

## BINARY SIGNED-DIGIT REPRESENTATIONS IN PAPERFOLDING

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### Abstract

When a page, represented by the interval  $[0, 1]$ , is folded right over left  $n$  times, the right-hand fold contains a sequence of points. We specify these points using two different representation techniques, both involving binary signed-digit representations.

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### 1. Introduction and preliminaries

When the line joining 0 to 1 (or a rectangular piece of paper with  $[0, 1]$  at its edge) is folded in half, right over left, the point  $1/2$  is at the fold. When the line is then folded in half again, right over left, the points  $1/4, 3/4$  are at this second fold, in that order, from the outside of the fold to its interior as shown in Figure 1, where the second fold sequence is shown by the horizontal arrow and the third fold sequence is shown by the vertical arrow.

The line when *unfolded* forms the paperfolding sequence, a sequence of  $\vee$  and  $\wedge$  creases. It has a rich history commencing with Davis and Knuth [4]. Other important contributions include Dekking *et al.* [5], Allouche and Shallit [1], Mendès France and van der Poorten [8] and Mendès France and Shallit [7]. Our interest, in this paper, is to explore paperfolding when it remains in its *folded* form.

After  $n$  folds, the paper strip is partitioned by the folds into  $2^n$  segments which we call *intervals*. In Bunder *et al.* [3], we identified the location of points in a paperfold. The present paper provides some new results by utilising two different and relatively unknown techniques that can be readily converted into algorithms. The first technique (Section 2) identifies the point at each location within a fold (Theorem 3.1). The second technique (Section 4) does the reverse: for each point of the form  $(2k + 1)/2^n < 1$ , it identifies its location within a fold (Theorem 5.1).

The first technique involves *normalised additive factorisation*, which has been used in the Stern–Brocot tree (Bates *et al.* [2]). Normalised additive factorisation is a *binary signed-digit* representation of an integer (see Monroe [9], Ebeid and Hasan [6],

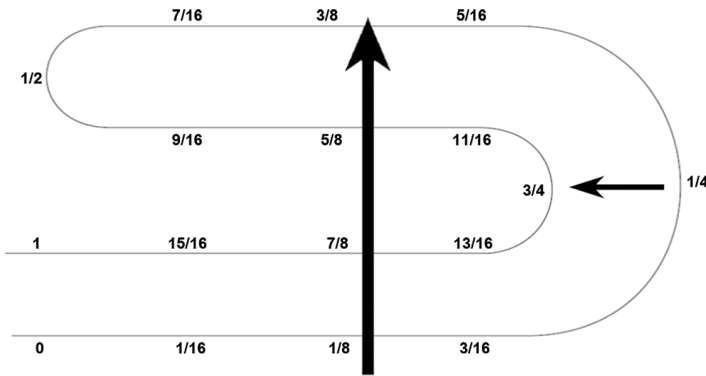


FIGURE 1. Points in a fold after two folds.

Tůma and Vábek [13] and Shallit [11]). The second technique is a different binary signed-digit representation of an integer. It makes use of what we have called *odd unit factorisation*, which is possibly novel, although it may exist in the literature under other guises.

We recall from Bunder *et al.* [3], the following definitions and results, with minor changes. In particular, Theorems 1.5–1.7 below are [3, Lemma 2.5 and Theorems 2.6 and 2.7] and Theorem 1.8 is a corollary of [3, Theorem 3.1].

**DEFINITION 1.1** (*n*th fold sequence). After we have folded the line *n* times, the sequence of points from the outside of the *n*th fold to the inside is the *n*th fold sequence,  $S_n = \langle S_{n,j} \rangle$ . We define the zeroth fold sequence,  $S_0 = \langle S_{0,1} \rangle = 1$ . We call  $S_{n,j}$  the *address* of the point found at  $S_{n,j}$  as well as the point itself.

**EXAMPLE 1.2.** The zeroth, first, second and third fold sequences are

$$S_0 = \langle S_{0,1} \rangle = 1,$$

$$S_1 = \langle S_{1,1} \rangle = \frac{1}{2},$$

$$S_2 = \langle S_{2,1}, S_{2,2} \rangle = \langle \frac{1}{4}, \frac{3}{4} \rangle \quad (\text{horizontal arrow in Figure 1}),$$

$$S_3 = \langle S_{3,1}, S_{3,2}, S_{3,3}, S_{3,4} \rangle = \langle \frac{1}{8}, \frac{7}{8}, \frac{5}{8}, \frac{3}{8} \rangle \quad (\text{vertical arrow in Figure 1}).$$

**DEFINITION 1.3** (*n*th prefold sequence). Fold the line from 0 to 1 *n* times, where  $n \geq 0$ . Let *j* be an integer with  $0 < j \leq 2^n$  and *a* a real number with  $0 < a < 1/2^n$ . We make the following definitions:

- (1)  $F_n(a, j)$ , the *j*th term from the bottom lying at or directly above *a*, is the *j*th term of the *n*th prefold sequence at *a* (in particular,  $F(a, 1) = a$ );
- (2)  $F_n(a) = \langle F_n(a, 1), F_n(a, 2), F_n(a, 3), \dots, F_n(a, 2^n) \rangle$  is the sequence of all terms in the *n*th prefold sequence at *a*;
- (3)  $\bar{F}_n(a) = \langle F_n(a, 2^n), \dots, F_n(a, 3), F_n(a, 2), F_n(a, 1) \rangle$  is the reverse *n*th prefold sequence at *a*.

**LEMMA 1.4.** For  $n > 0$ ,  $S_n = F_{n-1}(1/2^n)$  and  $S_n$  has  $2^{n-1}$  terms.

**PROOF.** At the  $(n - 1)$ th fold, there are  $2^{n-1}$  intervals each of size  $1/2^{n-1}$  units. By Definition 1.3,  $F_{n-1}(1/2^n)$  represents the sequence of consecutive midpoints of consecutive intervals in this  $(n - 1)$ th fold. By Definition 1.1, at the  $n$ th fold, this sequence of consecutive midpoints becomes the  $n$ th fold sequence, giving the result.  $\square$

**THEOREM 1.5 (Form of  $F_n(a)$ ).** For  $0 < a < 1/2^n$ ,

$$F_n(a) = \langle F_{n-1}(a), \widetilde{F}_{n-1}((1/2^{n-1}) - a) \rangle.$$

**THEOREM 1.6 (Recursive expression of prefolds).** We have

$$F_n(a, j) = \begin{cases} F_{n-1}(a, j) & \text{if } 0 < j \leq 2^{n-1}, \\ F_{n-1}((1/2^{n-1}) - a, 2^n - j + 1) & \text{if } 2^{n-1} < j \leq 2^n. \end{cases}$$

**THEOREM 1.7 (Prefold expression of folds).** For  $n > 1$ ,

$$S_n = \left\langle F_{n-2}\left(\frac{1}{2^n}, 1\right), F_{n-2}\left(\frac{1}{2^n}, 2\right), \dots, F_{n-2}\left(\frac{1}{2^n}, 2^{n-2}\right), F_{n-2}\left(\frac{3}{2^n}, 2^{n-2}\right), F_{n-2}\left(\frac{3}{2^n}, 2^{n-2} - 1\right), \dots, F_{n-2}\left(\frac{3}{2^n}, 1\right) \right\rangle.$$

**THEOREM 1.8 (Alternating differences within prefolds).** For  $n \geq 0$  and  $0 < j \leq 2^n$ ,

$$F_n\left(\frac{1}{2^{n+2}}, j\right) = F_n\left(\frac{3}{2^{n+2}}, j\right) + \frac{(-1)^j}{2^{n+1}}.$$

**THEOREM 1.9 (Alternating differences within folds).** For  $n \geq 2$ ,  $0 < j \leq 2^{n-2}$ ,

$$S_{n,j} = S_{n,2^{n-1}-j+1} + \frac{(-1)^j}{2^{n-1}}.$$

**PROOF.** Apply Theorem 1.8 to  $F_{n-2}$  for  $0 < j \leq 2^{n-2}$  and use Theorem 1.7.  $\square$

**THEOREM 1.10 (Couplet addition within prefolds).** For  $n \geq 1$  and  $k = 1, 2, 3, \dots, 2^{n-1}$ ,

$$F_n(a, 2k - 1) + F_n(a, 2k) = 1.$$

**PROOF.** We prove the result by induction on  $n$ . The result holds for  $n = 1$  because

$$F_1(a, 1) + F_1(a, 2) = a + (1 - a) = 1.$$

Suppose the result is true for some  $n$ , that is, it is true for both  $F_n(a)$  and  $F_n(1/2^n - a)$ . By Theorem 1.5,

$$F_{n+1}(a) = \left\langle F_n(a), \widetilde{F}_n\left(\frac{1}{2^n} - a\right) \right\rangle.$$

Here,  $\widetilde{F}_n(1/2^n - a)$  is the reverse of  $F_n(1/2^n - a)$ , and so even terms of  $F_n(1/2^n - a)$  become odd terms in  $\widetilde{F}_n(1/2^n - a)$ . This means that the result is true for both  $F_n(a)$  and  $\widetilde{F}_n(1/2^n - a)$ . It follows that the result is true for  $F_{n+1}(a)$  and therefore by induction for all  $n \geq 1$ .  $\square$

**THEOREM 1.11 (Couplet addition within folds).** For  $n \geq 2$  and  $k = 1, 2, 3, \dots, 2^{n-1}$ ,

$$S_{n,2k-1} + S_{n,2k} = 1.$$

**PROOF.** The result follows from Theorem 1.10 and Lemma 1.4. □

**LEMMA 1.12 (Order of points in a fold).** For  $n \geq 0$  and  $0 < a < 1/2^n$ , let the  $j$ th interval of the  $n$ th fold be the interval containing  $F_n(a, j)$ . Then

- for  $j$  odd,  $F_n(a, j)$  increases as  $a$  increases, and
- for  $j$  even,  $F_n(a, j)$  decreases as  $a$  increases.

**PROOF.** We prove the result by induction on  $n$ . For  $n = 0$  and  $0 < a < b < 1$ ,  $F_0(a, 1) = a$  and  $F_0(b, 1) = b$  and so the result holds for  $n = 0$ .

Suppose the result is true for some  $n$ . For  $0 < j \leq 2^n$ , at the  $(n + 1)$ th fold, the rightmost half of the  $j$ th interval in the  $n$ th fold is rotated  $180^\circ$  anticlockwise around  $1/2^{n+1}$  to become the  $(2^{n+1} - j + 1)$ th interval in the  $(n + 1)$ th fold. This rotation reverses the order of the points in this interval. That is, if we have  $0 < j \leq 2^n$  and  $0 < a < 1/2^{n+1}$ ,

- for  $j$  odd,  $F_{n+1}(a, 2^{n+1} - j + 1)$  decreases as  $a$  increases,
- for  $j$  even,  $F_{n+1}(a, 2^{n+1} - j + 1)$  increases as  $a$  increases.

Since  $2^{n+1} - j + 1$  has opposite parity to  $j$ , it follows that the result holds for  $n + 1$  and so for all  $n$ . □

**THEOREM 1.13.** For  $0 < j \leq 2^{n-1}$ ,

$$F_{n-1}\left(\frac{1}{2^{n+1}}, j\right) = F_{n-1}\left(\frac{1}{2^n}, j\right) + \frac{(-1)^j}{2^{n+1}}.$$

**PROOF.** By Lemma 1.12, since  $1/2^{n+1} < 1/2^n$ , for  $j$  odd,  $F_{n-1}(1/2^{n+1}, j)$  must be *less* than  $F_{n-1}(1/2^n, j)$  by  $1/2^n - 1/2^{n+1} = 1/2^{n+1}$  units. Similarly, for  $j$  even,  $F_{n-1}(1/2^{n+1}, j)$  must be *greater* than  $F_{n-1}(1/2^n, j)$  by  $1/2^{n+1}$  units. □

The next theorem shows the relationship between terms found in *different* folds.

**THEOREM 1.14 (Alternating differences between folds).** For  $n > 0$  and  $0 < j \leq 2^{n-1}$ ,

$$S_{n+1,j} = S_{n,j} + \frac{(-1)^j}{2^{n+1}} \quad \text{and} \quad S_{n+1,2^n-j+1} = S_{n,j} + \frac{(-1)^{j+1}}{2^{n+1}}.$$

**PROOF.** Suppose  $0 < j \leq 2^{n-1}$ . By Lemma 1.4, Theorems 1.5 and 1.13 and Lemma 1.4 again,

$$S_{n+1,j} = F_n\left(\frac{1}{2^{n+1}}, j\right) = F_{n-1}\left(\frac{1}{2^{n+1}}, j\right) = F_{n-1}\left(\frac{1}{2^n}, j\right) + \frac{(-1)^j}{2^{n+1}} = S_{n,j} + \frac{(-1)^j}{2^{n+1}}$$

and, by Theorem 1.9 and the result just established,

$$S_{n+1,2^n-j+1} = S_{n+1,j} + \frac{(-1)^{j+1}}{2^n} = S_{n,j} + \frac{(-1)^j}{2^{n+1}} + \frac{(-1)^{j+1}}{2^n} = S_{n,j} + \frac{(-1)^{j+1}}{2^{n+1}}. \quad \square$$

## 2. Normalised additive factorisation

From [2], each positive integer can be represented by its unique set of *normalised additive factors*. This representation is part of a wider family of *binary signed-digit representations* of an integer defined as follows.

**DEFINITION 2.1** (After Tůma and Vábek [13]). A *binary signed-digit representation* of an integer  $z$  is a string  $\beta = b_0, b_1, \dots, b_k$  with  $b_i \in \{-1, 0, 1\}$  such that  $z = \sum_{i=0}^k b_i 2^i$ .

We will use such representations in which  $k$  is even and the nonzero  $b_i$  values are alternately positive and negative. It is easy to see that such a representation is unique.

Normalised additive factorisation is used to represent a positive integer  $j$  as an alternating series that contains an odd number of terms such that consecutive terms involve the value 2 raised to consecutive normalised additive factors of  $j$ . We designate the ordered set of normalised additive factors of  $j$  as  $j_{\text{NAF}}$ , where, for  $k$  even,  $j_{\text{NAF}} = \{b_{j,0}, b_{j,1}, \dots, b_{j,k}\}$ .

The algorithm for obtaining the normalised additive factors of  $j$  is as follows.

- Let  $b_{j,0}$  be the *smallest nonnegative integer* for which  $j \leq 2^{b_{j,0}}$ .
- If  $j = 2^{b_{j,0}}$ , then  $b_{j,0}$  is the only normalised additive factor of  $j$ .
- If  $j < 2^{b_{j,0}}$ , then  $b_{j,0}$  is a positive integer. Let  $b_{j,1}$  be the *smallest positive integer less than  $b_{j,0}$* , for which  $j > 2^{b_{j,0}} - 2^{b_{j,1}}$ .
- Let  $b_{j,2}$  be the *smallest nonnegative integer* for which  $j \leq 2^{b_{j,0}} - 2^{b_{j,1}} + 2^{b_{j,2}}$ .
- If  $j = 2^{b_{j,0}} - 2^{b_{j,1}} + 2^{b_{j,2}}$ , then  $\{b_{j,0}, b_{j,1}, b_{j,2}\}$  is the set of normalised additive factors of  $j$ .
- If  $j < 2^{b_{j,0}} - 2^{b_{j,1}} + 2^{b_{j,2}}$ , let  $b_{j,3}$  be the *smallest positive integer less than  $b_{j,2}$* , for which  $j > 2^{b_{j,0}} - 2^{b_{j,1}} + 2^{b_{j,2}} - 2^{b_{j,3}}$ .
- Let  $b_{j,4}$  be the *smallest positive integer* for which  $j \leq 2^{b_{j,0}} - 2^{b_{j,1}} + 2^{b_{j,2}} - 2^{b_{j,3}} + 2^{b_{j,4}}$ .
- Continuing in this way, the process must eventually terminate at  $b_{j,k}$ , where  $k$  is even, such that  $j = \sum_{i=0}^k (-1)^i 2^{b_{j,i}}$ . The ordered set of normalised additive factors of  $j$  is then  $j_{\text{NAF}} = \{b_{j,0}, b_{j,1}, \dots, b_{j,k}\}$  and has odd cardinality. Note that for  $j$  odd,  $b_{j,k} = 0$  and for  $j$  even,  $b_{j,k} > 0$ .

The Maple code for this algorithm appears at the end of this section.

**EXAMPLE 2.2.** The set of normalised additive factors of  $12 = 2^4 - 2^3 + 2^2$  is  $\{4, 3, 2\}$ , not  $\{4, 2\}$ , since the set must have odd cardinality. Similarly, the set of normalised additive factors of 15 is  $\{4, 1, 0\}$ , not  $\{4, 0\}$ .

**THEOREM 2.3** (Normalised additive factors). If  $j_{\text{NAF}} = \{b_{j,0}, b_{j,1}, \dots, b_{j,k}\}$ , then

$$(2^{b_{j,0}} - j + 1)_{\text{NAF}} = \begin{cases} \{b_{j,1}, b_{j,2}, \dots, b_{j,k}, 0\} & \text{for } b_{j,k} > 0, \text{ that is, for } j \text{ even,} \\ \{b_{j,1}, b_{j,2}, \dots, b_{j,k-1}\} & \text{for } b_{j,k} = 0, \text{ that is, for } j \text{ odd.} \end{cases}$$

**PROOF.** Since  $j = 2^{b_{j,0}} - 2^{b_{j,1}} + 2^{b_{j,2}} - \dots + 2^{b_{j,k}}$ , for  $j$  even, we must have  $b_{j,k} > 0$ , so

$$2^{b_{j,0}} - j + 1 = 2^{b_{j,1}} - 2^{b_{j,2}} + 2^{b_{j,3}} - \dots - 2^{b_{j,k}} + 2^0.$$

That is,  $(2^{b_{j,0}} - j + 1)_{\text{NAF}} = \{b_{j,1}, b_{j,2}, \dots, b_{j,k}, 0\}$ . Similarly, for  $j$  odd, we must have  $b_{j,k} = 0$ , so

$$\begin{aligned} 2^{b_{j,0}} - j + 1 &= 2^{b_{j,1}} - 2^{b_{j,2}} + 2^{b_{j,3}} - \dots + 2^{b_{j,k-1}} - 2^{b_{j,k}} + 2^0 \\ &= 2^{b_{j,1}} - 2^{b_{j,2}} + 2^{b_{j,3}} - \dots + 2^{b_{j,k-1}}. \end{aligned}$$

That is,  $(2^{b_{j,0}} - j + 1)_{\text{NAF}} = \{b_{j,1}, b_{j,2}, \dots, b_{j,k-1}\}$ . □

**2.1. Maple code for the normalised additive factors of  $j$ .** The method for obtaining the normalised additive factors of  $j$  is translated into Maple code, graciously provided by the referee. Let  $b$  be the least integer such that  $2^b \geq x$ . That is,  $b$  is the ceiling of the base 2 logarithm of  $x$  and, for integers  $b$  and  $x$ ,  $b < x$  which is the same as  $b \leq x + 1$ .

```

for j to 5 do m := 0;
i := 0;
finished := false;
while finished = false do
  b[2 * i] := ceil(log[2](j - m));
  m := m + 2^b[2 * i];
  if m = j then finished := true end if;
  if finished = false then b[2 * i + 1] := ceil(log[2](m - j + 1));
  m := m - 2^b[2 * i + 1];
  i := i + 1
end if
end do;
printf('j = %2d###', j);
for k from 0 to 2 * i do printf('%2d', b[k])
end do;
printf('\n', x)
end do;
j = 1 ### 0
j = 2 ### 1
j = 3 ### 210
j = 4 ### 2
j = 5 ### 320

```

### 3. Finding points from folds

We now show that  $S_{n,j}$  can be represented using an alternating series of unit fractions based on  $n$  and the set of normalised additive factors of  $j$ . Thus we can find every point in every fold using only  $n$  and  $j$ .

**THEOREM 3.1 (Points from addresses).** *Let  $j_{\text{NAF}} = \{b_{j,0}, b_{j,1}, \dots, b_{j,k}\}$  and  $n \geq 1$ . Then*

$$S_{n,j} = \sum_{i=0}^k \frac{(-1)^i}{2^{b_{j,i}-1}} - \frac{1}{2^n} \quad \text{if } j \text{ is even,}$$

$$S_{n,j} = \sum_{i=0}^{k-1} \frac{(-1)^{i+1}}{2^{b_{j,i}-1}} + \frac{1}{2^n} \quad \text{if } j \text{ is odd.}$$

**PROOF.** We prove the result by induction on  $n$ . We have  $1_{\text{NAF}} = \{0\}$  and the result holds for  $n = 1$  because

$$S_{1,1} = \frac{1}{2} = \sum_{i=0}^{0-1} \frac{(-1)^{i+1}}{2^{b_{j,i}-1}} + \frac{1}{2^1} = 0 + \frac{1}{2}.$$

Suppose the result holds for  $n$ . We consider two cases.

*Case 1:*  $0 < j \leq 2^{n-1}$ . We use Theorem 1.14 and the induction hypothesis. For  $j$  even,

$$S_{n+1,j} = S_{n,j} + \frac{1}{2^{n+1}} = \sum_{i=0}^k \frac{(-1)^i}{2^{b_{j,i}-1}} - \frac{1}{2^n} + \frac{1}{2^{n+1}} = \sum_{i=0}^k \frac{(-1)^i}{2^{b_{j,i}-1}} - \frac{1}{2^{n+1}}.$$

For  $j$  odd,

$$S_{n+1,j} = S_{n,j} - \frac{1}{2^{n+1}} = \sum_{i=0}^{k-1} \frac{(-1)^{i+1}}{2^{b_{j,i}-1}} + \frac{1}{2^n} - \frac{1}{2^{n+1}} = \sum_{i=0}^{k-1} \frac{(-1)^{i+1}}{2^{b_{j,i}-1}} + \frac{1}{2^{n+1}}.$$

Thus, for  $0 < j \leq 2^{n-1}$ , our result is true for  $n + 1$ .

*Case 2:*  $2^{n-1} < j \leq 2^n$ . For this case, we have  $b_{j,0} = n$ . There are two subcases. For  $j$  even,  $2^n - j + 1 \leq 2^{n-1}$  is odd. By Theorem 2.3,

$$(2^n - j + 1)_{\text{NAF}} = \{b_{j,1}, b_{j,2}, \dots, b_{j,k}, 0\}$$

and so by Case 1 and Theorem 1.14,

$$S_{n+1,j} = S_{n,2^n-j+1} + \frac{1}{2^{n+1}} = \sum_{i=1}^k \frac{(-1)^i}{2^{b_{j,i}-1}} + \frac{1}{2^n} + \frac{1}{2^{n+1}} = \sum_{i=0}^k \frac{(-1)^i}{2^{b_{j,i}-1}} - \frac{1}{2^{n+1}},$$

since  $1/2^{b_{j,0}-1} = 1/2^{n-1}$ . For  $j$  odd,  $2^n - j + 1 \leq 2^{n-1}$  is even. By Theorem 2.3,

$$(2^n - j + 1)_{\text{NAF}} = \{b_{j,1}, b_{j,2}, \dots, b_{j,k-1}\},$$

and so by Case 1 and Theorem 1.14,

$$S_{n+1,j} = S_{n,2^n-j+1} - \frac{1}{2^{n+1}} = \sum_{i=1}^{k-1} \frac{(-1)^{i+1}}{2^{b_{j,i}-1}} - \frac{1}{2^n} - \frac{1}{2^{n+1}} = \sum_{i=0}^{k-1} \frac{(-1)^{i+1}}{2^{b_{j,i}-1}} + \frac{1}{2^{n+1}},$$

where again  $1/2^{b_{j,0}-1} = 1/2^{n-1}$ . Thus, for  $2^{n-1} < j \leq 2^n$ , the result is true for  $n + 1$ . By induction the result is true for all  $n$ . □

**EXAMPLE 3.2.** We have  $11_{\text{NAF}} = \{b_{11,0}, b_{11,1}, b_{11,2}, b_{11,3}, b_{11,4}\} = \{4, 3, 2, 1, 0\}$ . From Theorem 3.1,

$$S_{6,11} = \sum_{i=0}^3 \frac{(-1)^{i+1}}{2^{b_{11,i}-1}} + \frac{1}{2^6} = \frac{41}{64}.$$

**REMARK 3.3.** Appendix A identifies all points from addresses for  $n = 6$  and illustrates many of the results obtained so far. The numerators of the sequence of addresses  $(1, 63, 33, 31, \dots)$  are a subset of sequence A119608 in the on-line Encyclopedia of Integer Sequences [10]. In fact, A119608 concatenates the sequence of numerators for each  $n$ .

### 4. Odd unit factorisation

We now examine the *reverse* situation and identify the address (fold  $n$  and location  $j$ ) of the point  $(2k + 1)/2^n < 1$ .

**DEFINITION 4.1 (Initial odd factorisation).** If  $p$  is an odd integer, then there is an odd integer  $q$  such that either  $p = 2q + 1$  or  $p = 2q - 1$ . This  $q$  is called the *initial odd factor* of  $p$ .

**THEOREM 4.2 (Odd factorisation).** Let  $p$  be an odd integer.

- (i) If  $p \equiv 1 \pmod{4}$ , then  $p$  has initial odd factorisation  $p = 2\lceil p/2 \rceil - 1$ .
- (ii) If  $p \equiv 3 \pmod{4}$ , then  $p$  has initial odd factorisation  $p = 2\lfloor p/2 \rfloor + 1$ .

**PROOF.** For  $p \equiv 1 \pmod{4}$ ,  $\lfloor p/2 \rfloor$  is even and  $\lceil p/2 \rceil$  is odd. It follows that the initial odd factorisation of  $p$  is given by  $p = 2\lceil p/2 \rceil - 1$ .

For  $p \equiv 3 \pmod{4}$ ,  $\lceil p/2 \rceil$  is even and  $\lfloor p/2 \rfloor$  is odd. It follows that the initial odd factorisation of  $p$  is given by  $p = 2\lfloor p/2 \rfloor + 1$ . □

For  $p$  odd, we obtain the odd factorisation of  $p$  as follows. (Note that, for each iteration,  $\pm$  denotes  $+$  or  $-$  but not both, and is determined via Theorem 4.2.)

- Let  $q_0$  be the initial odd factor of  $p$ . Then  $p = 2q_0 \pm 1$ .
- Let  $q_1$  be the initial odd factor of  $q_0$ . Then  $q_0 = 2q_1 \pm 1$  and  $p = 2(2q_1 \pm 1) \pm 1$ .
- Continuing in this way, the process terminates at  $q_t = 1$  and

$$p = 2(\dots(2(2(\mathbf{1} + 1) \pm 1) \pm 1)\dots) \pm 1. \tag{4.1}$$

If at this stage,  $t$  of these factorisation steps have occurred, we call this expression the  $t$ th odd unit factorisation of  $p$ .

The  $\mathbf{1}$  in the middle of (4.1) can be replaced with

$$\mathbf{1} = 2\mathbf{1} - 1 = 2(2\mathbf{1} - 1) - 1 = \dots = (2(\dots(2(2(2\mathbf{1} - 1) - 1) - 1)\dots) - 1). \tag{4.2}$$

Thus we can extend (4.1) indefinitely by replacing the  $\mathbf{1}$  in the middle of (4.1) with (4.2). With these replacements we obtain the  $(t + 1)$ th,  $(t + 2)$ th, ... odd unit factorisations of  $p$ .



**DEFINITION 4.3 (Odd unit factorisations of  $p$ ).** For  $p$  an odd integer, let  $p_{(n)} = \underbrace{(2 \cdots (2 \cdots (2 \cdots (2 \cdots (2(2.1 - 1) - 1) \cdots - 1) + 1) \cdots + 1) - 1) \cdots - 1) \cdots \pm 1)}_{n \text{ pairs of brackets}}$

be the  $n$ th odd unit factorisation of  $p$ . The  $n$ th odd unit set of  $p$ , styled  $p_{\text{oun}}$ , is a set  $p_{\text{oun}} = \{p_0, p_1, \dots, p_k\}$  where  $p_i > 0$  for  $i > 0$  and  $n = \sum_{i=0}^k p_i$ , such that:

- $p_0$  is the number of consecutive innermost brackets ending in  $-1$ ;
- $p_1$  is the number of consecutive innermost brackets ending in  $+1$  enclosing the innermost  $p_0$  brackets;
- $p_2$  is the number of consecutive brackets ending in  $-1$  enclosing the innermost  $(p_0 + p_1)$  brackets, and so on until
- $p_k$  is the number of consecutive brackets ending in  $+1$  for  $k$  odd, or  $-1$  for  $k$  even, enclosing the innermost  $\sum_{i=0}^{k-1} p_i$  brackets

**EXAMPLE 4.4.** What are the odd unit sets of 17? We progressively odd-factorise 17 until we obtain its odd unit factorisation:

$$17 = 2.9 - 1 = 2(2.5 - 1) - 1 = 2(2(2.3 - 1) - 1) - 1 = 2(2(2(2.1 + 1) - 1) - 1) - 1.$$

So the odd unit factorisations of 17 are

$$\begin{aligned} 17_{(4)} &= (2(2(2(2.1 + 1) - 1) - 1) - 1), \\ 17_{(5)} &= (2(2(2(2(2.1 - 1) + 1) - 1) - 1) - 1), \\ 17_{(6)} &= (2(2(2(2(2(2.1 - 1) - 1) + 1) - 1) - 1) - 1), \\ &\vdots \end{aligned}$$

Accordingly, the odd unit sets of 17 are

$$17_{\text{ou}4} = \{0, 1, 3\}, \quad 17_{\text{ou}5} = \{1, 1, 3\}, \quad 17_{\text{ou}6} = \{2, 1, 3\}, \quad \dots, \quad 17_{\text{ou}n} = \{n - 4, 1, 3\}.$$

### 5. Finding folds from points

We now find the address of the point  $p/2^n < 1$ , where  $p$  is an odd positive integer, by examining the odd unit sets of  $p$ . Theorem 5.1 contains two binary signed-digit expansions: in (i),  $j - 1$  is written as an alternating series where each term is 2 raised to a power that is the sum of the members of a subset of an odd unit set; while in (ii), similar comments apply for  $j$ .

**THEOREM 5.1 (Addresses from points).** Let  $p < 2^n$  be an odd positive integer where  $p_{\text{oun}} = \{p_0, p_1, \dots, p_k\}$  and  $n = \sum_{i=0}^k p_i$ . The point  $p/2^n$  is the point  $S_{n,j}$  where:

- (i) if  $k$  is even, then  $j$  is odd and  $j = \sum_{i=0}^{k-1} (-1)^{i+1} 2^{p_0+p_1+\dots+p_i} + 1$ ;
- (ii) if  $k$  is odd, then  $j$  is even and  $j = \sum_{i=0}^{k-1} (-1)^i 2^{p_0+p_1+\dots+p_i}$ .

**PROOF.** We prove the result by induction on  $n$ . For  $n = 1$ , the point  $\frac{1}{2}$  has  $1_{\text{ou}1} = \{1\}$  and so  $k = 0$ , which is even. So (i) holds with  $j = 1$ , that is,  $\frac{1}{2} = S_{1,1}$ .

Suppose the result is true for  $S_n$ , where  $0 < j \leq 2^{n-1}$ . We consider two cases.

*Case (i):*  $j$  odd. Then  $k$  is even. Let  $S_{n+1,j} = p' / 2^{n+1}$ .

For  $0 < j \leq 2^{n-1}$ , by Theorem 1.14,  $p' = 2p - 1$ , so  $p'_i = p_i$  for  $0 \leq i \leq k - 1$  and by the induction hypothesis,

$$j = \sum_{i=0}^{k-1} (-1)^{i+1} 2^{p'_0+p'_1+\dots+p'_i} + 1.$$

For  $2^{n-1} < j \leq 2^n$ , by Theorem 1.14,  $S_{n+1,j} = S_{n,2^n-j+1} + 1/2^{n+1}$ . If  $S_{n,2^n-j+1} = p'' / 2^n$ , then  $p' = 2p'' + 1$ ,  $p'_i = p''_i$  for  $0 \leq i \leq k$  and  $p'_{k+1} = 1$ . By the induction hypothesis, as  $j$  is odd,  $2^n - j + 1$  is even and

$$2^n - j + 1 = \sum_{i=0}^k (-1)^i 2^{p'_0+p'_1+\dots+p'_i}.$$

Since  $n = p'_0 + p'_1 + \dots + p'_k$ , this again gives the required result for  $j$ .

Thus for  $0 < j \leq 2^n$ , the result is true for all odd values of  $j$  in  $S_{n+1}$ .

*Case (ii):*  $j$  even. Then  $k$  is odd. This case proceeds identically as for (i).

It follows that if our result is true for each  $S_{n,j}$  where  $0 < j \leq 2^{n-1}$ , it is also true for each  $S_{n+1,j}$  where  $0 < j \leq 2^n$ . □

**EXAMPLE 5.2.** What is the address of the point  $25/64$ ? Since  $25/64 = 25/2^6$ , it is found in the 6th fold. We have

$$25_{(6)} = 2(2(2(2(2(2 \cdot 1 - 1) - 1) + 1) + 1) - 1) - 1.$$

Thus  $25_{\text{out}6} = \{2, 2, 2\}$ . By Theorem 5.1(i), for  $k$  even,  $25/64$  must be at the position

$$2^{2+2} - 2^2 + 1 = 16 - 4 + 1 = 13.$$

Thus  $25/64$  is the point  $S_{6,13}$ .

**REMARK 5.3.** Appendix B, for  $n = 6$ , identifies each address that is generated from each point. The numerators of the sequence of addresses  $(1, 32, 17, 16, \dots)$  are a subset of the paperfolding sequence A281589 in the on-line Encyclopedia of Integer Sequences [12]. The comments in [12] link the sequence to a 2-regular sequence.

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My (BB) introduction to paperfolding occurred as an undergraduate student of Alf van der Poorten. He casually mentioned it at a residential school. It never occurred to me that I would become fascinated by it.

APPENDIX A. Identifying points from addresses for  $n = 6$ .

Address, $S_{n,j}$	$j_{NAF}$	Theorem 3.1	Point at $S_{n,j}$
$S_{n,1}$	0	$j$ odd	$\frac{1}{64}$
$S_{n,2}$	1	$j$ even	$1 - \frac{1}{64} = \frac{63}{64}$
$S_{n,3}$	2, 1, 0	$j$ odd	$-\frac{1}{2} + 1 + \frac{1}{64} = \frac{33}{64}$
$S_{n,4}$	2	$j$ even	$\frac{1}{2} - \frac{1}{64} = \frac{31}{64}$
$S_{n,5}$	3, 2, 0	$j$ odd	$-\frac{1}{4} + \frac{1}{2} + \frac{1}{64} = \frac{17}{64}$
$S_{n,6}$	3, 2, 1	$j$ even	$\frac{1}{4} - \frac{1}{2} + 1 - \frac{1}{64} = \frac{47}{64}$
$S_{n,7}$	3, 1, 0	$j$ odd	$-\frac{1}{4} + 1 + \frac{1}{64} = \frac{49}{64}$
$S_{n,8}$	3	$j$ even	$\frac{1}{4} - \frac{1}{64} = \frac{15}{64}$
$S_{n,9}$	4, 3, 0	$j$ odd	$-\frac{1}{8} + \frac{1}{4} + \frac{1}{64} = \frac{9}{64}$
$S_{n,10}$	4, 3, 1	$j$ even	$\frac{1}{8} - \frac{1}{4} + 1 - \frac{1}{64} = \frac{55}{64}$
$S_{n,11}$	4, 3, 2, 1, 0	$j$ odd	$-\frac{1}{8} + \frac{1}{4} - \frac{1}{2} + 1 + \frac{1}{64} = \frac{41}{64}$
$S_{n,12}$	4, 3, 2	$j$ even	$\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - \frac{1}{64} = \frac{23}{64}$
$S_{n,13}$	4, 2, 0	$j$ odd	$-\frac{1}{8} + \frac{1}{2} + \frac{1}{64} = \frac{25}{64}$
$S_{n,14}$	4, 2, 1	$j$ even	$\frac{1}{8} - \frac{1}{2} + 1 - \frac{1}{64} = \frac{39}{64}$
$S_{n,15}$	4, 1, 0	$j$ odd	$-\frac{1}{8} + 1 + \frac{1}{64} = \frac{57}{64}$
$S_{n,16}$	4	$j$ even	$\frac{1}{8} - \frac{1}{64} = \frac{7}{64}$
$S_{n,17}$	5, 4, 0	$j$ odd	$-\frac{1}{16} + \frac{1}{8} + \frac{1}{64} = \frac{5}{64}$
$S_{n,18}$	5, 4, 1	$j$ even	$\frac{1}{16} - \frac{1}{8} + 1 - \frac{1}{64} = \frac{59}{64}$
$S_{n,19}$	5, 4, 2, 1, 0	$j$ odd	$-\frac{1}{16} + \frac{1}{8} - \frac{1}{2} + 1 + \frac{1}{64} = \frac{37}{64}$
$S_{n,20}$	5, 4, 2	$j$ even	$\frac{1}{16} - \frac{1}{8} + \frac{1}{2} - \frac{1}{64} = \frac{27}{64}$
$S_{n,21}$	5, 4, 3, 2, 0	$j$ odd	$-\frac{1}{16} + \frac{1}{8} - \frac{1}{4} + \frac{1}{2} + \frac{1}{64} = \frac{21}{64}$
$S_{n,22}$	5, 4, 3, 2, 1	$j$ even	$\frac{1}{16} - \frac{1}{8} + \frac{1}{4} - \frac{1}{2} + 1 - \frac{1}{64} = \frac{43}{64}$
$S_{n,23}$	5, 4, 3, 1, 0	$j$ odd	$-\frac{1}{16} + \frac{1}{8} - \frac{1}{4} + 1 + \frac{1}{64} = \frac{53}{64}$
$S_{n,24}$	5, 4, 3	$j$ even	$\frac{1}{16} - \frac{1}{8} + \frac{1}{4} - \frac{1}{64} = \frac{11}{64}$
$S_{n,25}$	5, 3, 0	$j$ odd	$-\frac{1}{16} + \frac{1}{4} + \frac{1}{64} = \frac{13}{64}$
$S_{n,26}$	5, 3, 1	$j$ even	$\frac{1}{16} - \frac{1}{4} + 1 - \frac{1}{64} = \frac{51}{64}$
$S_{n,27}$	5, 3, 2, 1, 0	$j$ odd	$-\frac{1}{16} + \frac{1}{4} - \frac{1}{2} + 1 + \frac{1}{64} = \frac{45}{64}$
$S_{n,28}$	5, 3, 2	$j$ even	$\frac{1}{16} - \frac{1}{4} + \frac{1}{2} - \frac{1}{64} = \frac{19}{64}$
$S_{n,29}$	5, 2, 0	$j$ odd	$-\frac{1}{16} + \frac{1}{2} + \frac{1}{64} = \frac{29}{64}$
$S_{n,30}$	5, 2, 1	$j$ even	$\frac{1}{16} - \frac{1}{2} + 1 - \frac{1}{64} = \frac{35}{64}$
$S_{n,31}$	5, 1, 0	$j$ odd	$-\frac{1}{16} + 1 + \frac{1}{64} = \frac{61}{64}$
$S_{n,32}$	5	$j$ even	$\frac{1}{16} - \frac{1}{64} = \frac{3}{64}$

APPENDIX B. Identifying addresses from points for  $n = 6$ .

Point $\frac{p}{64}$	$p_{\text{ou}6}$	$j$	Address
$\frac{1}{64}$	6	1	$S_{6,1}$
$\frac{3}{64}$	5, 1	$2^5 = 32$	$S_{6,32}$
$\frac{5}{64}$	4, 1, 1	$2^5 - 2^4 + 1 = 17$	$S_{6,17}$
$\frac{7}{64}$	4, 2	$2^4 = 16$	$S_{6,16}$
$\frac{9}{64}$	3, 1, 2	$2^4 - 2^3 + 1 = 9$	$S_{6,9}$
$\frac{11}{64}$	3, 1, 1, 1	$2^5 - 2^4 + 2^3 = 24$	$S_{6,24}$
$\frac{13}{64}$	3, 2, 1	$2^5 - 2^3 + 1 = 25$	$S_{6,25}$
$\frac{15}{64}$	3, 3	$2^3 = 8$	$S_{6,8}$
$\frac{17}{64}$	2, 1, 3	$2^3 - 2^2 + 1 = 5$	$S_{6,5}$
$\frac{19}{64}$	2, 1, 2, 1	$2^5 - 2^3 + 2^2 = 28$	$S_{6,28}$
$\frac{21}{64}$	2, 1, 1, 1, 1	$2^5 - 2^4 + 2^3 - 2^2 + 1 = 21$	$S_{6,21}$
$\frac{23}{64}$	2, 1, 1, 2	$2^4 - 2^3 + 2^2 = 12$	$S_{6,12}$
$\frac{25}{64}$	2, 2, 2	$2^4 - 2^2 + 1 = 13$	$S_{6,13}$
$\frac{27}{64}$	2, 2, 1, 1	$2^5 - 2^4 + 2^2 = 20$	$S_{6,20}$
$\frac{29}{64}$	2, 3, 1	$2^5 - 2^2 + 1 = 29$	$S_{6,29}$
$\frac{31}{64}$	2, 4	$2^2 = 4$	$S_{6,4}$
$\frac{33}{64}$	1, 1, 4	$2^2 - 2^1 + 1 = 3$	$S_{6,3}$
$\frac{35}{64}$	1, 1, 3, 1	$2^5 - 2^2 + 2^1 = 30$	$S_{6,30}$
$\frac{37}{64}$	1, 1, 2, 1, 1	$2^5 - 2^4 + 2^2 - 2^1 + 1 = 19$	$S_{6,19}$
$\frac{39}{64}$	1, 1, 2, 2	$2^4 - 2^2 + 2^1 = 14$	$S_{6,14}$
$\frac{41}{64}$	1, 1, 1, 1, 2	$2^4 - 2^3 + 2^2 - 2^1 + 1 = 11$	$S_{6,11}$
$\frac{43}{64}$	1, 1, 1, 1, 1, 1	$2^5 - 2^4 + 2^3 - 2^2 + 2^1 = 22$	$S_{6,22}$
$\frac{45}{64}$	1, 1, 1, 2, 1	$2^5 - 2^3 + 2^2 - 2^1 + 1 = 27$	$S_{6,27}$
$\frac{47}{64}$	1, 1, 1, 3	$2^3 - 2^2 + 2^1 = 6$	$S_{6,6}$
$\frac{49}{64}$	1, 2, 3	$2^3 - 2^1 + 1 = 7$	$S_{6,7}$
$\frac{51}{64}$	1, 2, 2, 1	$2^5 - 2^3 + 2^1 = 26$	$S_{6,26}$
$\frac{53}{64}$	1, 2, 1, 1, 1	$2^5 - 2^4 + 2^3 - 2^1 + 1 = 23$	$S_{6,23}$
$\frac{55}{64}$	1, 2, 1, 2	$2^4 - 2^3 + 2^1 = 10$	$S_{6,10}$
$\frac{57}{64}$	1, 3, 2	$2^4 - 2^1 + 1 = 15$	$S_{6,15}$
$\frac{59}{64}$	1, 3, 1, 1	$2^5 - 2^4 + 2^1 = 18$	$S_{6,18}$
$\frac{61}{64}$	1, 4, 1	$2^5 - 2^1 + 1 = 31$	$S_{6,31}$
$\frac{63}{64}$	1, 5	$2^1 = 2$	$S_{6,2}$

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