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Characters on $C(X)$

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Abstract. The precise condition on a completely regular space X for every character on $C(X)$ to be an evaluation at some point in X is that X be realcompact. Usually, this classical result is obtained by relying heavily on involved (and even nonconstructive) extension arguments. This note provides a direct proof that is accessible to a large audience.

Throughout this note, X stands for a completely regular space and $C(X)$ denotes the algebra of all real-valued continuous functions on X. By a *character* on $C(X)$ we mean a nonzero algebra homomorphism from $C(X)$ onto $\mathbb{R}$. For instance, the evaluation δ_w at a point w in X, which is defined by

$$\delta_w(f) = f(w) \quad \text{for all } f \in C(X),$$

is a character on $C(X)$. Conversely, it is well known that the precise condition on X for every character on $C(X)$ to be an evaluation at some point in X is that X be *realcompact* (that is, a closed set in an appropriate product $\mathbb{R}^I$ of real lines). Usually, this classical result is gotten via involved and even nonconstructive (*i.e.*, in ZFC set theory) extension arguments (see, for instance, [3–5,7]). Nonetheless, in his remarkable note [6], Ransford proved in a simple way that if X is a Lindelöf space, then every character on $C(X)$ is an evaluation at some point in X. In this regard, he observed that his proof is simpler than most, even for the special case $X = \mathbb{R}$. It seems that Ransford was right at that time. However, it turns out that the case $X = \mathbb{R}$ can be obtained much more easily than is apparent at first sight. Indeed, let φ be a character on $C(\mathbb{R})$ and put $w = \varphi(\mathrm{id}_\mathbb{R})$, where $\mathrm{id}_\mathbb{R}$ is the identity function on $\mathbb{R}$. Pick $f \in C(\mathbb{R})$ and define

$$g = \big(f - \varphi(f)\big)^2 + (\mathrm{id}_\mathbb{R} - w)^2 \in C(\mathbb{R}).$$

Obviously, $\varphi(g) = 0$ and so g has no inverse in $C(\mathbb{R})$. It follows that $g(r) = 0$ for some $r \in \mathbb{R}$. But then $\varphi(f) = f(r)$ and $w = \mathrm{id}_\mathbb{R}(r) = r$. We derive that $\varphi(f) = f(w)$ for all $f \in C(\mathbb{R})$, so $\varphi = \delta_w$. This quick proof (which can be compared with [1] by Aron and Frike) is derived from our recent note [2], though it was not stated explicitly.

The main objective of this short paper is to use the above idea to get the result in its most general version. We think that the following proof, which seems to be strangely unknown, may be of interest because is accessible to a large audience and can be understood by readers with a standard first-year graduate background. All we need is the very first properties of product topology.

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Theorem 1 *Let X be a realcompact space and φ be a character on $C(X)$. Then there exists $w \in X$ such that $\varphi(f) = f(w)$ for all $f \in C(X)$.*

Proof As observed earlier, X is a closed set in a product $\mathbb{R}^I$ of real lines. For every $i \in I$, the projection $p_i \in C(X)$ is defined by $p_i(x) = x_i$ for all $x = (x_i) \in \mathbb{R}^I$ (the idea of using projections comes from [4]). Put $w = \big(\varphi(p_i)\big) \in \mathbb{R}^I$. First, we claim that $w \in X$. Otherwise, since X is a closed set in $\mathbb{R}^I$, there would exist $\epsilon \in (0, \infty)$ and a non-empty finite subset J of I such that the set

$$\Omega = \bigcap_{j \in J} \{(x_i)_{i \in I} \in X : |x_j - \varphi(p_j)| < \epsilon\}$$

is empty. Define

$$g = \sum_{j \in J} \big(p_j - \varphi(p_j)\big)^2 \in C(X)$$

and observe that $\varphi(g) = 0$. Hence, g has no inverse in $C(X)$ and thus $g(w_g) = 0$ for some $w_g \in X$. Then, $|p_j(w_g) - \varphi(p_j)| = 0$ for all $j \in J$ and so $w_g \in \Omega = \varnothing$, which is an obvious contradiction. We derive that $w \in X$, as desired.

Now, pick $f \in C(X)$ and $\epsilon > 0$. Since $w \in X$ and f is continuous on X, there exists $\eta > 0$ and a non-empty finite subset J of I such that, whenever $x \in X$,

$$(*) \qquad |p_j(x) - \varphi(p_j)| < \eta \quad \text{for all } j \in J \text{ implies } |f(x) - f(w)| < \epsilon.$$

Define

$$h = \big(f - \varphi(f)\big)^2 + \sum_{j \in J} \big(p_j - \varphi(p_j)\big)^2 \in C(X).$$

Clearly, $\varphi(h) = 0$ and so $h(w_h) = 0$ for some $w_h \in X$. Therefore, $f(w_h) = \varphi(f)$ and $p_j(w_h) = \varphi(p_j)$ for all $j \in J$. These equalities together with $(*)$ yield that

$$|\varphi(f) - f(w)| = |f(w_h) - f(w)| < \epsilon.$$

But then $\varphi(f) = f(w)$, completing the proof of the theorem. ∎

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