J. Fluid Mech. (2018), *vol.* 854, *pp.* 146–163.
doi:10.1017/jfm.2018.627

A nonlinear Schrödinger equation for gravity–capillary water waves on arbitrary depth with constant vorticity. Part 1

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(Received 24 December 2017; revised 27 April 2018; accepted 31 July 2018; first published online 3 September 2018)

A nonlinear Schrödinger equation for the envelope of two-dimensional gravity–capillary waves propagating at the free surface of a vertically sheared current of constant vorticity is derived. In this paper we extend to gravity–capillary wave trains the results of Thomas *et al.* (*Phys. Fluids*, 2012, 127102) and complete the stability analysis and stability diagram of Djordjevic & Redekopp (*J. Fluid Mech.*, vol. 79, 1977, pp. 703–714) in the presence of vorticity. The vorticity effect on the modulational instability of weakly nonlinear gravity–capillary wave packets is investigated. It is shown that the vorticity modifies significantly the modulational instability of gravity–capillary wave trains, namely the growth rate and instability bandwidth. It is found that the rate of growth of modulational instability of short gravity waves influenced by surface tension behaves like pure gravity waves: (i) in infinite depth, the growth rate is reduced in the presence of positive vorticity and amplified in the presence of negative vorticity; (ii) in finite depth, it is reduced when the vorticity is positive and amplified and finally reduced when the vorticity is negative. The combined effect of vorticity and surface tension is to increase the rate of growth of modulational instability of short gravity waves influenced by surface tension, namely when the vorticity is negative. The rate of growth of modulational instability of capillary waves is amplified by negative vorticity and attenuated by positive vorticity. Stability diagrams are plotted and it is shown that they are significantly modified by the introduction of the vorticity.

Key words: capillary waves, surface gravity waves, waves/free-surface flows

1. Introduction

Generally, gravity–capillary waves are produced by wind which generates a shear flow in the uppermost layer of the water and consequently these waves propagate in the presence of vorticity. These short waves play an important role in the initial development of wind waves, contribute to some extent to the sea surface stress and consequently participate in air–sea momentum transfer. Accurate representation of

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the surface stress is important in modelling and forecasting ocean wave dynamics. Furthermore, the knowledge of their dynamics at the sea surface is crucial for satellite remote sensing applications.

In this paper we consider both the effect of surface tension and vorticity due to a vertically sheared current on the modulational instability of a weakly nonlinear periodic short wave trains. Recently, Thomas, Kharif & Manna (2012) have derived a nonlinear Schrödinger (NLS) equation for pure gravity water waves on finite depth with constant vorticity. Their main findings were (i) a restabilization of the modulational instability for waves propagating in the presence of positive vorticity whatever the depth and (ii) the importance of the nonlinear coupling between the mean flow induced by the modulation and the vorticity. One of our aims is to extend Thomas' investigation to the case of gravity–capillary waves propagating on a vertically sheared current.

The number of studies on the computation of steadily propagating periodic gravity waves on a vertically sheared current is quite large. For a brief review one can refer to the paper by Thomas *et al.* (2012). On the other hand, investigations devoted to the calculation of nonlinear gravity–capillary waves in the presence of horizontal vorticity are rather scarce. One can cite Bratenberg & Brevik (1993) who used a third-order Stokes expansion for periodic gravity–capillary waves travelling on an opposing current and Hsu *et al.* (2016) who extended this work to the case of co- and counter-propagating waves. Kang & Vanden-Broeck (2000) computed periodic and solitary gravity–capillary waves in the presence of constant vorticity on finite depth. They derived analytical solutions for small-amplitude waves and numerical solutions for steeper waves. Using the bifurcation theory, Wahlen (2006*a*,*b*) proved the rigorous existence of periodic capillary waves and capillary–gravity waves in the presence of an arbitrary vorticity distribution. For a review on this aspect of the problem, one can refer to the paper by Wahlen (2007). More recently, Martin & Matioc (2014) demonstrated the existence of steady periodic capillary–gravity waves with a piecewise constant vorticity distribution.

To our knowledge, the unique study concerning the modulational instability of gravity–capillary waves travelling on a vertically sheared current is that of Hur (2017). The stability of irrotational gravity–capillary waves has been deeply investigated by several authors. Djordjevic & Redekopp (1977) and Hogan (1985) derived nonlinear envelope equations and considered the modulational instability of periodic gravity–capillary waves. Note that in the gravity–capillary range, three-wave interaction is possible whereas modulational instability corresponds to a four-wave resonant interaction. The numerical computations were extended to capillary waves by Chen & Saffman (1985) and Tiron & Choi (2012). Zhang & Melville (1986) investigated numerically the stability of gravity–capillary waves including, besides the four-wave resonant interaction, three-wave and five-wave resonant interactions. For a review on the stability of irrotational gravity–capillary waves, one can refer to the review paper by Dias & Kharif (1999).

This study is devoted to the modulational instability of weakly nonlinear gravity–capillary wave packets propagating at the surface of a vertically sheared current of finite depth. In § 2, the governing equations are given and the nonlinear Schrödinger equation in the presence of surface tension and constant vorticity is derived by using a multiple scale method. In § 3, the linear stability analysis of a weakly nonlinear wave train is carried out as a function of the Bond number, the dispersive parameter and the intensity of the vertically sheared current.

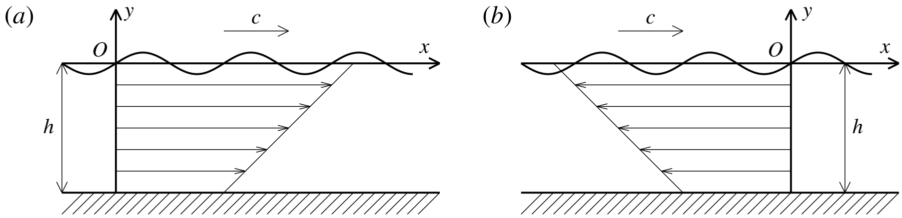

FIGURE 1. Shear flow in the fixed frame where c is the wave velocity. (a) Waves propagating downstream ($\Omega > 0$). (b) Waves propagating upstream ($\Omega < 0$).

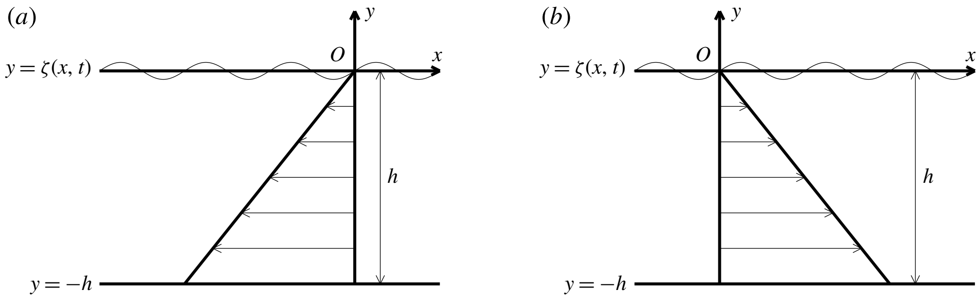

FIGURE 2. Shear flow in the reference frame moving with the surface current velocity. (a) Waves propagating downstream ($\Omega > 0$). (b) Waves propagating upstream ($\Omega < 0$).

2. Derivation of the NLS equation in the presence of surface tension and vorticity

We consider the modulational instability of weakly nonlinear surface gravity–capillary wave trains in the presence of vorticity. Our investigation is confined to two-dimensional water waves propagating in finite depth. Viscosity is disregarded and the fluid is considered incompressible. The geometry configuration is presented in figures 1 and 2. We choose an Eulerian frame ($Oxyz$) with unit vectors ($\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$). The vector $\boldsymbol{e}_y$ is oriented upwards so that the gravity is $\boldsymbol{g} = -g\boldsymbol{e}_y$ with $g > 0$. The equation of the undisturbed free surface is $y = 0$ whereas the disturbed free surface is $y = \zeta(x, t)$. The bottom is located at $y = -h$.

The waves are travelling at the surface of a vertically sheared current of constant vorticity. In the fixed frame, the underlying current is given by $\boldsymbol{u}_0(y) = (U_0 + \Omega y)\boldsymbol{e}_x$ where Ω is the current intensity and U_0 is the current velocity at the surface. Note that the vorticity is $-\Omega$. We choose a reference frame moving with the horizontal velocity U_0. Consequently, in this frame of reference, the current at the surface vanishes. In the moving frame the fluid velocity is given by

$$\boldsymbol{u}(x, y) = \Omega y \boldsymbol{e}_x + \nabla\phi, \tag{2.1}$$

where $\nabla\phi(x, y, t)$ is the wave induced velocity. The waves are potential due to the Kelvin theorem which states that vorticity is conserved for a two-dimensional flow of an incompressible and inviscid fluid with external forces derived from a potential. There is no loss of generality if the study is restricted to carrier waves with positive phase speeds so long as both positive and negative values of Ω are considered.

The potential ϕ satisfies the Laplace equation

$$\nabla^2\phi = 0, \tag{2.2}$$

and the Euler equation can be written as follows

$$\boldsymbol{\nabla}\left(\phi_t + \frac{1}{2}u^2 + \frac{P}{\rho_w} + gy\right) = \boldsymbol{u}\wedge\boldsymbol{\omega}, \tag{2.3}$$

with $\boldsymbol{\omega}$ the vorticity vector along z, P the pressure and ρ_w the water density. Subscripts stand for derivatives of corresponding variables.

Using the Cauchy–Riemann relations

$$\psi_y = \phi_x, \quad \psi_x = -\phi_y, \tag{2.4a,b}$$

where ψ is the streamfunction

$$\boldsymbol{u}\wedge\boldsymbol{\omega} = \boldsymbol{\nabla}\left(\tfrac{1}{2}\Omega^2y^2 + \Omega\psi\right). \tag{2.5}$$

The Euler equation (2.3) can be rewritten as follows

$$\boldsymbol{\nabla}\left(\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + \Omega y\phi_x + gy - \Omega\psi + \frac{P}{\rho_w}\right) = 0. \tag{2.6}$$

Spatial integration gives the Bernoulli equation

$$\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + \Omega y\phi_x + gy - \Omega\psi + \frac{P}{\rho_w} = C(t). \tag{2.7}$$

In the presence of surface tension at the free surface $y = \zeta(x, t)$, the Laplace law is written as

$$P = P_a - T\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}}, \tag{2.8}$$

where P_a is the atmospheric pressure and T surface tension.

The dynamic boundary condition at the free surface $y = \zeta$ is

$$\phi_t + \frac{1}{2}\phi_x^2 + \frac{1}{2}\phi_y^2 + \Omega\zeta\phi_x + g\zeta - \Omega\psi - \frac{T}{\rho_w}\frac{\zeta_{xx}}{(1+\zeta_x^2)^{3/2}} = 0. \tag{2.9}$$

Without loss of generality, we set $P_a = 0$ and incorporate $C(t)$ into the potential ϕ.

Along with these, we have the kinematic free surface boundary condition

$$\zeta_t + \zeta_x(\phi_x + \Omega y) - \phi_y = 0, \quad y = \zeta(x, t), \tag{2.10}$$

and the bottom boundary condition

$$\phi_y = 0, \quad y = -h. \tag{2.11}$$

Following Thomas *et al.* (2012) we can remove ψ by differentiating (2.9) with respect to x. Then using relations (2.4), keeping in mind that we are dealing with weakly nonlinear waves (low wave steepness), and that (2.9) is evaluated at $y=\zeta$, we get the equation

$$\begin{aligned}&\phi_{tx}+\phi_{ty}\zeta_x+\phi_x(\phi_{xx}+\phi_{xy}\zeta_x)+\phi_y(\phi_{xy}+\phi_{yy}\zeta_x)+\Omega\zeta_x\phi_x\\&\quad+\Omega\zeta(\phi_{xx}+\phi_{xy}\zeta_x)+g\zeta_x+\Omega(\phi_y-\phi_x\zeta_x)\\&\quad-\frac{T}{\rho_w}\left(\zeta_{xxx}-\frac{3}{2}\zeta_x^2\zeta_{xxx}-3\zeta_{xx}^2\zeta_x\right)=0,\quad y=\zeta(x,t),\end{aligned}\tag{2.12}$$

that matches the one derived by Thomas *et al.* (2012) for $T=0$.

Following Davey & Stewartson (1974), we look for solutions depending on slow variables $(\xi,\tau)=(\varepsilon(x-c_gt),\varepsilon^2t)$ where $\varepsilon=ak$ ($\varepsilon\ll 1$) and a, k and c_g are the amplitude, wavenumber and group velocity of the carrier wave, respectively. The system of governing equations becomes

$$\varepsilon^2\phi_{\xi\xi}+\phi_{yy}=0,\quad -h\leqslant y\leqslant\zeta(\xi,\tau),\tag{2.13a,b}$$

$$\phi_y=0,\quad y=-h,\tag{2.14}$$

$$\varepsilon^2\zeta_\tau-\varepsilon c_g\zeta_\xi+\varepsilon\zeta_\xi(\varepsilon\phi_\xi+\Omega[y+h])-\phi_y=0,\quad y=\zeta(\xi,\tau),\tag{2.15}$$

$$\begin{aligned}&\varepsilon^3\phi_{\tau\xi}-\varepsilon^2c_g\zeta_\xi+\varepsilon^3\phi_{\tau y}\zeta_\xi-\varepsilon^2c_g\phi_{\xi y}\zeta_\xi+\varepsilon^3\phi_\xi(\phi_{\xi\xi}+\phi_{\xi y}\zeta_\xi)\\&\quad+\varepsilon\phi_y(\phi_{\xi y}+\phi_{yy}\zeta_\xi)+\varepsilon^2\Omega\zeta_\xi\phi_\xi+\varepsilon^2\Omega\zeta(\phi_{\xi\xi}+\phi_{\xi y}\zeta_\xi)+\varepsilon g\zeta_\xi\\&\quad+\Omega(\phi_y-\varepsilon^2\phi_\xi\zeta_\xi)-\varepsilon^3\frac{T}{\rho_w}\left(\zeta_{\xi\xi\xi}-\frac{3}{2}\varepsilon^2\zeta_\xi^2\zeta_{\xi\xi\xi}-3\varepsilon^2\zeta_{\xi\xi}^2\zeta_\xi\right)=0,\quad y=\zeta(\xi,\tau).\end{aligned}\tag{2.16}$$

An asymptotic solution to the system (2.13)–(2.16) is sought in the following form

$$\phi=\sum_{n=-\infty}^{+\infty}\phi_nE^n,\quad \zeta=\sum_{n=-\infty}^{+\infty}\zeta_nE^n,\tag{2.17a,b}$$

where $E=\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ is a plane wave with ω the frequency of the carrier wave. We impose that $\phi_{-n}=\bar{\phi}_n$ and $\zeta_{-n}=\bar{\zeta}_n$ where the bar denotes complex conjugate, so that the functions are real. The amplitudes ϕ_n and ζ_n are then expanded in a perturbation series in terms of $\varepsilon=ak$

$$\phi_n=\sum_{j=n}^{+\infty}\varepsilon^j\phi_{nj},\quad \zeta_n=\sum_{j=n}^{+\infty}\varepsilon^j\zeta_{nj}.\tag{2.18a,b}$$

The terms depending on surface tension occur only at a higher order. The expansions (2.18) are substituted into the system of equations. The linear Laplace equation (2.13) is easier to handle, since solutions can be derived iteratively. Here we will simply write the first-order solution for ϕ_{11} that is obtained by using the bottom boundary condition (2.14)

$$\phi_{11}=A(\xi,\tau)\frac{\cosh[k(y+h)]}{\cosh(kh)},\tag{2.19}$$

where the slow-varying function $A(\xi,\tau)$ will be used to express all other terms. Higher-order expansions of the Laplace equation introduce more unknown functions into the solutions. Nevertheless, through expansions of the boundary conditions they can all be combined to $A(\xi, \tau)$.

The evolution of this unknown will depend on the initial condition $A(\xi, 0)$. We then use (2.18) in the dynamic and kinematic free surface boundary conditions, and collect terms of equal power in ε and E, which allows the expressions for the ζ_{ij} and ϕ_{ij} to be found successively.

The calculations are somewhat tedious but some steps are of interest. At first, the linear dispersion relation is derived

$$\omega^2 + \sigma \Omega \omega - \sigma g k(1+\kappa) = 0, \tag{2.20}$$

where $\sigma = \tanh(\mu)$ with $\mu = kh$ and $\kappa = Tk^2/\rho_w g$.

As mentioned above, we consider a carrier wave travelling from left to right whose frequency ω, phase velocity c_p and group velocity c_g are

$$\omega = -\frac{\Omega\sigma}{2} + \sqrt{\left(\frac{\Omega\sigma}{2}\right)^2 + g\sigma k(1+\kappa)}, \tag{2.21}$$

$$c_p = -\frac{\Omega\sigma}{2k} + \sqrt{\left(\frac{\Omega\sigma}{2k}\right)^2 + \frac{g\sigma}{k}(1+\kappa)}, \tag{2.22}$$

$$c_g = -\frac{\Omega h}{2}(1-\sigma^2) + \frac{(1-\sigma^2)(\Omega^2 h\sigma/2 + g\mu(1+\kappa)) + g\sigma(1+3\kappa)}{\sqrt{\Omega^2\sigma^2 + 4gk\sigma(1+\kappa)}}. \tag{2.23}$$

The relation between $A(\xi, \tau)$ and ζ_{11} is the following

$$\zeta_{11} = \mathrm{i}\frac{\omega(1+X)}{g(1+\kappa)} A(\xi, \tau), \tag{2.24}$$

where $X = \sigma\Omega/\omega$.

From the above dispersion relation we can show easily that $X > -1$. We note that X depends also on the surface tension through ω and its associated dispersion relation. It is also to be noted that the expression of the mean-flow term, which is important in the development of the modulational instability, is similar to that of Thomas *et al.* (2012). Nevertheless, surface tension takes place through the phase velocity c_p, the group velocity c_g and ω.

$$(c_g(c_g + \Omega h) - gh)\phi_{01,\xi} = \left(\frac{g\sigma}{c_p^2}(2\omega + \sigma\Omega) + k^2 c_g(1-\sigma^2)\right)|A|^2, \tag{2.25}$$

and

$$g\zeta_{02} = (c_g + \Omega h)\phi_{01,\xi} - k^2(1-\sigma^2)|A|^2. \tag{2.26}$$

Although the expressions are identical to those of Thomas *et al.* (2012), it should be noted that the surface tension acts through the dispersion relation, affecting ω, c_p and c_g.

At order $O(\varepsilon^3 E)$ the nonlinear Schrödinger equation is found for the potential envelope A, so that

$$\mathrm{i}A_\tau + \alpha A_{\xi\xi} = \gamma |A|^2 A, \tag{2.27}$$

where the coefficients α and γ depend on (κ, Ω, kh).

Then the dispersion coefficient reads

$$\alpha = \frac{-\omega}{k^2\sigma(2+X)}\left[\sigma\rho^2+\mu\frac{1+X}{1+\kappa}(\sigma[\sigma+\mu(1-\sigma^2)]-1)\right. \\ \left. +\mu(1-\sigma^2)(\rho-\mu\sigma)X-\frac{\kappa}{1+\kappa}\alpha_1\right], \tag{2.28}$$

with

$$\alpha_1 = -\mu(1+X)(1-\sigma^2)(\mu\sigma-1)+\sigma(1+X)(1+2\rho) \\ +2\left(\sigma\rho+\mu(1-\sigma^2)X-2\frac{\sigma\kappa}{1+\kappa}(1+X)\right), \tag{2.29}$$

where $\rho = c_g/c_p$ is here the ratio of the group velocity to the phase velocity of the carrier. It can be expressed in a concise form

$$\rho = \frac{(1-\sigma^2)\mu+(1+X)\left(\sigma+\dfrac{2\sigma\kappa}{1+\kappa}\right)}{\sigma(2+X)}, \tag{2.30}$$

which depends only on μ, κ and $X=\sigma\Omega/\omega$.

The nonlinear coefficient is

$$\gamma = \frac{k^4}{2\omega(1+X)(2+X)}\left[-\frac{3\sigma^2\kappa}{1+\kappa}(1+X)^2-2(1+\kappa)(1-\sigma^2)[(1+X)^2-\sigma^2]\right. \\ +\sigma^2(1+X)(8+6X)+\frac{1+X}{\sigma^2-\kappa(3-\sigma^2+3X)}\gamma_1 \\ \left. +2\frac{(1+X)(2+X)+\rho(1+\kappa)(1-\sigma^2)}{(1+\kappa)\left(\rho^2+\mu\rho\dfrac{X}{\sigma}-\dfrac{\mu(1+X)}{\sigma(1+\kappa)}\right)}\gamma_2\right], \tag{2.31}$$

with

$$\gamma_1 = 9-10\sigma^2+\sigma^4+(18-4\sigma^2-4\sigma^4)X+(15+3\sigma^2)X^2 \\ +(6+2\sigma^2)X^3+X^4 \\ +\kappa\left[21-10\sigma^2+\sigma^4+(42+2\sigma^2-4\sigma^4)X\right. \\ \left. +(30+12\sigma^2)X^2+(9+5\sigma^2)X^3+X^4\right], \tag{2.32}$$

and finally

$$\gamma_2 = (1+\kappa)\left[(1+X)^2\left(1+\rho+\frac{\mu X}{\sigma}\right)+1+X-\sigma(\rho\sigma+\mu X)\right] \\ -\kappa(1+X)(2+X). \tag{2.33}$$

We can check that these coefficients reduce to those of Djordjevic & Redekopp (1977), or Hogan (1985) in deep water, if $\Omega=0$ and to those of Thomas *et al.* (2012) if $\kappa=0$.

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The last term in brackets of (2.31) corresponds to the coupling between the mean flow due to the modulation and the vorticity which occurs at third order. This coupling was found by Thomas *et al.* (2012) for the case of pure gravity waves and has an important impact on the stability analysis of progressive wave trains.

We can see that in (2.31) there are two possible singularities that one should avoid, either

$$\sigma^2 - \kappa(3 - \sigma^2 + 3X) = 0, \tag{2.34}$$

which corresponds to the first gravity–capillary resonance at $\kappa_c = \sigma^2/(3 - \sigma^2)$ without vorticity, associated with the Wilton ripple phenomenon, or

$$\rho^2 + \rho\frac{\mu X}{\sigma} - \frac{\mu(1+X)}{\sigma(1+\kappa)} = 0, \tag{2.35}$$

which is rewritten as follows

$$c_g^2 + \frac{g\mu}{\omega}X\frac{1+\kappa}{1+X}c_g - \frac{g^2\mu\sigma}{\omega^2}\frac{1+\kappa}{1+X} = 0. \tag{2.36}$$

In the absence of vorticity, the latter condition reduces to $c_g^2 = gh$ which matches the long wave/short wave resonance as shown by Davey & Stewartson (1974) and Djordjevic & Redekopp (1977). From equation (2.21) we can derive the following relation

$$\omega^2 = g\sigma k\frac{1+\kappa}{1+X}. \tag{2.37}$$

Substituting this expression for ω^2 into (2.36) gives

$$c_g^2 + \varOmega h c_g - gh = 0, \tag{2.38}$$

which admits as solution the group velocity of a carrier wave travelling from left to right

$$c_g = -\frac{\varOmega h}{2} + \sqrt{gh + \varOmega^2 h/4}. \tag{2.39}$$

The right-hand side of (2.39) is the phase velocity of a long gravity wave propagating on a linear shear current of intensity $\varOmega$. Consequently, the long wave/short wave resonance persists in the presence of vorticity.

3. Stability analysis and results

Let us write ζ in the form

$$\zeta = \tfrac{1}{2}(\epsilon a \mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.}) + O(\epsilon^2), \tag{3.1}$$

where $a = 2\zeta_{11}$ is the envelope of the free surface elevation and c.c. denotes complex conjugation. Using (2.24) the NLS equation (2.27) is rewritten for the complex envelope $a(\xi, \tau)$ as follows

$$\mathrm{i}a_\tau + \alpha a_{\xi\xi} = \tilde{\gamma}|a|^2 a, \tag{3.2}$$

where

$$\tilde{\gamma} = \frac{g^2}{4\omega^2}\left(\frac{1+\kappa}{1+X}\right)^2 \gamma. \tag{3.3}$$

The nonlinear coefficient $\tilde{\gamma}$ can be written in a more compact form

$$\tilde{\gamma} = \frac{\omega^2}{4k^2\sigma^2}\gamma. \tag{3.4}$$

In this section we consider the stability of a Stokes wave solution of the NLS equation (3.2) to infinitesimal disturbances.

Equation (3.2) admits the following solution

$$a_s(\tau) = a_0 \mathrm{e}^{-\mathrm{i}\tilde{\gamma} a_0^2 \tau}, \tag{3.5}$$

with the initial condition a_0.

We consider infinitesimal perturbations to this solution, in amplitude $\delta_a(\xi, \tau)$ and in phase $\delta_w(\xi, \tau)$, so that the perturbed solution a_s' is written as

$$a_s' = a_s(1+\delta_a)\mathrm{e}^{\mathrm{i}\delta_w}. \tag{3.6}$$

Substituting this expression in the NLS equation (3.2), linearizing and separating between real and imaginary parts, yields to a system of linear coupled partial differential equations with constant coefficients. Then, this system admits solutions of the form

$$\left.\begin{aligned} \delta_a &= \delta_{a_0}\mathrm{e}^{\mathrm{i}(p\xi - \Gamma\tau)}, \\ \delta_w &= \delta_{w_0}\mathrm{e}^{\mathrm{i}(p\xi - \Gamma\tau)}. \end{aligned}\right\} \tag{3.7}$$

The necessary and sufficient condition for the existence of non-trivial solutions is

$$\Gamma^2 = \alpha p^2 (2\tilde{\gamma} a_0^2 + \alpha p^2). \tag{3.8}$$

The Stokes wave solution is stable when $\alpha(2\tilde{\gamma}a_0^2 + \alpha p^2) \geqslant 0$ and unstable when $\alpha(2\tilde{\gamma}a_0^2 + \alpha p^2) < 0$.

The growth rate of instability is then

$$\Gamma_i = p(-2\tilde{\gamma}\alpha a_0^2 - \alpha^2 p^2)^{1/2}. \tag{3.9}$$

We set $\alpha = \omega\alpha_2/k^2$ and $\tilde{\gamma} = \omega k^2 \tilde{\gamma}_1$, so that α_2 and $\tilde{\gamma}_1$ are dimensionless functions of $\mu = kh$, $X = \sigma\Omega/\omega$ and κ only. The growth rate of instability becomes

$$\Gamma_i = \frac{\omega p}{k^2}(-2\tilde{\gamma}_1\alpha_2 a_0^2 k^4 - \alpha_2^2 p^2)^{1/2}. \tag{3.10}$$

The maximal growth rate is obtained for $p = \sqrt{-\tilde{\gamma}_1/\alpha_2}\, a_0 k^2$ and its expression is $\Gamma_{imax} = \sqrt{-\tilde{\gamma}_1/\alpha_2}\sqrt{-\tilde{\gamma}_1\alpha_2}\,\omega(a_0 k)^2$. Note that instability occurs when $\tilde{\gamma}_1$ and α_2 have opposite signs.

The growth rate of instability is written in the following dimensionless form

$$\frac{\Gamma_i}{\omega a_0^2 k^2} = \tilde{p}\,(-2\tilde{\gamma}_1\alpha_2 - \alpha_2^2\tilde{p}^2)^{1/2}, \tag{3.11}$$

where $\tilde{p} = p/(a_0 k^2)$.

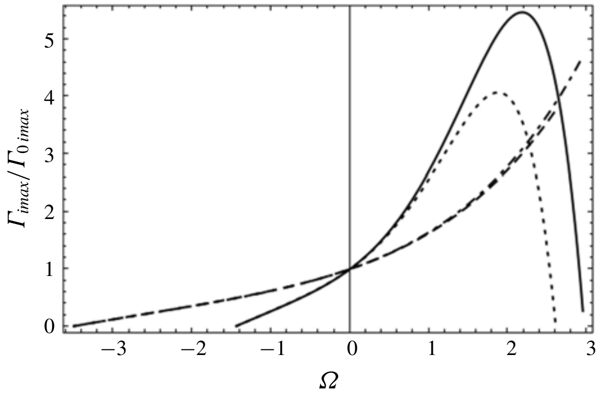

FIGURE 3. Dimensionless maximal growth rate of modulational instability as a function of Ω in finite depth ($\mu = 2$) and deep water ($\mu = \infty$). Solid line ($\kappa = 0.005$, $\mu = 2$); dot-dashed line ($\kappa = 0.005$, $\mu = \infty$); dotted line ($\kappa = 0$, $\mu = 2$); dashed line ($\kappa = 0$, $\mu = \infty$). $\Gamma_{0\,imax}$ is the maximal growth rate for $\Omega = 0$.

The dimensionless bandwidth of instability is $\Delta\tilde{p} = \sqrt{-2\tilde{\gamma}_1/\alpha_2}$ and $\Delta p/k = \sqrt{-2\tilde{\gamma}_1/\alpha_2}\,a_0k$.

For $\kappa = 0$ and $\Omega \neq 0$, equation (3.11) gives the rate of growth of Thomas *et al.* (2012). Figure 3 shows plots of the dimensionless maximal growth rate of the modulational instability of pure gravity waves and gravity waves influenced by the surface tension effect ($\kappa = 0.005$) as a function of Ω for infinite and finite depths. We can observe that the combined effect of surface tension and vorticity increases significantly the rate of growth of the modulational instability of short gravity waves propagating in finite depth and in the presence of negative vorticity ($\Omega > 0$) whereas the effect is insignificant in deep water. For positive vorticity ($\Omega < 0$) the curves almost coincide at finite depth and deep water and the increase of the rate of growth due to surface tension is of the order of κ.

For $\Omega = 0$ and $\kappa \neq 0$, equation (2.20) of Djordjevic & Redekopp (1977) becomes

$$\mathrm{i}a_\tau - \frac{\omega}{8k^2}\frac{1-6\kappa-3\kappa^2}{(1+\kappa)^2}a_{\xi\xi} = \frac{k^2\omega}{16}\frac{8+\kappa+2\kappa^2}{(1-2\kappa)(1+\kappa)}|a|^2a, \tag{3.12}$$

for the envelope of the surface elevation in deep water.

The coefficients $\tilde{\gamma}_1$ and α_2 corresponding to this NLS equation are

$$\tilde{\gamma}_1 = \frac{1}{16}\frac{8+\kappa+2\kappa^2}{(1-2\kappa)(1+\kappa)}, \quad \alpha_2 = -\frac{1}{8}\frac{1-6\kappa-3\kappa^2}{(1+\kappa)^2}. \tag{3.13a,b}$$

Consequently, the rate of growth of the modulational instability of pure capillary wave trains on an infinite depth, obtained for $\kappa \to \infty$, is

$$\Gamma_i \to \frac{\omega}{8k^2}(3a_0^2k^4p^2 - 9p^4)^{1/2} \quad \text{as } \kappa \to \infty, \tag{3.14}$$

which can be found in Chen & Saffman (1985). The wavenumber of the fastest-growing modulational instability is $p_{max} = a_0k^2/\sqrt{6}$ and the maximum growth rate is $\omega(a_0k)^2/16$. Tiron & Choi (2012) have extended the linear stability of finite-amplitude capillary waves on deep water subject to superharmonic and subharmonic perturbations without a vorticity effect.

We have considered the case of pure capillary waves on deep water ($\kappa \rightarrow \infty$ and $\mu \rightarrow \infty$) in the presence of vorticity ($\Omega \neq 0$). The corresponding analytic expressions of $\tilde{\gamma}_1$ and α_2 are

$$\tilde{\gamma}_1 = -\frac{3 + 14X + 23X^2 + 11X^3 - 3X^4}{24(X+1)(3X+2)}, \tag{3.15}$$

$$\alpha_2 = \frac{3(X+1)(X^2+X+1)}{(2+X)^3}, \tag{3.16}$$

where $X = \Omega/\omega$ and $\omega = -\Omega/2 \pm \sqrt{(\Omega/2)^2 + k^3 T/\rho_w}$.

Due to the high wave frequency of capillaries on deep water we assume $|X| \ll 1$. The coefficients $\tilde{\gamma}_1$ and α_2 become

$$\tilde{\gamma}_1 = -\tfrac{1}{16}\left(1 + \tfrac{13}{6}X\right) + O(X^2), \tag{3.17}$$

$$\alpha_2 = \frac{3}{8}\left(1 + \frac{X}{2}\right) + O(X^2). \tag{3.18}$$

The rate of growth of the modulational instability of capillary waves on deep water in the presence of vorticity is

$$\Gamma_i = \frac{\omega p}{8k^2}\sqrt{3a_0^2k^4 - 9p^2 + (8a_0^2k^4 - 9p^2)X + O(X^2)}, \tag{3.19}$$

and in dimensionless form

$$\frac{\Gamma_i}{\omega\, a_0^2 k^2} = \frac{\tilde{p}}{8}\sqrt{3 - 9\tilde{p}^2 + (8 - 9\tilde{p})X + O(X^2)}. \tag{3.20}$$

The maximal growth rate of instability is obtained for $p = (1 + 5X/6)a_0k^2/\sqrt{6} + O(X^2)$ and its value is $(1 + 13X/6)\omega a_0^2 k^2/16 + O(X^2)$. The bandwidth of modulational instability is $\Delta p = (1 + 5X/6)a_0k^2/\sqrt{3}$.

Consequently, the rate of growth of modulational instability of capillary waves in deep water is larger for negative vorticity ($X > 0$) than for positive vorticity ($X < 0$). The bandwidth of the instability presents the same trend.

In figure 4 we show the dimensionless rate of growth of the modulational instability of pure capillary waves in finite depth as a function of the wavenumber of the perturbation, for several values of Ω. The rate of growth of the instability increases as Ω increases, as with infinite depth.

The sign of the product $\alpha\tilde{\gamma}$ determines the stability of the solution under infinitesimal perturbations. If the product is positive then the solutions are modulationally stable, otherwise they are modulationally unstable and grow exponentially with time. Davey & Stewartson (1974) and Djordjevic & Redekopp (1977) showed that this criterion, which works for one-dimensional propagation, can be extended to the case of two-dimensional propagation. In this way, our stability diagrams could be compared to those of Djordjevic & Redekopp (1977) when $\Omega = 0$. The linear stability analysis only captures the linear part of the instability, and thus its onset. We plot in the ($\mu = kh$, κ)-plane, for fixed values of the vorticity Ω, the unstable and stable regions. As a check, the instability diagrams we obtain are compared

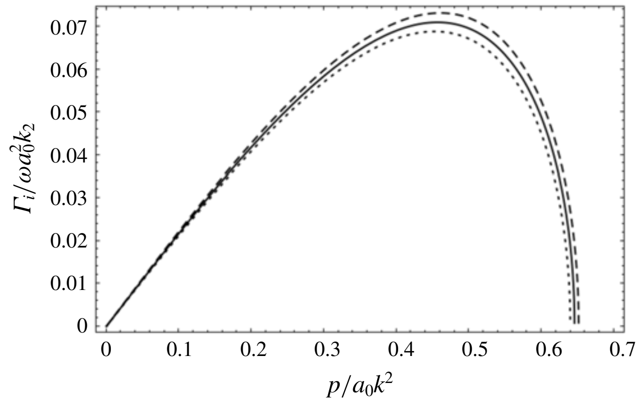

FIGURE 4. Dimensionless growth rate of the modulational instability of pure capillary waves in finite depth ($\mu = 2$) as a function of the dimensionless wavenumber of the perturbation for several values of Ω. $\Omega = 0$ (solid line); $\Omega = 2$ (dashed line); $\Omega = -2$ (dotted line).

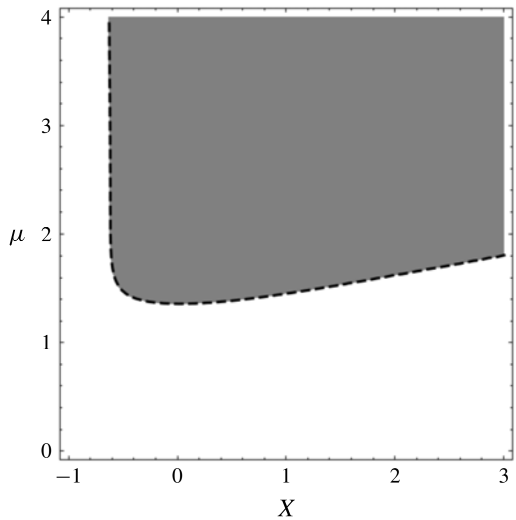

FIGURE 5. The (μ, X)-instability diagram for gravity waves, matching the results of Thomas *et al.* (2012) (dashed lines). Here, there is no surface tension. The unstable regions are in grey whereas stable regions are in white. For $X = 0$ (or $\Omega = 0$) the value $kh \approx 1.363$ is found, below which there is no instability.

in figures 5 and 6 with those obtained by Thomas *et al.* (2012) for $\kappa = 0$ and Djordjevic & Redekopp (1977) for $\Omega = 0$. In that way, we can verify that these limiting cases are reproduced correctly. Following Djordjevic & Redekopp (1977), the boundaries of the unstable regions have been numbered from 1 to 5. Curve 1 crosses the μ-axis at the point corresponding to restabilization of the modulational instability. Note that this feature holds for two-dimensional water waves. Curve 2 corresponds to vanishing of the dispersive coefficient α and minimum phase velocity ($c_g = c_p$) whereas along curves 3 and 4 the nonlinear coefficient $\tilde{\gamma}$ is singular. These

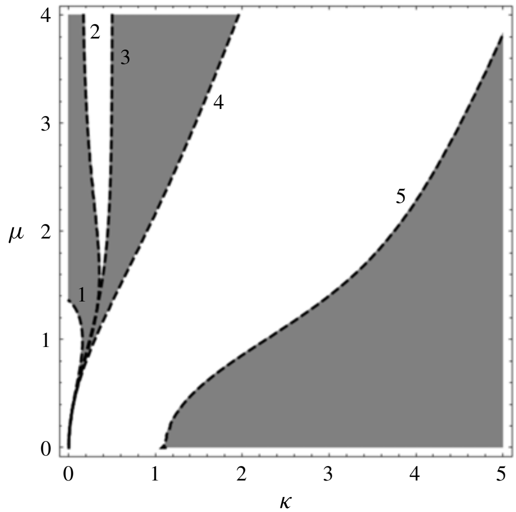

FIGURE 6. The (μ, κ)-instability diagram for gravity–capillary waves, matching the results from Djordjevic & Redekopp (1977) (dashed lines). Here, there is no vorticity. The unstable regions are in grey whereas stable regions are in white.

singularities define Wilton and long wave/short wave resonances, respectively. The Wilton ripple phenomenon considered herein corresponds to a bifurcation in which a steady progressive gravity–capillary wave can double its wavelength and the associated value of κ is given implicitly by

$$\kappa = \frac{\sigma^2\omega(\kappa)}{(3-\sigma^2)\omega(\kappa)+3\sigma\Omega}. \tag{3.21}$$

The critical value depends strongly on Ω and consequently curve 3 becomes distorted as $|\Omega|$ increases, this trend is amplified when vorticity intensity is large.

Curve 4 corresponds to the long wave/short wave resonance. The group velocity c_g of the short gravity–capillary wave matches the phase velocity of the long gravity wave influenced by vorticity (see (2.39)). Significant resonant wave interaction is to be expected in this case. Note that because (3.2) breaks down for these two resonances a different analysis and scaling is required.

Curves 1 and 5 correspond to simple zeros of the nonlinear coefficient $\tilde{\gamma}$.

Curve 4 has the following asymptote

$$\mu = \left(1+\frac{\Omega^2}{2}-\sqrt{\frac{\Omega^2}{4}(4+\Omega^2)}\right)\left(\frac{9}{4}\kappa-\frac{3}{4}\right), \quad \mu \gg 1, \tag{3.22}$$

whereas curve 5 has the asymptote

$$\mu = \frac{9}{4}\left(1+\frac{\Omega^2}{2}-\sqrt{\frac{\Omega^2}{4}(4+\Omega^2)}\right)\kappa+\frac{1}{4}\left(-35+3\Omega^2+\frac{29\Omega}{\sqrt{4+\Omega^2}}-\frac{3\Omega^3}{\sqrt{4+\Omega^2}}\right), \quad \mu \gg 1. \tag{3.23}$$

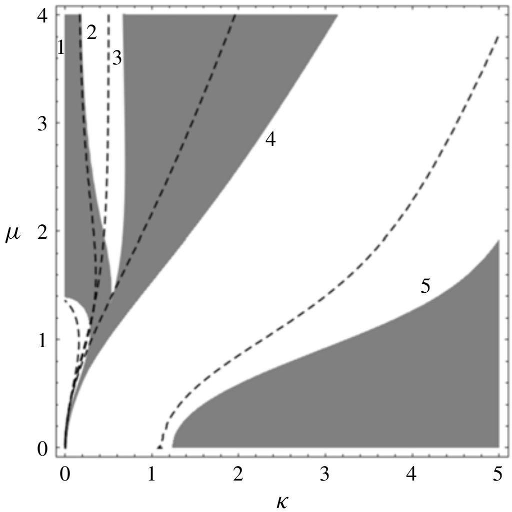

FIGURE 7. The (μ, κ)-instability diagram for $\Omega = -0.5$ (positive vorticity). The dashed lines correspond to $\Omega = 0$.

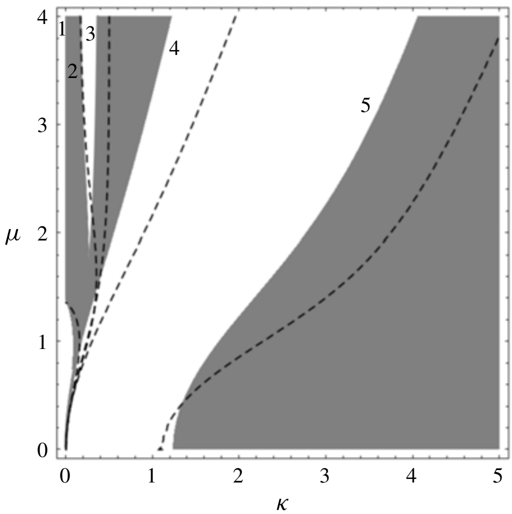

FIGURE 8. The (μ, κ)-instability diagram for $\Omega = 0.5$ (negative vorticity). The dashed lines correspond to $\Omega = 0$.

For $\Omega = 0$, the equations of Djordjevic & Redekopp (1977) are rediscovered except that instead of $-61/8$ we found $-35/4$ which is slightly different. The asymptotes have the same slope. In the region between these two asymptotes the capillary waves ($\kappa \gg 1$) are modulationally stable. This feature was emphasized by Djordjevic & Redekopp (1977) in the absence of vorticity.

In figures 7–14 the effect of positive and negative vorticity on the ($\mu = kh$, κ) diagrams is investigated. The curves of Djordjevic & Redekopp (1977) have been plotted to show the effect of the vorticity. As it can be observed, the vorticity has a significant effect on stability diagrams of gravity–capillary waves. Very recently, this

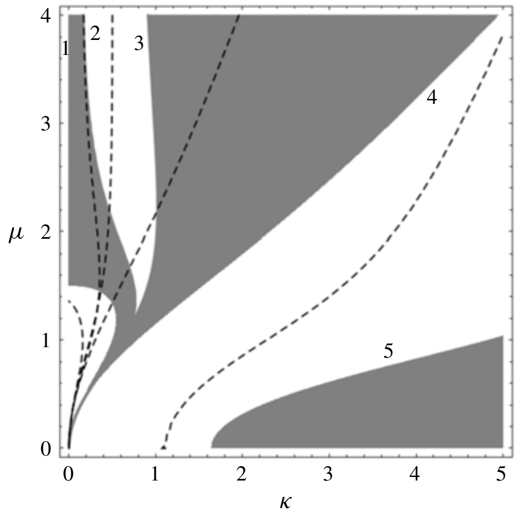

FIGURE 9. Same as figure 7 for $\varOmega = -1$.

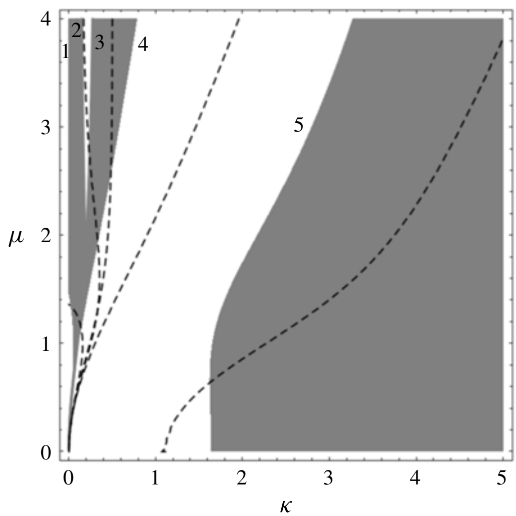

FIGURE 10. Same as figure 8 for $\varOmega = 1$.

feature was emphasized by Hur (2017) who proposed a shallow water wave model with constant vorticity and surface tension. This model presents some differences with our approach: (i) dispersion is introduced heuristically and is fully linear; (ii) nonlinear terms due to surface tension effect are ignored; (iii) the coupling between nonlinearity and dispersion is not taken into account.

As positive vorticity ($\varOmega < 0$) increases, we observe in figures 7, 9, 11 and 13 along the μ-axis in the vicinity of $\kappa = 0$ an increase of the region where the Stokes gravity–capillary wave train is modulationally stable. Consequently, gravity waves influenced by surface tension behave as pure gravity waves (see Thomas *et al.* (2012)). Nevertheless, a very thin tongue of instability persists, near $\kappa = 0$, in the shallow water regime.

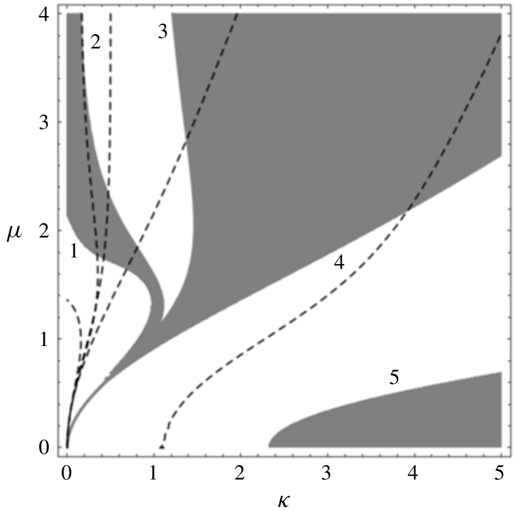

FIGURE 11. Same as figure 7 for $\Omega = -1.5$.

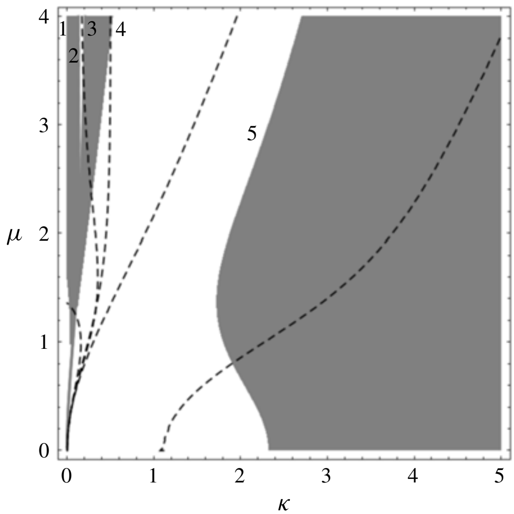

FIGURE 12. Same as figure 8 for $\Omega = 1.5$.

As the intensity of negative vorticity ($\Omega > 0$) increases, the band of instability along the μ-axis that corresponds to small values of κ becomes narrower, as shown in figures 8, 10, 12 and 14. Contrary to the case of positive vorticity, the region of restabilization along the μ-axis does not increase in the vicinity of $\kappa = 0$.

4. Conclusion

A nonlinear Schrödinger equation for capillary–gravity waves in finite depth with a linear shear current has been derived which extends the work of Thomas *et al.* (2012). The combined effect of vorticity and surface tension on the modulational instability properties of weakly nonlinear gravity–capillary and capillary wave trains has been

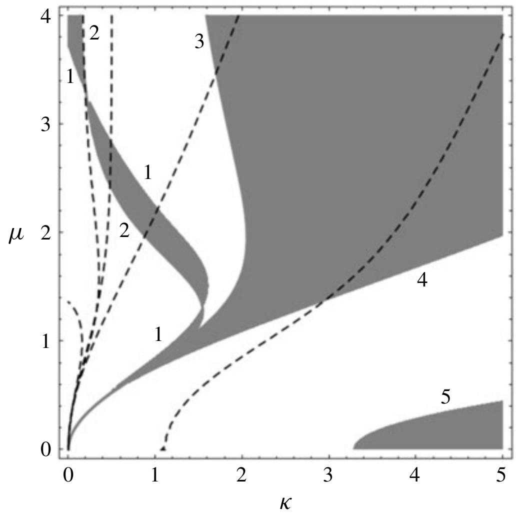

FIGURE 13. Same as figure 7 for $\Omega = -2$.

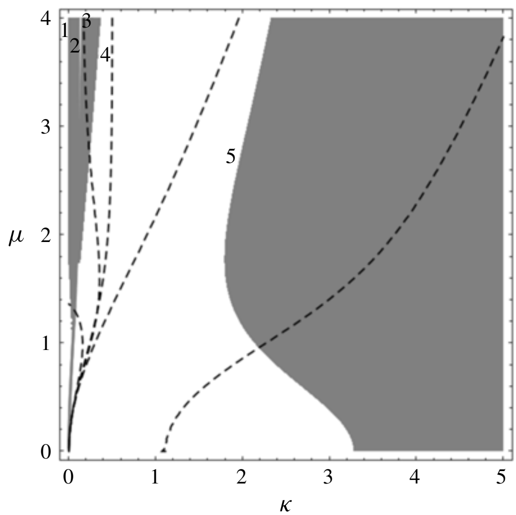

FIGURE 14. Same as figure 8 for $\Omega = 2$.

investigated. The explicit expressions of the dispersive and nonlinear coefficients are given as a function of the frequency and wavenumber of the carrier wave, the vorticity, the surface tension and the depth. The linear stability to modulational perturbations of the Stokes wave solution of the NLS equation has been carried out. Two kinds of waves have been especially investigated that concern short gravity waves influenced by surface tension and pure capillary waves. In both cases, the effect of vorticity is to modify the rate of growth of the modulational instability and the instability bandwidth. Furthermore, it is shown that the vorticity effect modifies significantly the stability diagrams of the gravity–capillary waves.

REFERENCES

BRATENBERG, C. & BREVIK, I. 1993 Higher-order water waves in currents of uniform vorticity in the presence of surface tension. *Phys. Scr.* **47**, 383–393.

CHEN, B. & SAFFMAN, P. G. 1985 Three-dimensional stability and bifurcation of capillary and gravity waves on deep water. *Stud. Appl. Maths* **72**, 125–147.

DAVEY, A. & STEWARTSON, K. 1974 On three-dimensional packets of surface waves. *Proc. R. Soc. Lond.* A **338**, 101–110.

DIAS, F. & KHARIF, C. 1999 Nonlinear gravity and capillary-gravity waves. *Annu. Rev. Fluid Mech.* **31**, 301–346.

DJORDJEVIC, V. D. & REDEKOPP, L. G. 1977 On two-dimensional packets of capillary-gravity waves. *J. Fluid Mech.* **79**, 703–714.

HOGAN, S. J. 1985 The fourth-order evolution equation for deep-water gravity–capillary waves. *Proc. R. Soc. Lond.* A **402**, 359–372.

HSU, H. C., FRANCIUS, M., MONTALVO, P. & KHARIF, C. 2016 Gravity–capillary waves in finite-depth on flows of constant vorticity. *Proc. R. Soc. Lond.* A **472**, 20160363.

HUR, V. M. 2017 Shallow water models with constant vorticity. *Eur. J. Mech.* (B*/Fluids)*; (in press) 10.1016.

KANG, Y. & VANDEN-BROECK, J.-M. 2000 Gravity–capillary waves in the presence of constant vorticity. *Eur. J. Mech.* (B*/Fluids)* **19**, 253–268.

MARTIN, C. I. & MATIOC, B. V. 2014 Existence of capillary-gravity water waves with piecewise constant vorticity. *J. Differ. Equ.* **256**, 3086–3114.

THOMAS, R., KHARIF, C. & MANNA, M. A. 2012 A nonlinear schrodinger equation for water waves on finite depth with constant vorticity. *Phys. Fluids*, 127102.

TIRON, R. & CHOI, W. 2012 Linear stability of finite-amplitude capillary waves on water of infinite depth. *J. Fluid Mech.* **696**, 402–422.

WAHLEN, E. 2006*a* Steady periodic capillary-gravity waves with vorticity. *SIAM J. Math. Anal.* **38** (3), 921–943.

WAHLEN, E. 2006*b* Steady periodic capillary waves with vorticity. *Ark. Mat.* **44**, 367–387.

WAHLEN, E. 2007 On rotational water waves with surface tension. *Phil. Trans. R. Soc. Lond.* A **365**, 2215–2225.

ZHANG, J. & MELVILLE, W. K. 1986 On the stability of weakly-nonlinear gravity–capillary waves. *Wave Motion* **8**, 439–454.