbb{C}_{\alpha_1} \times \mathbb{C}_{\alpha_1,\kappa} * \mathbb{P}_{\alpha_2} * \mathbb{P}_{[\alpha_2,\kappa)}$ and $\mathbb{C}_{\alpha_1} * \mathbb{P}_{\alpha_2} * \mathbb{C}_{\alpha_1,\kappa} * \mathbb{P}_{[\alpha_2,\kappa)}$ are equivalent forcing notions.

For any $\alpha < \kappa$, we say that $\mathbb{C}_\kappa * \mathbb{P}_e$ is *factorable* at α if $\mathbb{C}_\kappa * \mathbb{P}_e = \mathbb{C}_\alpha * \mathbb{P}_\alpha * \mathbb{C}_{\alpha,\kappa} * \mathbb{P}_{[\alpha,\kappa)}$. We will show there are club many $\alpha < \kappa$ such that $\mathbb{C}_\kappa$ is factorable at α.

Proof of Lemma 4.4. Work in $V[G]$ where G is a $\mathbb{C}_\kappa * \mathbb{P}_e$-generic filter. Assume for a contradiction that $\vec{C} = \langle C_\alpha \mid \alpha \in Lim(\omega_2^{V[G]})\rangle$ is a square sequence. Write G as $G^1 * G^2$, where G^1 is $\mathbb{C}_\kappa$-generic over V and G^2 is $\mathbb{P}_e$-generic over $V[G^1]$. For any $\alpha < \kappa$, we factor $\mathbb{C}_\kappa$ as $\mathbb{C}_\alpha \times \mathbb{C}_{\alpha,\kappa}$ and let $G^1 = G^1_\alpha \times G^1_{[\alpha,\kappa)}$ be the corresponding generic filter. In $V[G^1]$, we factor $\mathbb{P}_e$ as $\mathbb{P}_\alpha * \mathbb{P}_{[\alpha,\kappa)}$ and let $G^2 = G^2_\alpha \times G^2_{[\alpha,\kappa)}$ be the corresponding generic filter.

Claim 4.7. *Let D consist of all $\alpha \in \kappa$ such that the following hold:*

(1) $\mathbb{P}_\alpha \in V^{\mathbb{C}_\alpha}$.
(2) $\vec{C} \upharpoonright \alpha \subset V^{\mathbb{C}_\alpha * \mathbb{P}_\alpha}$.

Then D is a club.

Proof. We construct two functions $f_1, f_2 : \kappa \to \kappa$ such that if α closed under f_1 and f_2, then $\alpha \in D$. Fix $\beta < \kappa$. In $V[G^1]$, $\mathbb{P}_\beta$ is an ω_1 sized subset of V. By the κ-c.c. of $\mathbb{C}_\kappa$, there is a nice name t for $\mathbb{P}_\beta$ in V_κ. Let $f_1(\beta)$ be least such that

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$t \in V_{f_1(\beta)}$. Then t is a $\mathbb{C}_{f_1(\beta)}$-name and thus $\mathbb{P}_\beta \in V[G^1_{f_1(\beta)}]$. In $V[G]$, C_β is a subset of β. Since $\mathbb{C}_\kappa * \mathbb{P}_e$ is κ-c.c., there is a nice name t for C_β in V_κ. Let $f_2(\beta)$ be least such that t is a $\mathbb{C}_\kappa * \mathbb{P}_{f_2(\beta)}$-name in $V_{f_2(\beta)}$, then $C_\beta \in V[G^1_{f_1(f_2(\beta))} * G^2_{f_2(\beta)}]$.[17] ⊣

For any $\alpha \in D$, $\mathbb{C}_\kappa * \mathbb{P}_e$ is factorable at α. Recall that $S = \{\alpha < \kappa \mid \alpha \text{ is measurable}\}$ is a stationary subset of κ. Fix $\alpha \in D \cap S$, then $\vec{C} \restriction \alpha \in V[G^1_\alpha * G^2_\alpha]$ and α is measurable. We show that no $C \in V[G]$ can thread $\vec{C} \restriction \alpha$.[18] As C_α must thread $\vec{C} \restriction \alpha$, this leads to a contradiction and thus ends the proof of Lemma 4.4.

Case 1) $C \in V[G^1_\alpha * G^2_\alpha]$.

As α is $\omega_2^{V[G^1_\alpha * G^2_\alpha]}$, the cofinality of C is larger than ω_1. Hence there is a $\beta < \alpha$ such that $\beta \in Lim(C)$ and $C \cap \beta$ is of cofinality greater than ω_1. As C threads $\vec{C} \restriction \alpha$, $C_\beta = C \cap \beta$. However, by the definition of a square sequence, the order type of $C_\beta = C \cap \beta$ must be less than ω_1 in $V[G]$. Hence C cannot thread $\vec{C} \restriction \alpha$.

Case 2) C is not in $V[G^1_\alpha * G^2_\alpha]$.

Let $\dot{C}$ be a $\mathbb{C}_\kappa * \mathbb{P}_e$-name for C. Fix any $\mathbb{C}_\alpha * \mathbb{P}_\alpha$-condition $t \in G^1_\alpha * G^2_\alpha$ such that there are $\mathbb{C}_\kappa * \mathbb{P}_e$-conditions p and q such that $p \restriction \mathbb{C}_\alpha * \mathbb{P}_\alpha = q \restriction \mathbb{C}_\alpha * \mathbb{P}_\alpha = t$, $p \Vdash \gamma \in \dot{C}$, and $q \Vdash \gamma \notin \dot{C}$. Via a density argument, for any condition $s < t$, it suffices to find $r < s$ which forces a contradiction. For simplicity, we assume $s = t$.

Fix a sufficiently large $\theta > \kappa$ such that the above is expressible in $H(\theta)$. Consider the structure $N = \langle H(\theta), \mathbb{C}_\kappa * \mathbb{P}_e, \dot{C}, \dot{\vec{C}}, p, q, \gamma\rangle$. Let $\langle M_0, M_1\rangle$ be the pair of elementary submodels of N constructed by applying Lemma 4.2 to α, θ and the above mentioned parameters in $H(\theta)$. We denote $\sup(M_0 \cap \alpha) = \sup(M_1 \cap \alpha)$ by α' and $\sup(M_0 \cap M_1 \cap \alpha)$ by $\bar{\alpha}$. Since M_0 is a countable elementary submodel of $H(\theta)$, we can apply Lemma 4.6 to get a flat $(M_0, \mathbb{C}_\kappa * \mathbb{P}_e)$-complete condition $\bar{p} = \bar{p}_0 * \bar{p}_1$ extending p. We claim that $\bar{p} \cap V_{\bar{\alpha}} \in M_1$. This is because $\bar{p} = \bar{p}_1(\omega_1^-) \cup \bar{p} \setminus \bar{p}_1(\omega_1^-)$, where $\bar{p}_1(\omega_1^-) \in H(\omega_1) \subset M_1$ and $(\bar{p} \cap V_{\bar{\alpha}}) \setminus \bar{p}(\omega_1^-) \in P_{\omega_1}(M_0 \cap V_{\bar{\alpha}}) \subset M_1$. Fix a countable elementary submodel $\bar{M}$ of M_1 such that $\bar{p} \cap V_{\bar{\alpha}} \in \bar{M}$ and $\sup(\bar{M} \cap \alpha) = \sup(M_1 \cap \alpha) = \alpha'$. This is possible since α' has countable cofinality. Note that $\bar{p} \cap V_{\bar{\alpha}}$ and q are compatible, let $q' \in \bar{M}$ witness this. Applying Lemma 4.6, let $\bar{q} < q'$ be a flat $(\bar{M}, \mathbb{C}_\kappa * \mathbb{P}_e)$-complete condition. Note that $\mathrm{spt}(\bar{p}) \subset M_0$ and $\mathrm{spt}(\bar{q}) \subset \bar{M}$. Via a classical argument, we can show that

Claim 4.8. *$\bar{p} \Vdash \gamma \in C_{\sup(\alpha \cap M_0)} = C_{\alpha'}$ and $\bar{q} \Vdash \gamma \notin C_{\sup(\alpha \cap \bar{M})} = C_{\alpha'}$.*

Proof. We only verify the claim for $\bar{p}$. The verification process for $\bar{q}$ is the same. Since $\bar{p}$ is $(M_0, \mathbb{C}_\kappa * \mathbb{P}_e)$-complete, it forces that $\dot{C}$ is unbounded in $M_0 \cap \alpha$ and thus $\alpha' = \sup(M_0 \cap \alpha)$ is a limit point of $\dot{C}$. Note that $\gamma \in M \cap \alpha$. Hence, $\bar{p} \Vdash \gamma \in \dot{C} \cap \alpha' = C_{\alpha'}$. ⊣

The following is the key of the argument:

Claim 4.9. *$\bar{p} \restriction \alpha$ and $\bar{q} \restriction \alpha$ are compatible.*

Proof. Inductively on $\gamma \leq \alpha$, we construct conditions r_γ extending both $\bar{p} \restriction \gamma$ and $\bar{q} \restriction \gamma$ such that

[17] The appearance of f_1 ensures that $V[G^1_{f_1(f_2(\beta))} * G^2_{f_2(\beta)}]$ is meaningful.

[18] We say a set C *threads* $\vec{C} \restriction \alpha$ if C is a club subset of α, $\mathrm{ot}(C) \leq |\alpha|$ and for all $\beta \in Lim(C)$, $C \cap \beta = C_\beta$.

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(1) $r_{\gamma_1} \restriction \gamma_2 = r_{\gamma_2}$ for $\gamma \geq \gamma_1 > \gamma_2$.
(2) $\mathrm{spt}(r_\gamma) = \mathrm{spt}(\bar{p} \restriction \gamma) \cup \mathrm{spt}(\bar{q} \restriction \gamma)$.

For $\gamma < \bar{\alpha}$, $\bar{q} \restriction \gamma < q' \restriction \gamma < \bar{p} \restriction \gamma$. We choose r_γ to be $\bar{q} \restriction \gamma$.

Now suppose $\gamma = \beta + 1 > \bar{\alpha}$. If $\beta \notin \mathrm{spt}(\bar{p} \restriction \alpha) \cup \mathrm{spt}(\bar{q} \restriction \alpha)$, then let

$$r_\gamma(\delta) = \begin{cases} r_\beta(\delta), & \text{if } \delta < \beta, \\ 1, & \text{if } \delta = \beta. \end{cases}$$

Otherwise, $\beta \in \mathrm{spt}(\bar{p} \restriction \alpha) \cup \mathrm{spt}(\bar{q} \restriction \alpha)$. Assume $\beta \in \mathrm{spt}(\bar{p} \restriction \alpha) \subset M_0$. It follows that $\beta \in (M_0 \cap \alpha) \setminus \bar{\alpha}$. Let

$$r_\gamma(\delta) = \begin{cases} r_\beta(\delta), & \text{if } \delta < \beta, \\ \bar{p}(\delta), & \text{if } \delta = \beta. \end{cases}$$

Then r_γ is as desired as $r_\beta < \bar{q} \restriction \beta \Vdash \bar{q}(\beta)$ is trivial. The case $\beta \in \mathrm{spt}(\bar{q} \restriction \alpha) \subset \bar{M}$ is similar.

Finally for $\gamma \geq \bar{\alpha}$ limit, we take r_γ to be the greatest lower bound of all r_β with $\beta < \gamma$. Clearly r_γ is as desired. ⊣

Let r be a common extension of $\bar{p} \restriction \alpha$ and $\bar{q} \restriction \alpha$. Let r_p be a common extension of r and $\bar{p}$ and r_q be a common extension of r and $\bar{q}$. Then $r_p \Vdash \gamma \in C_{\alpha'}$ and $r_q \Vdash \gamma \notin C_{\alpha'}$. Let G_α be an arbitrary $\mathbb{C}_\alpha * \mathbb{P}_\alpha$-generic filter containing r. Then by Claim 4.7, whether $\gamma \in C_{\alpha'}$ or not is decided in $V[G_\alpha]$. If $\gamma \in C_{\alpha'}$ and G is any generic filter extending G_α and r_q, then $V[G] \models \gamma \notin C_{\alpha'}$. Similar for the case $\gamma \notin C_{\alpha'}$. Hence r forces a contradiction. It also clear that r is stronger than t.

This ends Case 2) and the proof of Lemma 4.4. ⊣

§5. **An application on precipitous ideals over** ω_1**.** In this section, we study the relationship between SC and precipitous ideals. Recall that an ideal I on κ is *precipitous* iff for all generic $G \subseteq P(\kappa)/I$, the ultrapower V^κ/G is well-founded. As mentioned in Section 2, ASC refutes the existence of a precipitous ideal. The following general fact is due to Woodin. The special case when $\kappa = \omega_1$ is also proved by Schimmerling–Velickovic ([11]). We include a proof for completeness.

PROPOSITION 5.1. *Suppose κ is regular and SC_{κ^+} holds and is witnessed by $F : \kappa^+ \to H(\kappa^+)$. Then no ideal over κ definable in $\langle H(\kappa^+), F\rangle$ is precipitous.*

PROOF. Suppose the proposition fails at κ, and let I be a precipitous ideal over κ definable in $\langle H(\kappa^+), F\rangle$. Without loss of generality, we can assume that the critical point of the corresponding generic embedding j_G is forced to be η by the trivial condition κ, where $\eta \in [\omega_1, \kappa]$ is a regular cardinal. Since $I \in L[F]$ and the precipitousness of an ideal is a Π_1 property, $L[F] \models I$ is a precipitous ideal over κ such that $\kappa \Vdash_{P(\gamma)/I} cpt(j_G) = \eta$.

Let $\psi(\alpha, \beta, F)$ be the following sentence: F witnesses SC_{γ^+}, where γ is a regular cardinal such that $\alpha = \gamma^+$, and there is a precipitous ideal I over γ such that $\Vdash_{P(\gamma)/I} cpt(j_G) = \beta$. In $L[F]$, $\psi(\kappa^+, \eta, F)$ holds. Let (α, β) be the lexicographically least pair such that $L[F \restriction \alpha] \models \psi(\alpha, \beta, F \restriction \alpha)$. In $L[F \restriction \alpha]$, we have $V = L[F \restriction \alpha]$ and $V \models (\forall (\bar{\alpha}, \bar{\beta}) <_{lex} (\alpha, \beta))(L[F \restriction \bar{\alpha}] \models \neg\psi(\bar{\alpha}, \bar{\beta}, F \restriction \bar{\alpha}))$.

Hence we may assume there is an $F : \alpha \to F[\alpha]$ and a regular cardinal β such that $V = L[F] \models \psi(\alpha, \beta, F)$ and for all $(\bar{\alpha}, \bar{\beta}) <_{lex} (\alpha, \beta)$, $L[F \restriction \bar{\alpha}] \models \neg\psi(\bar{\alpha}, \bar{\beta}, F \restriction \bar{\alpha})$.

Let I be a precipitous ideal on γ such that $\Vdash_{P(\gamma)/I} cpt(j_G) = \beta$, where $\gamma^+ = \alpha$. Let $j : V \to M \subset V[G]$ be the valuation of j_G in $V[G]$, where G is $P(\gamma)/I$-generic.

By elementarity $M \models$ "$j(F)$ witnesses SC holds for $j(F[\alpha])$". By Theorem 2.5, $V[G] \models$"$j(F)$ witnesses SC holds for $j(F[\alpha])$". Note that $j[F[\alpha]] \prec j(F[\alpha])$, which follows from Tarski criterion and the elementarity of j. Although $j[F[\alpha]]$ may not be in M, it is in $V[G]$. So applying SC for $j(F[\alpha])$ in $V[G]$, we get $F[\alpha] =$ the transitive collapse of $j[F[\alpha]] = j(F)[\mathrm{ot}(\mathrm{Ord} \cap j[F[\alpha]])] = j(F)[\alpha]$ and thus $F = j(F) \restriction \alpha$. By elementarity, $M \models V = L[j(F)]$ and for all $(\bar{\alpha}, \bar{\beta}) <_{lex} (j(\alpha), j(\beta))$, $L[j(F) \restriction \bar{\alpha}] \models \neg\psi(\bar{\alpha}, \bar{\beta}, j(F) \restriction \bar{\alpha})$. However, $L[j(F) \restriction \alpha] = L[F] \models \psi(\alpha, \beta, j(F) \restriction \alpha)$. Note that $(\alpha, \beta) <_{lex} (j(\alpha), j(\beta))$. Contradiction. ⊣

Remark 5.2. In the above model for $\kappa = \omega_1$, we also have that any ideal on ω_1 is not ω_2-saturated. Assume there is a saturated ideal I on ω_1. Let $j : V \to M \subset V[G]$ be the generic embedding. Then $\omega_1^M = j(\omega_1) = \omega_2$. By elementarity, in M, $j(F)$ witnesses SC for $(H(\omega_2))^M = j(H(\omega_2))$. We have $(H(\omega_1))^M = j(F)[\omega_1^M] = j(F)[\omega_2] = H(\omega_2)$, which is impossible since their theories are different. This gives a different proof of a theorem of Baumgartner and Taylor ([1]) that there is a set-forcing which kills all saturated ideals on ω_1.

Corollary 5.3. *It is consistent relative to a supercompact cardinal that Con (there is a supercompact cardinal+no ideal on ω_1 definable over $H(\omega_2)$ is precipitous).*

Proof. Start with a ground model V where a supercompact cardinal κ exists and GCH holds. Let G be a $\mathbb{P}_e$-generic over V. Then in $V[G]$, κ remains supercompact as the size of $\mathbb{P}_e$ is small. On the other hand, by Proposition 5.1, there is no precipitous ideal on ω_1 which is definable over $H(\omega_2)$. ⊣

If a precipitous ideal over ω_1 exists in this model, then it cannot be the nonstationary ideal on ω_1 by definability. It is natural to ask whether SC over $H(\omega_2)$ is already strong enough to give a complete solution to Jech's question, which was stated in the introduction. This is refuted by the following theorem.

Theorem 5.4. *The following are equiconsistent*:

(1) *ZFC + there exists a measurable cardinal.*
(2) *ZFC + SC_{ω_2} + there is a precipitous ideal on ω_1.*

The proof heavily relies on master condition arguments. For an elementary embedding $j : V \to M$ and a P-generic filter G over V, a *j-G master condition* is a $j(P)$ condition p such that for all $q \in G$, $p < j(q)$. If a master condition p exists, then we can define the lifted embedding $j_G : V[G] \to M[H]$, where $H \ni p$ is any $j(P)$-generic filter over M. The general framework for constructing a precipitous ideal on ω_1 in an ω_2-length countable support iterated forcing model originates from [8] (also see [3]). We will follow this framework and make sufficient adaptions to fulfill our goal.

Proof. It is well known (see [8]) that Con(2) implies Con(1). We need to prove that Con(1) implies Con(2). Assume $V \models$GCH + κ is a measurable cardinal. Let U be a κ-complete normal measure over κ and $j : V \to M$ be the derived elementary embedding.

Consider the forcing $\mathrm{Col}(\omega, <\kappa) * \mathbb{P}_e$, where e is chosen in $M^{\mathrm{Col}(\omega,<\kappa)}$. Let $G * K$ be a $\mathrm{Col}(\omega, <\kappa) * \mathbb{P}_e$-generic filter over V. It follows that $\mathbb{P}_e \in M[G]$. By the discussion at the end of Section 3, $\mathbb{P}_e$ is also a forcing for SC_{ω_2} in $V[G]$. We will also assume that $\mathbb{P}_e$ consists only of flat conditions. Hence each condition in $\mathbb{P}_e$ is a countable set in $V^{\mathrm{Col}(\omega,<\kappa)}$. As $V \models$ GCH, GCH holds in $V[G]$. By Theorem 1.1, it follows that $V[G*K] \models \mathrm{SC}_{\omega_2}$. For any $\alpha \in [\kappa, \kappa^+)$, let K_α be the $\mathbb{P}_\alpha$-generic derived from K. Let H be the bijection from ω_2 to $(P(\omega_1))^{V[G*K]}$ as defined in Section 3. For $\alpha \in [\kappa, \kappa^+)$, let $H_\alpha = H \upharpoonright \alpha$. The rest of the proof is devoted to showing that there is a precipitous ideal over ω_1 in $V[G*K]$.

Denote $\mathrm{Col}(\omega, <\kappa)$ by $\mathbb{P}$. The proofs of the following facts are standard:

(1) $j(\mathbb{P}) \cong \mathbb{P} * \mathbb{Q}$, where $\mathbb{Q} = Col(\omega, [\kappa, < j(\kappa)))^M$. We identify these two posets without further comment.
(2) Let G' be $\mathbb{Q}$-generic over $V[G]$, then $G * G'$ is $j(\mathbb{P})$-generic over V. Also in $V[G*G']$, there is an elementary embedding $j_G : V[G] \to M[G*G']$ extending j, where $j_G(G) = G * G'$.
(3) $\omega_1^{V[G]} = \kappa$ and $\omega_1^{M[G*G']} = j(\kappa)$.
(4) In $V[G*G']$, let $U_G = \{X \in P(\kappa) \cap V[G] \mid \kappa \in j_G(X)\}$. Then U_G is a normal ultrafilter on $P(\kappa) \cap V[G]$ and $M[G*G'] = Ult(V[G], U_G)$.
(5) In $V[G]$, Let $I_G = \{X \in P(\kappa) \mid \Vdash_{\mathbb{Q}}^{V[G]} \kappa \notin j_G(X)\}$. Then I_G is a normal precipitous ideal over ω_1.

Since in $M[G]$, $\mathbb{P}_e$ is of size κ^+ and $j(\kappa)$ is inaccessible, $\mathbb{P}_e$ can be completely embedded into $\mathbb{Q}$ such that the quotient forcing is isomorphic to $\mathbb{Q}$. Let i_{ω_2} be such a complete embedding. In what follows, we always view G' as $K * r$, where r is $\mathbb{Q}/\mathbb{P}_e$ generic over $V[G*K]$, i.e., we always assume K is contained in G' via the embedding i_{ω_2}. As a result, we have $j_G : V[G] \to M[G*K*r]$, where $K * r = G'$.

We now deal with $\mathbb{P}_{\omega_1}$. In $M[G*G']$, $j_G(\mathbb{P}_{\omega_1})$ is the forcing to add a generic bijection from $j(\kappa)$ to $P_{j(\kappa)}(j(\kappa))$ using countable conditions. We need to build a master condition for j_G and K_{ω_1} in $M[G*G']$. By the definition of $\mathbb{P}_e$, H is a bijection from κ^+ to $P(\kappa) \cap M[G]$. As we assume that $K \in M[G*G']$, it follows that H is also in $M[G*G']$. Hence in $M[G*G']$, H is a countable injection into $P_{j(\kappa)}(j(\kappa))$ and thus a condition of $j_G(\mathbb{P}_{\omega_1})$. As $\bigcup j_G[K_{\omega_1}] = \bigcup K_{\omega_1} = H_{\omega_1} \subset H$, H is a master condition for j_G and K_{ω_1}. It will become clear in the proof of Claim 5.5 why we choose H instead of H_{ω_1} to be the master condition. Whenever K'_{ω_1} is $j_G(\mathbb{P}_{\omega_1})$-generic over $V[G*G']$ such that $H \in K'_{\omega_1}$, we can lift j_G to $j_{\omega_1} : V[G*K_{\omega_1}] \to M[G*G'*K'_{\omega_1}]$.

In $V[G*G'*K'_{\omega_1}]$, let $U_{\omega_1} = \{X \in P(\kappa) \cap V[G*K_{\omega_1}] \mid \kappa \in j_{\omega_1}(X)\}$. Then U_{ω_1} is a normal ultrafilter over $P(\kappa) \cap V[G*K_{\omega_1}]$. Let

$$I_{\omega_1} = \left\{ X \in P(\kappa) \cap V[G*K_{\omega_1}] \Vdash_{(\mathbb{Q}/K_{\omega_1})*(j_G(\mathbb{P}_{\omega_1})/H)}^{V[G*K_{\omega_1}]} \kappa \notin j_{\omega_1}(X) \right\}.$$

It is clear that $U_G \subset U_{\omega_1}$ and $I_G \subset I_{\omega_1}$. Moreover one can verify that $M[G*G'*K'_{\omega_1}] = Ult(V[G*K_{\omega_1}], U_{\omega_1})$ and I_{ω_1} is a precipitous ideal over ω_1.

We shall construct the following objects by induction on $\alpha \in [\kappa, \kappa^+)$ in $V[G]$:

(1) A $\mathbb{Q}$-name $\dot{D}_\alpha$ for a master condition appropriate for the embedding $j_G : V[G] \to M[G*G']$ and the forcing $j_G(\mathbb{P}_\alpha)$. $\dot{D}_\alpha$ will be a condition in $j_G(\mathbb{P}_\alpha)$ which is a lower bound for $j_G[K_\alpha]$.
(2) A $\mathbb{Q} * j_G(\mathbb{P}_\alpha)/D_\alpha$-name for $j_\alpha : V[G*K_\alpha] \to M[G*G'*K'_\alpha]$ extending j_G, where $K'_\alpha \ni \dot{D}_\alpha$ is $j_G(\mathbb{P}_\alpha)$-generic.

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(3) A $\mathbb{Q}$-name $\dot{I}_\alpha$ for a normal ideal on κ which is the set of those $X \subset \kappa$ in $V[G * K_\alpha]$ such that it is forced over $V[G * K_\alpha]$ by $(j(\mathbb{P})/(G * K_\alpha)) * (j_G(\mathbb{P}_\alpha)/D_\alpha)$ that $\kappa \notin j_\alpha(X)$.

For $\kappa \leq \alpha < \beta < \kappa^+$, we require that $D_\alpha(j_G(\omega_1^-)) = H$, $D_\beta \restriction j(\alpha) = D_\alpha$, $j_\beta \restriction V[G * K_\alpha] = j_\alpha$, and $I_\beta \cap V[G * K_\alpha] = I_\alpha$. Let $\langle C_\gamma \mid \gamma \in [\kappa, \alpha)\rangle$ be the sequence of club sets added by K_α in $V[G * K_\alpha]$. Then D_α is defined as:

$$D_\alpha(\gamma) = \begin{cases} H, & \text{if } \gamma = j_G(\omega_1^-), \\ C_\beta \cup \{\kappa\}, & \text{if } \gamma = j(\beta) \wedge \beta \neq \omega_1^-, \\ \emptyset, & \text{otherwise.} \end{cases}$$

By our assumption, $K_\alpha \in M[G * G']$. It follows that $\langle C_\gamma \mid \gamma \in [\kappa, \alpha)\rangle \in M[G * G']$. Also as $j[\alpha] \in M$, $D_\alpha \in M[G * G']$. Since $|\kappa|^{M[G*G']} = \omega$, the support of D_α is countable in $M[G * G']$.

By induction on $\alpha \in [\kappa, \kappa^+)$, we check that D_α is a master condition and define the embedding j_α. The case $\alpha = \kappa$ was already treated above. D_κ is a master condition for j_G and K_{ω_1}. If α is a limit ordinal, then it follows from the induction hypothesis that D_α is a condition. Now if $p \in K_\alpha$, then by the induction hypothesis for any $\beta < \alpha$, $j(p \restriction \beta) > D_\beta = D_\alpha \restriction \beta$. Hence $j(p) > D_\alpha$. It follows that $j_\alpha : V[G * K_\alpha] \to M[G * G' * K'_\alpha]$ can be defined whenever $K'_\alpha \ni D_\alpha$ is $j_G(\mathbb{P}_\alpha)$-generic over $M[G * G']$.

Now assume $\alpha = \beta + 1$ and the induction hypothesis holds at β. We may assume that $D_\alpha = D_\beta {}^\frown \langle j(\beta), C_\beta \cup \{\kappa\}\rangle$. We need to check that $D_\beta \Vdash C_\beta \cup \{\kappa\}$ is a $j_G(\dot{Q}_\beta)$-condition. Let $K'_\beta \ni D_\beta$ be an arbitrary $j(\mathbb{P}_\beta)$-generic filter over $V[G * G']$. From now on, we work in $V[G * G' * K'_\beta]$. Now $j_\beta : V[G * K_\beta] \to M[G * G' * K'_\beta]$ lifts j_{ω_1}. We need to prove $C_\beta \cup \{\kappa\} \subset j_\beta(S''_\beta)$. Since $C_\beta \subset S''_\beta \subset j_\beta(S''_\beta)$, we only need to check the following claim:

Claim 5.5. $\kappa \in j_\beta(S''_\beta)$.

Proof. By the definition of S''_β and j_β, $\kappa \in j_\beta(S''_\beta)$ if the structure $A = \langle j_\beta(e_\beta)[\kappa], j_\beta(H_\beta) \cap j_\beta(e_\beta)[\kappa], \in\rangle$ is a countable elementary substructure of $j_\beta(A_\beta)$ and

$$(j_\beta(H_\beta) \cap j_\beta(e_\beta)[\kappa])_{j_\beta(e_\beta)[\kappa]} \subset j_\beta(H_\beta).$$

It is clear that $j_\beta(e_\beta)[\kappa]$ is countable in $M[G * G' * K'_\beta]$. By the definition of e_β, $\langle e_\beta[\kappa], H_\beta, \in\rangle = A_\beta$. Now since j_β is elementary, applying the Tarski criterion, $\langle j_\beta[e_\beta[\kappa]], j_\beta[H_\beta], \in\rangle \prec j_\beta(A_\beta)$. It is also clear that $j_\beta[e_\beta[\kappa]] = j_\beta(e_\beta)[\kappa]$. Hence $A = \langle j_\beta[e_\beta[\kappa]], j_\beta[H_\beta], \in\rangle \prec j_\beta(A_\beta)$.

We finish the proof of the claim by showing that $(j_\beta(H_\beta) \cap j_\beta(e_\beta)[\kappa])_{j_\beta(e_\beta)[\kappa]} = H_\beta = j_\beta(H_\beta) \restriction \beta$. The first equality holds because $e_\beta[\kappa]$ is a transitive set isomorphic to $j_\beta[e_\beta[\kappa]]$. We show the second equality. Since $\beta < \kappa^+ < j(\kappa)$, $j_\beta(H_\beta) \restriction \beta \subset j_{\omega_1}(H_{\omega_1})$. By our construction, $H \in K'_{\omega_1}$. Hence $H_\beta \subset H \subset j_{\omega_1}(H_{\omega_1})$ and $H_\beta = j_{\omega_1}(H_{\omega_1}) \restriction \beta = j_\beta(H_\beta) \restriction \beta$. ⊣

We now check that D_α is a master condition. Fix $q \in K_\alpha$. As $q \restriction \beta \in K_\beta$, by induction hypothesis, D_β is stronger than $j_G(q) \restriction j(\beta)$. It remains to check that $j_G(q(\beta)) = q(\beta) \subset C_\beta$. The equality follows from the fact that q is flat. Hence D_α is a master condition.

It follows that we can lift j_{ω_1} to $j_\alpha : V[G*K_\alpha] \to M[G*G'*K'_\alpha]$, where $K'_\alpha \ni D_\alpha$ is a $j_G(\mathbb{P}_\alpha)$-generic over $V[G*G']$. Let $U_\alpha = \{X \in P(\kappa) \cap V[G*K_\alpha] \mid \kappa \in j_\alpha(X)\}$ and

$$I_\alpha = \left\{ X \in P(\kappa) \cap V[G*K_\alpha] \Vdash^{V[G*K_\alpha]}_{(\mathbb{Q}/K_\alpha)*(j_G(\mathbb{P}_\alpha)/D_\alpha)} \kappa \notin j_\alpha(X) \right\}.$$

As before, we have $M[G*G'*K'_\alpha] = Ult(V[G*K_\alpha], U_\alpha)$ and I_α is a precipitous ideal on ω_1. It is routine to verify that $I_\beta = I_\alpha \cap V[G*K_\beta]$ from the construction.

Let $I_{\omega_2} = \bigcup_{\alpha<\omega_2} I_\alpha$. Then I_{ω_2} is a proper normal ideal on ω_1. We claim that I_{ω_2} is actually a precipitous ideal and thus finish the proof.

Claim 5.6. *In $V[G*K]$, I_{ω_2} is a precipitous ideal.*

Proof. We first describe the general construction for a k which is $j_G(\mathbb{P}_{\omega_2})$-generic over $M[G*G']$. Let $\mathbb{P}'_{\omega_2} = \{t \in j_G(\mathbb{P}_{\omega_2}) \mid \exists\alpha(\mathrm{spt}(t) \subset j(\alpha) \wedge t < D_\alpha)\}$. Note that $\mathbb{P}'_{\omega_2}$ is a subset of $M[G*G']$ but is not in $M[G*G']$. However, we can force with $\mathbb{P}'_{\omega_2}$ over $V[G*G']$. Let $\bar{K}$ be a $\mathbb{P}'_{\omega_2}$-generic filter over $V[G*G']$. Let k be the $j_G(\mathbb{P}_{\omega_2})$-filter induced from $\bar{K}$.

We now show that k is in fact a $j_G(\mathbb{P}_{\omega_2})$-generic filter over $M[G*G']$. Suppose A is a maximal antichain in $M[G*G']$. By the $j(\kappa^+)$-c.c. and the fact that $j[\kappa^+]$ is cofinal in $j(\kappa^+)$, there is some $\alpha \in (\kappa, \kappa^+)$ such that A is a maximal antichain of $j_G(\mathbb{P}_{\omega_2})$ in $M[G*G']$. Let A' be the $j_G(\mathbb{P}_\alpha)$-dense open set generated by A. It is not difficult to verify that $A' \cap (\mathbb{P}'_{\omega_2} \restriction j(\alpha))$ is a dense open subset of $(\mathbb{P}'_{\omega_2} \restriction j(\alpha))$. By genericity, there is a $p \in A' \cap \bar{K} \restriction j(\alpha)$. Let $t \in A$ be the unique condition in $j_G(\mathbb{P}_\alpha)$ such that p is compatible with t. It follows that $t \in k$ and thus $k \cap A \neq \emptyset$. This ends the construction of k.

For any such $\bar{K}$ and derived k, as before, in $V[G*G'*\bar{K}]$, we can lift j_{ω_1} to $j_{\omega_2} : V[G*K] \to M[G*G'*k]$. Moreover for all $X \in P(\kappa) \cap V[G*K]$, it is routine to check that $X \in I_{\omega_2}$ iff for every P'_{ω_2}-generic filter $\bar{K}$ and derived embedding j_{ω_2}, $\kappa \notin j_{\omega_2}(X)$.

Now we can argue that I_{ω_2} is precipitous via a density argument. Fix $X \notin I_{\omega_2}$ and $X \subset \kappa$. We only need to show that X does not force that $Ult(V[G*K], \bar{U})$ is ill-founded, where $\bar{U}$ is the P'_{ω_2}-generic ultrafilter for $P(\kappa)/I_{\omega_2}$. By the last paragraph, there is a generic filter $\bar{K}$ such that in $V[G*G'*\bar{K}]$, $\kappa \in j_{\omega_2}(X)$. Let $U_{\omega_2} = \{X \in P(\kappa) \cap V[G*K] \mid \kappa \in j_{\omega_2}(X)\}$. As before, U_{ω_2} is a $P(\kappa) \cap V[G*K]$-ultrafilter and $M[G*G'*k] = Ult(V[G*K], U_{\omega_2})$. Hence $Ult(V[G*K], U_{\omega_2})$ is well-founded.

It remains to show that U_{ω_2} is $P(\kappa)/I_{\omega_2}$-generic. Let A be a maximal antichain of $P(\kappa)/I_{\omega_2}$ in $V[G*K]$. Suppose otherwise and let $p \in G*G'*\bar{K}$ be a condition such that $p \Vdash (\forall X \in A)(\kappa \notin j_{\omega_2}(X))$. Note that by the definition of $\mathbb{P}'_{\omega_2}$, p is also in $G*G'*k$.

We now work in $V[G*K]$. As p is in M, there is an $f : \kappa \to V$ representing p. Let $R = \{\alpha \in \kappa \mid f(\alpha) \in G*K\}$. Now in any $j_G(\mathbb{P}_{\omega_2})$-generic extension over $V[G*K]$ with generic k, $\kappa \in j_{\omega_2}(R)$ is equivalent to $p \in G*G'*k$. Hence $\kappa \notin j_{\omega_2}(X \cap R)$ and thus $X \cap R$ is in I_{ω_2} for all $X \in A$.

On the other hand, $R \notin I_{\omega_2}$ since $p \Vdash_{j_G(\mathbb{P}_{\omega_2})} \kappa \in j_{\omega_2}(R)$. This contradicts the fact that A is maximal. ⊣

§**6. Final Remarks.** This paper is the first attempt at the second objective we posed in Section 1. The result on square sequences gives us the impression that SC

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cannot replace the role of fine structure in any proof of theorems in L which uses fine structure essentially. However, it may still happen that there are truths in L all of whose available proofs involve some form of fine structure, yet an essentially different proof could be found. A natural candidate is the following question of Woodin ([14]) concerning an abstract analogue of Jensen's covering lemma for L.

QUESTION 6.1. *Suppose N is an inner model of ASC. Suppose that covering fails for N in V.*[19] *Must there exist a real x such that $N \subseteq L[x]$?*

On the other hand, a yet more difficult question is to construct model of larger fragments of SC:

QUESTION 6.2. *Is there a set sized forcing notion to obtain SC_{ω_3}?*

A positive answer to this question would be very plausible and would answer Jech's question negatively.

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[19] *Covering* means for any $Y \in V$, there is a $X \in N$ such that $Y \subset X$ and $|X| \leq |Y| + \omega_1$.

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