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A MARTINGALE VIEW OF BLACKWELL'S RENEWAL THEOREM AND ITS EXTENSIONS TO A GENERAL COUNTING PROCESS

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Abstract

Martingales constitute a basic tool in stochastic analysis; this paper considers their application to counting processes. We use this tool to revisit a renewal theorem and give extensions for various counting processes. We first consider a renewal process as a pilot example, deriving a new semimartingale representation that differs from the standard decomposition via the stochastic intensity function. We then revisit Blackwell's renewal theorem, its refinements and extensions. Based on these observations, we extend the semimartingale representation to a general counting process, and give conditions under which asymptotic behaviour similar to Blackwell's renewal theorem holds.

Keywords: Renewal process; semimartingale; Blackwell's renewal theorem; key renewal theorem; Wald's identity; stationary intervals; Palm distribution; modulated renewal process; Smith's asymptotic formulae

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1. Introduction

Let $N(t)$, $t \geq 0$, be a counting process with $\mathbb{E}[N(t)]$ finite for all $t \geq 0$. We are interested in the asymptotic behaviour of $\mathbb{E}[N(t)]$ for large t. For example, if $N(t)$ is a renewal process whose lifetime distribution is nonarithmetic and has a finite mean $1/\lambda$, Blackwell's renewal theorem, that $\mathbb{E}[N(t+h) - N(t)] \to \lambda h$ for $t \to \infty$ and any finite $h > 0$, is a well-known result, and has been extended to Markov renewal processes (see e.g. [1] and [6]). This theorem motivates us to consider under what conditions it may hold for a general counting process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

To answer this question, we need a suitable description for the dynamics of a general counting process. There have been various studies of refinements and extensions of Blackwell's renewal theorem (see e.g. [2], [8], [9], and [10]), but they are based on independence or Markov assumptions on the counting process. Hence, such traditional approaches may not be suitable for the present problem. In this paper we use martingales to study this question. In general, a martingale is used to construct an unbiased purely random component of a stochastic process. For *any* counting process $N(t)$ assumed to be right-continuous in t and with $\mathbb{E}[N(t)]$ finite for finite t, a martingale $M(t)$ typically arises in a semimartingale representation

$$N(t) = \Lambda(t) + M(t), \qquad t \geq 0, \tag{1.1}$$

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where $\Lambda(t)$ is a process of bounded variation. Here we must be careful about two things. One is the filtration $\mathbb{F} \equiv \{\mathcal{F}_t : t \geq 0\}$ to which $N(\cdot) \equiv \{N(t) : t \geq 0\}$ is adapted; the other is the predictability of $\Lambda(\cdot) \equiv \{\Lambda(t) : t \geq 0\}$ with respect to the filtration, where $\Lambda(\cdot)$ is said to be $\mathbb{F}$-predictable (or simply predictable when $\mathbb{F}$ is clear in the context) if it is measurable with respect to the σ-field $\mathcal{P}$ on $\Omega \times [0, \infty)$ generated by all left-continuous $\mathbb{F}$-adapted real-valued processes for time $t \in [0, \infty)$.

Remark 1.1. If a stochastic process $X(\cdot) \equiv \{X(t) : t \geq 0\}$ is $\mathbb{F}$-predictable, then $X(t)$ is $\mathcal{F}_{t-}$-measurable for all $t \geq 0$, where $\mathcal{F}_{t-} = \sigma(\cup_{s<t} \mathcal{F}_s)$ (see [12, Proposition I.2.4]). For convenience in this paper, we call $X(\cdot)$ satisfying the latter condition $\mathbb{F}$*-predictable in the weak sense.*

Consider the filtration $\mathbb{F}^N \equiv \{\mathcal{F}_t^N : t \geq 0\}$, where $\mathcal{F}_t^N = \sigma(\{N(u) : u \leq t\})$. When $\Lambda(\cdot)$ is $\mathbb{F}^N$-predictable, both it and therefore the martingale $M(\cdot)$ are a.s. uniquely determined by virtue of the Doob–Meyer decomposition because $N(\cdot)$ is a submartingale (see e.g. [13, Lemma 25.7]). Such predictable $\Lambda(\cdot)$ is called a *compensator*, and in this case $\Lambda(t)$ is nondecreasing in t. Consequently, if $\Lambda(t)$ is absolutely continuous with respect to Lebesgue measure, then we can write

$$\Lambda(t) = N(0) + \int_0^t \lambda_u \, \mathrm{d}u, \tag{1.2}$$

where λ_t is a nonnegative process and can be predictable with respect to $\mathbb{F}^N$, and called a stochastic intensity. In particular, λ_t is called the *hazard rate* function when $N(\cdot)$ is a renewal process (see e.g. [4] and [12]). However, such λ_t may not be amenable to our asymptotic analysis except when λ_t is a deterministic function of t or its randomization, namely, $N(\cdot)$ is a Poisson or doubly stochastic Poisson process, either of which is less interesting for us. This may be the reason why the semimartingale representation (1.1) is little used in renewal theory.

In this paper, we study counting processes using martingales. However, we do not use the filtration $\mathbb{F}^N$; rather, we consider nonpredictable $\Lambda(\cdot)$ that makes our asymptotic analysis tractable. To this end, we formally introduce the counting process and related notations. Let $N(\cdot)$ be a nonnegative integer-valued process such that $N(t)$ is finite, nondecreasing, right-continuous, has left-hand limits in t and $\Delta N(t) \equiv N(t) - N(t-) \leq 1$ at all $t \geq 0$, where $N(0-) = 0$. Define the nth counting time of $N(\cdot)$ by

$$t_{n-1} = \inf\{t \geq 0 : n \leq N(t)\}, \qquad n \geq 1.$$

Since $N(t)$ is finite for all $t \geq 0$, the set $\{t_{n-1}, n = 1, 2, \ldots\}$ has no finite accumulation point in $[0, \infty)$. Thus, $N(\cdot)$ is a general orderly counting process. Unless stated otherwise, assume that $t_0 = 0$, and speak of $N(\cdot)$ as being *nondelayed*. Otherwise, if $t_0 > 0$, $N(\cdot)$ is said to be delayed. In either case, $n \leq N(t)$ if and only if $t_{n-1} \leq t$. Let $T_0 = t_0$ and $T_n = t_n - t_{n-1}$ for $n \geq 1$. Let $R(t)$ be the residual time to the next counting instant at time t, that is,

$$R(t) = T_0 + \sum_{\ell=1}^{N(t)} T_\ell - t, \qquad t \geq 0, \tag{1.3}$$

and define $X(t) = (N(t), R(t))$ for $t \geq 0$, where

$$X(0) = \begin{cases} (1, T_1) & \text{if } t_0 = 0, \\ (0, T_0) & \text{if } t_0 > 0. \end{cases}$$

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From the definition of $N(\cdot)$, $X(t)$ is right-continuous for all $t \geq 0$, and has a left-limit for all $t > 0$. Hence, $N(t_n) = n+1$, $R(t_n-) = 0$ and $R(t_n) = T_{n+1}$. Let $\mathbb{F}^X \equiv \{\mathcal{F}_t^X : t \geq 0\}$, where

$$\mathcal{F}_t^X = \sigma(\{X(u) : u \leq t\}), \qquad t \geq 0. \tag{1.4}$$

Observe that $N(\cdot)$ is predictable under this filtration, but $R(\cdot)$ need not be because $R(t)$ cannot be $\mathcal{F}_{t-}^X$-measurable when $R(t-) = 0$ unless all T_n are deterministic. If we replace $R(t)$ by the attained time $A(t)$ since the last arrival instant, we gain nothing because $\mathbb{F}^X = \mathbb{F}^N$ in this case.

In what follows, we use a filtration $\mathbb{F} \equiv \{\mathcal{F}_t : t \geq 0\}$ to which $X(\cdot) \equiv \{X(t) : t \geq 0\}$ is adapted, i.e. $\mathcal{F}_t^X \subset \mathcal{F}_t$ for all $t \geq 0$; we denote this by $\mathbb{F}^X \preceq \mathbb{F}$. For convenience, we set $N(0-) = 0$ and $\mathcal{F}_{0-}^X = \mathcal{F}_{0-} = \sigma(R(0-))$ unless otherwise stated, where $R(0-) = R(0)\, 1(N(0) = 0)$. In most cases, $\mathbb{F} = \mathbb{F}^X$ is sufficient, but a larger $\mathbb{F}$ is needed in Sections 5.1 and 5.2. For a stopping time τ, define $\mathcal{F}_{\tau-}$ by

$$\mathcal{F}_{\tau-} = \sigma(\{A \cap \{t < \tau\} : A \in \mathcal{F}_t, t \geq 0\}).$$

Since t_n is a stopping time with respect to $\mathbb{F}^X$, it is an $\mathbb{F}$-stopping time. Hence, we have the following fact.

Lemma 1.1. (I.1.14 of [12].) *For each $n \geq 0$, t_n is a stopping time and $\mathcal{F}_{t_n-}$-measurable.*

We make the following two assumptions throughout the paper unless stated otherwise.

(A1) $X(0) = (1, T_1)$.

(A2) $\mathbb{E}[T_n] < \infty$ for $n \geq 1$, where $T_n > 0$ almost surely by the orderliness of $N(\cdot)$.

We first consider a renewal process, as a pilot example. In this case we assume the following in addition to (A1) and (A2).

(A3) For $n \geq 1$, T_n is independent of $\mathcal{F}_{t_{n-1}-}$, and $T_1, T_2, \ldots$ are identically distributed, with the common distribution denoted by F. If $\mathbb{F} = \mathbb{F}^X$, then this condition is the same as $T_1, T_2, \ldots$ being independent and identically distributed.

For such a renewal process, we derive a new semimartingale representation; it differs from the standard representation that uses the stochastic intensity function (see Theorem 2.1). This enables us to revisit Blackwell's renewal theorem and consider some of its refinements and extensions. Based on these observations, we extend the semimartingale representation to a general counting process, for which we replace the key renewal assumption (A3) by

(A4) $\mathbb{E}[N(t)] < \infty$ for finite $t \geq 0$.

We then give conditions under which an asymptotic result similar to Blackwell's renewal theorem holds. In particular, Blackwell's result extends quite naturally to the counting process which is generated by a stationary sequence of inter-arrival times, in other words, the counting process under a Palm distribution (see Corollary 5.2).

This paper has five more sections. In Section 2, we present a general framework for the new semimartingale representation. We apply it to a renewal process $N(t)$, and consider its interpretation. In Section 3, we revisit Blackwell's renewal theorem and some other asymptotic properties of $\mathbb{E}[N(t)]$. In Section 4, we show how the present approach can be used to derive limit properties of $\operatorname{var} N(t)$. In Section 5, a new semimartingale representation is derived for a general counting process, and Blackwell's renewal theorem is extended under various scenarios. We give concluding remarks in Section 6. Some proofs are deferred to an appendix.

2. A semimartingale representation

As discussed in Section 1, our aim is to derive the semimartingale representation (1.1) for a general counting process $N(\cdot)$. We first consider this problem in a broader context.

2.1. Martingales from a general setting

Let $\mathbb{F} \equiv \{\mathcal{F}_t : t \geq 0\}$ be a filtration, and let $N(\cdot)$ with $N(0-) = 0$ be the orderly counting process introduced in Section 1; let $Y(\cdot) \equiv \{Y(t) : t \geq 0\}$ with $Y(0-) \in \mathcal{F}_{0-}$ be a real-valued stochastic process which is right-continuous and has left-limits. Assume the following.

(B1) $N(t)$ is $\mathcal{F}_{t-}$-measurable for all $t \geq 0$, namely, $N(\cdot)$ is $\mathbb{F}$-predictable in the weak sense (see Remark 1.1), and $Y(\cdot)$ is $\mathbb{F}$-adapted.

(B2) For every $t \geq 0$, $N(t)$ increases if $Y(t) \neq Y(t-)$.

(B3) For every $t \geq 0$, $Y(t)$ has a right-hand derivative, denoted $Y'(t)$.

Note that (B2) does not exclude the case in which $N(t)$ increases when $Y(t) = Y(t-)$. Furthermore, define $R(t)$ by (1.3), and let $X(t) = (N(t), R(t))$. Then it is not hard to see that (B1) implies that $\mathbb{F}^X \preceq \mathbb{F}$.

Since $t_0 = 0$ only if $N(0) = 1$, it now follows easily from elementary calculus that

$$Y(t) = Y(0-) + \int_0^t Y'(u)\,\mathrm{d}u + \sum_{n=0}^{N(t)-1} \Delta Y(t_n), \qquad t \geq 0, \tag{2.1}$$

where $\Delta Y(t) = Y(t) - Y(t-)$. Let

$$M_Y(t) := \sum_{n=0}^{N(t)-1} (Y(t_n) - \mathbb{E}\,[Y(t_n) \mid \mathcal{F}_{t_n-}]), \tag{2.2}$$

$$D_Y(t) := \sum_{n=0}^{N(t)-1} (\mathbb{E}\,[Y(t_n) \mid \mathcal{F}_{t_n-}] - Y(t_n-)). \tag{2.3}$$

With these two functions we then have the following lemma.

Lemma 2.1. *Assume that $N(\cdot) \equiv \{N(t) : t \geq 0\}$ with $N(0-) = 0$ and $Y(\cdot)$ with $Y(0-) \in \mathcal{F}_{0-}$ satisfy conditions* (B1)–(B3). *If $\mathbb{E}(|M_Y(t)|) < \infty$ for all finite $t \geq 0$, then $M_Y(\cdot)$ is an $\mathbb{F}$-martingale, and*

$$Y(t) = Y(0-) + \int_0^t Y'(u)\,\mathrm{d}u + D_Y(t) + M_Y(t), \qquad t \geq 0. \tag{2.4}$$

Remark 2.1. Since in (2.4) both the integration term and $D_Y(t)$ are predictable in the weak sense at least, the representation (2.4) for $Y(t)$ may be called a special semimartingale, since this terminology is used when the bounded variation component of a semimartingale is predictable (see [12, Chapter 1, Section 4c]).

Proof. Equation (2.4) follows immediately from (2.1). Thus, we only need to prove that $M_Y(\cdot)$ is an $\mathbb{F}$-martingale. Since $\mathbb{E}\,[|M_Y(t)|] < \infty$ by assumption, this is done by showing that

$$\mathbb{E}\,[M_Y(t) \mid \mathcal{F}_s] = M_Y(s), \qquad 0 \leq s < t. \tag{2.5}$$

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To this end, recall that t_n is $\mathcal{F}_{t_n-}$-measurable by Lemma 1.1. Since $n+1 \le N(t)$ if and only if $t_n \le t$, we can write $M_Y(t)$ as

$$M_Y(t) = \sum_{n=0}^{\infty} (Y(t_n) - \mathbb{E}\,[Y(t_n) \mid \mathcal{F}_{t_n-}])1(t_n \le t)$$
$$= \sum_{n=0}^{\infty} (Y(t_n)1(t_n \le t) - \mathbb{E}\,[Y(t_n)1(t_n \le t) \mid \mathcal{F}_{t_n-}]),$$

where $1(\cdot)$ is the indicator function of the statement '$\cdot$'. Hence, $\mathbb{E}\,[M_Y(t) \mid \mathcal{F}_s]$ equals

$$M_Y(s) + \sum_{n=0}^{\infty} \mathbb{E}(Y(t_n)1(s < t_n \le t) - \mathbb{E}[Y(t_n)1(s < t_n \le t) \mid \mathcal{F}_{t_n-}] \mid \mathcal{F}_s) = M_Y(s).$$

This proves (2.5), and therefore $M_Y(\cdot)$ is an $\mathbb{F}$-martingale. □

In what follows, we apply Lemma 2.1 with $Y(\cdot)$ chosen appropriately. For example, we can set $Y(t) = N(t)$ because (B1)–(B3) are obviously satisfied. Then $D_Y(t) \equiv N(t)$ and $M_Y(t) \equiv 0$ because $N(\cdot)$ is $\mathbb{F}$-predictable, and substitution in (2.4) yields the identity $Y(t) = N(t)$, and we should learn nothing. Thus, it is important to choose $Y(\cdot)$ suitably when applying Lemma 2.1.

2.2. Application to a renewal process

In our first application of Lemma 2.1 we find the semimartingale representation (1.1) for a renewal process $N(\cdot)$ defined by (A1)–(A3). This representation is used in establishing asymptotic properties of moments of $N(t)$ including Blackwell's renewal theorem in Section 3, and extended to a general counting process in Section 5. We first note the following well-known fact (see e.g. [10, Lemma in XI.1]).

Lemma 2.2. *Conditions* (A1)–(A3) *imply* (A4).

We choose a filtration $\mathbb{F}$ such that $\mathbb{F}^X \preceq \mathbb{F}$, where $\mathbb{F}^X$ is defined through (1.4). Let $Y(t) = N(t) - \lambda R(t)$; such $Y(\cdot)$ clearly satisfies conditions (B1)–(B3). The idea behind this choice of $Y(\cdot)$ is to introduce a control parameter λ so that $D_Y(t)$ vanishes. A similar idea is used for the queue length process of a many-server queue in [16]. Indeed,

$$\mathbb{E}\,[Y(t_n) \mid \mathcal{F}_{t_n-}] = N(t_n-) + 1 - \lambda\mathbb{E}(T_{n+1}) = N(t_n-) = Y(t_n-), \qquad n \ge 0, \tag{2.6}$$

and therefore $D_Y(t_n)$ vanishes, where $N(0-) = 0$; recall that $\mathcal{F}_{0-} = \{\varnothing, \Omega\}$. Further, $\mathbb{E}\,[N(t)] < \infty$ by Lemma 2.2, and the bound

$$\mathbb{E}[|Y(t_n) - \mathbb{E}\,[Y(t_n) \mid \mathcal{F}_{t_n-}]|] = \mathbb{E}[|-\lambda T_{n+1} + 1|] \le 2 < \infty$$

implies that

$$\mathbb{E}[|M_Y(t)|] \le \sum_{n=1}^{\infty} \mathbb{E}\{|Y(t_n) - \mathbb{E}\,[Y(t_n) \mid \mathcal{F}_{t_n-}]|\}\,\mathbb{P}(t_{n-1} \le t) \le 2\mathbb{E}\,[N(t)] < \infty.$$

Hence, the next theorem follows from Lemma 2.1 because $Y(0-) = 0$ when $R(0-) = 0$.

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Theorem 2.1. *Let $\mathbb{F}$ be a filtration such that $\mathbb{F}^X \preceq \mathbb{F}$, and assume* (A1)–(A3). *Then the renewal process $N(t)$ is expressible as*

$$N(t) = \lambda(t + R(t)) + M(t), \qquad t \geq 0, \tag{2.7}$$

where

$$M(t) = \sum_{n=0}^{N(t)-1} (1 - \lambda T_{n+1}) = \sum_{n=1}^{N(t)} (1 - \lambda T_n), \qquad t \geq 0,$$

is an $\mathbb{F}$-martingale.

Remark 2.2. $N(\cdot)$ is called a delayed renewal process when condition (A1) is replaced by $X(0) = (0, T_0)$ with $T_0 > 0$, while (A2) and (A3) are unchanged. Let $Y(t) = N(t) - \lambda R(t)$ for $t \geq 0$ and $Y(0-) = R(0)$; then $Y(0-) = -\lambda T_0$, and for $t \geq 0$,

$$D_Y(t) = 0, \qquad M_Y(t) = M(t).$$

Hence, by Lemma 2.1, (2.7) for the delayed renewal process becomes

$$N(t) = \lambda(t + R(t) - T_0) + M(t), \qquad t \geq 0. \tag{2.8}$$

In particular, if $R(t)$ is stationary, then $N(t)$ has stationary increments because

$$N(t) = \sum_{0<u\leq t} 1(R(u-) \neq R(u)).$$

Hence, (2.8) and $T_0 = R(0)$ imply that $M(t)$ also has stationary increments.

This remark shows that the delayed renewal process only changes the semimartingale representation (2.7) to have the extra term $-\lambda T_0$ as in (2.8). Since T_0 is independent of all T_n for $n \geq 1$ by (A3) and the weak convergence of $R(t)$ to its stationary distribution as $t \to \infty$ is key to our later arguments, the asymptotic results in Sections 3 and 4 are also valid for the delayed renewal process if T_0 has an appropriate finite moment. Since such extensions are obvious, we do not discuss them further below.

2.3. Interpretation of the semimartingale representation

By Theorem 2.1, we have the semimartingale representation (1.1) with

$$\Lambda(t) = \lambda[t + R(t)],$$

which is different from the compensator (1.2). This is not surprising because we have made a special semimartingale not for $N(t)$ but for $Y(t) \equiv N(t) - \lambda R(t)$, where we recall the special martingale discussed in Remark 2.1. Further, the filtration is different, and $\Lambda(\cdot)$ is not predictable because $R(\cdot)$ need not be predictable. Nevertheless, it suggests that the asymptotics of $N(t)$ can be studied via a bias term $\lambda[t + R(t)]$ and a pure noise term $M(t)$.

Another feature of (2.7) is its relation to Wald's identity. Define $S_n = \sum_{\ell=1}^{n} T_\ell$ for $n \geq 1$; then $S_{N(t)} = t + R(t)$, and therefore (2.7) can be written as

$$S_{N(t)} - \mathbb{E}[T]N(t) = -\mathbb{E}[T]M(t), \qquad t \geq 0, \tag{2.9}$$

which immediately gives Wald's identity, $\mathbb{E}[S_{N(t)}] = \mathbb{E}[T]\mathbb{E}[N(t)]$, since $\mathbb{E}[M(t)] = \mathbb{E}[M(0)] = 0$. This type of Wald's identity is well known (see e.g. [2, Section V.6]).

It is interesting here that (2.9) says more. For example, the $\mathbb{F}$-martingale $-\mathbb{E}(T)M(t)$ is an error for estimating $S_{N(t)}$ by $[\mathbb{E}(T)]N(t)$. To evaluate this error, we use certain facts concerning the quadratic variations of $M(\cdot)$ (see [20] for their definitions).

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Lemma 2.3. *Under the assumptions of Theorem 2.1, if* $\mathbb{E}(T^2) < \infty$, *the optional and predictable quadratic variations of* $M(\cdot)$ *are given for* $t \geq 0$ *by*

$$[M](t) = \sum_{n=1}^{N(t)} (1 - \lambda T_n)^2, \tag{2.10}$$

$$\langle M \rangle(t) = \lambda^2 \sigma_T^2 N(t), \tag{2.11}$$

respectively, where σ_T^2 *is the variance of* T, *and therefore*

$$\mathbb{E}\,[M^2(t)] = \mathbb{E}\,[\langle M \rangle(t)] = \lambda^3 \sigma_T^2 (t + \mathbb{E}\,[R(t)]). \tag{2.12}$$

Proof. Since $M(t)$ is piecewise constant and discontinuous at increasing instants of $N(t)$, (2.10) is immediate from the definition of an optional quadratic variation (see e.g. [17, Theorem 3.1]). Since the predictable quadratic variation $\langle M \rangle(t)$ is defined as a predictable process for $M^2(t) - \langle M \rangle(t)$ to be a martingale, (2.11) is obtained from Lemma 2.1 with $Y(t) = M^2(t)$. Its proof is detailed in Appendix A.1. Finally, (2.12) follows from (2.11) and (2.7). □

Notice that $N(\cdot)$ is predictable, but $T_{N(\cdot)}$ is not in our filtration $\mathbb{F}$, while neither $N(\cdot)$ nor $T_{N(\cdot)}$ are predictable in the filtration $\mathbb{F}^N$ generated by $N(\cdot)$. This explains why $[M](t)$ of (2.10) differs from $\langle M \rangle(t)$ of (2.11).

If $\mathbb{E}\,[T^2] < \infty$, it follows from Lemma 2.3 and the inequality $\mathbb{E}\,[R(t)] \leq \lambda \mathbb{E}\,[T^2]$ for $t \geq 0$ (see e.g. [2, Proposition 6.2, p. 160]) that the expected quadratic error of (2.9) is

$$(\mathbb{E}\,[T])^2 \mathbb{E}\,[M^2(t)] = \sigma_T^2 \lambda (t + \mathbb{E}\,[R(t)]) \leq \sigma_T^2 (\lambda t + \lambda^2 \mathbb{E}\,[T^2]).$$

3. First moment asymptotics for a renewal process

We have asserted that the semimartingale representation (2.7) can be used to find the asymptotics of a renewal process $N(t)$ for large t. In this subsection, we do so for the first moment under two scenarios that depend on the finiteness or otherwise of $\mathbb{E}\,[T^2]$.

3.1. Blackwell's renewal theorem, revisited

The first moment asymptotic is well known as Blackwell's renewal theorem. In view of the representation (2.7), the asymptotic behaviour of $\mathbb{E}\,[N(t)]$ is determined by that of $\mathbb{E}\,[R(t)]$. Taking this into account, we reformulate Blackwell's renewal theorem as follows.

Lemma 3.1. *For a renewal process* $N(\cdot)$ *satisfying assumptions* (A1)–(A3), *the following three conditions are equivalent.*

(2a) *The distribution of* T *is nonarithmetic, i.e. there is no* $\delta > 0$ *such that* $\mathbb{P}(T \in \{n\delta : n \geq 1\}) = 1$.

(2b) *Blackwell's renewal theorem holds, i.e. for each* $h > 0$ *and* $\lambda = 1/\mathbb{E}\,[T]$,

$$\lim_{t \to \infty} \mathbb{E}\,[N(t+h) - N(t)] = \lambda h. \tag{3.1}$$

(2c) $R(t)$ *has a limiting distribution as* $t \to \infty$.

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When one of these conditions holds, the limiting distribution of $R(t)$ is given by

$$\lim_{t\to\infty} \mathbb{P}(R(t) \le x) = \lambda \int_0^x \mathbb{P}(T > u)\,\mathrm{d}u. \tag{3.2}$$

Remark 3.1. In the delayed case, the nonarithmetic condition in (2a) is not necessary for $R(t)$ to have a limiting distribution in (2c). Following equation (3.4) below, we give a direct proof that (2c) implies (2b), and hence of their equivalence in this case.

Proof. This lemma owes its proof to Feller's key renewal theorem [10, Chapter XI]. By Blackwell's renewal theorem, which is a special case of Feller's key renewal theorem, (2a) implies (2b).

By Feller's direct Riemann integrability argument, (2b) implies (2c) and (3.2) (see [10, Section XI.4]).

Finally, (2c) implies (2a) because (2c) does not hold if (2a) does not hold; equivalently, F is arithmetic. □

Up to this point, the asymptotic behaviour of $\mathbb{E}\,[N(t)]$ provided by Lemma 3.1 has nothing to do with the semimartingale representation (2.7). However, when seen from a sample path base, (2.7) can be viewed as a pre-limit renewal theorem. Taking its expectation yields

$$\mathbb{E}\,[N(t)] = \lambda t + \lambda \mathbb{E}\,[R(t)], \qquad t \ge 0, \tag{3.3}$$

which is equivalent to Wald's identity as discussed in Section 2.3. It is of interest here to see how Blackwell's renewal theorem (3.1) can be obtained directly from (3.3) which provides information on $R(t)$, namely, (3.1) holds if and only if

$$\lim_{t\to\infty} \mathbb{E}\,[R(t+h) - R(t)] = 0. \tag{3.4}$$

If $\mathbb{E}\,[T^2] < \infty$, then (3.4) is immediate from (2c) because $\mathbb{E}\,[R(t)] \to \frac{1}{2}\lambda\mathbb{E}(T^2)$ as $t \to \infty$. However, this argument fails when $\mathbb{E}\,[T^2] = \infty$. Nevertheless, (3.4), equivalently (2b), is still obtained directly from (2c) using another semimartingale representation of $N(t)$, as we now show.

Direct proof of (2b) *and* (3.2) *from* (2c). Let $\widetilde{R}$ be a random variable with the limit distribution of $R(t)$. Apply Lemma 2.1 with $Y(t) = N(t) - \lambda_v(v \wedge R(t))$ for each fixed $v \in C_{\widetilde{R}}$, where $a \wedge b = \min(a, b)$ for $a, b \in \mathbb{R}$, $\lambda_v = 1/\mathbb{E}\,[v \wedge T]$, and $C_{\widetilde{R}}$ is the set of all continuity points of the distribution of $\widetilde{R}$. Similarly to (2.6), we can check that $D_Y(t) = 0$ and $Y'(t) = \lambda_v 1(R(t) < v)$. Then

$$M_v(t) := \sum_{n=1}^{N(t)} (1 - \lambda_v(v \wedge T_n))$$

is an $\mathbb{F}$-martingale for the filtration $\mathbb{F}$ such that $\mathbb{F}^X \preceq \mathbb{F}$, and for $v > 0$,

$$N(t) = \lambda_v \int_0^t 1(R(u) < v)\,\mathrm{d}u + \lambda_v(v \wedge R(t)) + M_v(t). \tag{3.5}$$

Taking expectations in (3.5) yields

$$\mathbb{E}\,[N(t)] = \lambda_v \int_0^t \mathbb{P}(R(u) < v)\,\mathrm{d}u + \lambda_v \mathbb{E}\,[v \wedge R(t)], \qquad t \ge 0, \tag{3.6}$$

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and therefore, if (2c) holds, then, as $u \to \infty$, $\mathbb{P}(R(u) < v)$ converges to $\mathbb{P}(\widetilde{R} < v)$, which equals $\mathbb{P}(\widetilde{R} \le v)$ at continuity points $v \in C_{\widetilde{R}}$, so for such v,

$$\lim_{t\to\infty} \mathbb{E}\,[N(t)]\big/t = \lambda_v \mathbb{P}(\widetilde{R} \le v).$$

Since the left-hand side of this equation is independent of v, we have $\lambda_v \mathbb{P}(\widetilde{R} \le v) = \lambda^*$ for some λ^* and for all $v \in C_{\widetilde{R}}$. Hence, we have

$$\mathbb{P}(\widetilde{R} \le v) = \frac{\lambda^*}{\lambda_v} = \lambda^* \mathbb{E}\,[v \wedge T] = \lambda^* \mathbb{E}\left[\int_0^v 1(T > x)\,\mathrm{d}x\right], \tag{3.7}$$

and this equality holds for all $v \ge 0$ because the right-hand side is continuous in v. Letting $v \to \infty$ in (3.7) shows that $1 = \lambda^* \mathbb{E}\,[T]$. Hence, $\lambda^* = 1/\mathbb{E}\,[T] = \lambda$. This and (3.7) yield (3.2). It follows from (3.6) and (3.7) that, for each $h > 0$ and $v \in C_{\widetilde{R}}$,

$$\lim_{t\to\infty} (\mathbb{E}\,[N(t+h)] - \mathbb{E}\,[N(t)]) = \lambda_v \lim_{t\to\infty} \int_t^{t+h} \mathbb{P}(R(u) < v)\,\mathrm{d}u = \lambda h,$$

because $v \wedge R(t)$ is bounded by v. Thus, (2c) implies (2b) and (3.2). □

We show in Section 5 that the truncation technique used in this proof is useful for more general counting processes.

3.2. Infinite second moment case

When $\mathbb{E}\,[T^2] = \infty$, it is of interest to consider a refinement of the elementary renewal theorem $\mathbb{E}\,[N(t)] - \lambda t = \mathrm{o}(t)$. Sgibnev [18] studied this problem, starting with the case of an arithmetic lifetime distribution. Here we consider it through the asymptotic behaviour of $\mathbb{E}\,[R(t)]$ in (3.3). Recall first that, for some function $z(\cdot)\colon \mathbb{R}_+ \mapsto \mathbb{R}$, called a generator of $Z(\cdot)$, a solution Z of the general renewal equation of [10],

$$Z(t) = z(t) + \int_0^t Z(t-u)\,F(\mathrm{d}u), \qquad t \ge 0, \tag{3.8}$$

is given by

$$Z(t) = \mathbb{E}\left[\int_0^t z(t-u)\,N(\mathrm{d}u)\right].$$

We exhibit $\mathbb{E}\,[R(t)]$ as a solution of the general renewal equation. From (1.3), when $t_{n-1} \le t < t_n$, $R(t) = T_n - (t - t_{n-1})$, so choosing

$$z(t) = \mathbb{E}\,[(T-t)\,1(T > t)], \tag{3.9}$$

and noting the fact that t_{n-1} and T_n are independent, we have

$$\begin{aligned} \mathbb{E}\,[R(t)] &= \mathbb{E}\left[\sum_{n=1}^{\infty} (T_n - (t - t_{n-1}))1(t_{n-1} \le t < t_{n-1} + T_n)\right] \\ &= \mathbb{E}\left[\sum_{n=1}^{\infty} \mathbb{E}\,[(T_n - (t - t_{n-1}))\,1(t_{n-1} \le t < t_{n-1} + T_n) \,\Big|\, t_{n-1}]\right] \\ &= \mathbb{E}\left[\int_0^t z(t-u)\,N(\mathrm{d}u)\right]. \end{aligned} \tag{3.10}$$

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Thus, $z(\cdot)$ is indeed a generator for $\mathbb{E}\,[R(t)]$. To check the asymptotic behaviour of (3.10), the following lemma is useful (see also [18] or [9, Exercise 4.4.5(c)]).

Lemma 3.2. (Sgibnev [18], Theorem 4.) *If in* (3.8) *the generator $z(t)$ is nonnegative and nonincreasing in $t \geq 0$, then the solution $Z(\cdot)$ satisfies*

$$Z(t) \sim \lambda \int_0^t z(u)\,\mathrm{d}u,$$

where for functions $f, g \colon \mathbb{R}_+ \mapsto \mathbb{R}$, $f(t) \sim g(t)$ means $\lim_{t\to\infty} f(t)/g(t) = 1$.

It follows from Theorem 2.1 that (3.9), (3.10), and Lemma 3.2 yield

$$\mathbb{E}\,[N(t)] - \lambda t = \lambda \mathbb{E}\,[R(t)] \sim \lambda \int_0^t \int_u^\infty \mathbb{P}(T > x)\,\mathrm{d}x\,\mathrm{d}u \tag{3.11}$$

as shown in [18, Theorem 5]. The asymptotic behaviour of (3.11) may be viewed as a doubly integrated tail of the distribution F of T (see e.g. [11] for an integrated tail).

In this section, we have observed how the semimartingale representations (2.7) and (3.5) are helpful in elucidating the asymptotic behaviour of $\mathbb{E}\,[N(t)]$. One may wonder how the present approach might work for the asymptotic behaviour of higher moments of $N(t)$. This is considered in Section 4.

It is also of interest to see how the approach works for more general counting processes. Observe that (2.7) holds if $D_Y(t)$ of (2.3) vanishes, for which $N(t)$ need not necessarily be a renewal process; we discuss this extension in Section 5, where the exposition is independent of the results in Section 4.

4. Second moment asymptotics

We consider the variance of the renewal process $N(t)$, denoted $\operatorname{var} N(t)$. As shown below, the representation gives an alternative path for studying the asymptotic behaviour of $\operatorname{var} N(t)$. In particular, the martingale $M(t)$ plays an important role in this case, and this contrasts with the first moment case.

Begin by using (2.7) with $\mathbb{E}\,[M(t)] = 0$ to compute $\operatorname{var} N(t)$ in the form

$$\operatorname{var} N(t) = \lambda^2 \operatorname{var} R(t) + 2\lambda \mathbb{E}\,[R(t)M(t)] + \mathbb{E}\,[M^2(t)]. \tag{4.1}$$

From Lemma 2.3 we know that when $\mathbb{E}\,[T^2]$ is finite, $\mathbb{E}\,[M^2(t)] \sim \lambda^3 \sigma_T^2 t$. We therefore assume that $\mathbb{E}\,[T^2] < \infty$ because otherwise $\operatorname{var} N(t)$ is not finite. To study the asymptotic behaviour of $\operatorname{var} R(t)$, we consider $\mathbb{E}\,[R^2(t)]$; this function is the solution of the general renewal equation (3.8) with the generator (cf. around (3.9) above)

$$z(t) = z_2(t) := \mathbb{E}\,[(T-t)^2\, 1(T > t)] = \int_t^\infty 2x\,\mathbb{P}(T > x)\,\mathrm{d}x. \tag{4.2}$$

Let

$$h(t) = \int_0^t z_2(u)\,\mathrm{d}u = t\mathbb{E}\,[(T-t)T\, 1(T > t)] + \tfrac{1}{3}\mathbb{E}\,[(T \wedge t)^3].$$

Then Lemma 3.2 and $\mathbb{E}\,[T^2] < \infty$ yield, for $t \to \infty$,

$$\mathbb{E}\,[R^2(t)] \sim \lambda h(t) \sim \begin{cases} \lambda t z_2(t) = \mathrm{o}(t) & \text{if } \mathbb{E}\,[T^3] = \infty, \\ \frac{1}{3}\lambda \mathbb{E}\,[T^3] & \text{if } \mathbb{E}\,[T^3] < \infty. \end{cases} \tag{4.3}$$

Thus, $\mathbb{E}\,[R^2(t)] = \mathrm{o}(t)$. On the other hand, by the Cauchy–Schwarz inequality,

$$|\mathbb{E}[R(t)M(t)]| \le \sqrt{\mathbb{E}\,[R^2(t)]\,\mathbb{E}\,[M^2(t)]} \sim \lambda^2\sqrt{t\sigma_T^2 h(t)}. \tag{4.4}$$

Hence, because $\mathbb{E}\,[T^2] < \infty$, the relations (4.1) and $h(t) = \mathrm{o}(t)$ yield the known result (e.g. [7, Section 2])

$$\operatorname{var} N(t) = \lambda^3\sigma_T^2 t + \mathrm{o}(t), \qquad t \to \infty. \tag{4.5}$$

We can refine (4.5) by evaluating $\mathbb{E}[R(t)M(t)]$, namely, using (4.3) and (4.4), we have the next result.

Proposition 4.1. *Let N be a renewal process for which* (A1)–(A3) *hold. If* $\mathbb{E}\,[T^2] < \infty$ *and* $\mathbb{E}\,[T^3] = \infty$*, then with z_2 defined by* (4.2)*, the relation* (4.5) *is tightened to*

$$\operatorname{var} N(t) - \lambda^3\sigma_T^2 t = \mathrm{O}(t\sqrt{z_2(t)}).$$

Now consider the case when $\mathbb{E}\,[T^3] < \infty$. Then, from (4.3),

$$\lim_{t\to\infty} \operatorname{var} R(t) = \lim_{t\to\infty} h(t) - \lim_{t\to\infty} (\mathbb{E}\,[R(t)])^2 = \tfrac{1}{3}\lambda\mathbb{E}\,[T^3] - \tfrac{1}{4}\lambda^2(\mathbb{E}\,[T^2])^2. \tag{4.6}$$

To find the asymptotic behaviour of $\mathbb{E}[R(t)M(t)]$, we need the extra condition

(4a) $\mathbb{E}\,[R(t)] - C$ is directly Riemann integrable on $[0, \infty)$, (4.7)

where $C = \frac{1}{2}\lambda\mathbb{E}\,[T^2]$. Then the following holds (the proof is given in Appendix A.2).

Lemma 4.1. *Assume that* (A1)–(A3) *and condition* (4.7) *hold. Then, if* $\mathbb{E}\,[T^3] < \infty$,

$$\lim_{t\to\infty} \mathbb{E}\,[R(t)M(t)] = \tfrac{1}{2}(\lambda\mathbb{E}\,[T^2] - \lambda^2\mathbb{E}\,[T^3]) + \tfrac{1}{2}\lambda^3\sigma_T^2\mathbb{E}\,[T^2].$$

From this lemma and (4.6), equation (4.1) now yields (4.8).

Proposition 4.2. *Assume that* (A1)–(A3) *and condition* (4a) *hold. Then, if* $\mathbb{E}\,[T^3] < \infty$,

$$\operatorname{var} N(t) - \lambda^3\sigma_T^2 t = -\tfrac{2}{3}\lambda^3\mathbb{E}\,[T^3] + \tfrac{5}{4}\lambda^4(\mathbb{E}\,[T^2])^2 - \tfrac{1}{2}\lambda^2\mathbb{E}\,[T^2] + \mathrm{o}(1), \qquad t \to \infty. \tag{4.8}$$

Smith [19] first obtained this result under the condition that the distribution F of T is spread out.

Daley and Mohan [8] proposed two conditions A_ε and B_ρ as below.

Condition A_ε. For some $\varepsilon \ge 0$,

$$\mathbb{E}\,[R(t)] - C = \mathrm{o}(t^{-1-\varepsilon}), \qquad t \to \infty,$$

Condition B_ρ. F is strongly nonlattice, that is,

$$\liminf_{|\theta|\to\infty} |1 - \varphi_F(\theta)| > 0,$$

and $0 < \mathbb{E}\,[T^\rho] < \infty$ for some $\rho \ge 2$, where φ_F is the characteristic function of F, namely, $\varphi_F(\theta) = \mathbb{E}\,[\mathrm{e}^{\mathrm{i}\theta T}]$ for $\theta \in \mathbb{R}$ with $\mathrm{i} = \sqrt{-1}$.

Now the spread out condition implies that F is strongly nonlattice (see e.g. [2, Chapter VII, Proposition 1.6]), so when $\mathbb{E}[T^3] < \infty$, Condition B_ρ [8] is weaker than Smith's assumption [19].

It is easy to see that (4.7) is satisfied if either Condition A_ε holds for $\varepsilon > 0$ or Condition B_ρ holds (see e.g. [8, (2.5a)]). However a function $f(t) = o(t^{-1})$ need not be directly Riemann integrable. Hence, (4.7) may be stronger than Condition A_0, though it is unclear whether this case can occur. On the other hand, [7, Corollary 1] shows that

$$\lim_{t\to\infty} \int_0^t (\mathbb{E}[R(u)] - C)\, \mathrm{d}u \quad \text{exists and is finite,}$$

if and only if $\mathbb{E}[T^3] < \infty$. Thus, we may conjecture that $\mathbb{E}[T^3] < \infty$ implies (4.7), but this is a hard problem because $\mathbb{E}[R(t)] - C$ may oscillate wildly around the origin as $t \to \infty$. In other words, we do not know how to compare (4.7) with Condition A_0.

Thus, the semimartingale decomposition (2.7) can be used to study the asymptotic behaviour of a higher moment of $N(t)$, but it appears to require an extra condition such as (4.7).

5. Extension to a general counting process

The present martingale approach is easily adapted to a general counting process as long as $D_Y(t)$ of (2.3) vanishes. Here we consider such an extension, assuming (A1), (A2), and (A4). Recall that $\mathbb{F}^X \preceq \mathbb{F}$ means that $\mathcal{F}^X_t \subset \mathcal{F}_t$ for all $t \geq 0$, where $X(t) = (N(t), R(t))$. Our basic idea is to use a condition similar to (2c) (see condition (5a) later).

First we introduce a random function to replace λ in (2.7) for $v > 0$. Let $T^{(v)}_n = T_n \wedge v$; define $\widetilde{\lambda}^{(v)}(t)$ by

$$\widetilde{\lambda}^{(v)}(t) = \frac{1}{\mathbb{E}(T^{(v)}_{N(t)} \mid \mathcal{F}_{t_{N(t)-1}-})}, \qquad t \geq 0,$$

equivalently,

$$\widetilde{\lambda}^{(v)}(t) = \frac{1}{\mathbb{E}(T^{(v)}_n \mid \mathcal{F}_{t_{n-1}-})}, \qquad t \in [t_{n-1}, t_n),\ \ n = 1, 2, \ldots .$$

Condition (A2) implies that $\mathbb{E}[T^{(v)}_n \mid \mathcal{F}_{t_{n-1}-}]$ is finite and positive, so $\widetilde{\lambda}^{(v)}(t)$ is finite and bounded below by $1/v$, and is therefore well defined.

Lemma 5.1. *Let $\mathbb{F}$ be a filtration such that $\mathbb{F}^X \preceq \mathbb{F}$, and assume* (A1), (A2), *and* (A4). *Then the counting process $N(t)$ can be decomposed for each $v > 0$ via $R^{(v)}(t) = R(t) \wedge v$ as*

$$N(t) = \int_0^t \widetilde{\lambda}^{(v)}(s)\, 1(R(s) \leq v)\, \mathrm{d}s + \widetilde{\lambda}^{(v)}(t) R^{(v)}(t) + M^{(v)}(t), \tag{5.1}$$

where

$$M^{(v)}(t) = \sum_{n=1}^{N(t)} (1 - \widetilde{\lambda}^{(v)}(t_{n-1}) T^{(v)}_n), \qquad t \geq 0,$$

is an $\mathbb{F}$-martingale.

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Remark 5.1. The left-hand side of (5.1) does not depend on v, and therefore the right-hand side is also independent of v (see (3.5) and arguments below it). We note that Lemma 5.1 holds for $v = \infty$, and can be regarded as an extension of Theorem 2.1

Proof. Apply Lemma 2.1 with $Y(t) = N(t) - \widetilde{\lambda}^{(v)}(t)R^{(v)}(t)$. Note that

$$Y'(t) = \widetilde{\lambda}^{(v)}(t)\, 1(R(t) \le v), \qquad t_{n-1} < t < t_n,$$

because $\widetilde{\lambda}^{(v)}(t)$ is piecewise constant and $R'(t) = -1$ for $t \in (t_{n-1}, t_n)$. Then the facts that $R^{(v)}(t_n-) = 0$, $R^{(v)}(t_n) = T^{(v)}_{n+1}$, and $\widetilde{\lambda}^{(v)}(t_n)$ is $\mathcal{F}_{t_n-}$-measurable, imply that

$$D_Y(t) = \sum_{n=0}^{N(t)-1} \mathbb{E}(1 - \widetilde{\lambda}^{(v)}(t_n)T^{(v)}_{n+1} \mid \mathcal{F}_{t_n-}) = \sum_{n=0}^{N(t)-1} (1 - \widetilde{\lambda}^{(v)}(t_n)\mathbb{E}(T^{(v)}_{n+1} \mid \mathcal{F}_{t_n-})) = 0.$$

On the other hand,

$$M_Y(t) = 1 - \widetilde{\lambda}^{(v)}(t_0)T^{(v)}_1 + \sum_{n=1}^{N(t)-1} (1 - \widetilde{\lambda}^{(v)}(t_n)T^{(v)}_{n+1}) = \sum_{n=1}^{N(t)} (1 - \widetilde{\lambda}^{(v)}(t_{n-1})T^{(v)}_n).$$

Finally, since

$$\begin{aligned}\mathbb{E}\left[\sum_{n=1}^{N(t)} \widetilde{\lambda}^{(v)}(t_{n-1})T^{(v)}_n\right] &= \sum_{n=1}^{\infty} \mathbb{E}[\widetilde{\lambda}^{(v)}(t_{n-1})T^{(v)}_n\, 1(t_{n-1} \le t)] \\ &= \sum_{n=1}^{\infty} \mathbb{E}[\widetilde{\lambda}^{(v)}(t_{n-1})\, \mathbb{E}\,[T^{(v)}_n \mid \mathcal{F}_{t_{n-1}-}]\, 1(t_{n-1} \le t)] \\ &= \mathbb{E}\,[N(t)],\end{aligned}$$

then

$$\mathbb{E}[|M_Y(t)|] \le \mathbb{E}\,[N(t)] + \mathbb{E}\left[\sum_{n=1}^{N(t)} \widetilde{\lambda}^{(v)}(t_{n-1})T^{(v)}_n\right] = 2\mathbb{E}\,[N(t)] < \infty.$$

Hence, $M^{(v)}(\cdot) \equiv M_Y(\cdot)$ is an $\mathbb{F}$-martingale by Lemma 2.1, completing the proof of Lemma 5.1. □

Thus, we have derived the semimartingale representation (5.1) for $N(\cdot)$ under the assumptions (A1), (A2), and (A4). Using this representation, we extend Blackwell's renewal theorem to a general counting process. To do this, we focus attention on condition (2c) of Lemma 3.1, of which the following can be viewed as its extended version.

(5a) There exists $v > 0$ such that as $t \to \infty$, both $\mathbb{E}[\widetilde{\lambda}^{(v)}(t)\, 1(R(t) \le v)]$ and $\mathbb{E}[\widetilde{\lambda}^{(v)}(t)\, R^{(v)}(t)]$ converge to finite positive limits.

Since $\mathbb{E}\,[T^{(v)}_{N(t)} \mid \mathcal{F}_{t_{N(t)-1}-})$ is bounded by $v > 0$, it may be easier to check condition (5a) via the weak convergence of $\mathbb{E}\,[T^{(v)}_{N(t)} \mid \mathcal{F}_{t_{N(t)-1}-}]$ as $t \to \infty$, but to do this we need an extra condition of uniform integrability: the following is sufficient for (5a).

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(5b) There exists $v > 0$ such that:

(i) v is a continuity point of the limit distribution of $R^{(v)}(t)$, and

(ii) $(\mathbb{E}\,[T^{(v)}_{N(t)} \mid \mathcal{F}_{t_{N(t)-1}-}],\ R^{(v)}(t))$ has a limiting distribution as $t \to \infty$,

(iii) $\{\widetilde{\lambda}^{(v)}(t) \colon t \ge 0\}$ is uniformly integrable, i.e.

$$\lim_{a\to\infty} \sup_{t\ge 0} \mathbb{E}[\widetilde{\lambda}^{(v)}(t)\, 1(\widetilde{\lambda}^{(v)}(t) > a)] = 0.$$

We now present a general conclusion from (5a).

Theorem 5.1. *Under the assumptions of Lemma 5.1, if condition* (5a) *holds, then there exists* $\lambda > 0$ *such that*

$$\lim_{t\to\infty} \mathbb{E}\,[N(t)]\big/t = \lambda, \tag{5.2}$$

and

$$\lim_{t\to\infty} (\mathbb{E}[N(t+h)] - \mathbb{E}[N(t)]) = \lambda h, \qquad h \ge 0. \tag{5.3}$$

Proof. Let $v > 0$ be such that the expectations in condition (5a) converge; then by (5a) there exists finite $\lambda^{(v)} > 0$ such that

$$\lim_{t\to\infty} \mathbb{E}[\widetilde{\lambda}^{(v)}(t)\, 1(R(t) \le v)] = \lambda^{(v)}. \tag{5.4}$$

Apply Lemma 5.1. Taking the expectation of (5.1) yields

$$\mathbb{E}\,[N(t)] = \int_0^t \mathbb{E}[\widetilde{\lambda}^{(v)}(s)\, 1(R(s) \le v)]\, \mathrm{d}s + \mathbb{E}[\widetilde{\lambda}^{(v)}(t)\, R^{(v)}(t)]. \tag{5.5}$$

Divide both sides of this equation by t; letting $t \to \infty$ yields $\lim_{t\to\infty} \mathbb{E}\,[N(t)]/t = \lambda^{(v)}$. Now the left-hand side of this relation is independent of v, so $\lambda^{(v)}$ must also be independent of v: set $\lambda = \lambda^{(v)}$, giving (5.2). Equation (5.3) follows from (5.5) and (5a). □

In applying Theorem 5.1 it is important to check conditions (5a) or (5b). Obviously, conditions (5b) are satisfied by a nonarithmetic renewal process (see assumptions (A1)–(A3)), for which $T^{(v)}_n$ is identically distributed and independent of $\mathcal{F}_{t_{n-1}-}$. We sketch two scenarios in which the two conditions are relaxed.

5.1. Modulated inter-arrival times

Let $J(\cdot) \equiv \{J(t) \colon t \ge 0\}$ be a piecewise constant process on the state space S, which is a Polish space. Let $t_0 = 0$, and for $n = 1, 2, \ldots$ let t_n be the nth discontinuous instant of $J(t)$; these instants generate the counting process $N(\cdot)$. As usual, let $T_n = t_n - t_{n-1}$, and for $t \in [t_{n-1}, t_n)$ set $R(t) = t - t_{n-1}$ and $J(t) = J(t_{n-1})$. Define a joint process $U(\cdot)$ by

$$U(t) = (J(t), N(t), R(t)), \qquad t \ge 0.$$

Let $\mathcal{F}^U_t = \sigma(\{U(s) \colon s \le t\})$, and let $\mathbb{F}^U = \{\mathcal{F}^U_t \colon t \ge 0\}$; this is a filtration for $U(\cdot)$. Let $\mathbb{F} = \mathbb{F}^U$, then obviously $\mathbb{F}^X \preceq \mathbb{F}$ since $X(t) = (N(t), R(t))$. Assume the following conditions.

(M1) T_n is independent of $\mathcal{F}_{t_{n-1}-}$.

(M2) The distribution of T_n is nonarithmetic and determined by $J(t_{n-1}) \in S$.

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We refer to a process satisfying (M1) and (M2) as a *modulated renewal process*. A Markov-modulated renewal process is the special case in which $\{J(t_n)\colon n=0, 1, \dots\}$ is a Markov chain. Let $T^{(v)}(x)$ be the conditional expectation of $T_n^{(v)} \equiv v \wedge T_n$ given $J(t_{n-1})=x$, that is, $T^{(v)}(x) = \mathbb{E}\,[T_n^{(v)} \mid J(t_{n-1})=x]$.

Corollary 5.1. *For a modulated renewal process as defined above, if* (i) S *is countable,* (ii) $\inf_{x\in S} \mathbb{E}\,[T(x)] > 0$, *where* $T(x)=\mathbb{E}\,[T_n \mid J(t_{n-1})=x]$, *and* (iii) $(J(t), R(t))$ *has a limit distribution as* $t\to\infty$, *then both* (5.2) *and Blackwell's formula* (5.3) *hold with* λ *defined by*

$$\lambda = \mathbb{E}\left[\frac{1}{T^{(v)}(\widetilde{J})} 1(\widetilde{R} \le v)\right] = \mathbb{E}\left[\frac{1}{T(\widetilde{J})}\right], \tag{5.6}$$

where $(\widetilde{J}, \widetilde{R})$ *is a random variable with the limit distribution of* $(J(t), R(t))$, *and* v *is any continuity point of the distribution of* $\widetilde{R}$.

Proof. From condition (iii), for a continuity point v of the distribution of $\widetilde{R}$, and for a bounded function $f\colon S \mapsto \mathbb{R}$,

$$\lim_{t\to\infty} \mathbb{E}[f(J(t))\, 1(R(t)\le v)] = \mathbb{E}[f(\widetilde{J})\, 1(\widetilde{R}\le v)],$$
$$\lim_{t\to\infty} \mathbb{E}\,[f(J(t))\, R^{(v)}(t)] = \mathbb{E}\,[f(\widetilde{J})(\widetilde{R}\wedge v)].$$

By assumptions (i) and (ii) of the corollary, $f(x) := 1/\mathbb{E}\,[T^{(v)}(x)]$ is continuous, bounded, and positive, where x is discrete, so we take a discrete topology. Thus, condition (5a) is satisfied, and therefore (5.2) and (5.3) are obtained by Theorem 5.1. Here, (5.6) is immediate from (5.4) in the proof of Theorem 5.1. □

Remark 5.2. Under the conditions of Corollary 5.1, $J(\cdot)$ is piecewise continuous but no transition structure like that of a Markov chain is assumed: the restrictive conditions (i) and (ii) may be inconsistent with Markovianity. If S is a finite set, then (i) and (ii) automatically hold, and these may constitute circumstances when the present framework is useful. However, for a Markov-modulated renewal process, Blackwell's formula (5.3) is obtained under a certain recurrence condition of $J(t)$ without conditions (i) and (ii) (see e.g. [1]). In such a case the present approach would not be suitable.

5.2. Stationary inter-arrival times

Consider now the scenario in which $\{T_n\colon n\in\mathbb{Z}_+\}$ is a stationary sequence of positive reals with finite means, where $\mathbb{Z}_+$ is the set of all nonnegative integers. This sequence can be extended to a stationary sequence that starts at time $-\infty$, and is well described by the Palm distribution $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$. (We digress to note that in the point process literature, the Palm distribution is often notated as $\mathbb{P}_0$, and if need be, the distribution of a stationary point process, i.e. the distributions of counts on sets A_n are the same as for the translated sets A_n+t, are notated $\mathbb{P}$. To be consistent with Sections 1–3 of this paper, we retain the notation $\mathbb{P}$ for Palm distributions, and write $\overline{\mathbb{P}}$ for (count) stationary distributions as at (5.7) below.)

We introduce the standard formulation to describe $\{T_n\}$ by a point process under $(\Omega, \mathcal{F}, \mathbb{P})$ (see e.g. [3]). Let $\lambda = 1/\mathbb{E}(T_0)$, and let $\{t_n\}$ be a two-sided random sequence such that $t_0=0$ and

$$t_n = \begin{cases} T_1+\cdots+T_n & n>0, \\ -(T_{-1}+\cdots+T_{-|n|}) & n<0. \end{cases}$$

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Define a point process $N(\cdot)$ on $\mathbb{R}$ by

$$N(B) = \sum_{n=-\infty}^{\infty} 1(t_n \in B), \qquad B \in \mathcal{B}(\mathbb{R}).$$

Similarly to (1.3), define $R(t)$ by

$$R(t) = \begin{cases} \sum_{\ell=1}^{N([0,t])} T_\ell - t & t \geq 0, \\ \sum_{\ell=1}^{N((t,0))} (-T_{-\ell}) - t & t < 0. \end{cases}$$

We can then construct a shift operator group $\{\theta_t : t \in \mathbb{R}\}$ on Ω such that:

(S1) $\theta_t \circ A = \{\omega \in \Omega : \theta_t^{-1}(\omega) \in A\}$,

(S2) the point process N is consistent with θ_t, that is, $\theta_t \circ N(B) = N(B+t)$ for bounded $B \in \mathcal{B}(\mathbb{R})$ and $B + t = \{x + t \in \mathbb{R} : x \in B\}$, and

(S3) for $n \in \mathbb{Z}$, $\mathbb{P}(\theta_{t_n} \circ A) = \mathbb{P}(A)$ for $A \in \mathcal{F}$, where $\mathbb{Z}$ is the set of all integers.

Next define a probability measure $\overline{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ by

$$\overline{\mathbb{P}}(A) = \lambda \mathbb{E}\left[\int_0^{T_1} \theta_t \circ 1_A \,\mathrm{d}t\right], \qquad A \in \mathcal{F}. \tag{5.7}$$

It is well known (see e.g. [3]) that $N(\cdot)$ is a stationary point process under $\overline{\mathbb{P}}$, and $\overline{\mathbb{E}}[N(1)] = \lambda$. Furthermore, we recover $\mathbb{P}$ from $\overline{\mathbb{P}}$ by the so-called inversion formula: for each $\varepsilon > 0$,

$$\mathbb{P}(A) = \frac{1}{\lambda\varepsilon} \overline{\mathbb{P}}\left[\int_0^{\varepsilon} \theta_{-t} \circ 1_A \, N(\mathrm{d}t)\right], \qquad A \in \mathcal{F}. \tag{5.8}$$

We can now formulate Blackwell's renewal theorem for the stationary sequence.

Corollary 5.2. *Under assumptions* (A1)–(A2), *if* (i) $\{T_n : n \in \mathbb{Z}\}$ *is a stationary and ergodic sequence under the Palm distribution* $\mathbb{P}$, (ii) $\{\widetilde{\lambda}^{(v)}(t) : t \geq 0\}$ *is uniformly integrable under* $\mathbb{P}$, *and* (iii) *the mixing condition*

$$\lim_{t\to\infty} \overline{\mathbb{P}}(\theta_{-t} \circ A, t_1 \leq u) = \overline{\mathbb{P}}(A)\, \overline{\mathbb{P}}(t_1 \leq u), \qquad A \in \mathcal{F},\ u \geq 0, \tag{5.9}$$

holds, then Blackwell's formula (5.3) *holds together with* (5.2), *and*

$$\lambda = \frac{1}{\mathbb{E}\,[T_1]} = \mathbb{E}\left[\frac{1(R(0) \leq v)}{\mathbb{E}[T_1^{(v)} \mid \mathcal{F}_{0-}]}\right], \qquad v \geq 0, \tag{5.10}$$

where the sets $\mathcal{F}_t = \sigma(\{R(u) : u \leq t\} \cup \bigcup_{n=-\infty}^{\infty} \{t_n \leq t\})$ *define the filtration* $\mathbb{F} \equiv \{\mathcal{F}_t : t \in \mathbb{R}\}$.

Remark 5.3. The mixing condition (5.9) is used in [14, Theorem 3.2].

Proof. Let $\eta_n(\omega) = \theta_{t_n(\omega)}(\omega)$ be the shift operator on the sample space Ω; then

$$\begin{aligned}\eta_1 \circ (\widetilde{\lambda}^{(v)}(t), R^{(v)}(t)) &= \left(\frac{1}{\eta_1 \circ \mathbb{E}\,[T_n^{(v)} \mid \mathcal{F}_{t_{n-1}-}]},\ \eta_1 \circ T_n - (t - \eta_1 \circ t_{n-1})\right) \\ &= \left(\frac{1}{\mathbb{E}\,[T_n^{(v)} \mid \mathcal{F}_{t_n-}]},\ T_{n+1} - (t - t_n)\right), \qquad t_n \leq t < t_{n+1}.\end{aligned}$$

Hence for $n \in \mathbb{Z}$, $\{(\widetilde{\lambda}^{(v)}(t), R^{(v)}(t)) : t_{n-1} \leq t < t_n\}$ is a stationary sequence under $\mathbb{P}$, and

$$(\widetilde{\lambda}^{(v)}(t),\ R^{(v)}(t)) = \theta_t \circ (\widetilde{\lambda}^{(v)}(t) \circ \theta_{-t},\ R^{(v)}(t) \circ \theta_{-t}) = \theta_t \circ (\widetilde{\lambda}^{(v)}(0), R^{(v)}(0))$$

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is a stationary process under $\overline{\mathbb{P}}$. Let $f(x, y)$ be a nonnegative bounded continuous function on $\mathbb{R}_+^2$; then by (5.9), for $\varepsilon > 0$,

$$\lim_{t\to\infty} \overline{\mathbb{E}}[f(\widetilde{\lambda}^{(v)}(t),\ R^{(v)}(t))\, 1(t_1 \le \varepsilon)] = \overline{\mathbb{E}}[f(\widetilde{\lambda}^{(v)}(0),\ R^{(v)}(0))]\, \overline{\mathbb{P}}(t_1 \le \varepsilon).$$

On the other hand, by (5.8),

$$\left|\mathbb{E}\,[f(\widetilde{\lambda}^{(v)}(t),\ R^{(v)}(t))] - \frac{\overline{\mathbb{E}}[f(\widetilde{\lambda}^{(v)}(t+t_1),\ R^{(v)}(t+t_1))\, 1(t_1 \le \varepsilon)]}{\lambda\varepsilon}\right| \le \frac{\|f\|}{\lambda\varepsilon}\overline{\mathbb{E}}[N(\varepsilon)\, 1(N(\varepsilon) \ge 2)],$$

where $\|f\| = \sup_{(x,y)\in\mathbb{R}_+^2} |f(x, y)|$. This and (5.9) imply that

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbb{E}\,[f(\widetilde{\lambda}^{(v)}(t),\ R^{(v)}(t))] \\
&\quad\le \limsup_{t\to\infty} \frac{1}{\lambda\varepsilon}\overline{\mathbb{E}}\Big[\sup_{s\in[0,\varepsilon)} f(\widetilde{\lambda}^{(v)}(t+s),\ R^{(v)}(t+s))\, 1(t_1 \le \varepsilon)\Big] + \frac{\|f\|}{\lambda\varepsilon}\overline{\mathbb{E}}[N(\varepsilon)\, 1(N(\varepsilon) \ge 2)] \\
&\quad= \frac{1}{\lambda\varepsilon}\overline{\mathbb{E}}\Big[\sup_{s\in[0,\varepsilon)} f(\widetilde{\lambda}^{(v)}(s),\ R^{(v)}(s))\Big]\, \overline{\mathbb{P}}(t_1 \le \varepsilon) + \frac{\|f\|}{\lambda\varepsilon}\overline{\mathbb{E}}[N(\varepsilon)\, 1(N(\varepsilon) \ge 2)].
\end{aligned}$$

Now

$$\lim_{\varepsilon\downarrow 0} \frac{\overline{\mathbb{P}}(t_1 \le \varepsilon)}{\lambda\varepsilon} = 1 \quad \text{and} \quad \lim_{\varepsilon\downarrow 0} \frac{\overline{\mathbb{E}}[N(\varepsilon)\, 1(N(\varepsilon) \ge 2)]}{\lambda\varepsilon} = 0,$$

implying that

$$\limsup_{t\to\infty} \mathbb{E}[f(\widetilde{\lambda}^{(v)}(t),\ R^{(v)}(t))] \le \lim_{\varepsilon\downarrow 0} \overline{\mathbb{E}}\Big[\sup_{s\in[0,\varepsilon)} f(\widetilde{\lambda}^{(v)}(s),\ R^{(v)}(s))\Big] = \overline{\mathbb{E}}[f(\widetilde{\lambda}^{(v)}(0),\ R^{(v)}(0))],$$

by the right-continuity of $f(\widetilde{\lambda}^{(v)}(t),\ R^{(v)}(t))$. Similarly, we have

$$\liminf_{t\to\infty} \mathbb{E}[f(\widetilde{\lambda}^{(v)}(t),\ R^{(v)}(t))] \ge \overline{\mathbb{E}}[f(\widetilde{\lambda}^{(v)}(0),\ R^{(v)}(0))],$$

where

$$\overline{\mathbb{E}}[\widetilde{\lambda}^{(v)}(0)\, 1(R^{(v)}(0) \le v)] = \lambda\mathbb{E}\,[\widetilde{\lambda}^{(v)}(0)T_1^{(v)}] = \lambda\mathbb{E}\,[\widetilde{\lambda}^{(v)}(0)\, \mathbb{E}\,[t_1^{(v)} \mid \mathcal{F}_{0-}]] = \lambda < \infty.$$

Thus, by the uniform integrability assumption on $\widetilde{\lambda}^{(v)}(t)$, condition (5a) is satisfied. Equation (5.10) is an immediate consequence of (5.7), completing the proof. □

6. Concluding remarks

In this paper, we have used a certain martingale to give a new approach to Blackwell's renewal theorem and its extensions for general counting processes. One may envisage applying this approach to other problems.

For example, consider a diffusion approximation of the renewal process N of Section 1 for which $\mathbb{E}\,[T^2] < \infty$. Scale $N(t) - \lambda t$ as $\widetilde{N}_n(t) := n^{-1/2}(N(nt) - \lambda nt)$; this is called diffusion scaling. It is well known that $\widetilde{N}_n(\cdot)$ converges weakly to the Brownian motion $B(t)$ with $\operatorname{var} B(t) = \lambda^3\sigma_T^2 t$ in an appropriate function space with the Skorokhod topology. This is usually proved by the central limit theorem and a time change (see e.g. [5, Theorem 5.11] and [21, Corollary 7.3.1]). To derive this result in the framework of this paper, let $\widetilde{R}_n(t) =$

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$\lambda R(nt)/\sqrt{n}$ and $\widetilde{M}_n(t) = M(nt)/\sqrt{n}$; then, by Theorem 2.1,

$$\widetilde{N}_n(t) = \lambda\widetilde{R}_n(t) + \widetilde{M}_n(t). \tag{6.1}$$

Observe that as $n \to \infty$, $\widetilde{R}_n(t) \to 0$ in probability because

$$\limsup_{n\to\infty} \mathbb{E}\,[\widetilde{R}_n(t)] \le \limsup_{n\to\infty} \lambda\mathbb{E}\,[T^2]/\sqrt{n} = 0.$$

Further, Lemma 2.3 implies that

$$\lim_{n\to\infty} \langle \widetilde{M}_n(t)\rangle = \lim_{n\to\infty} \frac{N(nt)}{n}\,\lambda^2\sigma_T^2 = \lambda^3\sigma_T^2 t.$$

Hence (6.1) would imply that $\widetilde{N}_n(\cdot)$ converges weakly to the martingale with deterministic quadratic variation $\sigma_L^2 t$, and this is just the Brownian motion $B(t)$ with variance $\sigma_T^2 t$ if the limiting process of $\widetilde{N}_n(\cdot)$ is continuous in time. To convert this argument into a formal proof, we should need to verify a technical condition, called C-tightness (see e.g. [22, Theorem 2.1]); even without this, the above argument elucidates the mechanism of the diffusion approximation.

Thus, while the present approach is useful for studying counting processes, this may not be the case for studying stochastic models in applications. For example, counting processes appear as input data for stochastic models such as queueing and risk processes. In these applications, the semimartingale representation for a counting process may not be convenient because such stochastic processes are functionals of counting processes. In this situation, the general formulation in Section 2.1 would be useful if we can find an appropriate process $Y(t)$, which may not be a counting process but includes it as one of components. The second author recently studied this type of application in [16] and [15] for diffusion approximations and tail asymptotics of the stationary distributions for queues and their networks. This may be a direction for future study.

Appendix

A.1. Proof of (2.11)

Apply Lemma 2.1 with $Y(t) = M^2(t)$ for which $Y'(t) = 0$, and therefore

$$\begin{aligned}
\langle M\rangle(t) &= D_Y(t) \\
&= \sum_{n=0}^{N(t)-1} (\mathbb{E}\,[M^2(t_n) \mid \mathcal{F}_{t_n-}] - M^2(t_n-)) \\
&= \sum_{n=0}^{N(t)-1} (\mathbb{E}\,[(M(t_n-) + (1-\lambda T_{n+1}))^2 \mid \mathcal{F}_{t_n-}] - M^2(t_n-)) \\
&= \sum_{n=0}^{N(t)-1} \mathbb{E}\,[2M(t_n-)(1-\lambda T_{n+1}) + (1-\lambda T_{n+1})^2 \mid \mathcal{F}_{t_n-}] \\
&= \sum_{n=0}^{N(t)-1} \mathbb{E}\,[(1-\lambda T_{n+1})^2] \\
&= N(t)\mathbb{E}\,[(1-\lambda T)^2] = \lambda^2\sigma_T^2 N(t),
\end{aligned}$$

since $\mathbb{E}\,[1-\lambda T_{n+1} \mid \mathcal{F}_{t_n-}] = 0$. Thus, we have (2.11).

A.2. Proof of Lemma 4.1

The main idea of this proof is to apply the key renewal theorem. For this, recall that $t_{N(t)-1} \le t < t_{N(t)}$, and rewrite $R(t)M(t)$ as

$$R(t)M(t) = (t_{N(t)} - t)\sum_{\ell=1}^{N(t)}(1-\lambda T_\ell) = Z_1(t) + Z_2(t),$$

where

$$Z_1(t) = (T_{N(t)} + t_{N(t)-1} - t)(1-\lambda T_{N(t)}),$$
$$Z_2(t) = (T_{N(t)} + t_{N(t)-1} - t)\sum_{\ell=1}^{N(t)-1}(1-\lambda T_\ell).$$

We consider $\mathbb{E}(Z_1(t))$ and $\mathbb{E}(Z_2(t))$ separately. Let

$$z_3(t) = \mathbb{E}\,[(T-t)(1-\lambda T)\,1(T>t)], \qquad t \ge 0.$$

Then, much as for (3.10), the independence of t_{n-1} and T_n and the key renewal theorem (see e.g. [2, Example 2.6]) yield

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\,[Z_1(t)] &= \lim_{t\to\infty}\mathbb{E}\left[\sum_{n=1}^{\infty}(T_n - (t-t_{n-1}))(1-\lambda T_n)\,1(0 \le t - t_{n-1} < T_n)\right] \\
&= \lim_{t\to\infty}\mathbb{E}\left[\sum_{n=1}^{\infty} z_3(t-t_{n-1})\,1(t_{n-1}\le t)\right] \\
&= \lambda\int_0^\infty z_3(u)\,\mathrm{d}u = \lambda\mathbb{E}\left[\int_0^T (T-u)(1-\lambda T)\,\mathrm{d}u\right] \\
&= \tfrac{1}{2}\lambda(\mathbb{E}\,[T^2] - \lambda\mathbb{E}\,[T^3]).
\end{aligned}$$

In considering $\mathbb{E}\,[Z_2(t)]$, the limiting operations for the key renewal theorem are nested, so we use the extra condition (4.7). We prove that $\mathbb{E}\,[T^3] < \infty$ and that

$$\lim_{t\to\infty}\mathbb{E}\,[Z_2(t)] = \tfrac{1}{2}\lambda^3\mathbb{E}(T^2)\sigma_T^2. \tag{A.1}$$

First, rewrite $\mathbb{E}\,[Z_2(t)]$ as

$$\begin{aligned}
&\mathbb{E}\left[(T_{N(t)} + t_{N(t)-1} - t)\sum_{\ell=1}^{N(t)-1}(1-\lambda T_\ell)\right] \\
&\quad= \mathbb{E}\left[\sum_{n=1}^{\infty}(t_n - t)\sum_{\ell=1}^{n-1}(1-\lambda T_\ell)\,1(t_{n-1}\le t < t_n)\right] \\
&\quad= \mathbb{E}\left[\sum_{\ell=1}^{\infty}(1-\lambda T_\ell)\sum_{n=\ell+1}^{\infty}(t_n - t)\,1(t_{n-1}\le t< t_n)\right] \\
&\quad= \mathbb{E}\left[\sum_{\ell=1}^{\infty}(1-\lambda T_\ell)V_\ell(t)\,1(t_{\ell-1}\le t)\right],
\end{aligned} \tag{A.2}$$

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where

$$V_\ell(t) = \mathbb{E}\left[\sum_{n=\ell+1}^{\infty} (t_n - t)\, 1(t_{n-1} \le t < t_n) \,\Big|\, \mathcal{F}_{t_{\ell-1}}\right], \qquad t \ge 0,\ \ell \ge 1.$$

Let $\widetilde{N}(\cdot)$ be an independent copy of $N(\cdot)$, let $\widetilde{t}_n$ be the nth counting epoch of the renewal process $\widetilde{N}(\cdot)$ similar to $N(\cdot)$, and let $\widetilde{T}_n = \widetilde{t}_n - \widetilde{t}_{n-1}$ for $n \ge 1$, where $\widetilde{t}_0 = 0$. Similarly, let $\widetilde{R}(t)$ be the residual time to the next jump at time t. For $t \ge 0$, define

$$\begin{aligned}\widetilde{V}(t \mid x) &= \mathbb{E}\left[\sum_{n=2}^{\infty} (\widetilde{t}_n - t) 1(\widetilde{t}_{n-1} \le t < \widetilde{t}_n) \,\Big|\, \widetilde{T}_1 = x\right] \\ &= \mathbb{E}\left[\sum_{n=2}^{\infty} (\widetilde{t}_n - \widetilde{t}_1 - (t - \widetilde{t}_1)) 1(\widetilde{t}_{n-1} - \widetilde{t}_1 \le t - \widetilde{t}_1 < \widetilde{t}_n - \widetilde{t}_1) \,\Big|\, \widetilde{T}_1 = x\right].\end{aligned}$$

Since the last formula is independent of $\widetilde{T}_1$ and represents the residual time to the next jump at time $t - x$, it equals $\mathbb{E}\,[\widetilde{R}(t-x)]$.

For notational convenience in what follows, define a function $r(\cdot)$ by

$$r(t) = \begin{cases} \mathbb{E}\,[\widetilde{R}(t)] & t \ge 0, \\ 0 & t < 0, \end{cases}$$

and let $C = \frac{1}{2}\lambda \mathbb{E}(T^2)$. Since, for $n \ge \ell$, $t_n - t_\ell$ is independent of $\mathcal{F}_{t_{\ell-1}}$,

$$V_\ell(t) = \widetilde{V}(t - t_{\ell-1} \mid T_\ell) = r(t - (t_{\ell-1} + T_\ell)), \qquad t \ge 0.$$

Thus, (A.2) can be rewritten as

$$\mathbb{E}\left[(T_{N(t)} + t_{N(t)-1} - t) \sum_{\ell=1}^{N(t)-1} (1 - \lambda T_\ell)\right] = \mathbb{E}\left[\sum_{\ell=1}^{\infty} (1 - \lambda T_\ell) r(t - (t_{\ell-1} + T_\ell)) 1(t_{\ell-1} \le t)\right]. \tag{A.3}$$

Denote the right-hand side of (A.3) by $W(t)$, and decompose it as

$$W_1(t) = \mathbb{E}\left[\sum_{\ell=1}^{\infty} (\lambda T_\ell - 1)(C\, 1(T_\ell \le t - t_{\ell-1}) - r(t - (t_{\ell-1} + T_\ell)) 1(t_{\ell-1} \le t)\right],$$

$$W_2(t) = C\, \mathbb{E}\left[\sum_{\ell=1}^{\infty} (1 - \lambda T_\ell) 1(T_\ell \le t - t_{\ell-1})\right].$$

Define

$$w_1(t) = \mathbb{E}\,[(\lambda T - 1)(C - r(t-T)) 1(T \le t)] \quad \text{and} \quad w_2(t) = C\, \mathbb{E}\,[(1 - \lambda T) 1(T \le t)].$$

It is readily checked that, for $i = 1, 2$, $W_i(\cdot)$ is the solution of the general renewal equation with the generator $w_i(\cdot)$. Consider first $w_1(t)$, and introduce

$$g(t) = \begin{cases} C - r(t) & t \ge 0, \\ 0 & t < 0, \end{cases}$$

so that from the definition of w_1,

$$w_1(t) = \mathbb{E}\,[\lambda T g(t-T)1(T \le t)] - \mathbb{E}\,[g(t-T)1(T \le t)]$$
$$= \lambda \int_0^\infty u g(t-u)\, F(\mathrm{d}u) - \int_0^\infty g(t-u)\, F(\mathrm{d}u).$$

Using the assumption at (4.7), we show that the last two integrals are directly Riemann integrable. To this end, let $I_n^\delta(u) = (n\delta,\, (n+1)\delta]$ for $\delta > 0$ and $u \ge 0$. Then

$$\sup_{t \in I_n^\delta(0)} \int_0^\infty u\, g(t-u)\, F(\mathrm{d}u) \le \int_0^\infty u \sup_{t \in I_n^\delta} g(t-u)\, F(\mathrm{d}u).$$

Similarly,

$$\int_0^\infty u \inf_{t \in I_n^\delta} g(t-u)\, F(\mathrm{d}u) \le \inf_{t \in I_n^\delta(0)} \int_0^\infty u\, g(t-u)\, F(\mathrm{d}u).$$

Then, because $|g|$ is bounded, $\mathbb{E}\,[T] < \infty$, and for fixed $u \ge 0$, $g(t-u)$ is directly Riemann integrable for $t \ge u$, the first integral $\int_0^\infty u g(t-u)\, F(\mathrm{d}u)$ is directly Riemann integrable. Similarly, the second integral $\int_0^\infty g(t-u)\, F(\mathrm{d}u)$ is also directly Riemann integrable. Hence, $w_1(t)$ is directly Riemann integrable. We then compute the integration on w_1:

$$\int_0^s w_1(t)\, \mathrm{d}t = \mathbb{E}\left[\int_0^s (\lambda T - 1) g(t-T)\, 1(T \le t)\, \mathrm{d}t\right]$$
$$= \mathbb{E}\left[\int_0^{(s-T)^+} (\lambda T - 1) g(u)\, 1(u \ge 0)\, \mathrm{d}u\right]$$
$$= \mathbb{E}\left[\int_0^s (\lambda T - 1) g(u)\, \mathrm{d}u\right] - \mathbb{E}\left[\int_{(s-T)^+}^s (\lambda T - 1) g(u)\, \mathrm{d}u\right]$$
$$= \mathbb{E}\left[\int_{(s-T)^+}^s (1 - \lambda T) g(u)\, \mathrm{d}u\right]$$
$$= \int_0^s \mathbb{E}\,[(1-\lambda T)1(T > s-u)] g(u)\, \mathrm{d}u,$$

which is finite as $s \to \infty$ if $\mathbb{E}(T^2) < \infty$ because $|g(u)|$ is bounded by C. Furthermore, it is not hard to see that this integral converges to 0 as $s \to \infty$.

We next consider $w_2(t)$. Then, since $\mathbb{E}(T^2) < \infty$,

$$w_2(t) = C\,\mathbb{E}\,[(1-\lambda T)(1 - 1(T > t))] = -C\,\mathbb{E}\,[(1-\lambda T)1(T > t)],$$

is directly Riemann integrable, and

$$\int_0^\infty w_2(t)\, \mathrm{d}t = \lambda C\, \mathbb{E}\left[\int_0^\infty (T - \mathbb{E}\,[T])1(T > t)\, \mathrm{d}t\right] = \lambda C\, \sigma_T^2. \tag{A.4}$$

Hence, $w_1(t) + w_2(t)$ is directly Riemann integrable, and therefore the key renewal theorem and (A.4) yield

$$\lim_{t\to\infty} \mathbb{E}\left[(T_{N(t)} + t_{N(t)-1} - t) \sum_{\ell=1}^{N(t)-1} (1 - \lambda T_\ell)\right] = \lambda\left(\int_0^\infty w_1(t)\, \mathrm{d}t + \int_0^\infty w_2(t)\, \mathrm{d}t\right)$$
$$= \lambda \int_0^\infty w_2(t)\, \mathrm{d}t = \lambda^2 C\, \sigma_T^2.$$

Recalling that $C = \frac{1}{2}\lambda \mathbb{E}\,[T^2]$, (A.1) follows. This proves Lemma 4.1.

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