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Strong cohomological rigidity of toric varieties

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Every cohomology ring isomorphism between two non-singular complete toric varieties (respectively, two quasitoric manifolds), with second Betti number 2, is realizable by a diffeomorphism (respectively, homeomorphism).

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1. Introduction

A *toric variety* is a normal algebraic variety of complex dimension ℓ with an action of the algebraic torus $(\mathbb{C}^*)^\ell$ having an open dense orbit. A typical example of a non-singular complete toric variety is the projective space $\mathbb{C}P^\ell$ of complex dimension ℓ with the standard action of $(\mathbb{C}^*)^\ell$.

The *cohomological rigidity problem* for toric varieties poses the question as to whether two non-singular complete toric varieties are diffeomorphic if their cohomology rings are isomorphic as graded rings. Although a cohomology ring is known to be a weak invariant even under homotopy equivalence, no example able to refute the problem has been found yet. On the contrary, many results have been produced in support of the affirmative answer to the problem. One of the remarkable results on this topic is that two non-singular complete toric varieties with second Betti number 2 (or Picard number 2) are diffeomorphic if and only if their cohomology rings are isomorphic as graded rings (see [6]). We refer the reader to a survey paper [4] on this problem.

On the other hand, it is possible to pose a stronger version of the cohomological rigidity problem for toric varieties as follows. Throughout this paper, $H^*(X)$ denotes the integral cohomology ring of a topological space X.

Strong cohomological rigidity problem for toric varieties. *Let M and M' be non-singular complete toric varieties. If φ is a graded ring isomorphism from $H^*(M)$ to $H^*(M')$, does a diffeomorphism capable of inducing the isomorphism φ exist?*

The projective space $\mathbb{C}P^1$ is the only non-singular complete toric variety of complex dimension 1, and it is easy to show that every cohomology ring automorphism is realizable by a diffeomorphism. Note that every toric variety of complex dimension ℓ admits a canonical action of the ℓ-dimensional compact torus $T^\ell = (S^1)^\ell \subset (\mathbb{C}^*)^\ell$. Furthermore, Orlik and Raymond [14] showed that real four-dimensional compact manifolds that admit well-behaved actions of T^2 can be expressed as connected sums of copies of $\mathbb{C}P^2$, $\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$, and such manifolds are classified by their cohomology rings up to diffeomorphism, where $\overline{\mathbb{C}P^2}$ denotes $\mathbb{C}P^2$ with reversed orientation. According to Wall [16], each cohomology ring automorphism of such a manifold of real dimension 4 with second Betti number $\beta_2 \leqslant 10$ is induced by a diffeomorphism. Hence, one can conclude that the answer to the strong cohomological rigidity problem is affirmative for complex two-dimensional non-singular complete toric varieties with $\beta_2 \leqslant 10$.

However, the negative answer is also known. For instance, not every cohomology ring automorphism is realizable by diffeomorphism for complex two-dimensional non-singular complete toric varieties with $\beta_2 > 10$ (see [10]). Furthermore, this implies that the answer to the strong cohomological rigidity problem for toric varieties of arbitrary dimension with sufficiently large β_2 may be negative. Hence, it is reasonable to pose the strong cohomological rigidity problem for toric varieties of arbitrary complex dimension ℓ with small β_2. We note that, because a non-singular complete toric variety with $\beta_2 = 1$ is the complex projective space $\mathbb{C}P^\ell$, and every automorphism of $H^*(\mathbb{C}P^\ell)$ is induced by a diffeomorphism on $\mathbb{C}P^\ell$, the strong cohomological rigidity holds for non-singular complete toric varieties with $\beta_2 = 1$.

The aim of the work presented in this paper is to study the strong cohomological rigidity problem for non-singular complete toric varieties with $\beta_2 = 2$. We show that the problem can be solved by demonstrating that every cohomology ring automorphism of these toric varieties is realizable by a diffeomorphism. Combining our result with the fact that non-singular complete toric varieties with $\beta_2 = 2$ are smoothly classified by their cohomology rings [6], we have the following theorem.

Theorem 1.1. *Every cohomology ring isomorphism between two non-singular complete toric varieties with second Betti number 2 is realizable by a diffeomorphism.*

The notion of a quasitoric manifold was introduced in [9] as a topological analogue of a non-singular projective toric variety. A *quasitoric manifold M* is a real 2ℓ-dimensional compact smooth manifold with a locally standard T^ℓ-action whose orbit space can be identified with an ℓ-dimensional simple polytope P. Every complex ℓ-dimensional non-singular projective toric variety with a restricted action of $(\mathbb{C}^*)^\ell$ to T^ℓ is a quasitoric manifold of real dimension 2ℓ. It is noteworthy to remark that every non-singular complete toric variety with $\beta_2 = 2$ is projective, and hence is a quasitoric manifold. However, not all quasitoric manifolds can be toric varieties. For example, an equivariant connected sum $\mathbb{C}P^2 \# \mathbb{C}P^2$ of two $\mathbb{C}P^2$s with an appropriate T^2-action is a quasitoric manifold with orbit space a square

$\Delta^1 \times \Delta^1$, although it is not a toric variety because it does not admit an almost complex structure. Hence, the class of quasitoric manifolds is larger than that of non-singular projective toric varieties[1].

Note that quasitoric manifolds with $\beta_2 = 2$ are topologically classified by their cohomology rings [8]. In this work, we also investigate strong cohomological rigidity for quasitoric manifolds as follows.

THEOREM 1.2. *Every cohomology ring isomorphism between two quasitoric manifolds with second Betti number* 2 *is realizable by a homeomorphism.*

The remainder of this paper is organized as follows. In § 2, we review the properties of quasitoric manifolds and the topological classification of quasitoric manifolds with $\beta_2 = 2$. In § 3, we introduce the weighted projective space $\mathbb{C}P_a^{n+1}$ and obtain quasitoric manifolds over $\Delta^n \times \Delta^1$ by carrying out an equivariant connected sum

$$\mathbb{C}P_a^{n+1} \# \mathbb{C}P_a^{n+1} \quad \text{or} \quad \mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}}.$$

By using this, we show that every cohomology ring automorphism of such a quasitoric manifold is realizable by a diffeomorphism. In § 4 we discuss the realizability of a cohomology ring automorphism for a non-singular complete toric variety with $\beta_2 = 2$. In § 5, we consider quasitoric manifolds over the product of simplices $\Delta^n \times \Delta^m$ that are not non-singular complete toric varieties. Finally, we complete the proofs of theorems 1.1 and 1.2 in § 6.

2. Quasitoric manifolds with second Betti number 2

In this section, we first review the general properties of quasitoric manifolds from [1, 5, 9]. We partly focus on the case for which the second Betti number is 2. In addition, we recall the classification results in [6, 8].

Let M be a 2ℓ-dimensional quasitoric manifold over an ℓ-dimensional simple polytope P with d facets (codimension-1 faces). Let F be a k-dimensional face of P. Note that for the orbit map $\rho\colon M \to P$ and for a point $x \in \rho^{-1}(F^\circ)$, the isotropy subgroup at x is independent of the choice of x and is a codimension-k subtorus, which is not necessarily a coordinate subtorus, of T^ℓ, where F° denotes the interior of F. If F is a facet of P, then $\rho^{-1}(F)$ is fixed by a circle subgroup of T^ℓ. We define a function $\boldsymbol{\lambda}\colon \{F_1, \dots, F_d\} \to \mathrm{Hom}(S^1, T^\ell) \cong \mathbb{Z}^\ell$, known as the *characteristic function of* M, such that $\boldsymbol{\lambda}(F_i)$ fixes the *characteristic submanifold* $M_i := \rho^{-1}(F_i)$ for $i = 1, \dots, d$, where $\{F_1, \dots, F_d\}$ is the set of facets of P. We note that $\boldsymbol{\lambda}$ satisfies the following *non-singularity condition*:

$$\begin{aligned}&\boldsymbol{\lambda}(F_{i_1}), \dots, \boldsymbol{\lambda}(F_{i_\alpha}) \text{ form a part of an integral basis of } \mathbb{Z}^\ell \\ &\qquad\qquad \text{whenever the intersection } F_{i_1} \cap \cdots \cap F_{i_\alpha} \text{ is non-empty.}\end{aligned} \tag{2.1}$$

Conversely, let us consider a function $\boldsymbol{\lambda}\colon \{F_1, \dots, F_d\} \to \mathbb{Z}^\ell$ satisfying (2.1) and its matrix representation $\Lambda = \big(\boldsymbol{\lambda}(F_1) \cdots \boldsymbol{\lambda}(F_d)\big)$, called a *characteristic matrix*. For a characteristic matrix Λ and a face F of P, we denote by $T(F)$ the subgroup of

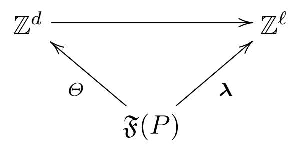
[1] In theory, a non-singular non-projective complete toric variety may fail to be a quasitoric manifold.

T^ℓ corresponding to the unimodular subspace of $\mathbb{Z}^\ell$ spanned by $\boldsymbol{\lambda}(F_{i_1}), \dots, \boldsymbol{\lambda}(F_{i_\alpha})$, where $F = F_{i_1} \cap \cdots \cap F_{i_\alpha}$. For each point $p \in P$, let $F(p)$ denote the face of P containing $p \in P$ in its relative interior. Then, we construct a manifold

$$M(P, \Lambda) := T^\ell \times P/\sim, \tag{2.2}$$

where $(t, p) \sim (s, q)$ if and only if $p = q$ and $t^{-1}s \in T(F(p))$. Then, the standard T^ℓ-action on T^ℓ induces a locally standard T^ℓ-action on $M(P, \Lambda)$, and $M(P, \Lambda)$ is indeed a quasitoric manifold over P whose characteristic function is $\boldsymbol{\lambda}$. Note that the two vectors $\boldsymbol{\lambda}(F_i)$ and $-\boldsymbol{\lambda}(F_i)$ determine the same circle subgroup of T^ℓ. Hence, if Λ' is a matrix obtained from Λ by changing the signs of some columns, then $M(P, \Lambda')$ is equal to $M(P, \Lambda)$.

Set $\mathfrak{F}(P) = \{F_1, \dots, F_d\}$ and define a map $\Theta\colon \mathfrak{F}(P) \to \mathbb{Z}^d$ by $\Theta(F_i) = \boldsymbol{e}_i$, where $\boldsymbol{e}_i$ is the ith standard basis vector. Using Θ, we can construct a T^d-manifold $\mathcal{Z}_P = T^d \times P/\sim$ as in (2.2). Then the dimension of $\mathcal{Z}_P$ is equal to $d + \ell$. The T^d-manifold $\mathcal{Z}_P$ is referred to as a *moment-angle manifold* of P. For instance, $\mathcal{Z}_{\Delta^\ell}$ is the $(2\ell+1)$-dimensional sphere $S^{2\ell+1}$. Note that for two simple polytopes P and Q, we have $\mathcal{Z}_{P\times Q} = \mathcal{Z}_P \times \mathcal{Z}_Q$. Hence, $\mathcal{Z}_{\Delta^n\times\Delta^m} = S^{2n+1} \times S^{2m+1}$. Let us consider the map $\mathbb{Z}^d \to \mathbb{Z}^\ell$ that makes the following diagram commute:



Then this map can be regarded as a homomorphism defined by $\boldsymbol{x} \mapsto \Lambda\boldsymbol{x}$ for every $\boldsymbol{x} \in \mathbb{Z}^d$. Henceforth, this homomorphism is denoted by $\boldsymbol{\lambda}$ unless this is confusing. Let K be the subtorus of T^d corresponding to $\ker \boldsymbol{\lambda}$. Then K acts freely on $\mathcal{Z}_P$, and the orbit space of K on $\mathcal{Z}_P$ is the quasitoric manifold $M(P, \Lambda)$.

Two quasitoric manifolds M and M' over P are said to be *equivalent* if there is a θ-equivariant homeomorphism $f\colon M \to M'$, i.e. $f(t \cdot x) = \theta(t) \cdot f(x)$ for $t \in T^\ell$ and $x \in M$, which covers the identity map on P for some automorphism θ of T^ℓ. Thus, $M(P, \Lambda)$ and $M(P, \Lambda')$ are equivalent if there is an element G in the general linear group $\mathrm{GL}(\ell, \mathbb{Z})$ of rank ℓ over $\mathbb{Z}$ such that $\Lambda' = G\Lambda$.

There is a well-known formula for the cohomology ring of a quasitoric manifold with $\mathbb{Z}$-coefficients. Let M be a quasitoric manifold over P with characteristic matrix $\Lambda = (\lambda_{ij})_{\substack{1\leqslant i\leqslant \ell,\\ 1\leqslant j\leqslant d}}$. Then,

$$H^*(M(P, \Lambda)) = \mathbb{Z}[x_1, \dots, x_d]/\mathcal{I}_P + \mathcal{J}, \tag{2.3}$$

where x_i is the degree-2 cohomology class dual to the characteristic submanifold M_i, $\mathcal{I}_P$ is the homogeneous ideal generated by all square-free monomials $x_{i_1} \cdots x_{i_\alpha}$ such that $F_{i_1} \cap \cdots \cap F_{i_\alpha}$ is empty, and $\mathcal{J}$ is the ideal generated by linear forms $\lambda_{i1}x_1 + \cdots + \lambda_{id}x_d$, $1 \leqslant i \leqslant \ell$. Note that the second Betti number of M is equal to $d - \ell$.

Let M be a quasitoric manifold with second Betti number $\beta_2 = 2$. Then the orbit space of M is a polytope of dimension ℓ with $\ell + 2$ facets. Hence, the orbit space is a product of two simplices $\Delta^n \times \Delta^m$ (see [12]) for some n and m satisfying

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$n+m=\ell$. Let $\{F_1,\dots,F_{n+1}\}$ and $\{F'_1,\dots,F'_{m+1}\}$ be the sets of facets of Δ^n and Δ^m, respectively. Then, each facet of $\Delta^n\times\Delta^m$ is either of the form $F_i\times\Delta^m$ or $\Delta^n\times F'_j$. We may assign an order to the facets of $\Delta^n\times\Delta^m$ as follows:

$$F_1\times\Delta^m,\ \Delta^n\times F'_1,\ F_2\times\Delta^m,\ \dots,\ F_{n+1}\times\Delta^m,\ \Delta^n\times F'_2,\ \dots,\ \Delta^n\times F'_{m+1}.$$

Since the last ℓ facets meet at a vertex, up to equivalence, we may assume that the last ℓ columns of the characteristic matrix Λ corresponding to M form an identity matrix. Furthermore, by the non-singularity condition (2.1), it becomes clear that

$$\Lambda=\begin{pmatrix} -1 & -b_1 & 1 & & & & \\ \vdots & \vdots & & \ddots & & 0 & \\ -1 & -b_n & & & 1 & & \\ -a_1 & -1 & & & & 1 & \\ \vdots & \vdots & & 0 & & & \ddots \\ -a_m & -1 & & & & & 1 \end{pmatrix}, \tag{2.4}$$

where $1-a_jb_i=\pm 1$ for $i=1,\dots,n$ and $j=1,\dots,m$. See [8] for more details. From now on, $M_{\boldsymbol{a},\boldsymbol{b}}$ denotes the quasitoric manifold $M(\Delta^n\times\Delta^m,\Lambda)$ for Λ in (2.4), where $\boldsymbol{a}=(a_1,\dots,a_m)$ and $\boldsymbol{b}=(b_1,\dots,b_n)$. By (2.3), the cohomology ring of $M_{\boldsymbol{a},\boldsymbol{b}}$ with $\mathbb{Z}$-coefficients is

$$H^*(M_{\boldsymbol{a},\boldsymbol{b}})=\mathbb{Z}[x_1,x_2]\Big/\Big\langle x_1\prod_{i=1}^{n}(x_1+b_ix_2),\, x_2\prod_{j=1}^{m}(a_jx_1+x_2)\Big\rangle. \tag{2.5}$$

A *generalized Bott tower* of height h, or an *h-stage generalized Bott tower*, is a sequence

$$B_h \xrightarrow{\pi_h} B_{h-1} \xrightarrow{\pi_{h-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0=\{\text{a point}\}$$

of manifolds $B_i=P(\underline{\mathbb{C}}\oplus\bigoplus_{j=1}^{n_i}\xi_{i,j})$, where $\underline{\mathbb{C}}$ is the trivial line bundle, $\xi_{i,j}$ is a complex line bundle over B_{i-1} for each $i=1,\dots,h$, and $P(\cdot)$ stands for the projectivization. We refer to B_i as an *i-stage generalized Bott manifold*. We remark that a two-stage generalized Bott manifold provided by $n=m=1$ is known as a *Hirzebruch surface* [9]. Note that h-stage generalized Bott manifolds are non-singular projective toric varieties with $\beta_2=h$, and are quasitoric manifolds over a product of h simplices. Moreover, by [5], a quasitoric manifold over a product of simplices has a non-singular complete toric variety structure if and only if it is equivalent to a generalized Bott manifold. Hence, every non-singular complete toric variety with $\beta_2=2$ is a two-stage generalized Bott manifold.

For simplicity, for every complex line bundle L over a base B, the a-times tensor bundle of L is denoted by L^a. If $\boldsymbol{b}=\mathbf{0}$, then $M_{\boldsymbol{a},\mathbf{0}}$ is equivalent to a two-stage generalized Bott manifold $P(\underline{\mathbb{C}}\oplus\bigoplus_{j=1}^{m}\gamma^{a_j})$, where γ is a tautological line bundle over $\mathbb{C}P^n$. Furthermore, in (2.5), the generator x_1 of $H^*(M_{\boldsymbol{a},\mathbf{0}})$ is $-c_1(\gamma)$, the negative of the first Chern class of γ, and the generator x_2 of $H^*(M_{\boldsymbol{a},\mathbf{0}})$ is the negative of the first Chern class of the tautological line bundle over $P(\underline{\mathbb{C}}\oplus\bigoplus_{j=1}^{m}\gamma^{a_j})$. On the other hand, if $\boldsymbol{a}=\mathbf{0}$, then a quasitoric manifold $M_{\mathbf{0},\boldsymbol{b}}$ is equivalent to a two-stage generalized Bott manifold $P(\underline{\mathbb{C}}\oplus\bigoplus_{i=1}^{n}\eta^{b_i})$, where η is the tautological line bundle over $\mathbb{C}P^m$ (see [5]). Similarly, in (2.5), the generator x_2 of $H^*(M_{\mathbf{0},\boldsymbol{b}})$ is

$-c_1(\eta)$ and the generator x_1 of $H^*(M_{\mathbf{0},\boldsymbol{b}})$ is the negative of the first Chern class of the tautological line bundle over $P(\underline{\mathbb{C}} \oplus \bigoplus_{i=1}^n \eta^{b_i})$.

The following theorem provides a smooth classification of two-stage generalized Bott manifolds.

Theorem 2.1 (Choi *et al.* [6]). *Let* $B_2 := P(\underline{\mathbb{C}} \oplus \bigoplus_{j=1}^m \gamma^{a_j})$ *and* $B_2' := P(\underline{\mathbb{C}} \oplus \bigoplus_{j=1}^m \gamma^{a_j'})$, *where* γ *denotes the tautological line bundle over* $B_1 = \mathbb{C}P^n$. *The following are equivalent.*

(1) *There exist* $\varepsilon = \pm 1$ *and* $w \in \mathbb{Z}$ *such that*

$$(1 + \varepsilon w x_1) \prod_{j=1}^{m} (1 + \varepsilon(a_j' + w)x_1) = \prod_{j=1}^{m} (1 + a_j x_1) \in H^*(B_1),$$

where $x_1 = -c_1(\gamma) \in H^2(B_1)$.

(2) *The generalized Bott manifolds* B_2 *and* B_2' *are diffeomorphic.*

(3) *The cohomology rings* $H^*(B_2)$ *and* $H^*(B_2')$ *are isomorphic as graded rings.*

If neither $\boldsymbol{a}$ nor $\boldsymbol{b}$ is the zero vector, then $M_{\boldsymbol{a},\boldsymbol{b}}$ cannot be equivalent to a two-stage generalized Bott manifold. Moreover, from the non-singularity condition of (2.4), either the non-zero entries of $\boldsymbol{a}$ are ± 2 and the non-zero entries of $\boldsymbol{b}$ are ± 1, or the non-zero entries of $\boldsymbol{a}$ are ± 1 and the non-zero entries of $\boldsymbol{b}$ are ± 2.

The following theorem gives a topological classification of quasitoric manifolds with $\beta_2 = 2$.

Theorem 2.2 (Choi *et al.* [8]). *Two quasitoric manifolds with second Betti number 2 are homeomorphic if and only if their integral cohomology rings are isomorphic as graded rings.*

Furthermore, a quasitoric manifold M *with* $\beta_2 = 2$ *that is not equivalent to a generalized Bott manifold is homeomorphic to* $M_{\boldsymbol{s},\boldsymbol{r}}$ *for some non-zero vectors*

$$\boldsymbol{s} := (\underbrace{2, \dots, 2}_{s}, 0, \dots, 0) \in \mathbb{Z}^m \quad \text{and} \quad \boldsymbol{r} := (\underbrace{1, \dots, 1}_{r}, 0, \dots, 0) \in \mathbb{Z}^n,$$

where $s \leqslant \lfloor (m+1)/2 \rfloor$ *and* $r \leqslant \lfloor (n+1)/2 \rfloor$. *In particular, in the case where* $n > 1$ *and* $m > 1$, *all* $M_{\boldsymbol{s},\boldsymbol{r}}$*s are distinct and they cannot be homeomorphic to generalized Bott manifolds. In other cases,* $M_{\boldsymbol{s},\boldsymbol{r}}$ *is homeomorphic to*

(1) $M_{\mathbf{0},1} = \mathbb{C}P^{m+1} \# \overline{\mathbb{C}P^{m+1}}$ *if* $n = 1$ *and* m *is even;*

(2) *either* $M_{\mathbf{0},1}$ *or* $M_{(2,0,\dots,0),1} = \mathbb{C}P^{m+1} \# \mathbb{C}P^{m+1}$ *if* $n = 1$ *and* m *is odd;*

(3) $M_{2,\mathbf{0}}$ *if* n *is even and* $m = 1$; *and*

(4) *either* $M_{2,\mathbf{0}}$ *or* $M_{2,(1,0,\dots,0)}$ *if* n *is odd and* $m = 1$,

where # *denotes an equivariant connected sum and* $\overline{\mathbb{C}P^{m+1}}$ *denotes* $\mathbb{C}P^{m+1}$ *with reversed orientation.*

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Since the orbit space of a quasitoric manifold $M_{\boldsymbol{a},\boldsymbol{b}}$ is $\Delta^n \times \Delta^m$, $M_{\boldsymbol{a},\boldsymbol{b}}$ is a quotient of $\mathcal{Z}_{\Delta^n\times\Delta^m} = S^{2n+1} \times S^{2m+1}$ by an action of T^2. More precisely, let us define a free action of the 2-torus $K_{\boldsymbol{a},\boldsymbol{b}}$ on $S^{2n+1} \times S^{2m+1}$ by

$$\begin{aligned}(t_1,t_2)\cdot((w_1,\dots,w_{n+1}),(z_1,\dots,z_{m+1}))\\ = ((t_1t_2^{b_1}w_1,\dots,t_1t_2^{b_n}w_n,t_1w_{n+1}),(t_1^{a_1}t_2z_1,\dots,t_1^{a_m}t_2z_m,t_2z_{m+1})).\end{aligned}$$

Then the orbit space $S^{2n+1} \times S^{2m+1}/K_{\boldsymbol{a},\boldsymbol{b}}$ is the quasitoric manifold $M_{\boldsymbol{a},\boldsymbol{b}}$.

REMARK 2.3. Let φ be a graded ring automorphism of $H^*(M_{\boldsymbol{a},\boldsymbol{b}})$. Then there is a matrix $(g_{ij})_{i,j=1,2}$ such that $g_{11}g_{22} - g_{12}g_{21} = \pm 1$ and

$$\begin{pmatrix}\varphi(x_1)\\ \varphi(x_2)\end{pmatrix} = \begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}.$$

Hence, $\mathrm{Aut}(H^*(M_{\boldsymbol{a},\boldsymbol{b}}))$ can be regarded as a subgroup of $\mathrm{GL}(2,\mathbb{Z})$.

3. Weighted projective spaces and their connected sums

It is well known that the quasitoric manifold $M_{1,\boldsymbol{0}}$ over $\Delta^n \times \Delta^1$ is the connected sum $\mathbb{C}P^{n+1}\#\overline{\mathbb{C}P^{n+1}}$, and the quasitoric manifold $M_{1,(2,0,\dots,0)}$ is the connected sum $\mathbb{C}P^{n+1}\#\mathbb{C}P^{n+1}$. In this section, we show that quasitoric manifolds $M_{2,\boldsymbol{0}}$ and $M_{2,(1,0,\dots,0)}$ over $\Delta^n \times \Delta^1$ can be expressed as equivariant connected sums of weighted projective spaces, before considering the realizability of the automorphism of $H^*(M_{a,\boldsymbol{b}})$ when $a = 1$ or $a = 2$.

Let us first consider the definitions and properties of weighted projective spaces.

DEFINITION 3.1. Let $\boldsymbol{q} = (q_0,\dots,q_\ell)$ be an $(\ell+1)$-tuple of positive integers, with $\gcd(q_0,\dots,q_\ell) = 1$. The (complex) *weighted projective space* of weight $\boldsymbol{q}$, denoted by $\mathbb{C}P^\ell_{\boldsymbol{q}}$, is defined as the quotient of $\mathbb{C}^{\ell+1}\setminus\{0\}$ by the weighted action of $\mathbb{C}^*$,

$$\zeta\cdot(z_0,\dots,z_\ell) \mapsto (\zeta^{q_0}z_0,\dots,\zeta^{q_\ell}z_\ell).$$

Alternatively, $\mathbb{C}P^\ell_{\boldsymbol{q}}$ can be realized as the quotient of the unit sphere $S^{2\ell+1} \subset \mathbb{C}^{\ell+1}$ by the action of S^1, which is obtained by the restriction of the above action of $\mathbb{C}^*$ to the unit circle S^1.

Note that if $q_0 = \cdots = q_\ell = 1$, then $\mathbb{C}P^\ell_{\boldsymbol{q}}$ is the ordinary projective space $\mathbb{C}P^\ell$. The image of

$$(\mathbb{C}^*)^{\ell+1} \subset \mathbb{C}^{\ell+1}\setminus\{0\}$$

in $\mathbb{C}P^\ell_{\boldsymbol{q}}$ is the quotient $(\mathbb{C}^*)^{\ell+1}/\mathbb{C}^*$, where we regard $\mathbb{C}^*$ as the subgroup of $(\mathbb{C}^*)^{\ell+1}$ via the map $\zeta \mapsto (\zeta^{q_0},\dots,\zeta^{q_\ell})$. Then, the action of $(\mathbb{C}^*)^{\ell+1}$ on $\mathbb{C}^{\ell+1}\setminus\{0\}$ descends to an action of $(\mathbb{C}^*)^\ell \cong (\mathbb{C}^*)^{\ell+1}/\mathbb{C}^*$ on $\mathbb{C}P^\ell_{\boldsymbol{q}}$. Furthermore, $\mathbb{C}P^\ell_{\boldsymbol{q}}$ is a projective toric variety that is not necessarily non-singular.

Note that $\mathbb{C}P^\ell_{\boldsymbol{q}}$ is equipped with an action of the ℓ-dimensional torus $T^\ell_{\boldsymbol{q}} = (S^1)^{\ell+1}/j_{\boldsymbol{q}}(S^1)$, where $j_{\boldsymbol{q}}\colon S^1 \to (S^1)^{\ell+1}$ is the embedding defined by $j_{\boldsymbol{q}}(\zeta) = (\zeta^{q_0},\dots,\zeta^{q_\ell})$. It is well known that $\mathbb{C}P^\ell_{\boldsymbol{q}}$ with this action of $T^\ell_{\boldsymbol{q}}$ is a toric Kähler orbifold; see [11] for more details.

We can also consider real weighted projective spaces as follows.

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Definition 3.2. Let $\boldsymbol{q} := (q_0, \ldots, q_\ell)$ be an $(\ell+1)$-tuple of positive integers, with $\gcd(q_0, \ldots, q_\ell) = 1$. The *real weighted projective space* of weight $\boldsymbol{q}$, denoted by $\mathbb{R}P^\ell_{\boldsymbol{q}}$, is defined as the quotient of $\mathbb{R}^{\ell+1} \setminus \{0\}$ by the weighted action of $\mathbb{R} \setminus \{0\}$,

$$\zeta \cdot (x_0, \ldots, x_\ell) = (\zeta^{q_0} x_0, \ldots, \zeta^{q_\ell} x_\ell).$$

Alternatively, $\mathbb{R}P^\ell_{\boldsymbol{q}}$ is also realized as the quotient of the unit sphere $S^\ell \subset \mathbb{R}^{\ell+1}$ by the action of $\mathbb{Z}_2 = \{\pm 1\}$, which is obtained by the restriction of the above action of $\mathbb{R} \setminus \{0\}$ to $\mathbb{Z}_2$. Hence, if all q_is are odd, then $\mathbb{R}P^\ell_{\boldsymbol{q}}$ is the ordinary real projective space $\mathbb{R}P^\ell$.

Note that the real weighted projective space $\mathbb{R}P^\ell_{\boldsymbol{q}}$ is the fixed set of the conjugation action on the weighted projective space $\mathbb{C}P^\ell_{\boldsymbol{q}}$.

As mentioned in the introduction, a quasitoric manifold is a topological generalization of a non-singular projective toric variety. The notion of a projective toric variety, which is not necessarily non-singular, is also topologically generalized to that of a *quasitoric orbifold*. This generalization was introduced by several authors in, for example, [9, 13, 15].

Suppose that P is a simple polytope of dimension ℓ with d facets $F_1, \ldots, F_d$. By relaxing the unimodality condition (2.1), we can define a *rational characteristic function* as follows. A function $\boldsymbol{\lambda}\colon \{F_1, \ldots, F_d\} \to \mathbb{Z}^\ell$ is called a rational characteristic function if $\boldsymbol{\lambda}(F_{i_1}), \ldots, \boldsymbol{\lambda}(F_{i_\alpha})$ are linearly independent over $\mathbb{Z}$ whenever the intersection $F_{i_1} \cap \cdots \cap F_{i_\alpha}$ is non-empty. Each vector $\boldsymbol{\lambda}(F_i)$ is the *rational characteristic vector* corresponding to F_i. Let K be the subtorus of T^d corresponding to the kernel of $\boldsymbol{\lambda}$. Then K acts on $\mathcal{Z}_P$ with finite isotropy groups. We denote by $Q(P, \boldsymbol{\lambda})$ the orbit space of K on $\mathcal{Z}_P$ and call it the *quasitoric orbifold* corresponding to $(P, \boldsymbol{\lambda})$. If we assign an order to the set of facets of P, the rational characteristic function $\boldsymbol{\lambda}$ can be represented by the *rational characteristic matrix* $\Lambda = \big(\boldsymbol{\lambda}(F_1) \cdots \boldsymbol{\lambda}(F_d)\big)$. For simplicity, we use the notation $Q(P, \Lambda)$ instead of $Q(P, \boldsymbol{\lambda})$ provided that this does not cause confusion.

Note that $\mathcal{Z}_{\Delta^\ell}$ is $S^{2\ell+1}$ and the subtorus K corresponding to the kernel of a rational characteristic function on Δ^ℓ is a circle with a suitable weight. Hence, the weighted projective space $\mathbb{C}P^\ell_{\boldsymbol{q}}$ is a quasitoric orbifold over Δ^ℓ.

In particular, for a positive integer a, let $\boldsymbol{q} := (1, \ldots, 1, a) \in \mathbb{Z}^{n+2}$. We specify $\mathbb{C}P^{n+1}_a := \mathbb{C}P^{n+1}_{\boldsymbol{q}}$ and $T^{n+1}_a := T^{n+1}_{\boldsymbol{q}}$. That is, $\mathbb{C}P^{n+1}_a$ is the quotient of S^{2n+3} by the action of S^1 with the weight $(1, \ldots, 1, a)$,

$$\zeta \cdot (z_0, \ldots, z_{n+1}) \mapsto (\zeta z_0, \ldots, \zeta z_n, \zeta^a z_{n+1}).$$

For each $\boldsymbol{z} = (z_0, \ldots, z_{n+1})$ in S^{2n+3}, the isotropy group of the action of S^1 at $\boldsymbol{z}$ is the identity except if $\boldsymbol{z} = (0, \ldots, 0, 1)$. The isotropy group at $\boldsymbol{z} = (0, \ldots, 0, 1)$ is μ_a, the group of the ath roots of 1. Therefore, the weighted projective space $\mathbb{C}P^{n+1}_a$ has a unique singularity at the point $[0, \ldots, 0, 1]$, modelled on $\mathbb{C}^{n+1}/\mu_a$. Let us find the rational characteristic function $\boldsymbol{\lambda}$ corresponding to $\mathbb{C}P^{n+1}_a$. Note that for each $i = 0, \ldots, n+1$, the suborbifold Q_i of $\mathbb{C}P^{n+1}_a$ described by $z_i = 0$ is fixed by the quotient of the $(i+1)$th coordinate circle of T^{n+2}. We identify T^{n+1}_a and T^{n+1} via the map that sends the $(i+1)$th coordinate circle of T^{n+2} to the ith coordinate circle of T^{n+1} for $i = 1, \ldots, n+1$. Since $[\zeta, 1, \ldots, 1] = [1, \zeta^{-1}, \ldots, \zeta^{-1}, \zeta^{-a}]$ in T^{n+1}_a, the first coordinate circle of T^{n+2} is identified with the circle subgroup of

T^{n+1} generated by $(\zeta^{-1},\dots,\zeta^{-1},\zeta^{-a})$. Moreover, the torus T^{n+1} acts on $\mathbb{C}P^{n+1}_a$ as follows:

$$(t_1,\dots,t_{n+1})\cdot[z_0,\dots,z_{n+1}]=[z_0,t_1z_1,\dots,t_{n+1}z_{n+1}].$$

Then, for each $i=1,\dots,n+1$, the suborbifold Q_i is fixed by the ith coordinate circle of T^{n+1}, and Q_0 is fixed by the circle generated by $(-1,\dots,-1,-a)$ in $\mathbb{Z}^{n+1}=\operatorname{Hom}(S^1,T^{n+1})$. Let us denote by F_i the facet of Δ^{n+1} corresponding to Q_i. Then $\boldsymbol{\lambda}(F_0)=(-1,\dots,-1,-a)$ and $\boldsymbol{\lambda}(F_i)=\boldsymbol{e}_i$ for $i=1,\dots,n$. Hence, the rational characteristic matrix corresponding to $\mathbb{C}P^{n+1}_a$ is

$$\Lambda_a := \big(\boldsymbol{\lambda}(F_0)\quad \boldsymbol{\lambda}(F_1)\quad\cdots\quad\boldsymbol{\lambda}(F_{n+1})\big)=\begin{pmatrix}-1 & 1 & & & \\ -1 & & 1 & & \\ \vdots & & & \ddots & \\ -1 & & & & 1 & \\ -a & & & & & 1\end{pmatrix}. \tag{3.1}$$

In particular, a fan[2] of $\mathbb{C}P^{n+1}_a$ as a projective toric variety is obtained by taking the cones generated by all proper subsets of

$$\{-\boldsymbol{e}_1-\cdots-\boldsymbol{e}_n-a\boldsymbol{e}_{n+1},\boldsymbol{e}_1,\dots,\boldsymbol{e}_{n+1}\}.$$

On the other hand, consider $(n+1)\times(n+2)$ matrices of the form

$$\Lambda=\begin{pmatrix}\pm1 & \pm1 & & & \\ \pm1 & & \pm1 & & \\ \vdots & & & \ddots & \\ \pm1 & & & & \pm1 & \\ \pm a & & & & & \pm1\end{pmatrix}.$$

Then Λ is a rational characteristic matrix on Δ^{n+1}. Because a row operation of Λ whose determinant is ±1 corresponds to an automorphism of T^{n+1}, and changing the signs of column vectors does not affect the subgroup generated by these column vectors, it is clear that $Q(\Delta^{n+1},\Lambda)$ is equivalent to the weighted projective space $\mathbb{C}P^{n+1}_a$ with a suitable action of T^{n+1}.

Now, let us consider a smooth manifold $\mathbb{C}P^{n+1}_a\#\overline{\mathbb{C}P^{n+1}_a}$ obtained by the (equivariant) connected sum of $\mathbb{C}P^{n+1}_a$ and $\overline{\mathbb{C}P^{n+1}_a}$ at their singular points. More precisely, let D' and D'' be closed balls in $\mathbb{C}P^{n+1}_a$ and $\overline{\mathbb{C}P^{n+1}_a}$, respectively, containing the singular point that is a suborbifold with a boundary diffeomorphic to $D^{2(n+1)}/\mu_a$, where $D^{2(n+1)}$ is the closed unit ball in $\mathbb{C}^{n+1}$. By deleting the interiors of the balls D' in $\mathbb{C}P^{n+1}_a$ and D'' in $\overline{\mathbb{C}P^{n+1}_a}$, and attaching the resulting punctured manifolds $\mathbb{C}P^{n+1}_a\setminus D'^{\circ}$ and $\overline{\mathbb{C}P^{n+1}_a}\setminus D''^{\circ}$ to each other by a diffeomor-

[2] A *fan* is a collection Σ of cones in $\mathbb{R}^\ell$ such that each face of a cone in Σ is also a cone in Σ, and the intersection of two cones in Σ is a face of each. It is well known that there is a one-to-one correspondence between fans in $\mathbb{R}^\ell$ and toric varieties of complex dimension ℓ.

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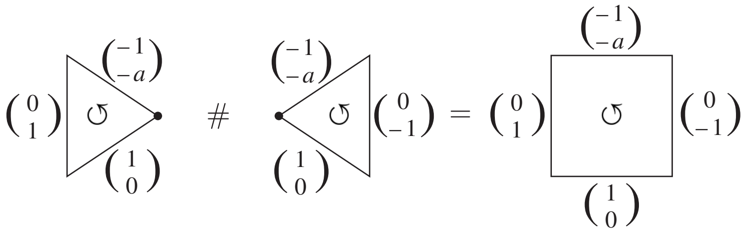

Figure 1. Illustration of $\mathbb{C}P^2_a\#\overline{\mathbb{C}P^2_a}$.

phism $\partial D' \cong S^{2n+1}/\mu_a \cong \partial D''$, we obtain a smooth manifold $\mathbb{C}P^{n+1}_a\#\overline{\mathbb{C}P^{n+1}_a}$.[3] This can be described in terms of toric topological language. Let us review the construction of the equivariant connected sum for quasitoric orbifolds $Q(P',\boldsymbol{\lambda}')$ and $Q(P'',\boldsymbol{\lambda}'')$. Let $v' = F'_1 \cap \cdots \cap F'_\ell$ and $v'' = F''_1 \cap \cdots \cap F''_\ell$ be the vertices of P' and P'', respectively, such that $\boldsymbol{\lambda}'(F'_i) = \boldsymbol{\lambda}''(F''_i)$ for $i = 1,\dots,\ell$. Then the *connected sum* of P' and P'' with respect to v' and v'' is combinatorially equivalent to the polytope formed by deleting the small balls of v' and v'' of P' and P'', respectively, and gluing together the resulting neighbourhoods. The hyperplanes containing F'_i and F''_i, respectively, must be attached to another for $i = 1,\dots,\ell$. When the choices of v' and v'' with the order of facets are clear, the connected sum is denoted by $P'\#P''$. Indeed, $P'\#P''$ is combinatorially equivalent to a simple polytope due to [2]. The *equivariant connected sum* of $Q(P',\boldsymbol{\lambda}')$ and $Q(P'',\boldsymbol{\lambda}'')$ is the quasitoric orbifold corresponding $Q(P'\#P'',\boldsymbol{\lambda})$, where $\boldsymbol{\lambda}$ is a characteristic function naturally defined by $\boldsymbol{\lambda}'$ and $\boldsymbol{\lambda}''$. Indeed, $\mathbb{C}P^{n+1}_a\#\overline{\mathbb{C}P^{n+1}_a}$ is the equivariant connected sum of two quasitoric orbifolds. Near the singular points $[0,\dots,0,1] \in \mathbb{C}P^{n+1}_a$ and $[0,\dots,0,1] \in \overline{\mathbb{C}P^{n+1}_a}$, both have the same singularity and the same characteristic vectors: $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$, and $-\boldsymbol{e}_1 - \cdots - \boldsymbol{e}_n - a\boldsymbol{e}_{n+1}$. Hence, we can carry out an equivariant connected sum $\mathbb{C}P^{n+1}_a\#\overline{\mathbb{C}P^{n+1}_a}$ by removing the singular points as in figure 1.

Note that the orientation of $\mathbb{C}P^{n+1}_a$ is associated with the orientation of $\Delta^{n+1} \subset \mathbb{R}^{n+1}$ and the columns of the characteristic matrix are determined up to sign. Then the characteristic matrix of $\mathbb{C}P^{n+1}_a\#\overline{\mathbb{C}P^{n+1}_a}$ is

$$\begin{pmatrix} -1 & & 1 & & & \\ -1 & & & 1 & & \\ \vdots & & & & \ddots & \\ -1 & & & & & 1 & \\ -a & -1 & & & & & 1 \end{pmatrix}. \tag{3.2}$$

Now, consider a two-stage generalized Bott manifold $P(\underline{\mathbb{C}} \oplus \gamma^a)$, where γ is the tautological line bundle over $\mathbb{C}P^n$. The relationship between the weighted projective

[3] Note that if we do a connected sum at a non-singular point, then the connected sum $\mathbb{C}P^{n+1}_a\#\overline{\mathbb{C}P^{n+1}_a}$ or $\mathbb{C}P^{n+1}_a\#\mathbb{C}P^{n+1}_a$ still has singular points. In this paper, we consider only the case in which both $\mathbb{C}P^{n+1}_a\#\overline{\mathbb{C}P^{n+1}_a}$ and $\mathbb{C}P^{n+1}_a\#\mathbb{C}P^{n+1}_a$ are smooth manifolds that are the connected sums at singular points.

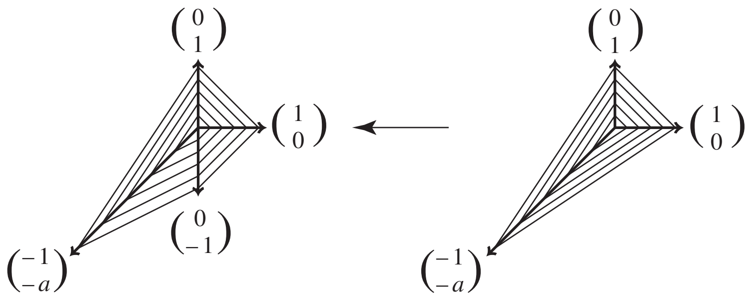

Figure 2. A Hirzebruch surface is a blow-up from $\mathbb{C}P^2_{(1,1,a)}$.

space $\mathbb{C}P_a^{n+1}$ and the projective bundle $P(\underline{\mathbb{C}} \oplus \gamma^a)$ over $\mathbb{C}P^n$ is provided by the following lemma.

Lemma 3.3. *Let $a > 1$. Then a projective bundle $P(\underline{\mathbb{C}} \oplus \gamma^a)$ over $\mathbb{C}P^n$ is the blow-up of $\mathbb{C}P_a^{n+1}$ at the singular point.*

Proof. Each one-dimensional cone in a fan of $P(\underline{\mathbb{C}} \oplus \gamma^a)$ is generated by one of the elements in

$$S := \{-\boldsymbol{e}_1 - \cdots - \boldsymbol{e}_n - a\boldsymbol{e}_{n+1}, -\boldsymbol{e}_{n+1}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}\}.$$

We obtain a fan of $\mathbb{C}P_a^{n+1}$ by taking the cones generated by all proper subsets of the set $S \setminus \{-\boldsymbol{e}_{n+1}\}$; see figure 2. Note that the cone generated by the set $\{-\boldsymbol{e}_1 - \cdots - \boldsymbol{e}_n - a\boldsymbol{e}_{n+1}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ corresponds to the singular point $[0, \ldots, 0, 1]$ of $\mathbb{C}P_a^{n+1}$, and $-a\boldsymbol{e}_{n+1} = (-\boldsymbol{e}_1 - \cdots - \boldsymbol{e}_n - a\boldsymbol{e}_{n+1}) + \boldsymbol{e}_1 + \cdots + \boldsymbol{e}_n$. Hence, $P(\underline{\mathbb{C}} \oplus \gamma^a)$ is the blow-up of $\mathbb{C}P_a^{n+1}$ at the singular point. See figure 2. □

Lemma 3.4.

(1) *For $a > 0$, $M_{a,\boldsymbol{0}}$ is homeomorphic to $\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}}$.*

(2) *Let*

$$\boldsymbol{r} = (\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0) \in \mathbb{Z}^n.$$

Then $M_{2,\boldsymbol{r}}$ is homeomorphic to

(a) $\mathbb{C}P_2^{n+1} \# \overline{\mathbb{C}P_2^{n+1}}$ *if r is even,*

(b) $\mathbb{C}P_2^{n+1} \# \mathbb{C}P_2^{n+1}$ *if r is odd.*

Proof. We make note of the fact that a blow-up of $\mathbb{C}P_a^{n+1}$ at the singular point is indeed $\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}}$. By comparing their characteristic functions, it follows from lemma 3.3 that $M_{a,\boldsymbol{0}}$ is equivalent to $\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}}$. Hence, statement (1) is proved.

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Now let us prove statement (2). Then, as in (2.4), the characteristic matrix of $M_{2,\boldsymbol{r}}$ is

$$\Lambda = \begin{pmatrix} -1 & -1 & 1 & & & & \\ \vdots & \vdots & & \ddots & & & \\ -1 & -1 & & & 1 & & & \\ -1 & 0 & & & & 1 & & \\ \vdots & \vdots & & & & & \ddots & \\ -1 & 0 & & & & & & 1 & \\ -2 & -1 & & & & & & & 1 \end{pmatrix},$$

where the order in $\mathfrak{F}(\Delta^n \times \Delta^1)$ is

$$F_1 \times \Delta^1,\ \Delta^n \times F_1',\ F_2 \times \Delta^1,\ \dots,\ F_{n+1} \times \Delta^1,\ \Delta^n \times F_2'.$$

We note that its orbit space $\Delta^n \times \Delta^1$ can be identified with $\Delta^{n+1} \# \Delta^{n+1}$. The columns of Λ corresponding to the (ordered) subset $\{F_1 \times \Delta^1, \dots, F_{n+1} \times \Delta^1, \Delta^n \times F_1'\}$ form the rational characteristic matrix $\tilde{\Lambda}$ on Δ^{n+1},

$$\tilde{\Lambda} = \begin{pmatrix} -1 & 1 & & & & & -1 \\ \vdots & & \ddots & & & & \vdots \\ -1 & & & 1 & & & -1 \\ -1 & & & & 1 & & 0 \\ \vdots & & & & & \ddots & \vdots \\ -1 & & & & & 1 & 0 \\ -2 & & & & & & -1 \end{pmatrix}.$$

Moreover, the columns of Λ corresponding to the subset $\{F_1 \times \Delta^1, \dots, F_{n+1} \times \Delta^1, \Delta^n \times F_2'\}$ also form the rational characteristic matrix Λ_2 on Δ^{n+1},

$$\Lambda_2 = \begin{pmatrix} -1 & 1 & & & & & \\ \vdots & & \ddots & & & & \\ -1 & & & 1 & & & \\ -1 & & & & 1 & & \\ \vdots & & & & & \ddots & \\ -1 & & & & & & 1 & \\ -2 & & & & & & & 1 \end{pmatrix}.$$

Note that the first $n+1$ columns of $\tilde{\Lambda}$ are equal to those of Λ_2, and they have the determinant 2. Furthermore, the characteristic matrix corresponding to

$$Q(\Delta^{n+1}, \tilde{\Lambda}) \# Q(\Delta^{n+1}, \Lambda_2)$$

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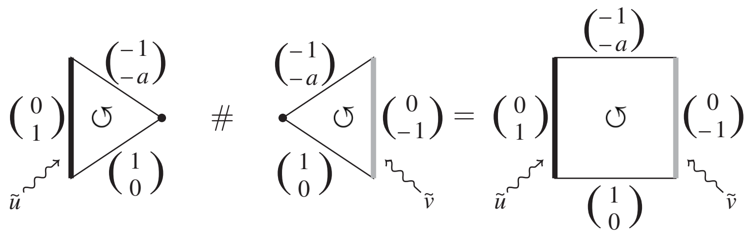

Figure 3. $\tilde{u}$ and $\tilde{v}$ in $H_2(\mathbb{C}P^2_a \# \overline{\mathbb{C}P^2_a})$.

coincides with Λ. Hence, $M_{2,\boldsymbol{r}}$ is equivalent to the equivariant connected sum of $Q(\Delta^{n+1}, \tilde{\Lambda})$ and $Q(\Delta^{n+1}, \Lambda_2)$ at the singular points. Note that

$$\tilde{\Lambda} = \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & & \vdots \\ & & 1 & & & & 1 \\ & & & 1 & & & 0 \\ & & & & \ddots & & \vdots \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & & & & & 0 \\ \vdots & & \ddots & & & & \vdots \\ 1 & & & 1 & & & 0 \\ -1 & & & & 1 & & 0 \\ \vdots & & & & & \ddots & \vdots \\ -1 & & & & & 1 & 0 \\ -2 & & & & & & -1 \end{pmatrix}. \qquad (3.3)$$

Since the first matrix on the right-hand side of (3.3) is in $\mathrm{GL}(n+1, \mathbb{Z})$, $Q(\Delta^{n+1}, \tilde{\Lambda})$ is equivalent to $\mathbb{C}P^{n+1}_2$. Therefore, $M_{2,\boldsymbol{r}}$ is equivalent to either

$$\mathbb{C}P^{n+1}_2 \# \overline{\mathbb{C}P^{n+1}_2} \quad \text{or} \quad \mathbb{C}P^{n+1}_2 \# \mathbb{C}P^{n+1}_2.$$

In particular, according to [8, theorem 5.5],

- if n is even, $M_{2,\boldsymbol{r}} = M_{2,\boldsymbol{0}} = \mathbb{C}P^{n+1}_2 \# \overline{\mathbb{C}P^{n+1}_2} = \mathbb{C}P^{n+1}_2 \# \mathbb{C}P^{n+1}_2$;
- if n is odd,
  - $M_{2,\boldsymbol{r}} = M_{2,\boldsymbol{0}} = \mathbb{C}P^{n+1}_2 \# \overline{\mathbb{C}P^{n+1}_2}$ for even r, and
  - $M_{2,\boldsymbol{r}} = M_{2,(1,0,\dots,0)} = \mathbb{C}P^{n+1}_2 \# \mathbb{C}P^{n+1}_2$ for odd r.

This proves statement (2). □

Now, let us consider the cohomology ring of $\mathbb{C}P^{n+1}_a \# \overline{\mathbb{C}P^{n+1}_a}$. Letting $X := \mathbb{C}P^{n+1}_a \setminus D^\circ$ and $Y := \overline{\mathbb{C}P^{n+1}_a} \setminus D'^\circ$, we can see that $X \cup Y = \mathbb{C}P^{n+1}_a \# \mathbb{C}P^{n+1}_a$ and $X \cap Y = \partial D = \partial D'$, where D° and D'° are the interiors of the closed balls D and D' containing the singular point in $\mathbb{C}P^{n+1}_a$ and $\overline{\mathbb{C}P^{n+1}_a}$, respectively.

Let $\tilde{u}$ and $\tilde{v}$ be the elements of $H_{2n}(X)$ and $H_{2n}(Y)$ represented by the submanifolds $\mathbb{C}P^n = \{z_{n+1} = 0\} \subset X$ and $\overline{\mathbb{C}P^n} = \{z_{n+1} = 0\} \subset Y$, respectively. Thus, $\tilde{u}$ and $\tilde{v}$ are elements of $H_{2n}(\mathbb{C}P^{n+1}_a \# \mathbb{C}P^{n+1}_a)$. See figure 3.

Recall the characteristic matrix (3.2) of $\mathbb{C}P^{n+1}_a \# \mathbb{C}P^{n+1}_a$. Then the cohomology ring is

$$H^*(\mathbb{C}P^{n+1}_a \# \overline{\mathbb{C}P^{n+1}_a}) = \mathbb{Z}[x_1, x_2] / \langle x_1^{n+1}, x_2(ax_1 + x_2) \rangle,$$

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where x_1 and x_2 correspond to the first and the second columns of (3.2), respectively. Because $\tilde{u}$ and $\tilde{v}$ represent the characteristic submanifolds associated with $\Delta^n \times F_1'$ and $\Delta^n \times F_2'$, through Poincaré duality, $\tilde{u}$ and $\tilde{v}$ correspond to $u = ax_1 + x_2$ and $v = x_2$ in $H^2(\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}})$, where the identities originate from $\mathcal{J}$ in (2.3).

Now assume that $ab = 2$. By lemma 3.4,

$$\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}} \cong M_{a,\mathbf{0}} \quad \text{and} \quad \mathbb{C}P_a^{n+1} \# \mathbb{C}P_a^{n+1} \cong M_{a,(b,0,\dots,0)}.$$

Hence, by using the cohomology formula (2.5), we compute their cohomology rings as follows:

$$H^*(\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}}) = \mathbb{Z}[x_1, x_2]/\langle x_1^{n+1}, x_2(ax_1 + x_2)\rangle,$$
$$H^*(\mathbb{C}P_a^{n+1} \# \mathbb{C}P_a^{n+1}) = \mathbb{Z}[x_1, x_2]/\langle x_1^{n}(x_1 + bx_2), x_2(ax_1 + x_2)\rangle.$$

Note that $u = ax_1 + x_2$ and $v = x_2$ correspond to the $(n+1)$th and $(n+3)$th columns of the characteristic matrix

$$\begin{pmatrix} -1 & 0 & 1 & & & \\ -1 & 0 & & 1 & & \\ \vdots & \vdots & & & \ddots & \\ -1 & 0 & & & & 1 & \\ -a & -1 & & & & & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & -b & 1 & & & \\ -1 & 0 & & 1 & & \\ \vdots & \vdots & & & \ddots & \\ -1 & 0 & & & & 1 & \\ -a & -1 & & & & & 1 \end{pmatrix}.$$

Proposition 3.5. *Assume that $a = 1$ or $a = 2$. Then the ring automorphism groups* $\mathrm{Aut}(H^*(\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}}))$ *and* $\mathrm{Aut}(H^*(\mathbb{C}P_a^{n+1} \# \mathbb{C}P_a^{n+1}))$ *are realizable by diffeomorphisms.*

Proof. If $n = 1$, then $\mathbb{C}P_a^2 \# \overline{\mathbb{C}P_a^2}$ is a Hirzebruch surface, and $\mathbb{C}P_2^2 \# \mathbb{C}P_2^2$ is diffeomorphic to $\mathbb{C}P^2 \# \mathbb{C}P^2$. According to [3] or [16], all ring automorphisms on their cohomology rings are realizable by diffeomorphisms. Henceforth, let us assume that $n > 1$.

We first compute the ring automorphism groups of

$$H^*(\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}}) \quad \text{and} \quad H^*(\mathbb{C}P_a^{n+1} \# \mathbb{C}P_a^{n+1})$$

as subgroups of $\mathrm{GL}(2, \mathbb{Z})$ (see remark 2.3). For each case, there is only one relation $x_2(ax_1 + x_2) = 0$ such that a product of two degree-2 elements is zero up to scalar multiplication. Accordingly, an automorphism should send $\{x_2, ax_1 + x_2\}$ to $\{x_2, ax_1 + x_2\}$ up to sign. Hence, there are at most eight automorphisms.

Let $u = ax_1 + x_2$ and $v = x_2$. Then we have

$$u^{n+1} = (-v)^{n+1} \qquad \text{in } H^*(\mathbb{C}P_a^{n+1} \# \overline{\mathbb{C}P_a^{n+1}})$$

and

$$u^{n+1} = (-1)^n v^{n+1} \quad \text{in } H^*(\mathbb{C}P_a^{n+1} \# \mathbb{C}P_a^{n+1}).$$

If n is even, neither $H^*(\mathbb{C}P_a^{n+1}\#\overline{\mathbb{C}P_a^{n+1}})$ nor $H^*(\mathbb{C}P_a^{n+1}\#\mathbb{C}P_a^{n+1})$ has any automorphism $(u,v)\mapsto\pm(u,-v)$. Hence,

$$\begin{aligned}\operatorname{Aut}(H^*(\mathbb{C}P_a^{n+1}\#\overline{\mathbb{C}P_a^{n+1}})) &= \left\{ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -a & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ a & 1\end{pmatrix} \right\}\\ &= \operatorname{Aut}(H^*(\mathbb{C}P_a^{n+1}\#\mathbb{C}P_a^{n+1}))\\ &\cong (\mathbb{Z}_2)^2.\end{aligned}$$

If n is odd, then

$$\begin{aligned}&\operatorname{Aut}(H^*(\mathbb{C}P_a^{n+1}\#\overline{\mathbb{C}P_a^{n+1}}))\\ &\quad= \left\{ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 2/a\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & -2/a\\ 0 & 1\end{pmatrix},\right.\\ &\qquad\left. \begin{pmatrix}-1 & 0\\ a & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -a & -1\end{pmatrix}, \begin{pmatrix}1 & 2/a\\ -1 & -1\end{pmatrix}, \begin{pmatrix}-1 & -2/a\\ a & 1\end{pmatrix} \right\}\\ &\quad= \operatorname{Aut}(H^*(\mathbb{C}P_a^{n+1}\#\mathbb{C}P_a^{n+1}))\\ &\quad\cong (\mathbb{Z}_2)^3.\end{aligned}$$

We consider an involution s on $\mathbb{C}P_a^{n+1}$ defined by

$$s\colon [z_0,\dots,z_{n+1}]\mapsto[\overline{z_0},\dots,\overline{z_{n+1}}].$$

For odd n, we consider another involution t defined by

$$t\colon [z_0,\dots,z_{n+1}]\mapsto[-z_0,\dots,-z_{k-1},z_k,\dots,z_{n+1}],$$

where $k=(n+1)/2$. Observe that

(1) the involution s reverses the orientation of the submanifold $\mathbb{C}P^n=\{z_{n+1}=0\}$, and fixes the real weighted projective space $\mathbb{R}P_a^{n+1}$;

(2) the fixed-point set of the involution t is the disjoint union of $\{z_k=\cdots=z_{n+1}=0\}=\mathbb{C}P^{k-1}$ and $\{z_0=\cdots=z_{k-1}=0\}=\mathbb{C}P_a^k$;

(3) the point $[0,\dots,0,1]$ is fixed by both s and t.

Note that if $a=1$, then $[0,\dots,0,1]$ is a smooth point. If $a=2$, both $[0,\dots,0,1]\in\mathbb{R}P_a^{n+1}$ and $[0,\dots,0,1]\in\mathbb{C}P_a^k$ have the same singularity, that is, $[0,\dots,0,1]\in\mathbb{R}P_a^{n+1}$ is locally modelled by $\mathbb{R}^{n+1}/\mu_2$, and $[0,\dots,0,1]\in\mathbb{C}P_a^k$ is locally modelled by $\mathbb{C}^k/\mu_2$.

Type 1. We consider the involution s on both $\mathbb{C}P_a^{n+1}$ and $\overline{\mathbb{C}P_a^{n+1}}$. Take the equivariant connected sum of $\mathbb{C}P_a^{n+1}$ and $\overline{\mathbb{C}P_a^{n+1}}$ at $[0,\dots,0,1]$. The resulting involution on $\mathbb{C}P_a^{n+1}\#\overline{\mathbb{C}P_a^{n+1}}$ then sends (u,v) to $(-u,-v)$.

Type 2. We consider the involution s on $\mathbb{C}P_a^{n+1}$ and t on $\overline{\mathbb{C}P_a^{n+1}}$. Take the equivariant connected sum of $\mathbb{C}P_a^{n+1}$ and $\overline{\mathbb{C}P_a^{n+1}}$ at $[0,\dots,0,1]$. The resulting involution on $\mathbb{C}P_a^{n+1}\#\overline{\mathbb{C}P_a^{n+1}}$ then sends (u,v) to $(-u,v)$.

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Type 3. Let D' (respectively, D'') be a suborbifold of $\mathbb{C}P_a^{n+1}$ (respectively, $\overline{\mathbb{C}P_a^{n+1}}$) with a boundary that is diffeomorphic to $D^{2(n+1)}/\mu_a$, where $D^{2(n+1)}$ is the closed unit ball. Then $\mathbb{C}P_a^{n+1}\#\overline{\mathbb{C}P_a^{n+1}}$ is obtained by deleting the interiors of the suborbifolds D' and D'' containing $[0,\dots,0,1]$ from $\mathbb{C}P_a^{n+1}$ and $\overline{\mathbb{C}P_a^{n+1}}$ and gluing the resulting boundaries $\partial D'$ and $\partial D''$. Hence, $\mathbb{C}P_a^{n+1}\#\overline{\mathbb{C}P_a^{n+1}}$ admits a reflection about $\partial D' = \partial D''$ that maps

$$\mathbb{C}P_a^{n+1} \setminus D' \quad \text{to} \quad \overline{\mathbb{C}P_a^{n+1}} \setminus D''.$$

This reflection sends (u, v) to (v, u).

Then type 1, type 2 and type 3 correspond to

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -1 & -2/a \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ a & 1 \end{pmatrix}$$

respectively.

Combining the diffeomorphisms of the three types above, it becomes possible to realize every element of

$$\mathrm{Aut}(H^*(\mathbb{C}P_a^{n+1}\#\overline{\mathbb{C}P_a^{n+1}}))$$

for $a = 1, 2$.

Because we can use the same process for $\mathbb{C}P_a^{n+1}\#\mathbb{C}P_a^{n+1}$, we can realize every element of $\mathrm{Aut}(H^*(\mathbb{C}P_a^{n+1}\#\mathbb{C}P_a^{n+1}))$ for $a = 1, 2$. □

Remark 3.6. In general, if a is odd, then $[0,\dots,0,1] \in \mathbb{R}P_a^{n+1}$ is smooth, and if a is even, then $[0,\dots,0,1] \in \mathbb{R}P_a^{n+1}$ has an isotropy group μ_2. On the other hand, $[0,\dots,0,1] \in \mathbb{C}P_a^k$ has an isotropy group μ_a. Hence, type 2 is only possible when $a = 1$ or 2.

Combining lemma 3.4 and proposition 3.5, we obtain the following.

Corollary 3.7. *Assume that $a = 1$ or 2. Then for a quasitoric manifold $M_{a,\boldsymbol{b}}$, every automorphism of $H^*(M_{a,\boldsymbol{b}})$ is realizable by a homeomorphism.*

4. Two-stage generalized Bott manifolds

In this section, we restrict our attention to two-stage generalized Bott manifolds. We show that every cohomology ring automorphism of a two-stage generalized Bott manifold is realizable by a diffeomorphism. We prepare the following two lemmas.

Lemma 4.1 (Choi *et al*. [6, lemma 5.2]). *Let E and E' be Whitney sums of complex line bundles over the complex projective space $\mathbb{C}P^n$ of the same dimension. If E and E' have the same total Chern classes, then E and E' are isomorphic.*

Lemma 4.2 (Choi *et al*. [6, lemma 6.2]). *Let $M = P(\underline{\mathbb{C}} \oplus \bigoplus_{i=1}^m \gamma^{a_i})$ and $M' = P(\underline{\mathbb{C}} \oplus \bigoplus_{i=1}^m \gamma^{a_i'})$ be projective bundles over the complex projective space $B = \mathbb{C}P^n$. Assume that m is greater than 1. Then every cohomology ring isomorphism $\varphi\colon H^*(M) \to H^*(M')$ preserves the subring $H^*(B)$ unless M is $\mathbb{C}P^n \times \mathbb{C}P^m$.*

Now we can show the realizability of a cohomology ring automorphism of a two-stage generalized Bott manifold.

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Proposition 4.3. *Let $E := \underline{\mathbb{C}} \oplus \bigoplus_{j=1}^{m} \gamma^{a_j}$ be the Whitney sum of complex line bundles over $\mathbb{C}P^n$. Then every graded ring automorphism of $H^*(P(E))$ is induced by a diffeomorphism.*

Proof. Since every cohomology ring automorphism of a product of complex projective spaces is induced by a diffeomorphism [7], we may assume that $P(E)$ is a non-trivial fibre bundle.

Note that $P(E)$ is a Hirzebruch surface if $n = m = 1$. Each cohomology ring automorphism of a Hirzebruch surface is realizable by a diffeomorphism by [3] or [16].

If $m = 1$ and $1 \leqslant a_1 \leqslant 2$, then $P(E)$ is diffeomorphic to $\mathbb{C}P^{n+1}_{a_1} \# \overline{\mathbb{C}P^{n+1}_{a_1}}$. Hence, by proposition 3.5, every automorphism of $H^*(P(E))$ is realizable by a diffeomorphism.

Then the remaining cases are (i) $m > 1$ and (ii) $m = 1$, $n > 1$, and $a_1 > 2$. Note that

$$\begin{aligned} H^*(P(E)) &= H^*(\mathbb{C}P^n)[x_2] \Big/ \Big\langle x_2 \prod_{j=1}^{m} (a_j x_1 + x_2) \Big\rangle \\ &= \mathbb{Z}[x_1, x_2] \Big/ \Big\langle x_1^{n+1}, x_2 \prod_{j=1}^{m} (a_j x_1 + x_2) \Big\rangle, \end{aligned}$$

where $x_1 = -c_1(\gamma) \in H^2(\mathbb{C}P^n) \subset H^2(P(E))$ and $x_2 \in H^2(P(E))$ is the negative of the first Chern class of the tautological line bundle over $P(E)$. We first claim that every cohomology ring automorphism of $H^*(P(E))$ preserves the subring $H^*(\mathbb{C}P^n)$ in each case.

In the first case, i.e. if $m > 1$, by lemma 4.2, every automorphism of $H^*(P(E))$ preserves the subring $H^*(\mathbb{C}P^n)$.

Now, we consider the second case, i.e. $m = 1$, $n > 1$, and $a_1 > 2$. Let φ be a ring automorphism of $H^*(P(E))$. Since $n > 1$, there is only one relation $x_2(a_1 x_1 + x_2) = 0$ such that a product of two degree-2 elements is zero up to scalar multiplication. Thus, φ should send $\{x_2, a_1 x_1 + x_2\}$ to $\{x_2, a_1 x_1 + x_2\}$ up to sign. Suppose that $\varphi(a_1 x_1 + x_2) = \pm(a_1 x_1 + x_2)$ and $\varphi(x_2) = \mp x_2$. Then $\varphi(x_1) = \pm(x_1 + (2/a_1)x_2)$. Because $a_1 > 2$, φ cannot be an isomorphism. Therefore, there are only four automorphisms of $H^*(P(\underline{\mathbb{C}} \oplus \gamma^{a_1}))$, those being as follows:

$$\operatorname{Aut}(H^*(P(E))) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a_1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ a_1 & 1 \end{pmatrix} \right\}.$$

Hence, in each case, every ring automorphism of $H^*(P(E))$ preserves the subring $H^*(\mathbb{C}P^n)$, which proves the claim.

Let φ be a ring automorphism of $H^*(P(E))$. By the above claim, $\varphi(x_1) = \pm x_1$. Since every automorphism of $H^*(\mathbb{C}P^n)$ is induced by a diffeomorphism, we may assume that $\varphi(x_1) = x_1$.

We write $\varphi(x_2) = \varepsilon x_2 + A x_1$, where $\varepsilon = \pm 1$ and $A \in \mathbb{Z}$. Then the map φ lifts to a grading preserving isomorphism $\bar{\varphi}\colon \mathbb{Z}[x_1, x_2] \to \mathbb{Z}[x_1, x_2]$ with $\bar{\varphi}(\tilde{\mathcal{J}}) = \tilde{\mathcal{J}}$, where $\tilde{\mathcal{J}} \subset \mathbb{Z}[x_1, x_2]$ is the ideal generated by the homogeneous polynomials x_1^{n+1} and $x_2 \prod_{j=1}^{m} (a_j x_1 + x_2)$.

(I) We assume that $\varphi(x_2) = x_2 + Ax_1$. Since $\bar{\varphi}(x_2 \prod_{j=1}^{m}(a_j x_1 + x_2)) \in \tilde{\mathcal{J}}$, we have

$$(x_2 + Ax_1)\prod_{j=1}^{m}(x_2 + (A + a_j)x_1) = f(x_1, x_2)x_1^{n+1} + \alpha x_2 \prod_{j=1}^{m}(x_2 + a_j x_1), \quad (4.1)$$

where $f(x_1, x_2)$ is a homogeneous polynomial of degree $m - n$ and α is an integer. Note that if $n \geqslant m$, then $f = 0$. By comparing the coefficients of x_2^{m+1} and $x_1 x_2^m$ on both sides of (4.1), it is clear that $A = 0$. Hence, φ is the identity that is obviously induced from the identity map of $P(E)$.

(II) Now assume that $\varphi(x_2) = -x_2 + Ax_1$. Since $\bar{\varphi}(x_2 \prod_{j=1}^{m}(a_j x_1 + x_2))$ belongs to $\tilde{\mathcal{J}}$, we have

$$(-x_2 + Ax_1)\prod_{j=1}^{m}(-x_2 + (A + a_j)x_1) = f(x_1, x_2)x_1^{n+1} + \alpha x_2 \prod_{j=1}^{m}(x_2 + a_j x_1), \quad (4.2)$$

where $f(x_1, x_2)$ is a homogeneous polynomial of degree $m - n$ and α is an integer. By comparing the coefficients of x_2^{m+1} on both sides of (4.2), it follows that $\alpha = (-1)^{m+1}$. By substituting $x_2 = 1$ into (4.2), we obtain

$$(1 - Ax_1)(1 - (A + a_1)x_1)\cdots(1 - (A + a_m)x_1) = (1 + a_1 x_1)\cdots(1 + a_m x_1) \quad (4.3)$$

in $H^*(\mathbb{C}P^n) = \mathbb{Z}[x_1]/\langle x_1^{n+1}\rangle$. Since E possesses a Hermitian metric, its dual bundle $E^* = \mathrm{Hom}(E, \mathbb{C})$ is canonically isomorphic to the conjugate bundle $\underline{\mathbb{C}} \oplus \gamma^{-a_1} \oplus \cdots \oplus \gamma^{-a_m}$. By lemma 4.1, equation (4.3) implies that

$$E^* \otimes \gamma^{-A} = \gamma^{-A} \oplus \gamma^{-A-a_1} \oplus \cdots \oplus \gamma^{-A-a_m} = \underline{\mathbb{C}} \oplus \gamma^{a_1} \oplus \cdots \oplus \gamma^{a_m} = E$$

as complex vector bundles over $\mathbb{C}P^n$.

Let $\langle\cdot,\cdot\rangle$ be a Hermitian metric on E. Then the map $\tilde{h}\colon E \to E^*$, $u \mapsto \langle u, \cdot\rangle$, induces the isomorphism $h\colon P(E) \to P(E^*)$ as fibre bundles. If y is the negative of the first Chern class of the tautological line bundle over $P(E^*)$, then $h^*(y) = -x_2$.

For each $q \in \mathbb{C}P^n$, we choose a non-zero vector v_q from the fibre of γ^{-A} over q and define a map $\tilde{g}\colon E^* \to E^* \otimes \gamma^{-A}$ by $\tilde{g}(u_q) = u_q \otimes v_q$, where u_q is an element of the fibre of E^* over q. The map $\tilde{g}$ depends on the choice of v_qs but the induced map $g\colon P(E^*) \to P(E^* \otimes \gamma^{-A})$ does not have this dependency because γ^{-A} is a line bundle. Then the map

$$g\colon P(E^*) \to P(E^* \otimes \gamma^{-A}) = P(E)$$

preserves the complex structure on each fibre. Therefore, it induces a complex vector bundle isomorphism $T_f P(E^*) \to T_f P(E^* \otimes \gamma^{-A})$ between their tangent bundles along the fibres. According to the Borel–Hirzebruch formula, their respective total Chern classes are

$$(1 + y)(1 - a_1 x_1 + y)\cdots(1 - a_m x_1 + y)$$

and

$$(1 - Ax_1 + x_2)(1 - Ax_1 - a_1 x_1 + x_2)\cdots(1 - Ax_1 - a_m x_1 + x_2).$$

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Since $g^*(c_1(T_f(P(E)))) = c_1(T_f(P(E^*)))$, we have

$$g^*\left((m+1)(x_2 - Ax_1) - \sum_{j=1}^{m} a_j x_1\right) = (m+1)y - \sum_{j=1}^{m} a_j x_1.$$

Furthermore, the map g covers the identity map on $\mathbb{C}P^n$; thus, $g^*(x_2) = y + Ax_1$. Therefore,

$$h^*(g^*(x_2)) = -x_2 + Ax_1 = \varphi(x_2).$$

By (I) and (II), every ring automorphism φ is induced by a diffeomorphism. □

5. Quasitoric manifolds over $\Delta^n \times \Delta^m$

In this section, we show that every cohomology ring automorphism of a quasitoric manifold with second Betti number 2 is realizable by a homeomorphism. As we have seen in the previous section, every cohomology ring automorphism of a two-stage generalized Bott manifold is realizable by a diffeomorphism. Hence, we only need to consider quasitoric manifolds over $\Delta^n \times \Delta^m$ that are not equivalent to a two-stage generalized Bott manifold.

Let $M_{\boldsymbol{a},\boldsymbol{b}}$ be a quasitoric manifold over $\Delta^n \times \Delta^m$. By theorem 2.2, it is sufficient to consider the case in which $\boldsymbol{a} = \boldsymbol{s} = (2,\dots,2,0,\dots,0) \neq \mathbf{0}$ and $\boldsymbol{b} = \boldsymbol{r} = (1,\dots,1,0,\dots,0) \neq \mathbf{0}$.

Proposition 5.1. *Let $M_{\boldsymbol{s},\boldsymbol{r}}$ be a quasitoric manifold over $\Delta^n \times \Delta^m$, where two non-zero vectors $\boldsymbol{s}$ and $\boldsymbol{r}$ have the forms*

$$\boldsymbol{s} := (\underbrace{2,\dots,2}_{s},0,\dots,0) \in \mathbb{Z}^m \quad \text{and} \quad \boldsymbol{r} := (\underbrace{1,\dots,1}_{r},0,\dots,0) \in \mathbb{Z}^n.$$

Then every element of $\operatorname{Aut}(H^*(M_{\boldsymbol{s},\boldsymbol{r}}))$ is induced by a homeomorphism.

Proof. The detailed computation of $\operatorname{Aut}(H^*(M_{\boldsymbol{s},\boldsymbol{r}}))$ can be found in the proof of theorem 6.2 in [8]. Even though it is one of key parts of this proof, the result is used here without detailed calculation to avoid repetition of the elementary computation.

If $n = 1$ or $m = 1$, every automorphism of $H^*(M_{\boldsymbol{s},\boldsymbol{r}})$ is realizable by a homeomorphism by corollary 3.7.

Now, assume that both n and m are greater than 1.

(I) If $s \neq (m+1)/2$ and $r \neq (n+1)/2$, then

$$\operatorname{Aut}(H^*(M_{\boldsymbol{s},\boldsymbol{r}})) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \cong \mathbb{Z}_2.$$

Define a homeomorphism $f\colon S^{2n+1} \times S^{2m+1} \to S^{2n+1} \times S^{2m+1}$ by

$$((w_1,\dots,w_{n+1}),(z_1,\dots,z_{m+1})) \mapsto ((\overline{w_1},\dots,\overline{w_{n+1}}),(\overline{z_1},\dots,\overline{z_{m+1}})).$$

Then f preserves the orbits of the action of $K_{\boldsymbol{s},\boldsymbol{r}}$ defined in § 2. Hence, f induces a homeomorphism from $M_{\boldsymbol{s},\boldsymbol{r}} = S^{2n+1} \times S^{2m+1}/K_{\boldsymbol{s},\boldsymbol{r}}$ to itself. Let $\bar{f}$ be the homeomorphism induced from f. Then $\bar{f}^*$ is represented by the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, and hence $\{\bar{f}^*\}$ generates $\operatorname{Aut}(M_{\boldsymbol{s},\boldsymbol{r}})$.

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(II) If $s = (m+1)/2$ and $r \neq (n+1)/2$, then

$$\begin{aligned}\mathrm{Aut}(H^*(M_{\boldsymbol{s},\boldsymbol{r}})) &= \left\{ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -2 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix} \right\}\\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2.\end{aligned}$$

Define a homeomorphism $g\colon S^{2n+1} \times S^{2m+1} \to S^{2n+1} \times S^{2m+1}$ by

$$\begin{aligned}&((w_1, \dots, w_{n+1}), (z_1, \dots, z_{m+1}))\\ &\qquad \mapsto ((w_1, \dots, w_r, \overline{w_{r+1}}, \dots, \overline{w_{n+1}}), (z_{s+1}, \dots, z_{m+1}, z_1, \dots, z_s)),\end{aligned}$$

and then g preserves the orbits of the action of $K_{\boldsymbol{s},\boldsymbol{r}}$ on $S^{2n+1} \times S^{2m+1}$.

Let $\bar{g}$ be the homeomorphism from $S^{2n+1} \times S^{2m+1}/K_{\boldsymbol{s},\boldsymbol{r}}$ to itself induced from g. Then $\bar{g}^*$ is represented by the matrix $\left(\begin{smallmatrix}-1 & 0\\ 2 & 1\end{smallmatrix}\right)$, and hence the set $\{\bar{f}^*, \bar{g}^*\}$ generates $\mathrm{Aut}(H^*(M_{\boldsymbol{s},\boldsymbol{r}}))$.

(III) If $s \neq (m+1)/2$ and $r = (n+1)/2$, then

$$\begin{aligned}\mathrm{Aut}(H^*(M_{\boldsymbol{s},\boldsymbol{r}})) &= \left\{ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix} \right\}\\ &\cong \mathbb{Z}_2 \times \mathbb{Z}_2.\end{aligned}$$

Define a homeomorphism $h\colon S^{2n+1} \times S^{2m+1} \to S^{2n+1} \times S^{2m+1}$ by

$$\begin{aligned}&((w_1, \dots, w_{n+1}), (z_1, \dots, z_{m+1}))\\ &\qquad \mapsto ((w_{r+1}, \dots, w_{n+1}, w_1, \dots, w_r), (z_1, \dots, z_s, \overline{z_{s+1}}, \dots, \overline{z_{m+1}})).\end{aligned}$$

and then h also preserves the orbits of the action of $K_{\boldsymbol{s},\boldsymbol{r}}$ on $S^{2n+1} \times S^{2m+1}$.

Let $\bar{h}$ be the homeomorphism from $S^{2n+1} \times S^{2m+1}/K_{\boldsymbol{s},\boldsymbol{r}}$ to itself induced from h. Then $\bar{h}^*$ is represented by the matrix $\left(\begin{smallmatrix}1 & 1\\ 0 & -1\end{smallmatrix}\right)$, and hence the set $\{\bar{f}^*, \bar{h}^*\}$ generates $\mathrm{Aut}(H^*(M_{\boldsymbol{s},\boldsymbol{r}}))$.

(IV) If $s = (m+1)/2$ and $r = (n+1)/2$, then the set $\{\bar{f}^*, \bar{g}^*, \bar{h}^*\}$ generates

$$\begin{aligned}\mathrm{Aut}(H^*(M_{\boldsymbol{s},\boldsymbol{r}})) = \Biggl\{ &\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ -2 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 2 & 1\end{pmatrix},\\ &\begin{pmatrix}-1 & -1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 1\\ -2 & -1\end{pmatrix}, \begin{pmatrix}-1 & -1\\ 2 & 1\end{pmatrix} \Biggr\}.\end{aligned}$$

□

6. Proofs of theorems 1.1 and 1.2

Let M and M' be quasitoric manifolds with second Betti number 2. If $\varphi\colon H^*(M) \to H^*(M')$ is an isomorphism as graded rings, then there exists a homeomorphism $f\colon M \to M'$ by theorem 2.2. Then f induces a graded ring isomorphism $f^*\colon H^*(M') \to H^*(M)$. Accordingly, $f^* \circ \varphi$ is a ring automorphism of $H^*(M)$. Hence, by proposition 4.3 and proposition 5.1, there exists a homeomorphism

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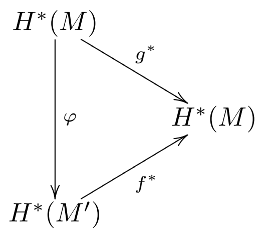
$g\colon M \to M$ such that $g^* = f^* \circ \varphi$. Hence, φ is realizable by a homeomorphism $g \circ f^{-1}$, as shown in the following diagram:



This proves theorem 1.2.

Furthermore, if M and M' are non-singular complete toric varieties, then f and g are diffeomorphisms by theorem 2.1 and proposition 4.3. This proves theorem 1.1.

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