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Component Games on Regular Graphs

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We study the $(1 : b)$ Maker–Breaker component game, played on the edge set of a d-regular graph. Maker's aim in this game is to build a large connected component, while Breaker's aim is to prevent him from doing so. For all values of Breaker's bias b, we determine whether Breaker wins (on any d-regular graph) or Maker wins (on almost every d-regular graph) and provide explicit winning strategies for both players.

To this end, we prove an extension of a theorem of Gallai, Hasse, Roy and Vitaver about graph orientations without long directed simple paths.

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1. Introduction

Let X be a finite set, let $\mathcal{F} \subseteq 2^X$ be a family of subsets of X, and let m, b be two positive integers. In the $(m : b)$ Maker–Breaker game $(X, \mathcal{F})$, two players, called Maker and Breaker, take turns at claiming previously unclaimed elements of X. On his move, Maker claims m elements of X, and Breaker, on his move, claims b elements. The player who makes the very last move may not be able to complete m (or b) steps, so he stops after claiming all remaining elements. The game ends when all of the elements have been claimed by either of the players. The description of the game is completed by stating which of the players is the first to move, though usually it makes little difference. Maker wins the game $(X, \mathcal{F})$ if by the end of the game he has claimed all the elements of some $F \in \mathcal{F}$; otherwise Breaker wins. For convenience, we typically assume that $\mathcal{F}$ is closed

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upwards, and specify only the inclusion-minimal elements of $\mathcal{F}$. Since these are finite, perfect information games with no possibility of a draw, for each set-up of $\mathcal{F}, m, b$ and the identity of the first player, one of the players has a strategy to win regardless of the other player's strategy. Therefore, for a given game we may say that the game is Maker's win, or alternatively that it is Breaker's win. The set X is referred to as the *board* of the game, and the elements of $\mathcal{F}$ are referred to as the *winning sets*.

When $m = b = 1$, we say that the game is *unbiased*; otherwise it is *biased*, and the positive integers m and b are called the *bias* of Maker and Breaker, respectively. Maker–Breaker games are *bias monotone*. It means that if Maker wins some game with bias $(m : b)$, he also wins this game with bias $(m' : b')$, for every $m' \geqslant m$, $b' \leqslant b$. Similarly, if Breaker wins a game with bias $(m : b)$, he also wins this game with bias $(m' : b')$, for every $m' \leqslant m$, $b' \geqslant b$. Indeed, suppose that some player has a winning strategy with bias c, and now he plays with bias $c' > c$. He can use his old strategy and in addition claim arbitrarily $c' - c$ extra elements per move and pretend he did not claim them; whenever his strategy tells him to claim some element he has previously claimed he just claims arbitrarily some unclaimed element. Similarly, if his opponent claims fewer elements, he can assign (in his mind) some extra elements to his opponent in each move, and continue with his strategy. The same reasoning shows that it is never a disadvantage in a Maker–Breaker game to be the first player, and a winning strategy as a second player can be used as a winning strategy as a first player. This bias monotonicity allows us to define the *threshold bias*: for a given game $\mathcal{F}$, the threshold bias b^* is the value for which Maker wins the game $\mathcal{F}$ with bias $(1 : b)$ for every $b \leqslant b^*$, and Breaker wins the game $\mathcal{F}$ with bias $(1 : b)$ for every $b > b^*$.

In this paper, our attention is dedicated to the $(1 : b)$ Maker–Breaker s-component game on regular graphs; that is, the board is the edge set of some d-regular graph G on n vertices and the winning sets are connected components of G with s vertices.

1.1. Previous results

A natural case to consider is $s = n$; that is, the winning sets are the spanning trees of G. This $(1 : b)$ n-component game is also known as the *connectivity* game.

The unbiased game was completely solved by Lehman [15], who showed that Maker wins the $(1 : 1)$ connectivity game on a graph G if and only if G contains two edge-disjoint spanning trees. It follows easily from [18, 21] that if G is $2k$-edge-connected then it contains k pairwise independent spanning trees; thus, Maker wins the $(1 : 1)$ connectivity game on 4-regular 4-edge-connected graphs, whereas Breaker trivially wins the $(1 : 1)$ connectivity game on graphs with less than $2n - 2$ edges, *i.e.*, average degree under $4 - O(1/n)$.

For denser graphs, since Maker wins the unbiased game by such a large margin, it only seems fair to even out the odds by strengthening Breaker, giving him a bias $b \geqslant 2$. First and most natural board to consider is the edge set of the complete graph K_n (*i.e.*, $d = n - 1$). Chvátal and Erdős [5] showed that $\left(\frac{1}{4} - o(1)\right)n/\log n \leqslant b^*(K_n) \leqslant (1 + o(1))n/\log n$; the upper bound was proved to be tight by Gebauer and Szabó [9]; that is, $b^*(K_n) = (1 + o(1))n/\log n$. The doubly biased connectivity game $(m : b)$ on K_n was considered by Hefetz, Mikalački and Stojaković [12], where the winner was determined for almost all values of m and b.

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Another natural board to consider is the edge set of a random graph. Stojaković and Szabó [20] considered the well-known Erdős–Rényi random graph $\mathcal{G}_{n,p}$, in which each of the $\binom{n}{2}$ possible edges appears independently with probability p. They showed that almost surely $b^*(\mathcal{G}_{n,p}) = \Theta(np/\log n)$, provided that $p \geqslant C \log n/n$ for some constant C (note that for $p < \log n/n$, the graph $\mathcal{G}_{n,p}$ itself is almost surely not connected and so Breaker wins no matter how he plays). It was later shown by Ferber, Glebov, Krivelevich and the second author in [7] that in fact, if $p = \omega(\log n/n)$ then $b^*(\mathcal{G}_{n,p}) = np/\log n$ almost surely.

A different random graph model, the random d-regular graph $\mathcal{G}_{n,d}$ on n vertices, was considered by Hefetz, Krivelevich, Stojaković and Szabó [11]. They showed that almost surely $b^*(\mathcal{G}_{n,d}) \geqslant (1-\epsilon)d/\log_2 n$ for $d = o(\sqrt{n})$. Note that when $d = \Omega(\sqrt{n})$, $\mathcal{G}_{n,d}$ is quite close to $\mathcal{G}_{n,p}$ for $p = d/n$, by concentration of the binomial distribution. Moreover, they showed that $b^*(G) \leqslant \max\{2, \bar{d}/\log n\}$ for a graph G of average degree $\bar{d}$, so the result is asymptotically tight.

Breaker's strategy in practically all results mentioned above is to deny connectivity by isolating a single vertex. Much less is known, however, for the case $s < n$. It seems that even if Breaker is able to isolate a vertex in a constant number of moves, it does little to prevent Maker from winning the s-component game for $s = \Omega(n)$.

Instead of considering the threshold bias b^*, we shift the focus to the maximal component size s achievable by Maker in the $(1:b)$ game, for a given bias b. Let us denote this quantity by $s^*_b(G)$. Bednarska and Łuczak considered in [3] the $(1:b)$ game on the complete graph. They showed that $s^*_b(K_n)$ undergoes a phase transition around $b = n$; namely, that $s^*_{n+t}(K_n) = (1-o(1))n/t$ for $\sqrt{n} \ll t \ll n$ but $s^*_{n-t}(K_n) = t + O(\sqrt{n})$ for $0 \leqslant t \leqslant n/100$.

1.2. Our results

For $d \geqslant 3$, let

$$s^*_b(n,d) = \max\{s^*_b(G) : G \text{ is a } d\text{-regular graph on } n \text{ vertices}\},$$

where $s^*_b(G)$, as above, is the maximal component size s achievable by Maker in the $(1:b)$ game on G.

For $b \geqslant 2d-2$, Breaker can immediately isolate each edge claimed by Maker in the $(1:b)$ game, so trivially $s^*_b(G) = 2$. Furthermore, Breaker can do something similar while $b > d-2$, as the following proposition shows.

Proposition 1.1. *For any positive k, $s^*_{d-2+k}(n,d) \leqslant 2\lceil d/k \rceil$.* □

In the $(1:d-2)$ game, Breaker can still restrict the size of Maker's connected components.

Theorem 1.2. *$s^*_{d-2}(n,d) \leqslant \alpha_d + \beta_d \log n$, where α_d and β_d depend only on d.*

Remark 1.3. Our proof yields $\alpha_d = O(d^2)$ and $\beta_d = O(d/\log\log d)$.

The proof of Theorem 1.2 relies on the following combinatorial lemma, which may be of independent interest.

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Lemma 1.4. *Let G be a graph on n vertices with minimal degree $\delta \geqslant 3$. Then, there exists an orientation D of G such that every vertex has a positive out-degree and all simple directed paths in D are of length at most $\chi(G) + \kappa_\delta \log n$, where $\kappa_\delta = O(1/\log\log\delta)$.*

Remark 1.5. Note that $s^*_b(T_{b+1}(k)) \geqslant k$, where $T_r(k)$ is the complete k-level r-ary tree (*i.e.*, every non-leaf vertex has r children), since Maker can easily build a path from the root to some leaf. Completing $T_{d-1}(k)$ to a d-regular graph on $n = (d-1)^k$ vertices thus shows that $s^*_{d-2}(n,d) \geqslant \log_{d-1} n$.

To complement Theorem 1.2, we prove that in the $(1 : d-3)$ game on almost every graph, Maker can already build a very large connected component.

Theorem 1.6. *Let $\mathcal{G}_{n,d}$ be the random d-regular graph on n vertices, where $d \geqslant 3$. Then, $s^*_{d-3}(\mathcal{G}_{n,d}) \geqslant \epsilon_d n$ almost surely, where $\epsilon_d > 0$ depends only on d. In particular, $s^*_{d-3}(n,d) \geqslant \epsilon_d n$.*

Remark 1.7. A quick calculation shows that $\epsilon_d \geqslant \mathrm{poly}(1/d)$.

When d is at most polylogarithmic in n, Theorems 1.2 and 1.6 show a phase transition phenomenon that occurs at $b = d-2$; instead of tiny, polylogarithmic-sized components, Maker is suddenly able to build a giant, almost linear-sized component. When d is constant, we even have a double-jump: from constant, through logarithmic, to linear-sized components. This is summarized in Figure 1.

This behaviour is somewhat consistent with the so-called random graph intuition in positional games: the outcome of a game between two intelligent players is often the same as the outcome of that game between two players acting randomly. Consider the bond percolation with parameter p (*i.e.*, each edge is deleted independently with probability $1-p$). It is known (see, *e.g.*, [2, 17]) that for well-expanding d-regular graphs, where d is constant, the size of the largest connected component has a double-jump at $p = \frac{1}{d-1}$: it is linear for $p \geqslant \frac{1+\epsilon}{d-1}$, logarithmic for $p \leqslant \frac{1-\epsilon}{d-1}$, and $\Theta(n^{2/3})$ for $p = \frac{1}{d-1}$.

Although the sizes of the components are different, both the bond percolation and $s^*_b(n,d)$ have a sharp threshold at the same point, since in a random play of a $(1 : b)$ game, Maker gets each edge with probability $\frac{1}{1+b} = \frac{1}{d-1}$ (this random graph intuition is not a formal argument, so we allow ourselves to neglect the dependence of these random choices).

1.3. Notation

We use standard graph-theoretic terminology, and in particular use the following.

For a given graph G we let $V(G)$ and $E(G)$ denote the set of its vertices and the set of its edges, respectively. We often just use V and E when there is no chance of confusion. For two disjoint sets of vertices $A, B \subseteq V$ we let $E(A,B)$ denote the set of all edges $(a,b) \in E$ with $a \in A$ and $b \in B$. For a connected component S in Maker's graph, and for an edge $e \in E$ we say that e is *incident* to S if at least one of its endpoints belongs to S; if both

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$$s_b^*(n,d) = \begin{cases} \Theta(1) & b > d-2, \\ \Theta(\log n) & b = d-2, \\ \Theta(n) & b < d-2. \end{cases}$$

(a) $d = \Theta(1)$

$$s_b^*(n,d) = \begin{cases} \Theta(\operatorname{poly}\log n) & b \geqslant d-2, \\ \Theta(n/\operatorname{poly}\log n) & b < d-2. \end{cases}$$

(b) $d = O(\operatorname{poly}\log n)$

Figure 1. Phase transition phenomenon at $b = d-2$.

endpoints of e belong to S, we say that e is *inside* S. When G is a directed graph, we say that a vertex v is *reachable* from a vertex u if there is a directed path in G from u to v.

An unclaimed edge is called *free*. The act of claiming one free edge by one of the players is called a *step*. Maker's m (Breaker's b) successive steps are called a *move*. A *round* in the game consists of one move of the first player, followed by one move of the second player. Whenever Maker claims a free edge, it becomes part of some connected component of his; we then say he *touched* that component. If a connected component in Maker's graph has at least one free edge adjacent to it, we say it is a *live* component.

As mentioned before, if one of the players has a winning strategy as a second player, he can use it to obtain a winning strategy as a first player. Hence, when we describe Maker's strategy we assume that he is the second player, implying that under the described conditions he can win as either a first or a second player. The same goes for Breaker's strategy.

2. Maker's strategy

Throughout this section we assume that the first player is Breaker.

In this section we describe and analyse a very basic strategy for Maker, to which we refer throughout the paper as *the tree strategy*. Maker's goal is to build a component of size s, and his strategy is to build a single connected component T. He starts from a single arbitrary vertex r, and in every move he adds a new vertex to T by claiming a free edge $e \in E(T, V \setminus T)$. If all edges in $E(T, V \setminus T)$ have already been claimed by Breaker, and Maker's component is of size strictly less than s, he forfeits the game. Note that indeed T is a tree throughout the game.

Definition. Let $G = (V, E)$ be a graph on n vertices. For an integer $k = 1, 2, \ldots, \lfloor n/2 \rfloor$, we define

$$\Psi_E(G,k) = \min\left\{ \frac{|E(S, V \setminus S)|}{|S|} : S \subseteq V, 1 \leqslant |S| \leqslant k \right\}.$$

Considered as a function of k, *i.e.*, when G is fixed, Ψ_E is sometimes called the *edge isoperimetric profile*.

The next proposition shows that if the graph has good expanding properties, then Breaker cannot separate T from $V \setminus T$ unless T is large enough.

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Proposition 2.1. *Assume $\Psi_E(G,k) > b$. Then Maker is able to carry out the tree strategy for at least k rounds in the $(1 : b)$ game on the graph G.* □

Proof. Consider the moment before Maker's jth move for some $1 \leqslant j \leqslant k$. We have $|T| = j \leqslant k$ and thus $|E(T, V \setminus T)| > |T|b = jb$. During j moves, Breaker could have claimed at most jb edges, so some edge of $E(T, V \setminus T)$ is still available for Maker to claim. □

Since we only need Maker's connected component to span a constant fraction of the graph, we can make use of the following result on the edge expansion of small sets in the random d-regular graph.

Lemma 2.2 ([13], Theorem 4.16). *Let $d \geqslant 3$ be an integer and let $\delta > 0$. Then there exists $\epsilon = \epsilon(d, \delta) > 0$ such that $\Psi_E(\mathcal{G}_{n,d}, \epsilon n) > d - 2 - \delta$ almost surely.*

Taking $\delta = 1$ in Lemma 2.2 and employing the tree strategy yields Theorem 1.6.

Remark 2.3. Lemma 2.2 is strong enough to render the tree strategy effective also in the doubly biased $(m : b)$ game, as long as $b/m < d - 2$. The proof is completely analogous to the proof of Proposition 2.1 when Maker is the first player, and very simple adjustments are needed when Breaker starts.

3. Breaker's strategy

Throughout this section we assume that the first player is Maker.

3.1. Reactive strategies

Definition. A strategy of Breaker is called *reactive* if the following holds: in each of his steps, if the connected component last touched by Maker is live, Breaker claims a free edge incident to it.

Note that there can be many reactive strategies for Breaker, varying in the way that he chooses which free edge to claim among those that are incident to Maker's last touched component. In this paper, we limit ourselves to reactive Breaker strategies; this allows Breaker to control the number of free edges incident to Maker's connected components, as the following claim shows.

Claim 3.1. *Let b, d be positive integers and let G be a d-regular graph. If Breaker uses a reactive strategy, then throughout the $(1 : b)$ Maker–Breaker game played on the edge set of G, at the beginning of each round every connected component S in Maker's graph is incident to at most $(d - 2 - b)|S| + b + 2$ free edges.*

Proof. The claim trivially holds at the beginning of the game, as every connected component is a single vertex, and vertex degrees in G are all equal to d. In every move, Maker either:

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(a) claims an edge inside some component; or
(b) merges two connected components, *i.e.*, claims a free edge between two connected components S_1 and S_2, creating a new connected component S of size $|S| = |S_1| + |S_2|$.

In the first case, the claim trivially holds no matter how Breaker plays. In the second case, S_i (for $i = 1, 2$) was incident before Maker's move to at most $(d-2-b)|S_i| + b + 2$ free edges. As Maker has just claimed an edge incident to both S_1 and S_2, after Maker's move at most $(d-2+b)|S| + 2(b+2) - 2$ free edges are incident to the merged component S. Breaker in his next move claims b of these edges (or simply all of them, if there are fewer than that), leaving at most $(d-2-b)|S| + 2(b+2) - 2 - b = (d-2-b)|S| + b + 2$ free edges incident to S, so the claim still holds. □

We can now prove Proposition 1.1.

Proof of Proposition 1.1. By Claim 3.1 we get that by using any reactive strategy, Breaker can make sure that every component S in Maker's graph will have at most $-k|S| + k + d$ free edges incident to it. In particular, $k + d > k|S|$ for any live component S, or equivalently $|S| < (d/k) + 1$. This last inequality may be rewritten as $|S| \leqslant \lceil d/k \rceil$. Since every component in Maker's graph was created by merging two live components, the result follows. □

Remark 3.2. For fixed d and k, the bound of Proposition 1.1 is tight for large enough n, via the tree strategy on $\mathcal{G}_{n,d}$.

Remark 3.3. Reactive strategies are effective for Breaker also in the doubly biased $(m : b)$ game, as long as $b/m > d - 2$. A proof very similar to the one above shows that Breaker can limit Maker in the $(m : m(d-2) + k)$ game to connected components of size at most $(m+1)\lceil md/k \rceil$.

3.2. Playing against the tree strategy

Before presenting a fully fledged strategy for Breaker in the $(1 : d-2)$ game, let us first consider a simplified version of it, which remains effective as long as Maker adheres to the tree strategy of Section 2. Taking Claim 3.1 one step further, Breaker needs to make sure that, before Maker's tree T grows too much, the only free edges incident to it will be edges inside T. This gives rise to the following definition.

Definition. Let G be a graph. A simple path $p = v_1 v_2 \cdots v_k$ in G is called *self-colliding* if v_k is adjacent to some v_i for $1 \leqslant i \leqslant k-2$. We could also view p as a simple path $v_1 v_2 \cdots v_{i-1}$, which we call the *tail*, leading to a simple cycle $v_i v_{i+1} \cdots v_k v_i$, which we call the *body*. It is possible that the tail is empty, that is, p is a cycle of length k.

We use the following variation of the Moore bound on the girth of graphs with minimum degree k.

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Lemma 3.4. *Let G be a graph on n vertices with minimum degree $\delta(G) \geqslant k$. Then, for every edge $(u,v) \in E$, there is a self-colliding path p starting with (u,v) of length at most $2\lceil \log_{k-1} n \rceil$. In particular, $g(G) \leqslant 2\lceil \log_{k-1} n \rceil$. Moreover, the distance along p from u to every body vertex is at most $\lceil \log_{k-1} n \rceil$.*

Proof. The number of non-backtracking walks of length $j+1$ starting with the edge (u,v) is $(k-1)^j$. Since the graph has only n vertices, there exist two distinct non-backtracking walks of lengths $i+1$ and $j+1$ ending at the same vertex, where $i \leqslant j \leqslant \lceil \log_{k-1} n \rceil$. Together, these walks form a (not necessarily simple) cycle of length at most $2\lceil \log_{k-1} n \rceil$ passing through v. Now take any simple subcycle of it to be p's body and connect it back to u via a simple path. □

We now describe Breaker's strategy. After Maker's first move, Breaker chooses arbitrarily one of the two vertices Maker has just touched and denotes it by u. Breaker then uses Lemma 3.4 to pick, for each neighbour v of u, a self-colliding path p_v of length at most $2\lceil \log_{d-1} n \rceil$ beginning with the edge (u,v). Note that the paths chosen for two neighbours v, v' are not necessarily disjoint.

Now Breaker's strategy is to allow Maker to claim only edges from $P = \cup\{p_v : (u,v) \in E\}$; this would limit the size of Maker's connected component to be at most $|P| \leqslant 2d\lceil \log_{d-1} n \rceil$.

Proposition 3.5. *In the $(1 : d-2)$ game on G, if Maker follows the tree strategy, Breaker is able to carry out the counter-strategy.* □

Proof. We show that Breaker can ensure that before every move of Maker, the only free edges in $E(T, V \setminus T)$ are in P; thus, Maker must claim an edge of P, advancing along some p_v. It is true at the beginning of the game as $T = \{u\}$. After Maker claims the edge $(v_{i-1}, v_i) \in p_v$, there are at most $d-2$ free edges incident to v_i in $E(T, V \setminus T) \setminus P$, since (v_{i-1}, v_i) has just been claimed and $(v_i, v_{i+1}) \in P$. Breaker can claim all of them (and, if necessary, some arbitrary extra edges outside P). Thus, after getting a spanning tree $T \subset P$, Maker forfeits. □

The counter-strategy is still effective when Maker builds a forest with many trees, as long as one of the connected components merged is always a single vertex; nevertheless, it breaks down when Maker builds up many small trees and connects them to one another, avoiding getting to the collision at the end of the self-colliding paths. Breaker could possibly deny a merge of two trees T and T' by forgoing the counter-strategy and claiming the free edge between T and T', but this might let Maker escape from the respective P or P'.

3.3. Playing against any strategy

We now describe a global strategy for Breaker, which copes well with Maker merging connected components of any size. Before starting the $(1 : d-2)$ game, Breaker uses Lemma 1.4 to pick an orientation D of the graph G such that every vertex has a positive

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out-degree and all simple directed paths in D are of length at most $d + \kappa_d \log n$. Note that Breaker may as well reveal D to Maker.

The strategy of Breaker goes as follows. Without loss of generality we may assume that Maker's strategy is always to build a forest, since claiming an edge within a connected component does not help Maker (formally, Maker claims edges inside his connected components only when all remaining free edges are such; by this time, the outcome of the game has already been determined). Thus, on each move Maker merges two trees T_1 and T_2 to a single tree T by claiming a free edge from T_1 to T_2. Breaker then claims $d-2$ free edges according to the following priorities:

(1) $E(V \setminus T, T_2)$,
(2) $E(V \setminus T, T_1)$,
(3) $E(T, V \setminus T)$.

In each step, Breaker claims an arbitrary free edge from the set with the smallest index. If there is no free edge among these sets, he just claims an arbitrary free edge.

Claim 3.6. *Each tree T in Maker's graph is a directed tree in D; that is, there is some $r \in T$ – which we call the root of T – such that every vertex in T is reachable from r. Moreover, at the beginning of each round (i.e., after Breaker's move), no free edges enter $T \setminus \{r\}$.*

Proof. The claim is trivially true at the beginning of the game, as the initial connected components are single vertices, so every vertex is the root and only member of its own directed tree. Suppose now that Maker merged T_1 and T_2, two trees with roots r_1 and r_2, respectively, by claiming an edge from T_1 to T_2. By our assumption, before the merge the only free edges entering T_1 and T_2 were into r_1 and r_2, respectively. Hence, Maker must have claimed an edge into r_2. Clearly, the merged component is a directed tree, and all vertices in $T_1 \cup T_2$ are now reachable from r_1, which becomes the root of the new tree. Furthermore, the in-degree of every vertex in D, and in particular of r_2, is at most $d-1$, so Breaker's preference towards $E(V \setminus T, T_2)$ ensures that all the edges entering r_2 are claimed after Breaker's move (one by the merge and all the rest by Breaker), and so all the free edges entering the new tree enter its root. □

It is beneficial to classify Maker's trees by the number of free in-edges.

Definition. The *type* of a tree T in Maker's graph is the number of free edges in $E(V \setminus T, T)$.

By Claim 3.6, the type of a tree is bounded by the in-degree of its root, so the possible types are $0, 1, \ldots, d-1$. Claim 3.6 also enables us to partially order the vertices in each tree, giving rise to the following definition.

Definition. Let T be a tree in Maker's graph. The *height* of a vertex $v \in T$, denoted by $h(v)$, is the length of the (unique) path $r \rightsquigarrow v$ in T, where r is the root of T; the height

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of an edge $(u,v) \in E(T, V \setminus T)$ is $h(u,v) = h(u)$; the height of T, denoted by $h(T)$, is the maximum height over all $v \in T$.

We wish to bound the size of Maker's trees. By the choice of D, we know that the trees are not too 'high', but we also need to ensure they do not become too 'wide'.

For this, we refine Breaker's strategy slightly. In the tree T just created by Maker, Breaker claims in-edges from highest to lowest, and then out-edges from lowest to highest. In more detail, in each step Breaker claims an incoming free edge $(x,y) \in E(V \setminus T, T)$ such that $h(y)$ is maximal, if possible; otherwise he claims an outgoing free edge $(x,y) \in E(T, V \setminus T)$ such that $h(x,y) = h(x)$ is minimal. In both cases, ties are broken arbitrarily.

Breaker's preference of claiming low out-edges gives the following.

Claim 3.7. *Let T be a tree in Maker's graph. If the edge $e \in E(T, V \setminus T)$ was claimed by Breaker, then $h(e') \geqslant h(e)$ for every free edge $e' \in E(T, V \setminus T)$.*

Proof. Note first that if Breaker has claimed an edge $(u,v) \in E(T, V \setminus T)$ for some tree T, then from that point until the end of the game u will only belong to trees of type zero. Indeed, according to his strategy, Breaker has claimed (u,v) only since there were no free edges entering T, so at that point T is of type zero. Furthermore, by Claim 3.6 we have that at any point until the end of the game u will only belong to trees rooted at T's root, implying that they will be of type zero as well. Therefore, Claim 3.7 trivially holds when the type of T is positive, since that implies that Breaker has claimed only edges entering T. We thus assume T is of type zero.

At the moment Breaker claims e, there is no edge lower than e among all free edges in $E(T, V \setminus T)$. In subsequent rounds, the only changes to $E(T, V \setminus T)$ (and to T) are when Maker claims some edge e' from T to another tree T'. The height of all vertices of T' in the merged tree, and thus also of all new edges in $E(T, V \setminus T)$, is at least $h(e') + 1 > h(e)$. □

Recall that in the counter-strategy to the tree strategy, Breaker only allowed Maker to pursue self-colliding paths, so Maker's final component consisted of d paths p_v sharing a root vertex. Here, similarly, Breaker's strategy allows Maker to extend every free edge in $E(T, V \setminus T)$ to a directed path. This motivates the following definition of width.

Definition. Let T be a tree in Maker's graph. The *i-width* of T, denoted $w_i(T)$, is the number of vertices in T of height i plus the number of free edges in $E(T, V \setminus T)$ of height strictly smaller than i. The width of T, denoted $w(T)$, is the maximum i-width in T, taken over $i = 0, 1, \ldots, h(T)$.

We are ready to prove the following proposition, which implies Theorem 1.2 since $|T| \leqslant 1 + h(T) \cdot w(T)$.

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Proposition 3.8. *Let T be a tree of type t in Maker's graph. Then,*

$$w(T) \leqslant \begin{cases} d-t & 1 \leqslant t \leqslant d-1, \\ 2d-2 & t=0. \end{cases}$$

□

Proof. We prove this by induction on the number of rounds in the game. The proposition holds for trivial trees. Assume Maker merges trees T_1 and T_2 of types t_1 and t_2, respectively, by claiming the edge (u,v), where v is the root of T_2. Then, the merged tree T has type $t = \max(0, t_1 + t_2 - d + 1)$ after Breaker's move. Note that necessarily $t_2 > 0$.

The vertices of T_1 maintain their height in T; vertices that had height j in T_2 now have height $h(u) + 1 + j$ in T. For $i \leqslant h(u)$, we have $w_i(T) = w_i(T_1) \leqslant w(T_1)$; for $i > h(v)$, the now-claimed edge (u,v) no longer counts for the i-width of T, so

$$w_i(T) = w_i(T_1) - 1 + w_{i-h(u)-1}(T_2) \leqslant w(T_1) + w(T_2) - 1. \tag{3.1}$$

If $t_1 > 0$ then, by the induction hypothesis, $w(T_1) \leqslant d - t_1$ and $w(T_2) \leqslant d - t_2$, so

$$w(T) \leqslant w(T_1) + w(T_2) - 1 \leqslant d - t_1 + d - t_2 - 1 = d - t.$$

If $t_1 = 0$ then $t = 0$ too; by the induction hypothesis, $w(T_1) \leqslant 2d-2$ and $w(T_2) \leqslant d - t_2 \leqslant d - 1$. For $i \leqslant h(u)$, as before, we have $w_i(T) \leqslant w(T_1) \leqslant 2d-2$; for $i > h(u)$, assuming we show that $w_i(T_1) \leqslant d$, the same calculation as in (3.1) yields $w_i(T) \leqslant w_i(T_1) + w(T_2) - 1 \leqslant d + (d-1) - 1 = 2d - 2$.

By the definition of $w_i(T_1)$, there exist a set $U \subseteq T_1$ of vertices of height i and a set $A \subseteq E(T, V \setminus T)$ of free edges of height less than i such that $w_i(T_1) = |U| + |A|$. For every vertex $x \in U$, pick a leaf $x' \in T_1$ reachable (in T_1) from x. The out-degree of x' in D is positive, so pick some edge $e = (x', y) \in E(D)$. If $y \in T_1$, no one will ever claim e; otherwise, $e \in E(T, V \setminus T)$ so Maker has not yet claimed it. By Claim 3.7, neither did Breaker since $h(e) = h(x') \geqslant h(x) = i > h(u)$ and $(u,v) \in E(T_1, V \setminus T_1)$ was free before Maker's move. Altogether, we have a set A' of $|A'| = |U|$ free edges coming out of T_1, disjoint from A since edge heights in A' are all at least i. By Claim 3.1, T_1 is incident to at most d free edges, so $w_i(T_1) = |A| + |A'| = |A \cup A'| \leqslant d$, establishing the proposition. □

Remark 3.9. In the previous subsection, using the counter-strategy to the tree strategy, Breaker could bound $w(T)$ by ensuring that, besides a single vertex of degree d, the degrees of all vertices in T were at most two. With the strategy presented in this subsection, Breaker cannot limit $w(T)$ by bounding the number of forks in T, *i.e.*, the number of vertices of out-degree at least 2. Indeed, already for $d = 3$, there exists a positive out-degree orientation D of a cubic graph G and a strategy for Maker to build a tree T with $\Omega(h(T))$ forks in a $(1:1)$ game on G.

4. Short graph orientations

In this section we discuss and prove Lemma 1.4. We begin by introducing the following notation.

Definition. For a directed graph D, we denote by $l(D)$ the maximal length of a simple directed path in D. For an undirected graph G and $j \in \{0,1\}$, we denote by $l_j(G)$ the minimum of $l(D)$ over all orientations D of G such that every vertex has out-degree at least j.

The case $j = 0$, that is, when we drop the positive out-degrees requirement, has been considered by (at least) four independent works, by Gallai [8], Hasse [10], Roy [19] and Vitaver [22].

Theorem 4.1 ([8, 10, 19, 22]). *For every graph G, $l_0(G) = \chi(G)$.*

We mention here only the easy side of the proof, which will be used shortly. To see that $l_0(G) \leqslant \chi(G)$, colour G properly with the colours $\{1,2,\ldots,\chi(G)\}$ and orient each edge $\{u,v\}$ from u to v if and only if u's colour is greater than v's colour.

Returning to the case $j = 1$, we cannot expect an orientation D with positive out-degrees for which $l(D)$ is independent of n. Indeed, when every vertex has a positive out-degree, D surely contains a directed cycle, so $l_1(G) \geqslant g(G)$. Known constructions of d-regular graphs of high girth (see, *e.g.*, [4, 6, 16]) yield families of graphs of order n, chromatic number $\Omega(d/\log d)$ and girth $\Omega(\log_{d-1} n)$. Thus, our best hope would be to show that $l_1(G) = O(\log n)$.

The main idea of the proof that follows is this: we find in G a set of disjoint short cycles, which we orient cyclically, and we orient the rest of the edges 'towards' the cycles. Lemma 3.4 will assist us in showing that simple directed paths outside the cycles are necessarily short.

Proof of Lemma 1.4. Fix $k = \max(3, \lceil \log\delta/\log\log\delta \rceil)$ and set

$$\gamma_\delta = \lceil \log_{\delta-1} n \rceil, \gamma_k = \lceil \log_{k-1} n \rceil.$$

Let $\mathcal{C}$ be a maximal collection of non-adjacent induced cycles of length at most $2\gamma_k$. That is, we begin with an empty collection $\mathcal{C} = \varnothing$ and, as long as there exists an induced cycle C in G of length $|C| \leqslant 2\gamma_k$ whose vertices have no neighbours among $V_{\mathcal{C}}$, the vertices of cycles in $\mathcal{C}$, we add C to $\mathcal{C}$. Note that $\mathcal{C}$ is non-empty since the girth of G is at most $2\gamma_\delta \leqslant 2\gamma_k$, by Lemma 3.4 (or the Moore bound).

Fix any cyclic orientation of the cycles in $\mathcal{C}$, and orient the edges of $E(V_{\mathcal{C}}, V \setminus V_{\mathcal{C}})$ into $\mathcal{C}$. All edges incident to $\mathcal{C}$ are thus oriented, as the cycles in $\mathcal{C}$ are induced and non-adjacent. Since no edges are leaving any cycle in $\mathcal{C}$, once we orient the rest of the graph, any simple directed path can contain at most $2\gamma_k$ vertices of $V_{\mathcal{C}}$, which form its suffix.

We now fuse all the vertices of cycles in $\mathcal{C}$ to a single vertex s. Let $G' = (V', E')$ be the resulting graph; make it simple by discarding loops and parallel edges incident to s.

For every vertex $v \in V'$ we denote its distance from s by $\rho(v)$. We claim that $\rho(v) \leqslant 1 + \gamma_\delta$; indeed, v is within distance γ_δ of some short cycle C by Lemma 3.4 (specifically, v is within distance γ_δ of any vertex on C), and C either intersects some cycle in $\mathcal{C}$, is adjacent to some cycle in $\mathcal{C}$, or simply $C \in \mathcal{C}$, by the maximality of $\mathcal{C}$.

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For $i \in \{1,2,\dots,1+\gamma_\delta\}$, consider the level set $V_i' = \{v \in V' : \rho(v) = i\}$ and the subgraph $G_i \subset G'$ it induces. As in the proof of Theorem 4.1, we orient edges between V_i' and V_{i+1}' 'downwards' (*i.e.*, from V_{i+1}' to V_i'). This ensures all vertices except s have a positive out-degree, via shortest paths to s.

By definition, every edge either lies inside a level set or connects two successive level sets. Therefore, it only remains to orient edges between same height vertices, which will be done using Theorem 4.1. For G_1 we have $l_0(G_1) = \chi(G_1) \leqslant \chi(G)$ since $G_1 \subset G$. By the maximality of $\mathcal{C}$, for all $i > 1$, G_i has no cycle of length at most $2\gamma_k$. Apply Lemma 3.4 to deduce that G_i cannot have a subgraph with minimum degree k; in other words, G_i is $(k-1)$-degenerate, and, in particular, k-colourable.

Altogether, we have an orientation D' of G' satisfying

$$l(D') \leqslant \sum_{i=1}^{1+\gamma_\delta} l_0(G_i) = \sum_{i=1}^{1+\gamma_\delta} \chi(G_i) \leqslant \chi(G) + k\gamma_\delta\,;$$

combined with the orientation of edges incident to $\mathcal{C}$ defined above, we get an orientation D of G with positive out-degrees and $l(D) \leqslant \chi(G) + k\gamma_\delta + 2\gamma_k$. Therefore,

$$l_1(G) \leqslant l(D) = \chi(G) + O(\log n/\log\log\delta),$$

as the lemma states. □

5. Concluding remarks and open problems

Component games on other graphs. For the sake of simplicity, we have presented our results in this paper only for regular graphs, but these stay put under the alternative definition

$$s_b^*(n,d) = \max\{s_b^*(G) : G \text{ is a graph on } n \text{ vertices and } \Delta(G) \leqslant d\}.$$

It would be interesting to consider the component game on families of sparse graphs of unbounded maximum degree. For instance, the Maker–Breaker component game on $\mathcal{G}_{n,p}$ is considered in [14].

Doubly biased games. As shown in Remark 2.3, Theorem 1.6 can be easily extended to the doubly biased game $(m : b)$ for $b/m < (d-2)$; similarly, Remark 3.3 extends Proposition 1.1 to the $(m : b)$ game for $b/m > (d-2)$. However, the strategy presented in Section 3.3 is inadequate in the $(m : (d-2)m)$ game. Indeed, already for $m = 2$, there exists a positive out-degree orientation D of a d-regular graph G and a strategy for Maker to build a connected component S of width $\Omega(d^{h(S)})$ in a $(2 : 2d-4)$ game on G. The key step in Maker's strategy is to merge connected components by claiming two out-edges entering the same vertex, nullifying Claims 3.6 and 3.7, and thus Proposition 3.8 no longer holds.

We believe that not all hope is lost for Breaker.

Conjecture 5.1. *Let G be a d-regular graph on n vertices, where $d \geqslant 3$, and let m be a positive integer. Then, in the $(m : (d-2)m)$ game on G, Breaker can force Maker to build only connected components of size $o(n)$, perhaps polylogarithmic (or even logarithmic) in n.*

Very large components. Recall the proof of Theorem 1.6 in Section 2, which combined the tree strategy with edge expansion via Proposition 2.1. How far can Proposition 2.1 push Maker? Can Maker use it to build a connected component of size $\lfloor n/2 \rfloor$? The following upper bound on the Cheeger constant of regular graphs, due to Alon [1], says that this is only possible when the bias is well below $d/2$.

Theorem 5.2 ([1]). *For every d-regular graph G, $\Psi_E(G, \lfloor n/2 \rfloor) \leqslant d/2 - \Omega(\sqrt{d})$.*

Proposition 2.1 poses a sufficient, but obviously not a necessary, condition for the tree strategy to succeed. It may be possible for Maker to build a connected component of size $\lfloor n/2 \rfloor$ via the tree strategy or some other strategy, without relying on expansion.

Short orientations. The proof of Lemma 1.4 shows that $l_1(G) \leqslant \chi(G) + O(\log n / \log\log d)$. On the other hand, $l_1(G) \geqslant g(G)$ and $l_1(G) \geqslant l_0(G) = \chi(G)$ and thus constructions of d-regular graphs of girth $\Omega(\log_{d-1} n)$ and chromatic number $\Omega(d/\log d)$ (see, *e.g.*, [4, 6, 16]) demonstrate that sometimes $l_1(G) \geqslant \chi(G) + \Omega(\log n/\log d)$.

We suspect the correct behaviour of $l_1(G)$ is actually the lower bound, as the following conjecture states.

Conjecture 5.3. *Let G be a d-regular graph on n vertices, where $d \geqslant 3$. Then,*

$$l_1(G) = \chi(G) + O(\log n/\log d).$$

One can also ask about the value of $l_j(G)$ for $j > 1$.

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