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The ground states of quasilinear Hénon equation with double weighted critical exponents

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We prove the existence of nontrivial ground state solutions of the critical quasilinear Hénon equation $-\Delta_p u = |x|^{\alpha_1}|u|^{p^*(\alpha_1)-2}u - |x|^{\alpha_2}|u|^{p^*(\alpha_2)-2}u$ in $\mathbb{R}^N$. It is a new problem in the sense that the signs of the coefficients of critical terms are opposite.

Keywords: Double weighted critical exponents; ground states; variational methods

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Secondary: 35H30, 35J75

1. Introduction

In this paper, we consider the p-Hénon equation

$$\begin{cases} -\Delta_p u = |x|^{\alpha_1}|u|^{p^*(\alpha_1)-2}u - |x|^{\alpha_2}|u|^{p^*(\alpha_2)-2}u \ \text{ in } \mathbb{R}^N, \\ u \in D_r^{1,p}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $1 < p < N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\alpha_1 > \alpha_2 > -p$, $p^*(\alpha_i) = \frac{p(N+\alpha_i)}{N-p}$ $(i = 1, 2)$, and $D_r^{1,p}(\mathbb{R}^N) = \{u \in D^{1,p}(\mathbb{R}^N) : u \text{ is radial}\}$, $D^{1,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|u\| := (\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x)^{1/p}$, $C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) : \ u \text{ is radial}\}$.

For $q \geqslant 1$, $\alpha \in \mathbb{R}$, let

$$L^q(\mathbb{R}^N; |x|^\alpha) := \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is Lebesgue measurable}, \ \int_{\mathbb{R}^N} |x|^\alpha |u|^q \mathrm{d}x < \infty \right\}$$

be the weighted Lebesgue space with the norm $\|u\|_{q,\alpha} := (\int_{\mathbb{R}^N} |x|^\alpha |u|^q \mathrm{d}x)^{1/q}$. For all $\alpha > -p$, the best weighted Sobolev constant

$$S_\alpha := \inf_{u \in D_r^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x}{\left(\int_{\mathbb{R}^N} |x|^\alpha |u|^{p^*(\alpha)} \mathrm{d}x\right)^{\frac{p}{p^*(\alpha)}}} \tag{1.2}$$

is achieved by the function (see [7, 24])

$$U_\alpha(x) = \frac{\left(\frac{(N-p)^{p-1}(N+\alpha)}{(p-1)^{p-1}}\right)^{\frac{N-p}{p(p+\alpha)}}}{\left(1+|x|^{\frac{p+\alpha}{p-1}}\right)^{\frac{N-p}{p+\alpha}}},$$

which is a positive solution of the critical equation

$$\begin{cases} -\Delta_p u = |x|^\alpha |u|^{p^*(\alpha)-2}u \text{ in } \mathbb{R}^N, \\ u \in D_r^{1,p}(\mathbb{R}^N). \end{cases} \tag{1.3}$$

The weighted Sobolev inequality (1.2) gives the weighted Sobolev embedding

$$D_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*(\alpha)}(\mathbb{R}^N; |x|^\alpha). \tag{1.4}$$

The number $p^*(\alpha) := \frac{p(N+\alpha)}{N-p}$ is named as the Sobolev (resp. Hardy–Sobolev, Hénon–Sobolev) critical exponent for $\alpha = 0$ (resp. $-p < \alpha < 0$ (cf. [10]), $\alpha > 0$ (cf. [20, 21, 23])). It should be pointed out that (1.2) and (1.4) are valid on $D^{1,p}(\mathbb{R}^N)$ for $-p < \alpha \leqslant 0$. Equation (1.3) with Hardy–Sobolev or Sobolev or Hénon–Sobolev critical exponent has been extensively investigated, we refer to [2, 5, 6, 8, 10–12, 15–18, 22] and some references therein.

In recent years the double critical elliptic equation

$$-\Delta_p u = |x|^{\alpha_1}|u|^{p^*(\alpha_1)-2}u + \lambda |x|^{\alpha_2}|u|^{p^*(\alpha_2)-2}u \text{ in } \mathbb{R}^N, \tag{1.5}$$

involving with Hardy–Sobolev and Sobolev critical exponents has been researched by a few of authors. Filippucci *et al.* [9, theorem 1] proved the existence of positive solutions of (1.5) for the case $\lambda = 1$, $\alpha_1 = 0$, $-p < \alpha_2 < 0$. Hsia*et al.* [13, theorem 1.2] established the ground state solutions for (1.5) as $p = 2$, $\lambda = 1$, $\alpha_1 = 0$, $-2 < \alpha_2 < 0$ in the half space $\mathbb{R}^N_+$. For (1.5) with $p = 2$, $\lambda \in \mathbb{R}$, $-2 < \alpha_2 < \alpha_1 < 0$, Li and Lin [19, theorems 1.3 and 1.4] found the ground state solutions in $\mathbb{R}^N_+$. More recently, we have established in [25] the positive ground state solutions of (1.5) as $p = 2$, $\lambda = 1$, $\alpha_1 > \alpha_2 > -2$ by using the ideas in [9]. To be more precise, the critical exponents in [25] include Hardy–Sobolev, Sobolev and Hénon–Sobolev critical exponents. In the case $p = 2$, $\alpha_i > 0$, we call (1.5) the Hénon equation which was raised by Hénon [14] in 1973 in studying the rotating stellar structures. Indeed, the results in [25] can be extended to the quasilinear case (1.5) with $1 < p < N$, $\alpha_1 > \alpha_2 > -p$. What is more interesting is that whether or not (1.5) with $\lambda = -1$ and $\alpha_1 > \alpha_2 > -p$ has nontrivial solutions. It is a new problem and has never been considered before. The following theorem gives a positive answer in radial case.

THEOREM 1.1. *Let $1 < p < N$ and $\alpha_1 > \alpha_2 > -p$. Then (1.1) has a nonnegative ground state solution.*

It is worth noting that the existence of nontrivial solutions for (1.1) with $\alpha_2 > \alpha_1 > -p$ is still an open problem. In § 2 we give the proof of theorem 1.1.

2. Proof of theorem 1.1

By the continuous embedding (1.4), weak solutions of (1.1) are exactly critical points of the C^1 functional

$$\Phi(u) = \frac{1}{p}A(u) - \frac{1}{p^*(\alpha_1)}B(u)\mathrm{d}x + \frac{1}{p^*(\alpha_2)}C(u), \quad u \in D_r^{1,p}(\mathbb{R}^N), \tag{2.1}$$

where

$$A(u) = \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x, \quad B(u) = \int_{\mathbb{R}^N} |x|^{\alpha_1}|u|^{p^*(\alpha_1)}\mathrm{d}x, \quad C(u) = \int_{\mathbb{R}^N} |x|^{\alpha_2}|u|^{p^*(\alpha_2)}\mathrm{d}x.$$

There exists a ground state solution of (1.1) provided the minimum

$$m := \inf_{u\in\mathcal{N}} \Phi(u) \tag{2.2}$$

can be achieved, where

$$\mathcal{N} := \left\{u \in D_r^{1,p}(\mathbb{R}^N)\backslash\{0\} : \ \langle \Phi'(u), u\rangle = 0\right\}$$

is the Nehari manifold for the functional Φ. Using the similar arguments in [**26**], we have the following properties about the manifold.

Lemma 2.1. *Let $\alpha_1 > \alpha_2 > -p$. For each $u \in D_r^{1,p}(\mathbb{R}^N)\backslash\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $\Phi(t_u u) = \max_{t\geqslant 0}\Phi(tu)$. The function $u \mapsto t_u$ is continuous and the map $u \mapsto t_u u$ is a homeomorphism of the unit sphere in $D_r^{1,p}(\mathbb{R}^N)$ with $\mathcal{N}$.*

Applying the mountain pass theorem in [**1**], we have the following lemma.

Lemma 2.2. *Let $\alpha_1 > \alpha_2 > -p$. There exists a sequence $\{u_n\} \subset D_r^{1,2}(\mathbb{R}^N)$ such that*

$$\Phi(u_n) \to \hat{c} > 0, \quad \Phi'(u_n) \to 0, \ n \to \infty \tag{2.3}$$

with

$$\hat{c} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)), \tag{2.4}$$

where $\Gamma := \left\{\gamma \in C([0,\, 1],\, D_r^{1,p}(\mathbb{R}^N)) : \gamma(0) = 0,\, \Phi(\gamma(1)) < 0\right\}$.

By the arguments in [**26**, chapter 4] and lemma 2.1, we get a key fact that

$$m = \hat{c}. \tag{2.5}$$

Now we analyse the properties of the $(\mathrm{PS})_{\hat{c}}$ sequence $\{u_n\}$ on the δ-ball $B_\delta := \{x \in \mathbb{R}^N : \ |x| < \delta\}$ and on the annular domain $B_{a,b} := \left\{x \in \mathbb{R}^N : \ 0 < a < |x| < b\right\}$ which are important to the proof of theorem 1.1. We remark that the discussion below will be carried out in the sense of subsequence which will be denoted by the original sequence.

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LEMMA 2.3. *Assume $u_n \rightharpoonup 0$ in $D_r^{1,p}(\mathbb{R}^N)$. Then for any annular domain $B_{a,b}$, we have*

$$\int_{B_{a,b}} |\nabla u_n|^p \mathrm{d}x \to 0, \quad \int_{B_{a,b}} |x|^{\alpha_i}|u_n|^{p^*(\alpha_i)}\mathrm{d}x \to 0 \ (i=1,2), \ n \to \infty. \tag{2.6}$$

Proof. Let $\eta \in C_{0,r}^{\infty}(\mathbb{R}^N)$ be such that $0 \leqslant \eta \leqslant 1$ and $\eta|_{B_{a,b}} \equiv 1$. Since

$$D_r^{1,p}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^q(B_R \backslash B_\rho;\ |x|^\alpha) \tag{2.7}$$

for any $R > \rho > 0$, $1 \leqslant q < \infty$ and $\alpha > -p$, see [**21**, lemma 6], it follows that

$$\int_{B_{a,b}} |x|^{\alpha_i}|u_n|^{p^*(\alpha_i)}\mathrm{d}x \to 0, \quad i=1,2,\ n\to\infty, \tag{2.8}$$

By Hölder inequality and (2.7), we get that

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-1}|\nabla(\eta^p)||u_n|\mathrm{d}x \leqslant \left(\int_{\mathbb{R}^N}|\nabla u_n|^p\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N}|\nabla(\eta^p)|^p|u_n|^p\mathrm{d}x\right)^{\frac{1}{p}} \to 0 \tag{2.9}$$

as $n \to \infty$. Furthermore, combining (2.3), (2.8), (2.9) and $\eta^p u_n \in D_r^{1,p}(\mathbb{R}^N)$, we get that

$$\begin{aligned} o(1) = \langle \Phi'(u_n), \eta^p u_n\rangle &= \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(\eta^p u_n)\mathrm{d}x + o(1) \\ &= \int_{\mathbb{R}^N} u_n|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(\eta^p) + |\eta\nabla u_n|^p\mathrm{d}x + o(1) \\ &= \int_{\mathbb{R}^N} |\eta\nabla u_n|^p\mathrm{d}x + o(1). \end{aligned}$$

It follows from $\eta|_{B_{a,b}} \equiv 1$ that

$$\int_{B_{a,b}} |\nabla u_n|^p\mathrm{d}x \to 0 \text{ as } n\to\infty,$$

and this completes the proof. □

For any $\delta > 0$, we set

$$\kappa := \lim_{n\to\infty}\int_{B_\delta}|\nabla u_n|^p\mathrm{d}x, \quad \kappa_i := \lim_{n\to\infty}\int_{B_\delta}|x|^{\alpha_i}|u_n|^{p^*(\alpha_i)}\mathrm{d}x,\ i=1,2.$$

From lemma 2.3 we see that these three quantities are well defined and are independent of the choice of $\delta > 0$. We have the following conclusion.

LEMMA 2.4. *Assume $u_n \rightharpoonup 0$ in $D_r^{1,p}(\mathbb{R}^N)$. Then*

$$\textit{either } \kappa_1 = 0 \textit{ or } \kappa_1 \geqslant S_{\alpha_1}^{\frac{p^*(\alpha_1)}{p^*(\alpha_1)-p}} \textit{ for all } \delta > 0.$$

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Proof. Let $\phi \in C^\infty_{0,r}(\mathbb{R}^N)$ satisfy $\phi|_{B_\delta} \equiv 1$. Since $\phi u_n \in D^{1,p}_r(\mathbb{R}^N)$,

$$\langle \Phi'(u_n), \phi u_n\rangle \to 0 \text{ as } n \to \infty. \tag{2.10}$$

According to lemma 2.3, we obtain that

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla(\phi u_n)\mathrm{d}x = \int_{B_\delta} |\nabla u_n|^2\mathrm{d}x + o(1),$$
$$\int_{\mathbb{R}^N} |x|^{\alpha_i}|u_n|^{p^*(\alpha_i)}\phi\mathrm{d}x = \int_{B_\delta} |x|^{\alpha_i}|u_n|^{p^*(\alpha_i)}\mathrm{d}x + o(1), \quad i = 1, 2.$$

Therefore (2.10) leads to

$$\kappa = \kappa_1 - \kappa_2. \tag{2.11}$$

The weighted Sobolev inequality (1.2) shows that

$$\left(\int_{\mathbb{R}^N} |x|^{\alpha_1}|\phi u_n|^{p^*(\alpha_1)}\mathrm{d}x\right)^{\frac{p}{p^*(\alpha_1)}} \leqslant S^{-1}_{\alpha_1}\int_{\mathbb{R}^N} |\nabla(\phi u_n)|^p\mathrm{d}x.$$

Using lemma 2.3 and (2.11), we get that

$$\kappa_1^{\frac{p}{p^*(\alpha_1)}} \leqslant S^{-1}_{\alpha_1}\kappa \leqslant S^{-1}_{\alpha_1}\kappa_1.$$

It follows that

$$\kappa_1 = 0 \text{ or } \kappa_1 \geqslant S_{\alpha_1}^{\frac{p^*(\alpha_1)}{p^*(\alpha_1)-p}},$$

and this completes the proof. □

We need the following interpolation inequality for proving lemma 2.6.

Lemma 2.5 [**24**, lemma 2.4]. *Assume* $1 < p < N$, $\alpha_1 > \alpha_2 > -p$. *For any* $u \in D^{1,p}_r(\mathbb{R}^N)$, *it holds that*

$$\|u\|_{p^*(\alpha_2),\alpha_2} \leqslant S_\theta^{-\frac{1-\tau}{p}}\|u\|^\tau_{p^*(\alpha_1),\alpha_1}\|u\|^{1-\tau},$$

where $\theta = \frac{p^*(\alpha_1)\alpha_2 - \nu\alpha_1}{p^*(\alpha_1)-\nu}$, $\tau = \frac{\nu}{p^*(\alpha_2)} \in (0, \frac{(p+\alpha_2)(N+\alpha_1)}{(p+\alpha_1)(N+\alpha_2)}]$, $0 < \nu \leqslant \frac{p+\alpha_2}{p+\alpha_1}p^*(\alpha_1)$.

Lemma 2.6. *There exist* $0 < \xi_1 < \frac{1}{2}S_{\alpha_1}^{\frac{p^*(\alpha_1)}{p^*(\alpha_1)-p}}$ *and a sequence* $\{r_n > 0\}$, *such that*

$$\tilde{u}_n(x) := r_n^{\frac{N-p}{p}}u_n(r_n x) \text{ for } x \in \mathbb{R}^N$$

verifies for all $\xi \in (0, \xi_1)$,

$$\int_{B_1} |x|^{\alpha_1}|\tilde{u}_n|^{p^*(\alpha_1)}\mathrm{d}x = \xi, \quad \forall\, n \in \mathbb{N}. \tag{2.12}$$

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Proof. It follows from $\hat{c} > 0$ and lemma 2.5 that $\kappa_\infty := \lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^{\alpha_1} |u_n|^{p^*(\alpha_1)} \mathrm{d}x > 0$. Let $\xi_1 := \min\{S_{\alpha_1}^{\frac{p^*(\alpha_1)}{p^*(\alpha_1)-p}}, \kappa_\infty\}$, for fixed $\xi \in (0, \xi_1)$ and any $n \in \mathbb{N}$, there exists $r_n > 0$ such that

$$\int_{B_{r_n}} |x|^{\alpha_1} |u_n|^{p^*(\alpha_1)} \mathrm{d}x = \xi.$$

By scaling, it is straightforward to check that $\{\tilde{u}_n\}$ satisfies (2.12). □

Proof of theorem 1.1. It is easy to see that $\{\tilde{u}_n\}$ satisfies (2.3). Since $p^*(\alpha_1) > p^*(\alpha_2) > p$, it follows from (2.3) that

$$\Phi(\tilde{u}_n) - \frac{1}{p^*(\alpha_2)} \langle \Phi'(\tilde{u}_n), \tilde{u}_n \rangle \geqslant \left(\frac{1}{p} - \frac{1}{p^*(\alpha_2)} \right) \|\tilde{u}_n\|^p. \tag{2.13}$$

Thus $\{\tilde{u}_n\}$ is bounded in $D_r^{1,p}(\mathbb{R}^N)$ and then there exists $\tilde{u} \in D_r^{1,p}(\mathbb{R}^N)$ such that

$$\begin{cases} \tilde{u}_n \rightharpoonup \tilde{u} \text{ in } D_r^{1,p}(\mathbb{R}^N); \\ \tilde{u}_n \rightharpoonup \tilde{u} \text{ in } L^{p^*(\alpha_i)}(\mathbb{R}^N; |x|^{\alpha_i}), \quad i = 1, 2. \\ \tilde{u}_n(x) \to \tilde{u}(x) \text{ a.e. on } \mathbb{R}^N. \end{cases}$$

Using the ideas of Boccardo and Murat [**3**] (see details in [**24**]), we can prove that $\nabla \tilde{u}_n(x) \to \nabla \tilde{u}(x)$ a.e. on $\mathbb{R}^N$. It follows that $\tilde{u}$ is a critical point of Φ and $\Phi(\tilde{u}) \geqslant 0$ by (2.13) again. Let $v_n := \tilde{u}_n - \tilde{u}$, then $\{v_n\}$ is bounded in $D_r^{1,p}(\mathbb{R}^N)$. Assume

$$A(v_n) \to A_\infty, \quad B(v_n) \to B_\infty, \; C(v_n) \to C_\infty.$$

Using Brezis–Lieb lemma[**4**], we get

$$\Phi(v_n) \to \frac{1}{p} A_\infty - \frac{1}{p^*(\alpha_1)} B_\infty + \frac{1}{p^*(\alpha_2)} C_\infty = \hat{c} - \Phi(\tilde{u}), \tag{2.14}$$

$$\langle \Phi'(v_n), v_n \rangle \to A_\infty - B_\infty + C_\infty = 0. \tag{2.15}$$

If $A_\infty = 0$, then $\tilde{u}$ is ground state solution of (1.1). Assume that $A_\infty > 0$ and $\tilde{u} = 0$. Then lemma 2.4 implies that

$$\text{either } \lim_{n\to\infty} \int_{B_1} |x|^{\alpha_1} |\tilde{u}_n|^{p^*(\alpha_1)} \mathrm{d}x = 0 \text{ or } \lim_{n\to\infty} \int_{B_1} |x|^{\alpha_1} |\tilde{u}_n|^{p^*(\alpha_1)} \mathrm{d}x \geqslant S_{\alpha_1}^{\frac{p^*(\alpha_1)}{p^*(\alpha_1)-p}}.$$

This contradicts (2.12) with $0 < \xi < \frac{1}{2} S_{\alpha_1}^{\frac{p^*(\alpha_1)}{p^*(\alpha_1)-p}}$. Thus $\tilde{u}$ is nontrivial. If $\Phi(\tilde{u}) = \hat{c}$, then we finish the proof with (2.5). Otherwise, we get that

$$\Phi(\tilde{u}) > m = \hat{c}.$$

Since

$$\Phi(v_n) - \frac{1}{p^*(\alpha_2)} \langle \Phi'(v_n), v_n \rangle \geqslant \left(\frac{1}{p} - \frac{1}{p^*(\alpha_2)} \right) A(v_n) \geqslant 0,$$

we get by (2.14) and (2.15) that

$$\Phi(\tilde{u}) \leqslant \hat{c},$$

which contradicts (2). It follows that $\tilde{u}$ is a ground state solution of (1.1).

By the structure of the manifold $\mathcal{N}$, we get that $|\tilde{u}| \in \mathcal{N}$, then a nonnegative ground state solution is established. □

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References

1 A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14** (1973), 349–381.

2 T. Aubin. Problèmes isopérimétriques et espaces de Sobolev (in French). *J. Differ. Geom.* **11** (1976), 573–598.

3 L. Boccardo and F. Murat. Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.* **19** (1992), 581–597.

4 H. Brézis and E. Lieb. A relation between pointwise convergence of functions and convergence of functionals. *Proc. Am. Math. Soc.* **88** (1983), 486–490.

5 L. Caffarelli, R. Kohn and L. Nirenberg. First order interpolation inequalities with weights. *Compositio Math.* **53** (1984), 259–275.

6 F. Catrina and Z.-Q. Wang. On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. *Commun. Pure Appl. Math.* **54** (2001), 229–258.

7 F. Catrina and Z.-Q. Wang, A one-dimensional nonlinear degenerate elliptic equation. *Proceedings of the USA-Chile Workshop on Nonlinear Analysis* (Viña del Mar-Valparaiso, 2000), 89–99, *Electron. J. Differ. Equ. Conf., 6*, Southwest Texas State Univ., San Marcos, TX, 2001.

8 K. S. Chou and C. W. Chu. On the best constant for a weighted Sobolev–Hardy inequality. *J. London Math. Soc.* **48** (1993), 137–151.

9 R. Filippucci, P. Pucci and F. Robert. On a p-Laplace equation with multiple critical nonlinearities. *J. Math. Pure Appl.* **91** (2009), 156–177.

10 N. Ghoussoub and C. Yuan. Multiple solutions for quasi-linear PDEs involving the critical Sobolev and Hardy exponents. *Trans. Am. Math. Soc.* **352** (2000), 5703–5743.

11 B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Commun. Pure Appl. Math.* **24** (1981), 525–598.

12 F. Gladiali, M. Grossi and S. L. N. Neves. Nonradial solutions for the Hénon equation in $\mathbb{R}^N$. *Adv. Math.* **249** (2013), 1–36.

13 C.-H. Hsia, C.-S. Lin and H. Wadade. Revisiting an idea of Brézis and Nirenberg. *J. Funct. Anal.* **259** (2010), 1816–1849.

14 M. Hénon. Numerical experiments on the stability of spherical stellar systems. *Astronom. Astrophys.* **24** (1973), 229–238.

15 T. Horiuchi. Best constant in weighted Sobolev inequality with weights being powers of distance from the origin. *J. Inequal. Appl.* **1** (1997), 275–292.

16 E. Lieb. Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. *Ann. Math.* **118** (1983), 349–374.

17 P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana* **1** (1985), 145–201.

18 P. -L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. II. *Rev. Mat. Iberoamericana* **1** (1985), 45–121.

19 Y. Y. Li and C.-S. Lin. A nonlinear elliptic PDE and two Sobolev–Hardy critical exponents. *Arch. Ration. Mech. Anal.* **203** (2012), 943–968.

20 J. Su, Z.-Q. Wang and M. Willem. Nonlinear Schröodinger equations with unbounded and decaying radial potentials. *Comm. Contemp. Math.* **9** (2007), 571–583.

21 J. Su, Z.-Q. Wang and M. Willem. Weighted Sobolev embedding with unbounded and decaying radial potential. *J. Differ. Equ.* **238** (2007), 201–219.

22 G. Talenti. Best constant in Sobolev inequality. *Ann. Math. Pura Appl.* **110** (1976), 353–372.

23 C. Wang and J. Su. Critical exponents of weighted Sobolev embeddings for radial functions. *Appl. Math. Lett.* **107** (2020), 106484.

24 C. Wang and J. Su, The ground states of Hénon equations for p-Laplacian in $\mathbb{R}^N$ involving upper weighted critical exponents. Preprint, 2020.

25 C. Wang and J. Su. The semilinear elliptic equations with double weighted critical exponents. *J. Math. Phys.* **63** (2022), 041505.

26 M. Willem. *Minimax Theorems* (Birkhuser Boston Inc., Boston, 1996).