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Non-classical global solutions for a class of scalar conservation laws

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We consider the Cauchy problem for a class of scalar conservation laws with flux having a single inflection point. We prove existence of global weak solutions satisfying a single entropy inequality together with a kinetic relation, in a class of bounded variation functions. The kinetic relation is obtained by the travelling-wave criterion for a regularization consisting of balanced diffusive and dispersive terms. The result is applied to the one-dimensional Buckley–Leverett equation.

1. Introduction

In this paper we establish an existence theorem for (non-classical) weak solutions of the Cauchy problem associated with a nonlinear hyperbolic conservation law,

$$\partial_t u + \partial_x f(u) = 0, \qquad u(t,x) \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}. \tag{1.2}$$

The flux-function $f : \mathbb{R} \to \mathbb{R}$ is *non-convex* and changes sign precisely at one point u_{I}. The initial data $u_0 : \mathbb{R} \to \mathbb{R}$ is a function of bounded variation. It is well known (see [14]) that smooth solutions of (1.1), (1.2) may develop discontinuities in finite time, so we are led to consider weak solutions in a distributional sense.

In recent years, many authors have shown interest in studying the presence of 'non-classical' shocks in the solutions of some scalars [1, 3, 10, 13] or systems of conservation laws [8, 12, 21], as, for example, in nonlinear elasticity and magnetohydrodynamics [5, 16, 17, 19, 22], as well as some numerical evidence of the presence of this kind of shock in the solutions of the equations [11, 13].

Non-classical shocks are discontinuities that do not satisfy the usual Lax inequalities (see § 2) on the speed of propagation. For $n \times n$ systems of conservation laws, they can be *overcompressive* when there are more than $n+1$ characteristics impinging on the shock, or *undercompressive* when there are less than $n+1$ characteristics impinging on it (see the book of Bressan [7] for an overview of the basics of hyperbolic systems of conservation laws). For scalar conservation laws, non-classical shocks are always undercompressive.

Non-classical solutions of (1.1), (1.2) are weak solutions that may contain non-classical shocks. Such solutions are usually related to zero diffusion-dispersion limits (see [15] and the references cited therein) such as

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_{xx} u + \gamma \varepsilon^2 \partial_{xxx} u, \quad \varepsilon \to 0 \text{ with } \gamma \text{ fixed.} \tag{1.3}$$

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Usually, every $n\times n$ physical system of conservation laws comes with an associated strictly convex *entropy*, that is, a strictly convex function $U:\mathbb{R}^n\to\mathbb{R}$, and an associated entropy flux $F:\mathbb{R}^n\to\mathbb{R}$ such that $\partial_t U(u)+\partial_x F(u)=0$ for all smooth solutions $u=u(t,x)$ of the system. For weak solutions, equality must be substituted by an *entropy inequality*, i.e.

$$\partial_t U(u)+\partial_x F(u)\leqslant 0, \tag{1.4}$$

in a distributional sense. For genuinely nonlinear systems, equation (1.4) selects the discontinuities that are admissible. In this way, every associated Riemann problem (see § 2) admits a unique self-similar solution that is the combination of a certain number of admissible waves. In turn, this allows us to construct a solution to the associated Cauchy problem, at least when the data are of small total variation.

On the other hand, for non-convex scalar equations, one entropy inequality is not enough to single out a unique way of solving the Riemann problem. This happens because there are too many non-classical shocks that are entropy admissible. So we need to supplement the entropy inequality by an additional *kinetic relation* (see [3, 10] and the references therein), or, equivalently, by using a *kinetic function* φ. For weak solutions obtained as limits of diffusive-dispersive approximations, the form of φ is related to the admissible travelling waves for the associated equation (1.3). The kinetic relation thus permits us to recover some information that was neglected at the hyperbolic level. By imposing both the entropy condition and the kinetic relation, it is possible to construct a unique solution of each associated Riemann problem.

In [1, 3], non-classical solutions are constructed by wave-front tracking methods [2,7] under some mild assumptions on the flux function f and the kinetic function φ. This technique consists of constructing a sequence of piecewise constant approximate solutions and then using a compactness argument to find a solution in the limit.

The purpose of this paper is to study the special case of the one-dimensional Buckley–Leverett equation (see § 6), applying techniques similar to those introduced in [1,3]. More precisely, we will study the class of equations presented in [13], to which the Buckley–Leverett equation belongs. Unfortunately, these equations do not satisfy the assumptions made in [1, 3, 4], and hence a new analysis is needed for this class. One of the main features is that it is possible to describe many properties of their kinetic function. As a matter of fact, in [13], the authors study the discontinuities that are admissible for a diffusive-dispersive regularization and prove, for the Buckley–Leverett equation under some additional hypotheses, the existence of a solution of every Riemann problem with data taking values in the interval $[0, 1]$. Starting from these solutions, we derive properties of the underlying kinetic function related to the travelling waves of the regularization. This allows us to use front-tracking techniques to prove the existence of weak non-classical solutions of the Cauchy problem (1.1), (1.2). Moreover, we will prove the existence under more general hypotheses than those in [13], also simplifying the description of the Riemann solutions.

The main difference between classical and non-classical solutions is that the latter ones may be non-monotone and not total variation diminishing. Indeed, interactions between relatively small waves can generate big non-classical shock waves with a consequent large increment of the total variation. To deal with this situation, for the class of fluxes considered, a new functional is introduced, equivalent to the total variation and decreasing along approximate solutions.

The paper is organized as follows. After some preliminaries, section 3 deals with the solution to the Riemann problem and its relations with the results in [13]. Then the wave-front-tracking algorithm is explained. Section 4 reports the complete list of all the possible interaction patterns under hypotheses (H1)–(H3), while the main existence result of this paper is stated and proved in § 5. The special case of the one-dimensional Buckley–Leverett equation is considered in § 6.

2. Preliminaries

We consider scalar equations (1.1) with the following hypotheses on the flux function f.

(H1) f is C^2.

(H2) It has a single inflection point at u_{I}. The function is convex up for $u < u_{\mathrm{I}}$ and convex down for $u > u_{\mathrm{I}}$, i.e. $f''(u)(u-u_{\mathrm{I}}) < 0$ for $u \neq u_{\mathrm{I}}$, $f'''(u) < 0$ for all u.

(H3) For u close to u_{I}, for some $K \in \mathbb{R}$, $H \neq 0$ and $p \in \mathbb{N}$, we have the expansion

$$f(u) = f(u_{\mathrm{I}}) + K(u - u_{\mathrm{I}}) + H(u - u_{\mathrm{I}})^{2p+1} + o((u - u_{\mathrm{I}})^{2p+1}).$$

We will also choose the usual entropy pair

$$U(u) = \tfrac{1}{2}u^2, \qquad F(u) = \int^u s f'(s)\,\mathrm{d}s. \tag{2.1}$$

In § 6, we will restrict our attention to the one-dimensional Buckley–Leverett equation.

We want to construct a sequence of piecewise constant approximate solutions $u_\nu(t, x)$ that converge to a solution $u(t, x)$ in the limit. For this purpose, we need strong convergence in L^1 and, as usual, this amounts to proving a uniform upper bound on the total variation of $u_\nu(t, \cdot)$ for all $t > 0$ and $\nu \in \mathbb{N}$ (see [7]).

The building block in the definition of u_ν is the Riemann Problem, i.e. the Cauchy problem when the initial datum is of the form

$$u(0, x) = \begin{cases} u_- & \text{if } x < 0, \\ u_+ & \text{if } x > 0. \end{cases} \tag{2.2}$$

The solution of the Riemann problem (1.1), (2.2) is a self-similar function $u(t, x) = v(x/t)$. More precisely, in the phase space, the solution consists of a certain number of constant states divided by elementary waves: shocks, rarefactions or one-sided contact discontinuities. *Rarefaction waves* are piecewise Lipschitz solutions of (1.1), (2.2) of the form

$$u(t, x) = \begin{cases} u_- & \text{if } x \leqslant f'(u_-)t, \\ u & \text{if } f'(u_-)t \leqslant x = f'(u)t \leqslant f'(u_+)t, \\ u_+ & \text{if } x \geqslant f'(u_+)t, \end{cases}$$

while a *shock wave* is a discontinuous solution of (1.1), (2.2) of the form

$$u(t,x) = \begin{cases} u_- & \text{if } x < \lambda t, \\ u_+ & \text{if } x > \lambda t, \end{cases}$$

where u_-, u_+ and the shock speed λ satisfy the Rankine–Hugoniot equation (see [7])

$$\lambda(u_+ - u_-) = f(u_+) - f(u_-).$$

A shock is *compressive* when its speed satisfies the Lax inequality $f'(u_+) \leqslant \lambda \leqslant f'(u_-)$. Otherwise, in the scalar case, it is *undercompressive*.

To solve the Riemann problem we need to decide which elementary waves should be considered to be admissible. For systems of conservation laws in the literature, there are several admissibility criteria, among which the more common are the vanishing viscosity, the entropy dissipation and the Lax stability criterion. For genuinely nonlinear systems and for small discontinuities, these three conditions are proved to be equivalent. Instead, for shocks with larger amplitude and non-convex fluxes, we can use the Oleinik–Liu criterion (see [18, 20]).

In [10], the authors analyse the shocks that are admissible under a vanishing viscosity-dispersion criterion. For $f(u) = u^3$, they prove the existence of travelling-wave solutions of the diffusive-dispersive approximate equation

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_{xx} u + \gamma \varepsilon^2 \partial_{xxx} u, \quad \gamma > 0, \tag{2.3}$$

corresponding to discontinuities of the conservation law (1.1) that violate Lax inequalities. These kind of discontinuities are commonly called 'non-classical shocks' and, for scalar equations, are undercompressive, i.e. characteristics enter from one side but exit from the other side of the discontinuity. Assuming these discontinuities to be admissible, and supplementing the equation by an additional entropy inequality and a 'nucleation criterion', Hayes and LeFloch prove the existence of a unique solution of the Riemann problem (1.1), (2.2) for every u_-, u_+, depending L^1-continuously on the left and right states.

In [1], the authors prove the existence of non-classical solutions (i.e. solutions that admit non-classical waves) to the Cauchy problem (1.1), (1.2) starting from the Riemann solver proposed in [10]. The results are then extended to a more general class of equations [3].

In another paper, Hayes and Shearer [13] (also see [6] for a generalization of their results) analyse the travelling waves for a diffusive-dispersive approximation for a class of fluxes satisfying hypotheses (H1), (H2) and prove the existence of a solution of the Riemann problem for equation (6.1) under some additional hypotheses (see assumptions 5.1, 5.2 in [13]). In the following sections, we will show how to prove the existence of non-classical solutions based on the Riemann solver introduced in [13], using an approach analogous to that in [1, 3].

3. The Riemann problem and approximate solutions

In this section we want to define a sequence of piecewise constant approximate solutions to the Cauchy problem (1.1), (1.2), applying a front tracking scheme. The starting point is the solution of the Riemann problem (1.1), (2.2). In [3], it is shown that in order to define the solution for every pair of left and right states u_-, u_+, it is sufficient to give an entropy pair U, F together with a *kinetic function* φ. The knowledge of φ is sufficient to describe the admissible waves that will be used to solve, in a unique way, every Riemann problem. Under the hypotheses made in [3], for each $u \in \mathbb{R}$, the value $\varphi(u)$ represents the unique state that can be connected to the right of u with an admissible non-classical undercompressive shock. In this paper and under hypotheses (H1)–(H3), it will represent the unique state that can be connected to the *left* of u with an admissible non-classical undercompressive shock.

The precise form of φ is recovered by the results in [13] (see also [6]), where a wave is considered admissible if its left and right states u_- and u_+ can be connected by a travelling wave for the diffusive-dispersive approximation

$$\partial_t u + \partial_x f(u) = \alpha\varepsilon\partial_{xx}u - \beta\varepsilon^2\partial_{xxx}u, \tag{3.1}$$

for some values of $\alpha, \beta \geqslant 0$. Denoting $\gamma = \alpha/\sqrt{\beta}$, this amounts to saying that, for some s, the couples $(u_-, 0)$ and $(u_+, 0)$ are two equilibria of the system of ordinary differential equations

$$\begin{aligned} u' &= v, \\ v' &= \gamma v - (f(u) - f(u_-) - s(u - u_-)), \end{aligned}$$

which can be connected by a heteroclinic orbit [13]. If, in addition, $(u_-, 0)$ and $(u_+, 0)$ are saddle points, then we say that $u_- \to u_+$ is a saddle-to-saddle connection. The main result in [13] (see § 4 and theorem 4.1 in [13]) is summarized in the following.

THEOREM 3.1. *The discontinuity connecting u_- and u_+ is an undercompressive shock if and only if $u_- \to u_+$ is a saddle-to-saddle connection and the Rankine Hugoniot condition holds with s equal to the speed of the travelling wave.*

Let $\gamma = \gamma_0 > 0$, and suppose that there is a saddle-to-saddle connection $u_-^0 \to u_+^0$ for some $u_-^0 < u_{\mathrm{I}} < u_+^0$. Then, for each $\gamma > 0$ near γ_0, there is $u^* = u^*(\gamma) < u_{\mathrm{I}}$ and a C^1 function $g(u_-)$, $u_- < u^*$, such that

(a) *$u_- \to u_+$ is a saddle-to-saddle connection with speed s if and only if*

$$u_- < u^* \quad \textit{and} \quad s = g(u_-) = \frac{f(u_+) - f(u_-)}{u_+ - u_-}.$$

(b) $\dfrac{\mathrm{d}g}{\mathrm{d}u_-}(u_-) > 0$ *for* $u_- < u^*$.

(c) *Writing $u_+ = u_+(u_-)$, we have*

$$\frac{\mathrm{d}u_+}{\mathrm{d}u_-}(u_-) < 0 \quad \textit{for } u_- < u^*.$$

(d) $\lim_{u_- \to u^*-} g(u_-) = f'(u^*)$.

A similar result holds for $u_-^0 > u_{\mathrm{I}}$ and some u^{**}.

Thanks to this result, Hayes and Shearer show the existence, under some additional hypotheses (assumptions 5.1, 5.2 in [13]), and provide the form of the solution of the Riemann problem for every choice of the left and right states, giving a complete description of the wave curve. The global overview is somewhat complex. Moreover, the list of all the possible interaction patterns would be long and complex to write. Here we adopt a different point of view. Given a state u_-, instead of the right wave curve starting from u_-, i.e. the collection of all the states that can be connected to u_- on the right, we study the left wave curve from u_-. We believe that this approach simplifies the computations, giving a clear idea of the geometry of the solutions and allowing the use of the machinery introduced in [1,3]. Moreover, it should clarify the choice of the additional hypotheses introduced by Hayes and Shearer and also provides the existence of the Riemann solver without assumption 5.2 of [13].

To describe the left wave curve, we first need some notations. For simplicity, assume that $f : \mathbb{R} \to \mathbb{R}$. For every $u \neq 0$, we denote by $\tau(u)$ the unique value such that

$$\frac{f(\tau(u)) - f(u)}{\tau(u) - u} = f'(\tau(u)).$$

In fact, the existence of τ for all u is not guaranteed by (H1)–(H3) only, but we have to put some additional assumptions on the first derivative of the flux, for example (if $f : \mathbb{R} \to \mathbb{R}$), $\lim_{|u|\to\infty} f'(u) = -\infty$. We do not want to specify such assumptions, but just assume the existence of τ.

From (d), we easily get that the limit $u_+^* := \lim_{u_- \to u^{*-}} u_+(u_-)$ must satisfy

$$\frac{f(u_+^*) - f(u^*)}{u_+^* - u^*} = f'(u^*),$$

and hence $u^* = \tau(u_+^*)$.

Analogously, we can see that the limit $u_+^{**} := \lim_{u_- \to u^{**+}} u_+(u_-)$ satisfies $u^{**} = \tau(u_+^{**})$.

By theorem 3.1, the function $u \mapsto u_+(u)$ is invertible. We call $\varphi(v) = (u_+)^{-1}(v)$ the inverse, which is defined on the set $]-\infty, u_+^{**}] \cup [u_+^*, +\infty[$. In the framework of [13], we call $\alpha(v)$ the middle equilibrium between $\varphi(v)$ and v such that

$$\frac{f(\varphi(v)) - f(v)}{\varphi(v) - v} = \frac{f(\alpha(v)) - f(v)}{\alpha(v) - v}.$$

Take $u_{\mathrm{I}} < u_+$. With the notations introduced, the solution of the Riemann problem (u_-, u_+) is described as follows.

(S1) If $u_{\mathrm{I}} \leqslant u_+ \leqslant u_+^*$, the solution coincides with the classical Oleinik–Liu solution [18,20].

(S2) If $u_+^* < u_+$, we have some subcases.

- (i) If $u_+ < u_-$, the solution consists of a rarefaction wave connecting u_- to u_+.
- (ii) If $\alpha(u_+) \leqslant u_- < u_+$, the solution consists of a classical Lax shock connecting u_- to u_+.

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(iii) If $\varphi(u_+) < u_- < \alpha(u_+)$, the solution is given by a classical shock connecting u_- to $\varphi(u_+)$ followed by a non-classical undercompressive shock connecting $\varphi(u_+)$ to u_+.

(iv) If $u_- \leqslant \varphi(u_+)$, the solution consists of a rarefaction wave connecting u_- to $\varphi(u_+)$ followed by a non-classical undercompressive shock connecting $\varphi(u_+)$ to u_+.

Notice that the obtained solution can be non-monotone. An analogous description holds for the case $u_+ < u_{\mathrm{I}}$, with all the signs reversed and u_+^{**} in place of u_+^*. Finally, if $u_+ = u_{\mathrm{I}}$, then the solution of the Riemann problem (u_-, u_+) is always given by a rarefaction wave.

Concerning assumption 5.2 of [13], as suggested by Hayes and Shearer and as seen above, it is not necessary to solve the Riemann problems.

For simplicity, we write (but in the following we shall drop the tilde accent)

$$\tilde{\varphi}(u) := \begin{cases} \varphi(u) & \text{if } u < u_+^{**} \text{ or } u_+^* < u, \\ \tau(u) & \text{if } u_+^{**} \leqslant u \leqslant u_+^*, \end{cases}$$

and define $\alpha(u) = \tau(u)$ for $u_+^{**} \leqslant u \leqslant u_+^*$. Then the solver (S1) can be viewed as a subcase of (S2) for which subcase (iii) never happens. In this case (or, generally, when $\varphi(u) = \tau(u)$), the discontinuity connecting $\varphi(u)$ to u is not properly a non-classical shock, but a one-sided contact discontinuity, i.e. its speed is equal to the characteristic speed of the left state. Nevertheless, for simplicity, it will be convenient to also consider these discontinuities as non-classical shocks. Notice that, by theorem 3.1 (c) and the definition of τ, it follows that φ is monotone decreasing.

Now we will describe how to construct piecewise constant approximate solutions of (1.1), (1.2) by wave-front tracking [2,7]. First we start to approximate the initial datum u_0 with a sequence of piecewise constant functions u_ν such that $u_\nu \to u_0$ in L^1 and $\mathrm{TV}(u_\nu) \leqslant C \cdot \mathrm{TV}(u_0)$. Next, for each ν, we locally solve each Riemann problem arising at each discontinuity point of u_ν by using the chosen Riemann solver. Since we want a piecewise constant function, due to the presence of rarefactions, we have to do it in an approximate way. More precisely, every rarefaction front is split into many small discontinuities, no larger than $\delta_\nu > 0$, where $\delta_\nu \to 0$ as $\nu \to \infty$. Each small jump travels with the characteristic speed of the left state (but, indeed, any speed between the characteristic speed on the left and right state will work). In this way, we define a piecewise constant function for small t. Now we prolong each front emerging from the Riemann problems at time 0 until two of these waves interact. A new Riemann problem is set and again we approximately solve it. The only difference is that, for positive times, the rarefaction fronts are substituted by a unique rarefaction shock with equal size. In this way, we define a function $u_\nu(t, x)$. To prove that $u_\nu(t, x)$ is globally defined for all positive times, we need to prove that this procedure can be iterated and this follows by proving that the total numbers of waves and interactions are finite (see § 5).

Next, to prove convergence of (a subsequence of) u_ν in $\mathrm{L}^1_{\mathrm{loc}}$, we need to supply a uniform bound on the total variation of $u_\nu(t, \cdot)$. Since $\mathrm{TV}(u_\nu(t, \cdot))$ is constant outside the interaction times, we are led to study how the total variation changes across each interaction. To this end, in [1, 3], a new functional V is introduced, equivalent to the total variation and such that $\mathrm{V}(u_\nu(t,\cdot))$ is decreasing in time. The precise form of V is obtained by listing all the possible interaction patterns and carefully studying how the total variation changes across each of them.

We want to use similar techniques to the class of equations satisfying (H1)–(H3) and, in particular, to (6.1).

4. List of wave interactions

In the following we shall list all the possible interaction patterns. For each of them, we will identify the incoming and outgoing wavefronts. Keeping in mind the point of view introduced in the previous section, we will consider a right incoming wave connecting the states u_m and u_r interacting with a left incoming wave connecting the states u_l and u_m. For simplicity, we will assume $u_\mathrm{r} > u_\mathrm{I}$ fixed (the case $u_\mathrm{r} < u_\mathrm{I}$ being similar) and let u_m and u_l vary. In the following, we will use R, C and NC to denote a rarefaction wave, a classical and a non-classical shock, respectively. Moreover, we will use, for example, the notation $(\mathrm{C},\mathrm{R}) \to (\mathrm{R})$ to denote an interaction between a left incoming classical shock and a right incoming rarefaction wave, producing an outgoing rarefaction wave.

The result is similar to that obtained in [1].

4.1. The case $u_\mathrm{m} > u_\mathrm{r}$

Since $u_\mathrm{m} > u_\mathrm{r}$, the right incoming wave is a rarefaction. The left incoming one cannot be a rarefaction too, otherwise they would not interact. Hence it must be a shock wave, classical or non-classical. Thus $u_\mathrm{l} < u_\mathrm{m}$.

There are various subcases.

Case 1 ($(\mathrm{C},\mathrm{R}) \to (\mathrm{R})$). When $u_\mathrm{l} \in [u_\mathrm{r}, u_\mathrm{m}]$. Since $u_\mathrm{l} \geqslant u_\mathrm{r}$, then the interaction produces a rarefaction wave from u_l to u_r.

Case 2 ($(\mathrm{C},\mathrm{R}) \to (\mathrm{C})$). When $u_\mathrm{l} \in [\alpha(u_\mathrm{r}), u_\mathrm{r})$. In this case, the incoming rarefaction cancels out with a part of the incoming shock.

Case 3 ($(\mathrm{C},\mathrm{R}) \to (\mathrm{C},\mathrm{NC})$). When $u_\mathrm{l} \in (\varphi(u_\mathrm{r}), \alpha(u_\mathrm{r}))$ and $u_\mathrm{l} \geqslant \alpha(u_\mathrm{m})$. In this case, $\alpha(u_\mathrm{m}) < \alpha(u_\mathrm{r})$ and there are two outgoing waves: a classical shock connecting u_l to $\varphi(u_\mathrm{r})$ followed by the non-classical shock connecting $\varphi(u_\mathrm{r})$ to u_r. Notice that the solution is non-monotone.

Case 4 ($(\mathrm{C},\mathrm{R}) \to (\mathrm{R},\mathrm{NC})$). In this case, $u_\mathrm{l} \in (\alpha(u_\mathrm{m}), \varphi(u_\mathrm{r})]$ and $\alpha(u_\mathrm{m}) < \varphi(u_\mathrm{r})$. The interaction produces a rarefaction between $\varphi(u_\mathrm{r})$ and u_l followed by a non-classical shock.

Case 5 ($(\mathrm{NC},\mathrm{R}) \to (\mathrm{C},\mathrm{NC})$). When $u_\mathrm{l} = \varphi(u_\mathrm{m})$ and $\varphi(u_\mathrm{r}) < \varphi(u_\mathrm{m})$. The outgoing waves are a classical shock connecting u_l to $\varphi(u_\mathrm{r})$ and a non-classical shock from $\varphi(u_\mathrm{r})$ to u_r.

Case 6 ($(\mathrm{NC},\mathrm{R}) \to (\mathrm{R},\mathrm{NC})$). When $u_\mathrm{l} = \varphi(u_\mathrm{m})$ and $\varphi(u_\mathrm{m}) \leqslant \varphi(u_\mathrm{r})$. The outgoing waves are a rarefaction connecting u_l to $\varphi(u_\mathrm{r})$ and a non-classical shock from $\varphi(u_\mathrm{r})$ to u_r.

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4.2. The case $u_{\mathrm{m}} \in [\alpha(u_{\mathrm{r}}), u_{\mathrm{r}})$

This means that the right incoming wave is a classical shock. The left incoming wave can be either a shock or a rarefaction. In the case $u_{\mathrm{m}} \geqslant u_{\mathrm{I}}$, we have the following subclasses.

Case 7 ((R, C) $\to$ (C)). When $u_{\mathrm{l}} \in (u_{\mathrm{m}}, u_{\mathrm{r}})$. The outgoing wave is a shock connecting u_{l} and u_{r}.

Case 8 ((C, C) $\to$ (C)). When $u_{\mathrm{m}} > u_{\mathrm{l}} \geqslant \max\{\alpha(u_{\mathrm{r}}), \alpha(u_{\mathrm{m}})\}$.

Case 9 ((C, C) $\to$ (C, NC)). When $u_{\mathrm{l}} \in (\alpha(u_{\mathrm{m}}), \alpha(u_{\mathrm{r}}))$ and $\alpha(u_{\mathrm{m}}) < \alpha(u_{\mathrm{r}})$. The interaction produces a classical shock connecting u_{l} to $\varphi(u_{\mathrm{r}})$ and a non-classical shock from $\varphi(u_{\mathrm{r}})$ to u_{r}.

Case 10 ((NC, C) $\to$ (C)). If $u_{\mathrm{l}} = \varphi(u_{\mathrm{m}})$ with $\varphi(u_{\mathrm{m}}) \geqslant \alpha(u_{\mathrm{r}})$.

Case 11 ((NC, C) $\to$ (C, NC)). If $u_{\mathrm{l}} = \varphi(u_{\mathrm{m}})$ with $\varphi(u_{\mathrm{r}}) < \varphi(u_{\mathrm{m}}) < \alpha(u_{\mathrm{r}})$. Again, a classical shock connecting u_{l} to $\varphi(u_{\mathrm{r}})$ and a non-classical shock from $\varphi(u_{\mathrm{r}})$ to u_{r} are produced.

Case 12 ((NC, C) $\to$ (R, NC)). If $u_{\mathrm{l}} = \varphi(u_{\mathrm{m}})$ with $\varphi(u_{\mathrm{r}}) \geqslant \varphi(u_{\mathrm{m}})$. In this case, the outgoing waves are a rarefaction connecting u_{l} to $\varphi(u_{\mathrm{r}})$ and a non-classical shock from $\varphi(u_{\mathrm{r}})$ to u_{r}.

The following deals with the case $\alpha(u_{\mathrm{r}}) \leqslant u_{\mathrm{m}} < u_{\mathrm{I}}$.

Case 13 ((NC, C) $\to$ (C)). If $u_{\mathrm{l}} = \varphi(u_{\mathrm{m}})$. The non-classical shock is cancelled out.

Case 14 ((C, C) $\to$ (C)). If $u_{\mathrm{m}} < u_{\mathrm{l}} \leqslant \alpha(u_{\mathrm{m}})$. Then the incoming classical shock is cancelled out.

Case 15 ((R, C) $\to$ (C)). If $\alpha(u_{\mathrm{r}}) \leqslant u_{\mathrm{l}} < u_{\mathrm{m}}$.

Case 16 ((R, C) $\to$ (C, NC)). If $\varphi(u_{\mathrm{r}}) < u_{\mathrm{l}} < \alpha(u_{\mathrm{r}})$. The interaction generates a classical shock from u_{l} to $\varphi(u_{\mathrm{r}})$ and a non-classical shock from $\varphi(u_{\mathrm{r}})$ to u_{r}.

Case 17 ((R, C) $\to$ (R, NC)). If $\varphi(u_{\mathrm{r}}) > u_{\mathrm{l}}$. The interaction generates a rarefaction connecting u_{l} to $\varphi(u_{\mathrm{r}})$ and a non-classical shock from $\varphi(u_{\mathrm{r}})$ to u_{r}.

4.3. Case $u_{\mathrm{m}} = \varphi(u_{\mathrm{r}})$

The right incoming front is a non-classical shock connecting u_{m} to u_{r}. The left incoming wave cannot be a rarefaction, since the two waves would not meet. Hence it must be a shock; classical when $u_{\mathrm{l}} \in (\alpha(u_{\mathrm{r}}), \alpha(u_{\mathrm{m}})]$ and non-classical when $u_{\mathrm{l}} = \varphi(u_{\mathrm{m}})$.

Case 18 ((C, NC) $\to$ (C)). If $\alpha(u_{\mathrm{r}}) < u_{\mathrm{l}} \leqslant \alpha(u_{\mathrm{m}})$. The non-classical shock cancels out and a classical shock is produced.

Case 19 ((NC, NC) $\to$ (C)). If $u_{\mathrm{l}} = \varphi(u_{\mathrm{m}})$. The two incoming non-classical shocks cancel out and a classical shock wave is generated.

5. Existence of solutions

In this section we will prove the existence of a non-classical solution of (1.1), (1.2) under hypotheses (H1)–(H3) and (H4) (see below).

It is easy to check that the total strength of waves (measured in the classical way) may well increase after an interaction, for example, in all cases in which a non-monotone profile is produced after the interaction (see also [1]). More precisely, it decreases in cases 1, 2, 4, 6, 7, 8, 10, 13, 14, 15, 17, 18 and 19, while it increases in cases 9, 11 and 16. This also implies that the total variation of the solutions may well increase in time. In case 16, it appears that the increment in the total strength of waves across the interaction is of the order of the total strength of the incoming ones. Hence it cannot be controlled by the decrease of a Glimm-type functional as in [9].

As in [3] (see also [1]), we define a new functional V, equivalent to the usual total variation functional TV, such that V decreases in time along the approximate solutions. More precisely, let $u : \mathbb{R} \mapsto \mathbb{R}$ be a piecewise constant function and let x_α, $\alpha = 1, \dots, N$, be the points of discontinuity of u. Let

$$\mathrm{V}(u) := \sum_{\alpha=1}^{N} \sigma(u(x_\alpha-), u(x_\alpha+)), \tag{5.1}$$

where $\sigma(u_\mathrm{l}, u_\mathrm{r})$ is a measure of the strength of the wave connecting u_l to u_r. In the case $\sigma(u_\mathrm{l}, u_\mathrm{r}) = |u_\mathrm{r} - u_\mathrm{l}|$, the functional $\mathrm{V}(u)$ is precisely the total variation of u. To compensate the increase of the total variation in some interactions, we have to redefine the strength of a wave by giving less weight to all non-classical shocks and classical shocks that cross the inflection point, i.e. for which $(u_\mathrm{l} - u_\mathrm{I})(u_\mathrm{r} - u_\mathrm{I}) < 0$.

For simplicity, by a linear change of variable in u, from now we can assume that $u_\mathrm{I} = 0$. Following [3], we define $c(u_\mathrm{l}, u_\mathrm{r}) := (u_\mathrm{l} - \varphi(u_\mathrm{r}))\operatorname{sgn}(u_\mathrm{r})$ and set

$$\sigma(u_\mathrm{l}, u_\mathrm{r}) := \begin{cases} (\psi(u_\mathrm{l}) - \psi(u_\mathrm{r}))\operatorname{sgn}(u_\mathrm{l} - u_\mathrm{r})\operatorname{sgn}(u_\mathrm{r}) & \text{if } c(u_\mathrm{l}, u_\mathrm{r}) \geqslant 0, \\ \psi(u_\mathrm{r}) + \psi(u_\mathrm{l}) - 2\psi(\varphi(u_\mathrm{r})) & \text{if } c(u_\mathrm{l}, u_\mathrm{r}) \leqslant 0, \end{cases} \tag{5.2}$$

where $\psi : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function that is increasing (respectively, decreasing) for u positive (respectively, negative) and such that $\psi(0) = 0$. It appears that the strength of a non-classical shock is counted *less* than what it would be with the standard total variation [3].

Let $\Psi(u) := \operatorname{sgn}(u)(\psi(u) - \psi(\varphi(u)))$. As in [3], it is possible to prove that when Ψ is monotone increasing, the piecewise constant approximate solutions $u_\nu(t)$ are well defined and uniformly bounded for all ν and all $t > 0$. Fix a bound $M > 0$ on the L^∞-norm of the initial data. Then we have the following.

PROPOSITION 5.1. *If Ψ is monotone increasing, then* V *is equivalent to* TV *in the sense that there exist $C_1, C_2 > 0$ such that $C_1\mathrm{V}(u) \leqslant \mathrm{TV}(u) \leqslant C_2\mathrm{V}(u)$ for all piecewise constant functions u with $\|u\|_{\mathrm{L}^\infty} \leqslant M$. Moreover, the function $t \mapsto \mathrm{V}(u_\nu(t))$ is monotone non-increasing for all $\nu \in \mathbb{N}$.*

Proof. For the first statement, see lemma 5.5 in [3]. Concerning the second one, it suffices to show that V decreases across every interaction. Let u_1, u_2 and u_3 be three states separated by two interacting waves of strength σ_1^- and σ_2^-. Assume, for simplicity, that $u_3 > 0$. By construction, there are at most two outgoing waves of strength σ_1^+ and (possibly) σ_2^+, say. Denote by

$$\Delta \mathrm{V} = (\sigma_1^+ + \sigma_2^+) - (\sigma_1^- + \sigma_2^-) =: \Sigma^+ - \Sigma^-$$

the variation of V across the interaction. The monotonicity property of V can be checked case by case following the complete list of interactions given above, or we can argue in the following way. First of all, by the very definition of σ, we have

$$\Sigma^+ = \sigma(u_1, u_3)$$

for every outgoing pattern. Moreover, if $\operatorname{sgn}(u_j)(u_i - \varphi(u_j)) \geqslant 0$ for all $i < j$, then $\sigma(u_i, u_j) = \pm(\psi(u_i) - \psi(u_j))$ for $i < j$, the sign depending on $\operatorname{sgn}(u_i - u_j)\operatorname{sgn}(u_j)$. By the definition of σ, it easily follows that whenever $s_{i,j} := \operatorname{sgn}(u_i - u_j)\operatorname{sgn}(u_j)$ is constant for all $i < j$, then

$$\sigma(u_1, u_3) = \sigma(u_1, u_2) + \sigma(u_2, u_3). \tag{5.3}$$

Now, first of all, notice that the monotonicity of φ excludes cases 5 and 12.

Cases 8, 9, 10, 11, 13, 14, 18 and 19 satisfy the previous property, and hence, by (5.3), it follows that $\Delta \mathrm{V} = 0$.

In cases 1, 2, 3, 7, 15 and 16, the value of $s_{i,j}$ is not constant. If, for example, $s_{1,3} = s_{1,2} = 1 = -s_{2,3}$, then we have

$$\sigma(u_1, u_3) = (\psi(u_1) - \psi(u_2)) + (\psi(u_2) - \psi(u_3)) = \sigma(u_1, u_2) - \sigma(u_2, u_3),$$

and hence $\Delta \mathrm{V} = -2\sigma(u_2, u_3)$. A similar result holds when $s_{1,3} = s_{2,3} = -s_{1,2}$.

It remains to check cases 4, 6 and 17, for which $\operatorname{sgn}(u_3)(u_1 - \varphi(u_3)) < 0$.

In case 4, we can compute

$$\begin{aligned}\Delta \mathrm{V} &= (\psi(u_1) + \psi(u_3) - 2\psi(\varphi(u_3))) - [(\psi(u_2) - \psi(u_1)) + (\psi(u_2) - \psi(u_3))] \\ &= -2(\psi(u_2) - \psi(u_3)) - 2(\psi(\varphi(u_3)) - \psi(u_1)) \\ &< 0.\end{aligned}$$

In case 17, we have

$$\begin{aligned}\Delta \mathrm{V} &= (\psi(u_1) + \psi(u_3) - 2\psi(\varphi(u_3))) - [(\psi(u_1) - \psi(u_2)) + (\psi(u_3) - \psi(u_2))] \\ &= -2(\psi(\varphi(u_3)) - \psi(u_2)) \\ &< 0.\end{aligned}$$

Finally, in case 6, we have

$$\Delta \mathrm{V} = -2[(\psi(u_2) - \psi(\varphi(u_2))) - (\psi(u_3) - \psi(\varphi(u_3)))],$$

which is non-positive thanks to the monotonicity property of Ψ. This completes the proof. □

Now the problem is to show the existence of a suitable function ψ with the desired properties. To this end, we shall make another assumption.

(H4) $u\alpha(u) \leqslant 0$ and $|\varphi^{[2]}(u)| < |u|$ for all $u \neq 0$, where $\varphi^{[2]}$ denotes the second iterate of φ (see [3]).

This assumption, which may seem technical, can be explained as follows. The condition $u\alpha(u) \leqslant 0$ (namely, $(u-u_{\mathrm{I}})(\alpha(u)-u_{\mathrm{I}}) \leqslant 0$ in the general case $u_{\mathrm{I}} \neq 0$) is justified by lemma 3.2 of [13], where it has been proved that, when $\gamma f''' < 0$, all the shocks connecting two states on the same side of u_{I} are admissible. If one had $u\alpha(u) > 0$ for some u, then, as in theorem 3.3 of [3], it would be possible to prove that the Riemann solver does not depend L^1-continuously on the left and right states, or even worse, it would be possible to find more than one solution to some Riemann problems. It can also be explained by saying that when the states live in the same region of convexity, then the solution must be classical. Secondly, $|\varphi^{[2]}(u)| < |u|$ is a strengthened version of (5.5) (see below). If one had $|\varphi^{[2]}(u)| = |u|$ for some $u \neq 0$, then it would be possible to construct, in the same spirit of example 7.2 in [3], a solution of (1.1), (1.2) whose total variation blows up in finite time. Since we are interested in BV solutions, the second assumption in (H4) seems natural.

THEOREM 5.2. *Under hypotheses (H1)–(H4) and if* $U(u) = \frac{1}{2}u^2$, *there exists a Lipschitz continuous function* ψ *that is increasing* (*respectively, decreasing*) *for* u *positive* (*respectively, negative*), $\psi(0) = 0$ *and such that* $\Psi(u) = \operatorname{sgn}(u)(\psi(u) - \psi(\varphi(u)))$ *is increasing.*

Proof. Basically, we want to use the contraction mapping principle applied to a suitable functional Banach space X and map T as in [3]. Let $M > 0$ be a bound on the L^∞-norm of the initial data. To apply the result in [3], we have to check the validity of the following hypotheses.

(i) The discontinuity connecting $\varphi(u)$ and u is an entropy-admissible non-classical shock.

(ii) φ is Lipschitz continuous and decreasing.

(iii) $u\alpha(u) \leqslant 0$.

(iv) There exists $\varepsilon_0 > 0$ such that

$$\operatorname{Lip}_{[-\varepsilon_0,\varepsilon_0]} \varphi^{[2]} < 1, \qquad \sup_{u \in [-M,M]\setminus\{0\}} \frac{\varphi^{[2]}(u)}{u} < 1. \tag{5.4}$$

Conditions (ii), (iii) are satisfied. Concerning (iv), by theorem 3.1, it follows that there exists a threshold under which the solution of the Riemann problem is classical. More precisely, if $u_+^{**} \leqslant u \leqslant u_+^*$, then $\varphi(u) = \tau(u)$. By (H3), proceeding as in lemma 2.1 of [3], it is possible to prove that $\varphi'(0) = \tau'(0) > -1$ and, by continuity, $(\varphi^{[2]})'(u) = \varphi'(\varphi(u))\varphi'(u) < 1$ if u is small enough. Hence the first inequality of (5.4) follows.

Moreover, in [3] it is proved that (i) implies

$$0 \leqslant \operatorname{sgn}(u)\varphi^{[2]}(u) \leqslant |u| \quad \text{for all } u \neq 0. \tag{5.5}$$

This, together with $\varphi'(0) > -1$ and (H4), implies the second relation of (5.4).

It remains to prove that the shock connecting $\varphi(u)$ and u is entropy admissible. This is guaranteed by [12], where it is proved that shock waves obtained as limits

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of the diffusive-dispersive approximation (1.3) satisfy the entropy inequality (1.4), where $U(u) = \frac{1}{2}u^2$. We remark that this could not be true for other entropies.

In the end, the results of [3, § 5] hold, thus proving the existence of a function ψ with the desired properties. □

We conclude with the main theorem.

THEOREM 5.3. *Let u_ν, $\nu \in \mathbb{N}$, be a sequence of piecewise constant approximate solutions constructed by wavefront tracking, as before. If hypotheses (H1)–(H4) hold and $U(u) = \frac{1}{2}u^2$, then (up to a subsequence) u_ν converges to a non-classical solution u of (1.1), (1.2), also satisfying the entropy inequality (1.4).*

Proof. By proposition 5.1 and theorem 5.2, it follows that $\mathrm{V}(u_\nu(t))$ is uniformly bounded for all ν and $t > 0$, and so is $\mathrm{TV}(u_\nu(t))$ thanks to proposition 5.1. By Helly's theorem, there exists a subsequence of u_ν converging to a function u in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$. As in [2,7], one shows that u is a weak solution of (1.1), (1.2), i.e. for every function ϕ with compact support in $[0, \infty) \times \mathbb{R}$, there holds

$$\int_0^\infty \int_{-\infty}^\infty [u(t,x)\partial_t\phi(t,x) + f(u(t,x))\partial_x\phi(t,x)]\,\mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty u_0(x)\phi(0,x)\,\mathrm{d}x = 0,$$

and that satisfies also the single entropy inequality (1.4), i.e. for every positive function ϕ with compact support in $\mathbb{R} \times \mathbb{R}$, it satisfies

$$\int_0^\infty \int_{-\infty}^\infty [U(u(t,x))\partial_t\phi(t,x) + F(u(t,x))\partial_x\phi(t,x)]\,\mathrm{d}x\mathrm{d}t \geqslant 0,$$

where U and F are as in (2.1). □

5.1. An alternative approach

The previous approach lacks in supplying a precise form for V (and σ). In [1] and [4], an explicit formula for σ is given, provided some additional hypotheses are satisfied. More precisely, we have the following.

(M1) $-1 \leqslant \varphi'(u) \leqslant 0$ and $|\varphi(u)| < |u|$ for all u.

(M2) There exists $\beta > 0$ such that $\varphi(u) = \tau(u)$ for all $|u| \leqslant \beta$.

(M3) $-1 \leqslant \alpha'(u) \leqslant 0$ for all $u \in \mathbb{R}$.

Then one can define

$$\sigma(u_\mathrm{l}, u_\mathrm{r}) := \begin{cases} |u_\mathrm{l} - u_\mathrm{r}| & \text{if } u_\mathrm{l}u_\mathrm{r} \geqslant 0, \\ |u_\mathrm{r} - (1 - K(u_\mathrm{r}))u_\mathrm{l}| & \text{if } u_\mathrm{l}u_\mathrm{r} < 0,\ |u_\mathrm{l}| \leqslant |\alpha(u_\mathrm{r})|, \\ |u_\mathrm{r} + \varphi(u_\mathrm{r}) - (2 - K(u_\mathrm{r}))\alpha(u_\mathrm{r})| & \text{if } u_\mathrm{l} = \varphi(u_\mathrm{r}), \end{cases}$$

where K is a Lipschitz continuous function satisfying the two conditions

$$\begin{aligned} K(u) &= 0 && \text{if } |u| \leqslant \beta, \\ K(u) &\in (0,2) && \text{if } |u| > \beta, \end{aligned}$$

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together with some differential inequalities. If we are interested in small data, then, as in [4], it is possible to find an explicit piecewise constant function K satisfying all the conditions and such that the corresponding functional V decreases in time along piecewise constant approximate solutions. Condition (M1) is satisfied by all u close to 0, while condition (M2) is satisfied by taking $\beta = \min\{|u_+^*|, |u_+^{**}|\}$. Because of the smallness assumption on the data, this approach cannot be used in the case of the one-dimensional Buckley–Leverett equation in the following section. Condition (M3) is not guaranteed to be satisfied, too. This is why we chose a more abstract approach, which indeed works for large L^∞ data.

6. The one-dimensional Buckley–Leverett equation

Here we will consider an application of the results presented in the previous sections to the one-dimensional inviscid Buckley–Leverett equation

$$\partial_t u + \partial_x\left(\frac{k_1(u)}{k_1(u) + k_2(1-u)}\right) = 0, \quad u \in [0,1], \tag{6.1}$$

in connection with a diffusive-dispersive approximation

$$\partial_t u + \partial_x\left(\frac{k_1(u)}{k_1(u) + k_2(1-u)}\right) = \alpha\varepsilon\partial_{xx}u - \beta\varepsilon^2\partial_{xxx}u, \tag{6.2}$$

with $\alpha > 0$ and $\beta \geqslant 0$. When $\beta = 0$, equation (6.2) models the flow of two immiscible fluids in porous media. Here, u and $1-u$ are the volume fractions of the two fluids, while $k_1(u)$ and $k_2(1-u)$ denote the respective permeabilities, divided by viscosity, of the medium to the fluids. The usual assumptions on k_i are, for $i = 1, 2$,

(i) k_i is monotonically increasing and convex; and

(ii) $k_i(0) = k_i'(0) = 0$, $k_i''(z) > 0$ for $0 \leqslant z \leqslant 1$ and $k_i(1) = 1/\mu_i$, where μ_1, μ_2 are the viscosities of the fluids.

We remark that the term $-\beta\varepsilon^2\partial_{xxx}u$ is not usually included in the model equation. Nevertheless, as observed in [13], it can be part of the truncation error of a numerical method. Indeed, the appearance of undercompressive shocks in simulations has been proved in [11] and, for the present model, investigated in [13]. In the latter, numerical evidences show that non-classical undercompressive shocks do appear in the solutions of the Riemann problem for (6.2). Furthermore, some second-order schemes applied to (6.1), as, for example, the two-step Richtmyer version of Lax–Wendroff, exhibit behaviour similar to (6.2). This justifies the analysis in case $\beta > 0$.

From the hypotheses on k_1, k_2, it is easily checked that the flux of (6.1) is non-convex, and usually it has an S-shape profile satisfying (H1)–(H3) (see [13]).

In the specific case of (6.1), where the u variable is restricted to take values in $[0,1]$, it is not clear that the non-classical solution of the Riemann problem will still take value in the same interval. In view of the monotonicity properties of the function φ that come from theorem 3.1, it is sufficient to ask that $\varphi(1) \in [0,1]$, more precisely, that it belongs to $[0, u^*]$, and symmetrically, $\varphi(0) \in [u^{**}, 1]$. This condition, which actually coincides with assumption 5.1 of [13], guarantees that the Riemann solution still takes values in $[0,1]$ and clarifies the choice made. As before, assumption 5.2 of [13] is not needed, and thus we prove existence under more general assumptions. The Riemann solution is given by (S1), (S2) in § 3, where now the variable $u \in [0,1]$.

In the present case, if theorem 3.1 holds, $\varphi(1), \varphi(0) \in [0,1]$ and (H4) is satisfied, so are theorems 5.2 and 5.3, thus providing an existence result for the Cauchy problem (6.1), (1.2), where the datum $u_0 : \mathbb{R} \to [0,1]$ is of bounded variation. The validity of the hypotheses is investigated numerically in [13] for the special case $k_i(u) = K_i u^2$, with $K_i > 0$ and $\alpha = \theta$, $\beta = 1 - \theta$ with $\theta \in [0,1]$, such that

$$f(u) = f(u;a) = \frac{u^2}{u^2 + a(1-u)^2},$$

with $a = K_2/K_1$.

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