THE JOURNAL OF NAVIGATION (2004), **57**, 311–325. © The Royal Institute of Navigation
Printed in the United Kingdom

Forum

DOI: 10.1017/S0373463304002644

A Novel Approach to Great Circle Sailings: The Great Circle Equation

Chih-Li Chen and Tien-Pen Hsu

(*Institute of Civil Engineering*, *National Taiwan University*)
(Email: clchen@mail.ntou.edu.tw)

Jiang-Ren Chang

(*Institute of System Engineering & Naval Architecture*, *National Taiwan Ocean University*)

In this paper, a novel approach using a Great Circle Equation (GCE) formulated by vector algebra is proposed to solve the problems of Great Circle Sailings (GCS). It is found that Great Circle Equation Method (GCEM) can calculate the latitude and longitude of the waypoints more effectively than conventional approaches. The methods of solving the waypoints of GCS are discussed and a technique using minimum error propagation in every step of the calculating procedure is suggested. Comparisons of the GCEM and the conventional computation approach are also conducted for further validation. Numerical results show that the GCEM is simpler and can solve the problems directly without requiring judgments from the solver.

KEY WORDS

1. Great circle sailing. 2. Great circle equation.

1. INTRODUCTION. It is widely known that the shortest distance between any two positions of the earth's surface lies on the great circle passing through them. Since sailing a vessel in a constantly varying course is not usually practical, a series of rhumb lines are followed to approximate the great circle track. In fact, a GCS is composed of segments of Rhumb Line Sailings. The navigator's main task in planning the use of GCS is to find the latitude and longitude of the waypoints along the great circle track from the initial conditions.

Basically, there are three alternatives to solve the problems of Great Circle Sailings. The first one is to make use of a great circle chart and the second is by using Sight Reduction Tables. Because of the mathematical errors present in both these methods, computation techniques may be preferred [1,2]. With increasing growth of computation technology, computation becomes more convenient and efficient. However, the conventional computation procedure is based on using the vertex to form numerous

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

right angled spherical triangles to give positions of waypoints along the great circle track; it therefore seems indirect and may be complicated. To overcome the drawbacks of this conventional computing approach, the proposed GCEM deals with these problems by using vector algebra, which is simpler and more direct. Because of the general nature of the solution employed in the GCEM, it may be possible to use a general commercial package to solve a specific navigation problem. Nevertheless, it cannot be emphasized too strongly that the only safe way to use computer software for the solution of navigation problems is with a full knowledge and understanding of the basic equation [2]. Hence, the theoretical background to the GCEM will be introduced. Besides, for the conventional computational procedure, different users may use non-vector calculation formulae [1,2] for the subsequent steps which may lead to step errors which will propagate into the calculation result. Therefore, the optimal formulae adopted, based on minimum error propagation in the steps of the computation procedure, are also suggested.

The paper is organized as follows. Following this introduction, Section 2 describes the conventional computation procedure and the choice of optimal formula. Derivations of the Great Circle Equation and its further applications are included in Section 3. Section 4 offers worked examples for comparisons of the GCEM and the conventional approach. A summary with some concluding remarks is in Section 5.

2. REVIEW OF THE CONVENTIONAL COMPUTATION PROCEDURE.

Dropping a perpendicular from the pole nearer the departure point to the great circle track defines the vertex, which forms numerous right angled spherical triangles with the pole and any waypoints along the great circle track as shown in Figure 1. From this perpendicular, a number of the waypoints on the great circle track can thus be found by using formulae derived from Napier's Rule. Therefore, finding the vertex is the key point in this computation procedure. Since

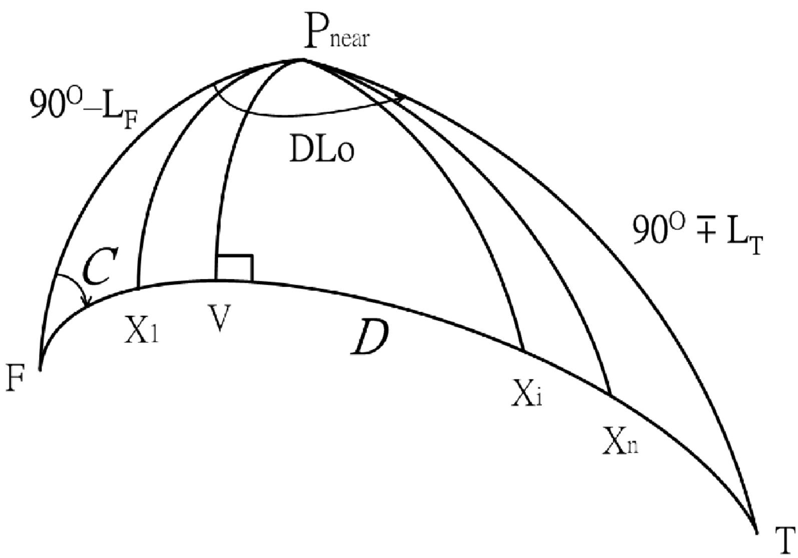

Figure 1. The navigational triangle used in the GCS.

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

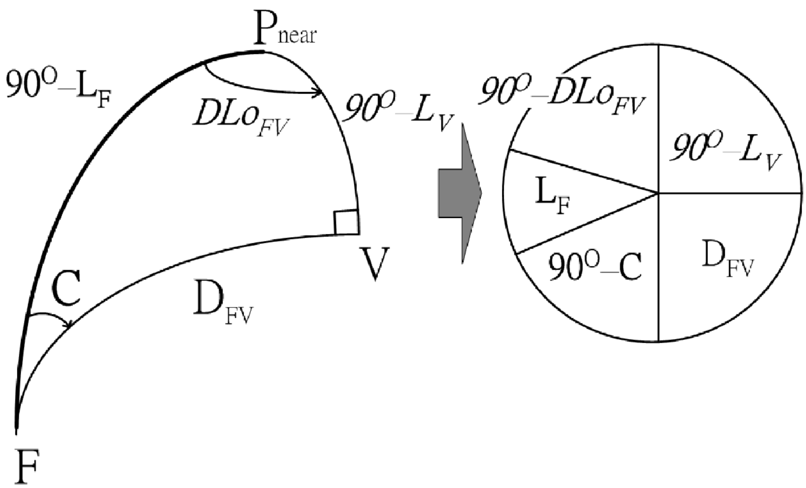

Figure 2. An illustration for finding the vertex on the great circle track using Napier's Rule for a right angled triangle.

non-vector formulae can be adopted in the subsequent steps of the calculation procedure, rounding errors present in each step of the procedure are propagated into the final result. Thus an optimal formula in each step to reduce the total error is necessary. The conventional computation procedure with the suggested formula in each step is summarized as follows. (All the symbols used in the following are listed in the appendix for quick reference.)

2.1. *Step 1. Finding the great circle distance* (D) *and the initial great circle course angle* (C). (See Figure 1). Based on identities of trigonometric functions, the great circle distance can be calculated by the cosine formula for the side of a spherical triangle as

$$\cos D = \sin L_F \cdot \sin L_T + \cos L_F \cdot \cos L_T \cdot \cos DLo \tag{1}$$

and the initial great circle course can be obtained by the four-part formula

$$\tan C = \frac{\sin DLo}{(\cos L_F \cdot \tan L_T) - (\sin L_F \cdot \cos DLo)}. \tag{2}$$

In the above equations if the hemisphere of the latitude of the destination, L_T, is the same as that of the departure point, L_F, it is treated as a positive quantity; if contrary, it is treated as negative.

2.2. *Step 2. Finding of the latitude and longitude of the vertex, namely* (L_V, λ_V). (Refer to Figure 2). By using Napier's Rule, the following two formulae can be yielded

$$\cos L_V = \cos L_F \cdot \sin C \tag{3}$$

and

$$\tan DLo_{FV} = \cot C \cdot \csc L_F. \tag{4}$$

In Equation (4) if the initial course angle C is less than 090°, the vertex is toward L_T, while if C is greater than 090°, the nearer vertex is in the opposite direction.

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

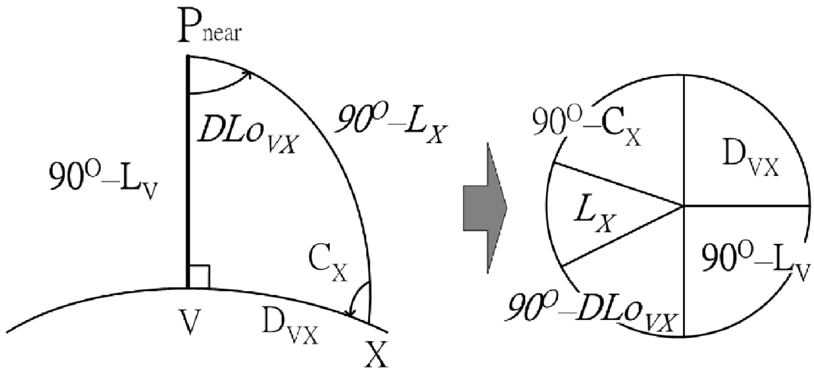

Figure 3. An illustration for finding the waypoints on the great circle track by using Napier's Rule.

2.3. *Step 3. Finding of the latitude and longitude of all waypoints, L_X and λ_X, along the great circle track*. (Refer to Figure 3). Since only the co-latitude of the vertex can be available, the initial condition is necessary for the positions of the waypoints according to Napier's Rule. Now,

- *Condition 1*: When λ_X is given, DLo_{VX} can be obtained and then, L_X can be calculated from the following formula

$$\tan L_X = \cos DLo_{VX} \cdot \tan L_V. \tag{5}$$

- *Condition 2*: When D_{VX} is given, let the great circle distance (D_{VX}) on either side of the vertex be a constant, then the positions of the waypoints can be obtained from the formula

$$\sin L_X = \sin L_V \cdot \cos D_{VX} \tag{6}$$

and

$$\tan DLo_{VX} = \tan D_{VX} \cdot \sec L_V. \tag{7}$$

2.4. *Formulae choice*. Discussions of the appropriate formulae in the computation procedure are given as follows. In Step 1, with an oblique spherical triangle of which two sides and the included angle are known it is necessary to determine the values of the third side and the outer angle. There are several methods [1,3] which can be adopted to solve the problems. These may be categorized into direct and indirect methods. For the indirect method, dropping a perpendicular from one of the apexes, the departure or destination point, to the opposite side forms two right angled spherical triangles, whilst for the direct one, the oblique spherical triangle is considered undivided. If using a calculator rather than programmable computer technology, the direct method is better than the indirect one. Equation (1) is known as the cosine formula for side of spherical triangle. Because the Haversine is not a built-in function, the Haversine formula, derived from the cosine formula for logarithmic work, is not ideal for the calculator. When rounding error and propagation in the procedure is considered, the four-part formula in Equation (2) is the best of the available formulae, including the sine and cosine formulae [2]. Similarly, those

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

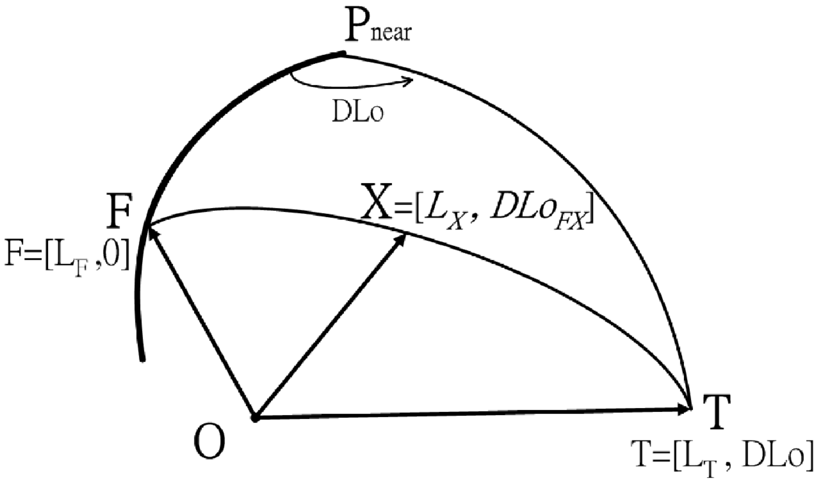

Figure 4. An illustration for three co-planar position vectors on the great circle track.

equations in Steps 2 and 3, which are all derived from Napier's Rule of the right angled spherical triangle, are suggested as the optimal choices.

3. DERIVATIONS OF THE GREAT CIRCLE EQUATION.

To begin with, the earth is treated as a unitary sphere. From the navigator's point of view, the earth's coordinate system can replace the spherical coordinates. Therefore, the vector for any point P on the earth's surface can be represented with the latitude L and longitude λ in a Cartesian coordinates system as

$$\vec{P}=[\cos L\cdot\cos\lambda,\ \cos L\cdot\sin\lambda,\ \sin L],\quad L=\left[-\frac{\pi}{2},\frac{\pi}{2}\right],\quad \lambda=[0,2\pi], \tag{8}$$

where the north latitude is treated as a positive value and the south latitude is treated as negative. By using the relative longitude concept and replacing the Greenwich meridian by the meridian of the departure point, as shown in Figure 4, the unit vectors for the departure point, the destination point and the waypoints on a great circle located on the earth's surface can be expressed as:

$$\vec{F}=[\cos L_F, 0,\ \sin L_F], \tag{9}$$

$$\vec{T}=[\cos L_T\cdot\cos DLo,\ \cos L_T\cdot\sin DLo,\ \sin L_T], \tag{10}$$

$$\vec{X}=[\cos L_X\cdot\cos DLo_{FX},\ \cos L_X\cdot\sin DLo_{FX},\ \sin L_X], \tag{11}$$

respectively. Since the three vectors are co-planar, the scalar triple product is equal to zero [4], that is,

$$\vec{X}\cdot(\vec{F}\times\vec{T})=0. \tag{12}$$

Now, assuming that

$$\vec{F}\times\vec{T}=[a,b,c], \tag{13}$$

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

Equation (13) can be viewed as a coefficient vector and $\vec{X}$ in Equation (12) is a variable vector. Substituting Equations (9) and (10) into Equation (13) yields

$$a = -\sin L_F \cdot \cos L_T \cdot \sin DLo, \tag{14}$$

$$b = \sin L_F \cdot \cos L_T \cdot \cos DLo - \cos L_F \cdot \sin L_T, \tag{15}$$

$$c = \cos L_F \cdot \cos L_T \cdot \sin DLo. \tag{16}$$

Finally, by using the scalar triple product, the GCE can be formulated as

$$a \cdot \cos L_X \cdot \cos DLo_{FX} + b \cdot \cos L_X \cdot \sin DLo_{FX} + c \cdot \sin L_X = 0. \tag{17}$$

In Equation (17), there are two variables only, DLo_{FX}, the difference of longitude and L_X, the latitude of the waypoints on a great circle track. Accordingly, if one of the variables is the entering argument, the other can be obtained by using the GCE.

- *Condition 1*: When λ_X is given first, DLo_{FX} can be obtained. Substituting it into Equation (17) and rearranging yields

$$\tan L_X = \frac{a \cdot \cos DLo_{FX} + b \cdot \sin DLo_{FX}}{-c} \tag{18}$$

If the solution of Equation (18), L_X, is positive, the position of the waypoint is in the Northern hemisphere; if negative, then the waypoint is a Southern latitude.

- *Condition 2*: When L_X is given first and assuming that

$$\tan \alpha = \frac{a}{b}, \tag{19}$$

where $\sin \alpha = (a/\sqrt{a^2 + b^2})$, $\cos \alpha = (b/\sqrt{a^2 + b^2})$. Substituting these results into Equation (17) gives us

$$\sin(DLo_{FX} + \alpha) = \frac{-c}{\sqrt{a^2 + b^2}} \cdot \tan L_X. \tag{20}$$

Once DLo_{FX} is obtained from Equations (19) and (20), λ_X can be calculated immediately.

Having constructed these equations, we can solve the waypoints on the great circle effectively. Several applications to the point crossing the equator and the vertex are illustrated as follows. Assuming that the point crossing the equator in the earth's coordinate system be expressed as (0, λ_E), since $L_E = 0$, substituting it into Equation (18) yields

$$\tan DLo_{FE} = -\frac{a}{b}. \tag{21}$$

Then, we can find the longitude at which a vessel crosses the equator. Again assuming that the vertex can be represented as (L_V, λ_V) at which L_V is the highest latitude for the great circle track, we have $(d \tan L/dL) = 0$ such that

$$\tan DLo_{FV} = \frac{b}{a}. \tag{22}$$

Substituting the above result into Equation (18) yields

$$\tan L_V = \frac{a \cdot \cos DLo_{FV} + b \cdot \sin DLo_{FV}}{-c}. \tag{23}$$

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

Therefore, a combined use of Equations (22) and (23) can easily obtain the vertex (L_V, λ_V). Although formulae derived by using plane analytic geometry for the vertex have been proposed [5], the complexity of the calculation procedure makes such techniques more time-consuming than our suggested methodology. It should be noted that when the product of Equations (21) and (22) is equal to -1, the difference of longitude between the point passing the equator and the vertex is 090°. From the spherical geometry, it means that two great circles bisect one another [3].

As for obtaining the great circle distance (D), as shown in Figure 4, it is actually the angle between the unit vectors $\vec{F}$ and $\vec{T}$; therefore,

$$\cos D = \vec{F} \cdot \vec{T} = \cos L_F \cdot \cos L_T \cdot \cos DLo + \sin L_F \cdot \sin L_T. \tag{24}$$

Equation (24) is the well-known cosine formula for side of spherical triangles and can be derived easily using vector algebra.

4. ILLUSTRATIVE EXAMPLES.

4.1. *Example 1. The captain decides to use great circle sailing from 33°51.5′S, 151°13.0′E (Sydney, Australia) to 08°53.0′N, 079°31.0′W (Balboa, Panama). Calculate the latitude and longitude of the waypoints on great circle at longitude 170°E and at each 20 degrees of longitude thereafter to longitude 90°W.*

Solutions of the GCS for a given λ_X using the conventional computation procedure and the proposed GCEM are shown in Tables 1 and 2, respectively. A comparison of the two tables shows that the proposed GCEM largely reduces the steps of the calculation procedure and the solution can be obtained in one step from the GCEM simply by using a scientific calculator with a memory function.

4.2. *Example 2. The same situation is given as Example 1. Calculate the latitude and longitude of the waypoints on the great circle track at equal interval of distance, 1200 nautical miles (20°), from the vertex.*

In this example, as the distance between the position of waypoint and the vertex is given in advance, the positions of the waypoints can be solved by the

Table 1. The conventional computation procedure of the GCS for a given latitude, λ_X.

	Item			
Process	Equation	Input	Output	Solution
1	(1)	L_F=33°51.5′ (S)	D=127.252415°	D=7635.2′
2	(2)	L_T=−08°53.0′ (N)	C=73.942671°	S73.9°E
		DLo=129°16.0′ (E)		C_n=106.1°
3	(3)	L_F=33°51.5′	L_V=37°03.5′	L_V=37°03.5′S
4	(4)	C=73.942671°	DLo_{FV}=27°19.3′	λ_V=178°32.3′E*
5	(5)	DLo_{VX}=178°32.3′E−170°E =8°32.3′W	L_X=36°45.1′	36°45.1′S 170°E
		DLo_{VX}=11°27.7′E	L_X=36°30.3′	36°30.3′S 170°W
		DLo_{VX}=31°27.7′E	L_X=32°47.2′	32°47.2′S 150°W
		DLo_{VX}=51°27.7′E	L_X=25°11.8′	25°11.8′S 130°W
		DLo_{VX}=71°27.7′E	L_X=13°30.0′	13°30.0′S 110°W
		DLo_{VX}=91°27.7′E	L_X=−01°06.2′	01°06.2′N 90°W

* λ_V is determined by the relative position of the vertex and the departure point, in which an artificial judgement of the initial great circle course is necessary.

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

Table 2. Use of the GCEM to solve the GCS for a given latitude λ_X.

Process	Item			
	Equation	Input	Output	Solution
1	(14)	$L_F=-33^\circ 51.5'$ (S)	$a=0.4261695713$	
2	(15)	$L_T=08^\circ 53.0'$ (N)	$b=0.2201663328$	
3	(16)	$DLo=129^\circ 16.0'$ (E)	$c=0.6352045779$	
4	(18)	$DLo_{FX}=151^\circ 13.0'\text{E}-170^\circ\text{E}$ $=18^\circ 47.0'\text{E}$	$L_X=-36^\circ 45.1'$	36°45.1′S 170°E
		$DLo_{FX}=38^\circ 47.0'\text{E}$	$L_X=-36^\circ 30.3'$	36°30.3′S 170°W
		$DLo_{FX}=58^\circ 47.0'\text{E}$	$L_X=-32^\circ 47.2'$	32°47.2′S 150°W
		$DLo_{FX}=78^\circ 47.0'\text{E}$	$L_X=-25^\circ 11.8'$	25°11.8′S 130°W
		$DLo_{FX}=98^\circ 47.0'\text{E}$	$L_X=-13^\circ 30.1'$	13°30.1′S 110°W
		$DLo_{FX}=118^\circ 47.0'\text{E}$	$L_X=1^\circ 06.2'$	01°06.2′N 90°W

Table 3. Use of the conventional computation procedure to solve the waypoints of the GCS when the distance from the vortex, D_{VX}, is known.

Process	Item			
	Equation	Input	Output	Solution
Process 1–4 is the same as Table 1. Preliminary: $L_V=37^\circ 03.5'\text{S}$; $\lambda_V=178^\circ 32.3'\text{E}$				
5	(6)	$D_{VX}=20^\circ$ (1200 nm)	$L_X=34^\circ 29.5'$	34°29.5′S 154°01.3′E
6	(7)		$DLo_{VX}=024^\circ 31.0'\,{}^{\text{W}}_{\text{E}}$	34°29.5′S 156°56.7′W
5	(6)	$D_{VX}=40^\circ$	$L_X=27^\circ 29.6'$	27°29.6′S 135°01.5′W
6	(7)		$DLo_{VX}=046^\circ 26.2'\text{E}$	
5	(6)	$D_{VX}=60^\circ$	$L_X=17^\circ 32.2'$	17°32.2′S 116°11.9′W
6	(7)		$DLo_{VX}=065^\circ 15.8'\text{E}$	
5	(6)	$D_{VX}=80^\circ$	$L_X=06^\circ 00.4'$	06°00.4′S 099°28.3′W
6	(7)		$DLo_{VX}=081^\circ 59.4'\text{E}$	
5	(6)	$D_{VX}=100^\circ$	$L_X=-06^\circ 00.4'$	06°00.4′N 083°27.1′W
6	(7)		$DLo_{VX}=-081^\circ 59.4'$ $=098^\circ 00.6'\text{E}$*	

* $\tan(-\theta)=\tan(180^\circ-\theta)$.

conventional procedure and the solving process with the suggested formulae are shown in Table 3.

4.3. *Example 3. The same situation is given as Example 1. Calculate the latitude and longitude of the waypoints on the great circle at latitude 35°S and at each 10 degrees of latitude northward to latitude 05°N.*

In this example, the latitude is given and the longitude cannot be solved by the conventional procedure; however, by using the proposed GCEM the solution can be obtained easily as shown in Table 4. In addition, for the solutions of the vertex and the point crossing the equator by the GCEM, as shown in Table 5, we find that the proposed method can easily solve any points along the great circle track, and the relationship of two great circles bisecting each other has been further verified through Example 1. All the formulae listed in the previous sections have been illustrated through these worked examples with tables. In practical situations, however, the leg of the rhumb line may be shorter than those of the examples. This will make no

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

Table 4. Use of the GCEM to solve the waypoints of the GCS given the latitude, L_X.

Process	Equation	Input	Output	Solution
	Item			
Process 1–3 is the same as Table 2. Preliminary: $a=0.4261695713$; $b=0.2201663328$; $c=0.6352045779$; $\lambda_F=151°13.0'E$				
4	(19)	a and b	$\alpha=62°40.7'$	
5	(20)	$L_X=-35°$	$DLo_{FX}+\alpha=068°00.4'$ and $111°59.6'$*	35°S 156°32.7′E
			$DLo_{FX}=005°19.7'E$ and $049°18.9'E$	35°S 159°28.1′W
		$L_X=-25°$	$DLo_{FX}+\alpha=141°52.0'$*	25°S 129°35.7′W
			$DLo_{FX}=079°11.3'E$	
		$L_X=-15°$	$DLo_{FX}+\alpha=159°13.0'$*	15°S 112°14.7′W
			$DLo_{FX}=096°32.3'E$	
		$L_X=-5°$	$DLo_{FX}+\alpha=173°20.8'$*	05°S 098°06.9′W
			$DLo_{FX}=110°40.1'E$	
		$L_X=5°$	$DLo_{FX}+\alpha=186°39.2'$**	05°N 084°48.5′W
			$DLo_{FX}=123°58.5'E$	

* $\sin(\theta)=\sin(180°-\theta)$.
** $\sin(-\theta)=\sin(180°+\theta)$.

Table 5. Use of the GCEM to solve the vertex and the equator crossing of the GCS.

Equation	Input	Output	Solution
(21)	$a=0.4261695713$	$DLo_{FE}=-62°40.7'$	$\lambda_E=091°27.7'W$
(22)	$b=0.2201663328$	$DLo_{FV}=27°19.3'$	$\lambda_V=178°32.3'E$
(23)	$DLo_{FV}=37°19.3'$	$L_V=-37°03.5'$	$L_V=37°03.5'S$
$DLo_{VE}=090°\Leftrightarrow$Two great circles bisect each other			

difference to the practical applicability of the GCEM because of the method's general nature.

5. CONCLUSION.

In this paper, the GCE based on the scalar triple product principle of vector algebra has been formulated and the GCEM is developed to calculate the latitude and longitude of the waypoints along the great circle track. Comparisons with conventional approaches show that the GCEM is not only simpler because of the reduced number of necessary steps, but is also as accurate as conventional methods. By considering the error propagation existing in the conventional calculating procedure, the optimal formulae are determined and suggested for potential users. Several worked examples validate the proposed GCEM.

REFERENCES

Bowditch, Nathaniel. (1981, 1984, 2002). *American Practical Navigator*. DMAH/TC, Washington.
Clough-Smith, J. H. (1966). *An Introduction to Spherical Trigonometry*. Brown, Son & Ferguson, Ltd., Glasgow.

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

Greenberg, Michael D. (1998). *Advanced Engineering Mathematics*. Prentice-Hall International, Inc., New Jersey.

Maloney, Elebert S. (1978, 1985). *Dutton's Navigation and Piloting*. Naval Institute Press, Annapolis, Maryland.

Prince, R. and Williams, Roy. (1995). *Sailing in Ever-Decreasing Circles*. This *Journal*, **48**, 307–313.

APPENDIX

The following symbols are used in this paper:

L	latitude
L_F	departure latitude
L_T	destination latitude
L_V	latitude of the vertex
L_X	latitude of great circle track waypoints
λ	longitude
λ_F	departure longitude
λ_T	destination longitude
λ_V	longitude of the vertex
λ_X	longitude of great circle track waypoints
λ_{E}	longitude of the equator crossing point on a great circle track
DLo	difference of longitude between departure and destination, $DLo = L_F \sim L_T$
DLo_{FV}	difference of longitude between departure and vertex
DLo_{VX}	difference of longitude between vertex and waypoints on the great circle track
DLo_{FX}	difference of longitude between departure and waypoints on the great circle track
DLo_{FE}	difference of longitude between departure and the equator crossing
DLo_{VE}	difference of longitude between vertex and equator crossing
D	great circle distance between the departure and the destination
D_{FV}	great circle distance between the departure and the vertex
D_{VX}	great circle distance between the vertex and the waypoints
C	initial great circle course angle from the departure to the destination

DOI: 10.1017/S0373463304002681

Great Circle Versus Rhumb Line Cross-track Distance at Mid-Longitude

Paul Hickley

(*Oxford Aviation Training*)

KEY WORDS

1. Great circle. 2. Rhumb line.

My present job requires me to teach airline pilot students up to ATPL standard, in order to prepare them for the JAA exams. In my opinion, the standard required of the candidates is quite high but, nevertheless, it is an undergraduate, not a post-graduate, qualification. As such, the stated JAR Learning Objectives do not include a requirement to know spherical trigonometry. The boys and girls are required to be able to calculate Rhumb Line tracks from one Lat and Long to another (from a

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

knowledge of change of latitude for N–S distance and from the departure formula for E–W distance, then the use of the tangent relationship to get the track angle). They could even be required to work out the Great Circle track given the Rhumb Line track (by use of the conversion angle formula: CA = (1/2)Change of Longitude × sine mean-Latitude). However, the sine rule, the cosine rule and Napier's Rules are not required knowledge and are not expected to be taught. It was therefore with some interest that we noted the following type of question appearing in the examinations recently:

Waypoint One is 60N 30W. Waypoint Two is 60N 20W. Your autopilot is coupled to the INS and you are steering from WP1 to WP2. What will be your latitude on passing 25W?

A	*6011N*
B	*6006N*
C	*6000N*
D	*5953N*

Bear in mind that this is two-hour exam worth 100 marks. Give them 5 minutes to read the paper at the start and five minutes to ensure that they have answered every question at the end and the candidates are looking at 110 minutes. This was a two-mark question. This means that the JAA expect them to be able to answer it in 2 minutes and 12 seconds. Now, I'm an experienced aviator and instructor and I'm not sure that I could have come up with an answer in that time without some sort of pre-warning of the type of question. How do we go about it?

Well, even at the most elementary level, the students should realise that an INS will steer you along the Great Circle, not the Rhumb Line track. They should also realise that the Great Circle track will be nearer the Pole (in whichever hemisphere) than the Rhumb Line track, so they can dismiss answers C and D above. That still leaves them to choose between 6011N and 6006N in this particular question.

Let's first sort out for ourselves what the answer is. Most readers of the *Journal* will probably have an advantage over the ATPL candidates in that they will be familiar with spherical trigonometry. So let's draw ourselves the spherical trig diagram in Figure 1. Position A is 60N 30W. Position B is the North Pole. Position C is wherever the Great Circle track crosses 25W. The examiners have made the problem relatively easy for a spherical trig solution by choosing the same latitude for start and finish and by putting the unknown latitude at the mid-longitude. You can either work out that the Rhumb Line track must be 090°T and at mid-meridian Great Circle and Rhumb Line track must be same, or you can realise that Point C is the vertex of a Great Circle and so its direction at that point must be 090°T. We also know that, as the latitude of A is 60N, the co-latitude must be 30°. We can also establish what angle A is, because the Rhumb Line track will be 090°T so the Great Circle track at that point must be (090° − conversion angle).

$$\begin{aligned}\text{CA} &= \tfrac{1}{2}\ \text{Change of Longitude} \times \text{sine mean-Latitude}\\ &= \tfrac{1}{2} \times 10^{\mathbf{0}} \times \sin 60^\circ\\ &= 4.33^\circ\end{aligned}$$

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

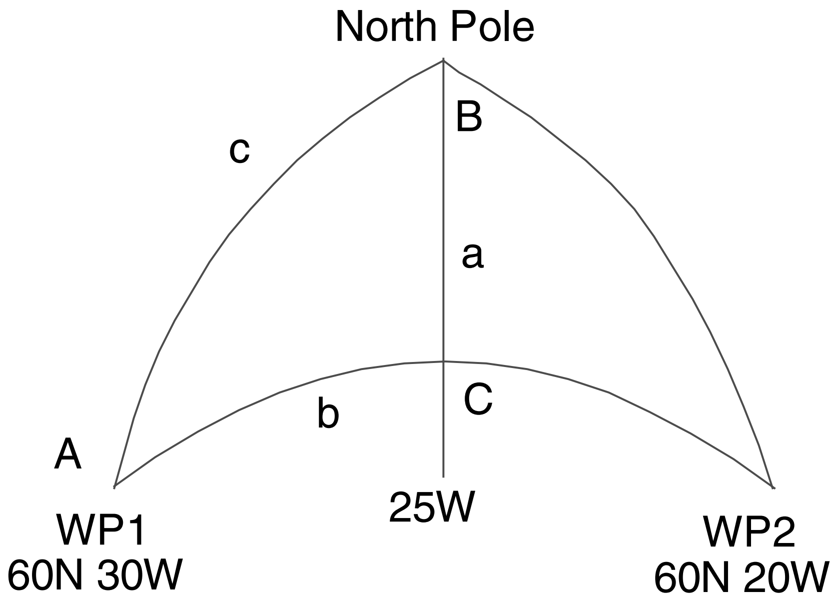

Figure 1.

$$\text{Therefore angle A} = 90 - 4.33^\circ$$
$$= 85.67^\circ.$$

Now apply the sine rule

$$\sin \mathrm{a} = \sin \mathrm{A} \times \frac{\sin c}{\sin C}$$
$$= \sin 85.67^\circ \times \frac{0.5}{1}$$

Therefore $a = 29^\circ 54.3'$, which is the co-latitude of Point C.

So C is at 60°05.7′N 025°00.0W – option (B) of the answers.

Well, yes – except that knowledge of some of the above formulae is not expected be available to my students. So, how are they supposed to answer it?

Method 1 – Half the Conversion Angle. One way might be by drawing it as a plane triangle, as in Figure 2. We can find the distance AD using the departure formula and, over this fairly small change of longitude and reasonably low latitude, this will be very close to the Great Circle distance AE.

$$\text{Departure} = \text{change of longitude (mins)} \times \cos \text{latitude}$$
$$= 5 \times 60 \times 0.5$$
$$= 150 \text{ nautical miles.}$$

(*Calculation of the Great Circle distance AE, using spherical trig, gives 149·86 nm, so it's pretty close.*)

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

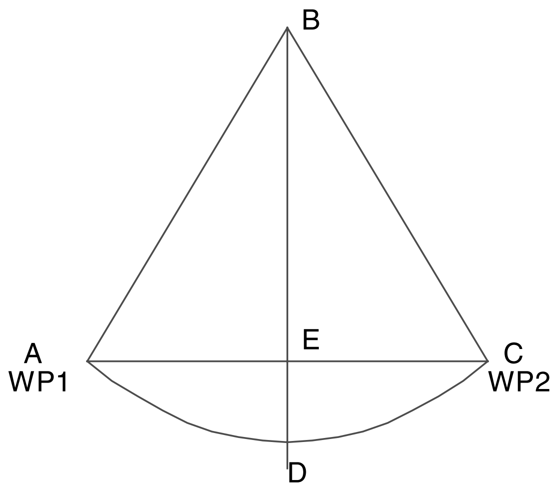

Figure 2.

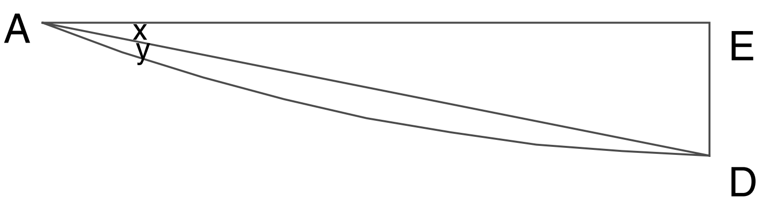

Figure 3.

We can find the conversion angle EAD. That is the conversion angle from WP1 to WP2. This will be as follows:

$$\begin{aligned} \text{CA (from A to C)} &= \tfrac{1}{2} \times \text{ch long} \times \sin \text{lat} \\ &= \tfrac{1}{2} \times 10 \times 0.8660 \\ &= 4.33^\circ. \end{aligned}$$

The problem is that the line AD is curved. If we are going to solve this using plane trigonometry, we need a straight line. But let's re-examine the previous diagram, blowing up the corner EAD as in Figure 3. 4.33° is the combination of angles x and y in Figure 3. We need the angle x on its own. If we can find y and subtract it from 4.33°, we can find x. That's not too difficult. y is the conversion angle between 60N 30W and 60N 25W.

$$\begin{aligned} \text{CA (from A to D)} &= \tfrac{1}{2} \times \text{ch long} \times \sin \text{lat} \\ &= \tfrac{1}{2} \times 5 \times 0.8660 \\ &= 2.165^\circ \\ \text{Therefore } x &= 4.33^\circ - 2.165^\circ \\ &= 2.165^\circ. \end{aligned}$$

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

So, the straight-line angle EAD is half the curved-line conversion angle EAD. This is a general result and is applicable to any such problem, not just at this latitude.

So now:

$$\text{sine } x = \frac{\text{DE}}{\text{AE}}$$

$$\text{and DE} = 150 \text{ nm} \times \sin 2.165^\circ$$

$$= 5.67 \text{ nm.}$$

Therefore the latitude at E is 6005.67N.
Answer B is correct.

Method 2. Method 1 is fine, but it does mean that you have to remember a one-off formula (half the conversion angle) to cover just one case. Another method is to treat it as a Lambert map projection. See Figure 2 again. Avoid the temptation to treat it as a Polar Stereo. If you do, the straight line AC is not a great circle. However, if you treat it as a Lambert projection, with the parallel of origin at 60N, then the Great Circle can be considered as a straight line.

The conversion angle EAD is 4.33° (see the argument in Method 1). Angle BAD is 90° (a parallel crossing a meridian on an orthomorphic chart). This means that the angle BAE is 85.67°. We know that A is 60N and B is 90N, so the distance AB is 1800 nautical miles. If we can find the distance BE we can subtract it from BD (also 1800 nm) to find DE.

Distance AE is 150 nm (see the argument in Method 1).

$$\text{Sine angle BAE} = \frac{\text{BE}}{\text{BA}}$$

$$\text{Sine } 85.67^\circ = \frac{\text{BE}}{1800}$$

$$\text{So BE} = 1794.86 \text{ nm.}$$

So DE is (1800 – 1794.86) = 5.14 nm, giving the latitude of E as 6005.14 N.

Method 3. There is a neater way of solving the above question, provided the Rhumb Line distance AD is fairly close to the Great Circle distance AE. In this example, it is. Therefore you can use Pythagorus. BAE is a right-angled triangle. Pythagorus's theorem states that:

$$\text{BA}^2 = \text{AE}^2 + \text{BE}^2$$

$$1800^2 = 150^2 + \text{BE}^2.$$

This gives the distance BE as 1793.74 nm. Subtracting this from 1800 gives 6.26 nm, making the latitude of E 6006.26N.

(*You can see that all of these methods give slightly different answers, because they are all approximations. The definitive spherical trigonometry answer is 60°05.6624′N. Nevertheless, they all give an answer that is close enough to separate the multiple-choice solutions and select answer B above.*)

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

Method 1 is reasonably accurate provided that the Great Circle Distance from the start point to the mid-longitude isn't too different from the Rhumb Line Departure distance. But it's a lot to do in 2 minutes and 12 seconds if you haven't been shown how to do it before. And when you are shown how to do it, it's a one-off formula which you don't use anywhere else and could lead to confusion with the Conversion Angle formula.

Method 2 is not too bad as long as you resist the temptation to treat it as a Polar Stereo chart.

Method 3 is quite quick, but does not have universal application. It, too, assumes that the Great Circle and the Rhumb line distance are fairly close. It would break down at near-Polar latitudes.

Can anyone suggest a neater and more elegant solution – without using spherical trig?

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press

https://doi.org/10.1017/S0373463304002681 Published online by Cambridge University Press