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Non-Bernoulli systems with completely positive entropy

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Abstract. A new approach to actions of countable amenable groups with completely positive entropy (cpe), allowing one to answer some basic questions in this field, was recently developed. The question of the existence of cpe actions which are not Bernoulli was raised. In this paper, we prove that every countable amenable group G, which contains an element of infinite order, has non-Bernoulli cpe actions. In fact we can produce, for any $h \in (0, \infty]$, an uncountable family of cpe actions of entropy h, which are pairwise automorphically non-isomorphic. These actions are given by a construction which we call co-induction. This construction is related to, but different from the standard induced action. We study the entropic properties of co-induction, proving that if α_G is co-induced from an action α_Γ of a subgroup Γ, then $h(\alpha_G) = h(\alpha_\Gamma)$. We also prove that if α_Γ is a non-Bernoulli cpe action of Γ, then α_G is also non-Bernoulli and cpe. Hence the problem of finding an uncountable family of pairwise non-isomorphic cpe actions of the same entropy is reduced to one of finding an uncountable family of non-Bernoulli cpe actions of $\mathbb{Z}$, which pairwise satisfy a property we call 'uniform somewhat disjointness'. We construct such a family using refinements of the classical cutting and stacking methods.

1. *Introduction*

A classical result of Ornstein [**24**] is that there exist non-Bernoulli K-automorphisms of any given entropy. This result was later improved by Ornstein and Shields [**26**] who produced an uncountable family of pairwise non-isomorphic K-automorphisms which are non-Bernoulli, but have the same positive entropy.

Another approach to constructing non-Bernoulli K-automorphisms was due to Feldman [**7**], who introduced the concept of a loosely Bernoulli system. A loosely Bernoulli action of positive entropy is one that is Kakutani equivalent to a Bernoulli shift. Feldman demonstrated the existence of K-automorphisms which are not loosely Bernoulli. This area was further investigated by Ornstein *et al* [**25**] who, in particular, produced an uncountable family of K-automorphisms which pairwise are not Kakutani equivalent. We mention also contributions of Katok to this program [**19, 20**]. Perhaps the simplest example of a K-automorphism S, which is not loosely Bernoulli, was given by Kalikow [**16**] in his famous study of T, T^{-1} actions. Kalikow's example has the property that S is isomorphic to S^{-1}.

Recently, Hoffman [**15**] developed a new and systematic approach to the problem of producing non-Bernoulli K-automorphisms, and many further properties of non-Bernoulli automorphisms have been demonstrated in the literature [**15, 35, 36**].

As we shall outline below, the theory of entropy and of cpe actions of amenable groups is now rather well developed. It is thus natural to ask whether an infinite amenable group must have non-Bernoulli cpe actions. In this article, we shall extend the theorems of Ornstein and Shields [**26**] to actions of amenable groups which have an element of infinite order. The question remains open for infinite amenable groups, all of whose elements are of finite order.

A constructive approach to the entropy of actions of locally-compact amenable groups is due to Ornstein and Weiss [**27**], Weiss [**40, 41**] and Lindenstrauss and Weiss [**21**]. They developed the theory of tiles and quasi-tiles in amenable groups, which allowed them to generalize some key results of Feldman, Ornstein, Rudolph, Sinai and others [**7, 25, 32, 34, 38**] for actions of classical groups $\mathbb{R}$ and $\mathbb{Z}$, to a broad class of amenable groups. We shall use their theories, particularly in §2 below, where we use generalized versions of Ornstein's and Sinai's theorems for countable amenable groups to study the entropy of an action as defined by Ollagnier [**23**].

The natural extension of K-automorphisms to this setting is the study of actions of amenable groups with completely positive entropy (cpe). For actions of $\mathbb{Z}$ on a Lebesgue space, this notion was introduced by Rokhlin and Sinai [**29**], who demonstrated that it is equivalent to the existence of perfect partitions with Kolmogorov's property K. They also proved, using perfect partitions, that cpe actions of $\mathbb{Z}$ have strong mixing properties (K-mixing) and a countable Lebesgue spectrum. Indeed, it was shown by Cornfeld *et al* [**3**] that K-mixing is equivalent to cpe for $\mathbb{Z}$-actions.

Kamiński [**17**] extended Rokhlin and Sinai's approach to actions of $\mathbb{Z}^d$, $d < \infty$, and Golodets and Sinel'shchikov [**12**] proved the existence of perfect partitions for actions of the group of upper triangular matrices over $\mathbb{Z}$ and its subgroups. However, it was demonstrated that the existence of such partitions for actions of the group $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ is a more difficult problem which remains unresolved [**18**]. In fact, the Rokhlin–Sinai approach cannot be applied to countable discrete amenable groups without past [**23**]. It seems that the notion of cpe actions is more appropriate for general amenable groups.

A new approach to the study of cpe actions, which may be applied to any free action of a countable amenable group, was introduced by Rudolph and Weiss [**37**]. In a well-known paper, Connes *et al* [**2**] proved that every free action of a countable amenable group G on a Lebesgue space is orbit equivalent to an action of $\mathbb{Z}$. It was shown [**37**] that the actions of G and $\mathbb{Z}$ have the same conditional mean entropy under certain additional assumptions. This allowed Rudolph and Weiss to prove that any cpe free action of a countable discrete amenable group is uniform mixing in the sense of Weiss [**40**], and indeed, that uniform mixing is equivalent to cpe [**13, 40**]. We will discuss the relationship between K-mixing and cpe in §4 and apply it in §5.

The results of Rudolph and Weiss [**37**] have caused heightened interest in cpe actions, and new results in this area have been obtained by Glasner *et al* [**11**], Golodets and Sinel'shchikov [**13**], Danilenko [**4**] and Dooley and Golodets [**6**]. Dooley and Golodets [**6**] proved that a cpe action of a countable amenable group has a countable Lebesgue spectrum, generalizing the result of Rokhlin and Sinai [**29**] for $\mathbb{Z}$. Avni [**1**] has recently announced new results on mixing and spectral properties of cpe actions of locally compact amenable groups with a good entropy theory, where he extended previously obtained methods and results [**6, 27, 37**].

We shall prove our version of the theorem of Ornstein and Weiss by constructing a non-Bernoulli cpe action of G starting from a non-Bernoulli cpe action of a subgroup isomorphic to $\mathbb{Z}$. To achieve this we present a construction which we call *co-induction*, which allows us to define an action α_G of G from a given action α_Γ of a subgroup Γ of G (see Definition 3.1 below). Co-induction is similar to, but differs from, induction in the sense of Mackey [**22**] and Zimmer [**42**].

We make a systematic study of the entropic properties of co-induced actions, in particular, establishing that $h(\alpha_G) = h(\alpha_\Gamma)$ (Proposition 3.4). Moreover, we show that if α_Γ is a cpe, non-Bernoulli action of Γ, then α_G is a cpe, non-Bernoulli action of G (Theorem 5.2). To prove this we need to establish some new estimates for the entropy of finite partitions (see Lemmas 5.1, 5.4 and 5.5), in order to show that the co-induced action α_G is uniform mixing. This guarantees that α_G has cpe in view of Theorem 4.2. Now as $\mathbb{Z}$ has non-Bernoulli cpe actions, the same is true for any countable amenable group containing $\mathbb{Z}$ as a subgroup (Corollaries 5.6 and 5.7). Conjecturally, this is the simplest class of non-Bernoulli cpe actions, and other interesting classes will be found. In Corollary 5.8 we considered the properties of cpe actions of an abelian group, co-induced from Kalikow's K-action of $\mathbb{Z}$.

The next problem we attack is to produce an uncountable family of cpe actions with the same positive entropy, as was done for automorphisms [**15, 26**].

Theorem 5.2 reduces this problem to the question of the existence of an uncountable family of K-automorphisms with a given entropy, which when co-induced create non-isomorphic actions of G. This level of rigidity is implied by a special property, uniform somewhat disjointness (Definition 6.3) which motivates the construction. Since we work with positive entropy, standard notions of disjointness, such as minimal self-joinings, are not available; however, this rather soft notion is, and is sufficient for our task. The main result of §6 is given in Corollary 6.33. We expect that the family of K-automorphisms we construct and the notion of uniform somewhat disjointness introduced will be of independent interest and application. The ideas of Hoffman [**15**] have been influential here, but his approach does not completely suffice for our purposes.

In §7, we use these results to construct, for any countable amenable group G containing $\mathbb{Z}$ as a subgroup, and for any $t \in (0, \infty]$, an uncountable family A_t of cpe actions satisfying:

- if $\alpha \in A_t$, then $h(\alpha) = t$;
- if $\alpha_1, \alpha_2 \in A_t$, $\alpha_1 \neq \alpha_2$, then α_1 and α_2 are not isomorphic; moreover
- α_1 and α_2 are not even automorphically isomorphic† (Theorem 7.5).

To illustrate our theorem, recall Grigorchuk's example [**14**] of a countable amenable finitely generated group, which is not elementary amenable. Since Grigorchuk's group contains elements of infinite order, our result implies that it has infinitely many non-automorphically isomorphic cpe actions of any given entropy (see Example 7.3.6).

2. *Entropy of an action of a countable discrete amenable group*

In this section, we introduce some basic notions on entropy of actions of a countable amenable group. In particular, we describe the relationship between the approach of Ornstein and Weiss [**27**] and that of Ollagnier [**23**], using the approach and methods of [**27**].

Let $(X, \mathcal{B}, \mu)$ be a standard Lebesgue space and $T \in \mathrm{Aut}(X, \mathcal{B}, \mu)$, with $\mu \circ T = \mu$. A partition $\mathbf{P}$ is a finite disjoint collection of sets from $\mathcal{B}(X)$ whose union is X. If $\mathbf{P}$ and $\mathbf{Q}$ are partitions then their join is

$$\mathbf{P} \vee \mathbf{Q} = \{P \cap Q \mid P \in \mathbf{P},\ Q \in \mathbf{Q}\}.$$

Similarly, we denote the multiple joining of $\{\mathbf{P}_{-m}, \ldots, \mathbf{P}_n\}$ by

$$\bigvee_{-m}^{n} \mathbf{P}_i = \mathbf{P}_{-m} \vee \mathbf{P}_{-m+1} \vee \cdots \vee \mathbf{P}_n.$$

The smallest complete σ-subalgebra of $\mathcal{B}(X)$ containing the sets of all the partitions $T^i\mathbf{P}$, $i \in \mathbb{Z}$, will be denoted by $\sigma(T, \mathbf{P})$. If $\sigma(T, \mathbf{P})$ is dense in $\mathcal{B}(X)$ with respect to the distance $d(A, B) = \mu(A \Delta B)$ then $\mathbf{P}$ is called a *generator* for T, or a *generating partition*. If $\mathbf{P}$ is a partition, we define the entropy $H(\mathbf{P})$ of $\mathbf{P}$ as

$$H(\mathbf{P}) = -\sum_{P \in \mathbf{P}} \mu(P) \log \mu(P).$$

The entropy of the partition $\mathbf{P}$ with respect to T is defined by

$$h(\mathbf{P}, T) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{0}^{n-1} T^i\mathbf{P}\right), \quad h(\mathbf{P}, T) \leq H(\mathbf{P}).$$

(e.g. Glasner [**10**]). Then $h(\mathbf{P}, T)$ is sometimes called the *mean entropy* of the process $(\mathbf{P}, T)$.

An automorphism T is said to have cpe if

$$h(\mathbf{P}, T) > 0$$

for any finite partition $\mathbf{P}$. Such an automorphism is called also a *K-automorphism*.

† I.e. not isomorphic by a group automorphism (Definition 7.4).

The *entropy of the automorphism* $h(T)$ is defined by

$$h(\mathbf{P}) = \sup_{\mathbf{P}} h(\mathbf{P}, T)$$

where sup is taken over all finite partition $\mathbf{P}$ of X.

Ornstein and Weiss [**27**] and Weiss [**40**] extended the above notions to the setting of a free action of a countable amenable group G on a Lebesgue space $(X, \mathcal{B}, \mu)$. Here, one defines the mean entropy of a finite partition $\mathbf{P}$ of X, relative to F, a finite subset of G, by taking $\mathbf{P}^F = \bigvee_{g\in F} g\mathbf{P}$, and defining

$$h(P, F) = \frac{1}{|F|} H(\mathbf{P}^F).$$

Recall that a subset T of G *tiles* G if there is a set $C \subset G$ such that CT is a partition of G, i.e. $G = \bigcup_{c\in C} cT$, $c_1T \cap c_2T = \emptyset$ for $c_1 \neq c_2$.

Further, a sequence of finite subsets $\{F_n\}_{n\in\mathbb{N}}$ of G is called a *Følner sequence* in G if $\lim |gF_n \Delta F_n|/|F_n| = 0$, for all $g \in G$. It is well known that a countable amenable group G has a Følner sequence†.

Ornstein and Weiss [**27**] showed that if $\{F_n\}_{n\in\mathbb{N}}$ is a Følner sequence in G and each F_n tiles G then for any finite $\mathbf{P}$ partition of X, the limit $h(\mathbf{P}, G) = \lim |F_n|^{(-1)} H(\mathbf{P}^{F_n})$ exists and is independent of the particular Følner sequence; it is said to define the mean entropy of $(\mathbf{P}, G)$. Given that entropy decreases as one refines the partition, one has

$$h(\mathbf{P}, G) = \lim_{n\to\infty} \frac{1}{|F_n|} H(\mathbf{P}^{F_n}) = \inf_n \frac{1}{|F_n|} H(\mathbf{P}^{F_n}).$$

Weiss [**41**] proved that any solvable countable group has a Følner sequence with some additional properties. In the general case, one needs to use the machinery of quasi-tiles to define the mean entropy $h(\mathbf{P}, G)$ of the process $(\mathbf{P}, G)$ [**21, 27**].

We say that the action of a countable amenable group G has cpe if

$$h(\mathbf{P}, G) > 0$$

for any finite partition $\mathbf{P}$. The entropy of the G-action is defined by $h(G) = \sup_{\mathbf{P}} h(\mathbf{P}, G)$ where the supremum is taken over all finite partitions $\mathbf{P}$.

If G acts freely and ergodically, we call a generating partition $\mathbf{P} = (P_i)$ a *Bernoulli partition* for the action of G if the partitions $g\mathbf{P}$ are independent for $g \in G$, $g \neq e$, i.e.

$$\mu(gP_iP_j) = \mu(P_i)\mu(P_j).$$

We say that the action is *Bernoulli* if there exists a Bernoulli partition.

It is easy to construct Bernoulli actions of countable amenable groups G. For each $g \in G$, let $X_g = \{0, 1, \ldots, n-1\}$, and define $\mu_g(i) = p_i$, such that $\sum_{i=0}^{n-1} p_i = 1$. We define

$$Y = \bigotimes_{g\in G} X_g, \quad \mu = \bigotimes_g \mu_g.$$

† Indeed, this is often taken as the definition of amenability.

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Then we can define an action of G on (Y, μ) by $(g \cdot y)_h = y_{gh}$. It is well known that this is a Bernoulli action and that

$$h(G) = -\sum_i p_i \log p_i = H(\mathbf{P}).$$

A celebrated result of Ornstein and Weiss [**27**] is that entropy is a full invariant of the Bernoulli action of G. More exactly, two Bernoulli actions of G with the same entropy are metrically isomorphic.

Another result which we will use below is Sinai's factor theorem (see Glasner [**10**]).

THEOREM 2.1. *If a countable discrete amenable group G acts freely and ergodically on $(X, \mathcal{B}, \mu)$ with positive entropy $h(G)$, then for any real number $0 < h \leq h(G)$ there exists a G-invariant factor-space X_h of X such that the restriction of the action of G on X_h is a Bernoulli action of G with $h(G|_{X_h}) = h$.*

A different approach to the entropy of actions of countable amenable groups is due to Ollagnier [**23**]. We explain the relationship between the two approaches.

Definition 2.2. Let G be as above, acting freely and ergodically on $(X, \mathcal{B}, \mu)$. Suppose that $\mathbf{P}$ is a finite partition of X.

The *Ollagnier mean entropy* $h_o(\mathbf{P}, G)$ of the process $(\mathbf{P}, G)$ is defined as

$$h_o(\mathbf{P}, G) = \inf_F \frac{1}{|F|} H(\mathbf{P}^F)$$

where the infimum is taken over all finite subsets F of G, see Ollagnier [**23,** Definition 4.3.1].

The *Ollagnier entropy* $h_o(G)$ of the action of G on $(X, \mathcal{B}, \mu)$ is defined by

$$h_o(G) = \sup_{\mathbf{P}} h_o(\mathbf{P}, G)$$

where the supremum is taken over all finite subsets of G, see Ollagnier [**23,** Definition 4.3.2].

Ollagnier [**23,** Theorem 4.3.14] proved that if $\mathbf{P}$ is a generating partition for an action of G then $h_o(G) = h_o(\mathbf{P}, G)$.

The following theorem is surely known to the experts in the field, but we have not been able to find a suitable reference, so we give a proof.

THEOREM 2.3. *Let G be a countable amenable group acting freely and ergodically on $(X, \mathcal{B}, \mu)$. Then*

$$h(G) = h_o(G),$$

where $h(G)$ is the entropy of the G-action in the sense of Ollagnier [**27**]*, and $h_o(G)$ is the Ollagnier entropy of the G-action.*

Proof. It follows from the definition of $h(\mathbf{P}, G)$ above and the definition in Lindenstrauss and Weiss [**21**] that

$$h_0(\mathbf{P}, G) \leq h(\mathbf{P}, G)$$

for any finite partition $\mathbf{P}$ of X. Hence $h_o(G) \leq h(G)$.

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Assume now that G has a Bernoulli action on $(X, \mathcal{B}, \mu)$ of finite entropy $h(G)$. Then there is a finite Bernoulli partition $\mathbf{P} = \{P_i\}$ and an easy calculation shows that

$$h(\mathbf{P}, G) = h_o(\mathbf{P}, G) = -\sum_i \mu(P_i) \log \mu(P_i).$$

Hence $h(G) = h_o(G)$ in this case. A similar argument shows that the equality also holds for the case of a Bernoulli action of infinite entropy.

Now suppose that G has action which is not necessarily Bernoulli on $(X, \mathcal{B}, \mu)$ with $0 < h(G) < \infty$. By Theorem 2.1, this action of G has a Bernoulli factor action on a G-invariant subspace X_h with entropy $h(G|_{X_h}) = h$ for $0 < h \leq h(G)$. Let $\mathbf{Q}$ be a Bernoulli partition of X_h with $H(\mathbf{Q}) = -\sum_i \mu(Q_i) \log \mu(Q_i) = h$. As above, we have $h_o(\mathbf{Q}, G) = h$, and hence $h \leq h_o(G) \leq h(G)$. Since $h \in (0, h(G)]$, it follows from Definition 2.2 that $h_o(G) = h(G)$ if $0 < h(G) < \infty$.

The same argument can be applied for $h(G) = \infty$. Finally we consider $h_o(G) = 0$. In this situation, it follows immediately from Theorem 2.1 that $h(G) = 0$. □

We will use the following consequences in §3.

COROLLARY 2.4. *Let* G, $(X, \mathcal{B}, \mu)$ *be as in the statement of Theorem 2.3, and* $\mathbf{P}$ *a finite partition of* X. *Then*

$$h_o(\mathbf{P}, G) = h(\mathbf{P}, G).$$

Proof. Consider first the special case where $\mathbf{P}$ is a generator for the action of G on $(X, \mathcal{B}, \mu)$. It follows from the remark after Definition 2.2 that $h_o(\mathbf{P}, G) = h_o(G)$, and one can see from the proof of Theorem 2.3 that $h_o(G) = h(G)$ and $h_o(\mathbf{P}, G) \leq h(\mathbf{P}, G)$. Hence

$$h_o(\mathbf{P}, G) \leq h(\mathbf{P}, G) \leq h(G) = h_o(\mathbf{P}, G),$$

and $h_o(\mathbf{P}, G) = h(\mathbf{P}, G)$.

In general, there exists a G-invariant factor space Y of $(X, \mathcal{B}, \mu)$ such that $\mathbf{P}$ is a generating partition for action of G on Y. □

3. *Co-induction and its properties*

In this section we introduce a construction which will allow us to obtain a non-Bernoulli cpe action of a countable amenable group G from a non-Bernoulli cpe action of a subgroup Γ.

Definition 3.1. Let G be a countable amenable group and Γ a subgroup of G. Let $(X, \mathcal{B}, \mu)$ be a (left) Γ-space.

Fix a section $s : \Gamma\backslash G \to G$ of the homogeneous space $\Gamma\backslash G$ with $s([e]) = e$. Consider the product space $Y = \prod_{\Gamma\backslash G}(X, \mathcal{B}, \mu)$, and equip Y with the associated product measure $\nu = \otimes_{\Gamma\backslash G}\mu$. We write the elements of Y as $(y_\theta)_{\theta\in\Gamma\backslash G}$.

Define an action of G on Y by

$$(gy)_\theta = (s(\theta)gs(\theta g)^{-1})y_{\theta g}, \quad y = (y_\theta) \in Y, \ \ y_\theta \in X, \ \theta \in \Gamma\backslash G, \ g \in G, \tag{3.1}$$

where Γ acts in each coordinate of Y by its action on X. We shall say that this G action is *co-induced from the action* of Γ. An easy calculation shows that this action is well defined; in particular, $s(\theta)gs(\theta g)^{-1} \in \Gamma$. It is also clear that the action preserves ν.

As is the case with the standard inducing procedure, the above action may be defined independently of the particular choice of section. In order to see this, let

$$\tilde{Y} = \{f : G \to (X, \mathcal{B}, \mu) : \forall \gamma \in \Gamma,\ g \in G\ f(\gamma g) = \gamma(f(g))\}$$

and define an action of G on $\tilde{Y}$ by

$$(g_0 \cdot f)(g) = f(gg_0).$$

Notice that the values of $f \in \tilde{Y}$ depend only on its values on a section. We can thus identify $\tilde{Y}$ with

$$Y_0 = \{h : \Gamma\backslash G \to (X, \mathcal{B}, \mu)\}$$

where, to a given $f \in \tilde{Y}$, we may associate $h \in Y_0$ defined by $h(\Gamma g) = f(s(\Gamma g))$. Now we can identify Y_0 with the space $Y = \prod_{\Gamma\backslash G}(X, \mathcal{B}, \mu)$ of Definition 3.1, and an easy calculation shows that the action on Y is given by (3.1).

Our definition is related to, but different from the well-known Mackey–Zimmer definition of an induced action (e.g. [**42**], [**43,** p. 75]). In particular, let G, Γ and X be as in Definition 3.1. The standard induced action of G may be realized on the space $\Gamma\backslash G \times X$, equipped with the product Borel σ-algebra and the product measure λ, by

$$g \cdot_{MZ} (\theta, x) = (\theta g, s(\theta g)^{-1} g^{-1}(s(\theta))x).$$

By contrast, the co-induced action acts on the space of measurable functions $h : \Gamma\backslash G \to X$.

Danilenko [**5**] used this construction in an investigation of spectral properties of ergodic actions of discrete groups. Gaboriau [**9**] used it in orbit equivalence theory.

It is not difficult to check that the action of G is non-Bernoulli, provided that the Γ-action is non-Bernoulli. To see this, we need the following simple proposition.

PROPOSITION 3.2. *The restriction of a Bernoulli action of a countable discrete amenable group G to an infinite subgroup Γ is also Bernoulli.*

Proof. Consider a Bernoulli action of G on $(X, \mathcal{B}, \mu)$. Then there exists a measurable generating partition ζ of $(X, \mathcal{B}, \mu)$ such that the family of partitions $\{g\zeta, g \in G\}$ is independent and generates B. Let $A \subset G$ be a set which meets each left Γ-coset Γg in exactly one point. Form the measurable partition $\eta = \bigcup_{g\in A} g\zeta$. It is easy to check that η is a generating partition for the action Γ, and its shifts by elements of Γ are independent. This proves the statement. □

COROLLARY 3.3. *Let G and Γ be as in Proposition 3.2. Suppose that the G-action on Y is co-induced from the Γ-action on X. Then the action of G will be Bernoulli on (Y, ν) if and only if the action of Γ is a Bernoulli on (X, μ).*

Proof. Consider the restriction of the co-induced action of G to the subgroup Γ. This restriction has the original Γ-action as a factor action given by the Γ-equivariant projection $Y \to X : y \mapsto y_{[e]}$. As this factor action of Γ is non-Bernoulli, the Γ-action on Y is non-Bernoulli [**27,** III, §6, Theorem 4] and the entire G-action is also non-Bernoulli by Proposition 3.2. □

We demonstrate now that co-induction behaves well with respect to entropy, and prove some results which we will apply later.

PROPOSITION 3.4. *Let G and Γ be as above, and suppose we are given an action of Γ on X. Consider the G-action on Y given by equation (3.1). Then $h_Y(G) = h_X(\Gamma)$ where $h_X(\Gamma)$ is the entropy of the Γ-action on X.*

Proof. Let $s : \Gamma\backslash G \to G$ be a section of the quotient map $\pi : G \to \Gamma\backslash G$ with $s([e]) = e$, where e is the identity element of G. Let ξ' be a finite partition of X, and ξ be its lift to a partition ξ of Y by identifying X as a factor space of Y, $Y \to X$ via the map $y \mapsto y_{[e]}$. Clearly ξ is generating partition for the G-action on Y if ξ' is a generator for the action Γ on X. One has, applying Definition 2.2,

$$\begin{aligned}
h_o(\xi, G) &= \inf_F \frac{1}{|F|} H\Big(\bigvee_{g\in F} g\xi\Big) = \inf_F \frac{1}{|F|} H\Big(\bigvee_{\theta\in\pi(F)} \bigvee_{g\in F\cap\pi^{-1}(\theta)} g\xi\Big)\\
&= \inf_F \frac{1}{|F|} \sum_{\theta\in\pi(F)} H\Big(\bigvee_{g\in F\cap\pi^{-1}(\theta)} g\xi\Big)\\
&= \inf_F \frac{1}{|F|} \sum_{\theta\in\pi(F)} \Big|F\cap\pi^{-1}(\theta)\Big| \frac{1}{|F_n\cap\pi^{-1}(\theta)|} H\Big(\bigvee_{g\in F\cap\pi^{-1}(\theta)} s(\theta)^{-1} g\xi\Big)\\
&\geq \inf_F \frac{1}{|F|} \sum_{\theta\in\pi(F)} \Big|F\cap\pi^{-1}(\theta)\Big| h_o(\xi', \Gamma) = h_o(\xi', \Gamma),
\end{aligned}$$

where F is a finite subset of G and we have taken into account the choice of ξ, the fact that $s(\theta)^{-1}g \in \Gamma$ if $\pi(g) = \theta$ and Definition 2.2.

Thus $h_o(\xi, G) \geq h_o(\xi', \Gamma)$. On the other hand, the properties of ξ imply (cf. Definition 2.2)

$$\begin{aligned}
h_o(\xi, G) &= \inf_{F\subset G} \frac{1}{|F|} H\Big(\bigvee_{g\in F} g\xi\Big) \leq \inf_{F\subset\Gamma} \frac{1}{|F|} H\Big(\bigvee_{g\in F} g\xi\Big)\\
&= \inf_{F\subset\Gamma} \frac{1}{|F|} H\Big(\bigvee_{g\in F} g\xi'\Big) = h_o(\xi', \Gamma),
\end{aligned}$$

where the infimum is over all finite subsets F. Thus $h_o(\xi, G) \leq h_o(\xi', \Gamma)$, and hence $h_o(\xi, G) = h_o(\xi', \Gamma)$. As $h(\xi, G) = h_o(\xi, G)$ and $h(\xi', \Gamma) = h_o(\xi', \Gamma)$ by Corollary 2.4 then we have $h(\xi, G) = h(\xi', \Gamma)$ for any finite partition ξ' of X.

Now let $h_X(\Gamma) < \infty$. Then there exists a finite generating partition ξ' of X for the action Γ on X such that $h_X(\Gamma) = h(\xi', \Gamma)$ according to Rosenthal [**30**]. However, ξ is a generating partition for the co-induced action of G on Y, by the construction, hence $h(\xi, G) = h_Y(G)$ and $h_X(\Gamma) = h_Y(G)$ in this case.

If $h_X(\Gamma) = \infty$ then $h_X(\Gamma) = \sup_{\xi'} h(\xi', \Gamma) = \sup_{\xi} h(\xi, G) \leq h_Y(G)$, and hence $h_Y(G) = \infty$. □

We now present a simple consequence of Proposition 3.4. Suppose that a countable amenable group Γ acts freely on $(X, \mathcal{B}, \mu)$. Recall that the σ-subalgebra $\Pi(\Gamma)$ of B is called the *Pinsker subalgebra* of the Γ-action on (X, μ), [**6, 11**]. If $\Pi(\Gamma)$ is a maximal

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Γ-invariant subalgebra of B such that for any finite partition $\mathbf{P}$ of X from $\Pi(\Gamma)$ one has $h(\mathbf{P}, G) = 0$. The algebra $\Pi(\Gamma)$ is called *trivial* if it contains only X and the empty set. It is clear if $\Pi(\Gamma)$ is trivial then Γ has cpe action on $(X, \mathcal{B}, \mu)$.

COROLLARY 3.5. *Let G, Γ, (X, μ) and (Y, ν) be as in Proposition 3.4. Let $\Pi(\Gamma)$ be the Pinsker algebra of Γ-action on (X, μ), and $\Pi(G)$ be the Pinsker algebra of G-action on (Y, ν). If $\Pi(\Gamma)$ is non-trivial then $\Pi(G)$ is also non-trivial. In other words, if the co-induced action of G on (Y, ν) has cpe then the action of Γ on (X, μ) must have cpe too.*

Proof. This follows directly from the Proof of Proposition 3.4. □

The above results reduce the problem of co-inducing non-Bernoulli cpe actions of G from those of Γ to the problem of showing that the co-induced action has cpe. We will return to this problem in §5.

We shall also use the following properties of co-induction.

PROPOSITION 3.6. *Let G, Γ, (X, μ) and (Y, ν) be as in Definition 3.1. If $[G : \Gamma] = \infty$ then the co-induced action of G on Y is ergodic. If $[G : \Gamma] < \infty$ then the co-induced action of G on Y is ergodic if and only if the action of Γ on X is ergodic.*

Proof. We will not use this assertion below, and give only some remarks about its proof. If G is abelian then to prove the ergodicity of the co-induced action of G, one can use the same standard argument as in the case when G has a Bernoulli action. In the general case, it is not too hard to extend this approach. □

A free action of a countable amenable group Γ on $(X, \mathcal{B}, \mu)$ satisfies the *weak Pinsker property* if for every $\delta > 0$ there is Γ-invariant Bernoulli factor Z_1 of X and independent Γ-invariant factor Z_2 of X with $h(\Gamma|_{Z_2}) < \delta$ such that $X = Z_1 \otimes Z_2$ [**8, 39**].

A well-known open problem in ergodic theory is the following [**10**]. Does every positive entropy ergodic free action of a countable amenable group Γ have the weak Pinsker property?

PROPOSITION 3.7. *Let G, Γ, $(X, \mathcal{B}, \mu)$ and (Y, ν) be as in Definition 3.1. If an action of Γ on $(X, \mathcal{B}, \mu)$ has the weak Pinsker property then the co-induced action of G on (Y, ν) also has this property.*

Proof. The proposition follows directly from Corollary 3.3 and Proposition 3.4. □

Notice also that examples of K-automorphisms constructed in §6 below have the weak Pinsker property. Hence, the co-induced actions of countable amenable groups defined in §7 also have this property.

We believe that the converse of Proposition 3.7 also holds.

4. *The Rudolph–Weiss mixing property and cpe*

Rudolph and Weiss [**37**] proved that a cpe action of a countable amenable group is *uniform mixing*: see Definition 4.1 below. The converse statement also holds [**13, 40**]. In this section we present a simple proof of the converse (Theorem 4.2) which, unlike the proof by Golodets and Sinel'shchikov [**13**], avoids the use of technical results on quasi-tiles [**37**]. We then apply this theorem to prove that the direct product of cpe actions of a

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countable amenable group G is again a cpe action of G (Theorem 4.4). The most important applications of Theorem 4.2 are given in the next section.

Let K be a finite subset of G. A finite set $S \subset G$ is said to be K*-spread* if for all $s_1 \neq s_2$ from S the element $s_1^{-1}s_2 \notin K$. The following definition was introduced by Weiss [**40**].

Definition 4.1. A free action of amenable countable group G on a Lebesgue space (X, μ) is called *uniform mixing* if, for any finite partition ξ of X and any $\varepsilon > 0$, there exists a finite subset $K \in G$ such that for any finite subset S of G, which is K-spread, we have

$$H(\xi) - \frac{1}{|S|} H\biggl(\bigvee_{g \in S} g\xi\biggr) < \varepsilon.$$

Notice that for the group $\mathbb{Z}$, uniform mixing in this sense is equivalent to the notion of uniform mixing in the terminology of Glasner [**10**], and to K-mixing in the terminology of Cornfeld *et al* [**3**].

We present a simple proof of the following theorem.

THEOREM 4.2. *Let G be a countable amenable group and (X, μ) a Lebesgue free G-space. Suppose that the action of G is uniform mixing. Then this action has cpe.*

Proof. Let G, (X, μ), ξ, K and ε be as in Definition 4.1, and suppose that $0 < \varepsilon < H(\xi)/2$. Fix a finite subset $F \subset G$ and consider a finite subset $S \subset F$ which is K-spread. If there is no $f \in F$ such that f does not belong to S but $S \cup \{f\}$ is again K-spread then we call S a *maximal K-spread subset* of F. It is clear that any K-spread subset $S' \subset F$ is contained in a maximal K-spread subset $S \subset F$. Moreover, it is obvious that a K-spread subset S is maximal if and only if any $f \in F$ is either contained in S or has the form $f = ks$ where $s \in S$ and $k \in K$. This gives the following estimates for $|S|$:

$$|S| \leq |F|,$$
$$|F| \leq |S| \cdot (|K| + 1).$$

Hence

$$\frac{1}{|F|} H\biggl(\bigvee_{f \in F} f\xi\biggr) \geq \frac{1}{|S| \cdot (|K| + 1)} H\biggl(\bigvee_{f \in S} f\xi\biggr) \geq \frac{1}{(|K| + 1)} (H(\xi) - \varepsilon) \geq \frac{H(\xi)}{2(|K| + 1)}.$$

It follows immediately from Definition 2.2 that

$$h_o(\xi, G) = \inf_F \frac{1}{|F|} H\biggl(\bigvee_{f \in F} g\xi\biggr) \geq \frac{H(\xi)}{2(|K| + 1)} > 0,$$

where the infimum is taken over all finite subsets F of G. As $h(\xi, G) = h_o(\xi, G)$ by Corollary 2.4, we have

$$h(\xi, G) > \frac{H(\xi)}{2(|K| + 1)} > 0.$$

The following statement now follows easily. □

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COROLLARY 4.3. *A free action of a countable amenable group G on a Lebesgue space (X, μ) has cpe if and only if for any finite partition ξ and any $\varepsilon > 0$ there exists a finite subset $K \subset G$ such that for any set S which is K-spread,*

$$\left| \frac{1}{|S|} H\left(\bigvee_{g \in S} g\xi \right) - H(\xi) \right| < \varepsilon.$$

Theorem 4.2 allows us to strengthen Theorem 4 from Glasner *et al* [**11**], where it was proved for I finite.

THEOREM 4.4. *Let Γ be an infinite countable discrete amenable group.*

Let I be a finite or countable set and for each $i \in I$, let $(X_i, \mathcal{B}_i, \mu_i)$ be a measure space on which Γ has a cpe action α_i.

Let $(Y, \nu) = \prod_{i \in I}(X_i, \mathcal{B}_i, \mu_i)$ and let α be the product actions of Γ on (Y, ν), given by

$$\alpha(\gamma)y = ((\alpha_i(\gamma)x_i))_{i \in I}, \quad \gamma \in \Gamma,$$

where $y = (x_i) \in Y$, $x_i \in X_i$, $i \in I$. Then α is a cpe action of Γ on (Y, ν).

Proof. The fact that the theorem holds for I finite [**11**], means that any partition measurable with respect to only finitely-many terms in the product space satisfies the mixing property in Theorem 4.2. Such partitions form a dense class in the full product algebra and hence the countable product action is also cpe. □

COROLLARY 4.5. *Let G, Γ, (X, μ) and (Y, ν) be as in Definition 3.1. Suppose that G is abelian and Γ is infinite. If Γ has a cpe action on (X, μ) then Γ acts ergodically on (Y, ν).*

Proof. Note that if G is abelian then $s(\theta) = s(\theta\gamma)$ for any $\gamma \in G$ and $s(\theta)\gamma s(\theta\gamma) = \gamma$. Hence

$$(\gamma y)_\theta = \gamma y_\theta.$$

Corollary 4.5 is now immediate from Theorem 4.4. □

5. *The existence of non-Bernoulli cpe actions*

In this section, we will show that a co-induced action of a non-Bernoulli cpe action is also cpe and non-Bernoulli. We can then use results of Ornstein and Shields [**26**], Feldman [**7**] and others to see that if G contains a subgroup isomorphic to $\mathbb{Z}$ (i.e. an element of infinite order), then G has a non-Bernoulli cpe action. As noted in §3, Proposition 3.4 and Corollary 3.3 reduce the problem to showing that the co-induced action has cpe.

LEMMA 5.1. *Let ξ and η be finite partition of X, $\eta < \xi$, and $S \subset G$ a finite subset. Then*

$$H(\eta) - \frac{1}{|S|} H\left(\bigvee_{g \in S} g\eta \right) \le H(\xi) - \frac{1}{|S|} H\left(\bigvee_{g \in S} g\xi \right).$$

Proof. Suppose $S = \{g_1, g_2, \ldots, g_n\}$. A multiple application of the relation $H(Q \vee P|Z) = H(Q|Z) + H(P|Q \vee Z)$ yields

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$$|S| \cdot H(\xi) - H\bigg(\bigvee_{g\in S} g\xi\bigg) = |S| \cdot H(\xi \vee \eta) - \sum_{i=1}^{n} H\left(\xi \vee \eta \,\middle|\, \bigvee_{j=1}^{i-1} g_i^{-1} g_j \xi\right)$$

$$= \left(|S| \cdot H(\eta) - \sum_{i=1}^{n} H\left(\eta \,\middle|\, \bigvee_{j=1}^{i-1} g_i^{-1} g_j \xi\right)\right) + \left(|S| \cdot H(\xi|\eta) - \sum_{i=1}^{n} H\left(\xi \,\middle|\, \eta \vee \bigvee_{j=1}^{i-1} g_i^{-1} g_j \xi\right)\right)$$

$$\geq \left(|S| \cdot H(\eta) - \sum_{i=1}^{n} H\left(\eta \,\middle|\, \bigvee_{j=1}^{i-1} g_i^{-1} g_j \eta\right)\right) + \left(|S| \cdot H(\xi|\eta) - \sum_{i=1}^{n} H\left(\xi \,\middle|\, \eta \vee \bigvee_{j=1}^{i-1} g_i^{-1} g_j \xi\right)\right).$$

Since $|S| \cdot H(\xi|\eta) - \sum_{i=1}^{n} H\big(\xi|\eta \vee \bigvee_{j=1}^{i-1} g_i^{-1} g_j \xi\big) \geq 0$, we deduce that

$$|S| \cdot H(\xi) - H\bigg(\bigvee_{g\in S} g\xi\bigg) \geq |S| \cdot H(\eta) - \sum_{i=1}^{n} H\left(\eta \,\middle|\, \bigvee_{j=1}^{i-1} g_i^{-1} g_j \eta\right)$$

$$= |S| \cdot H(\eta) - H\left(\bigvee_{g\in S} g\eta\right),$$

which is clearly equivalent to our statement. □

We now come to the main theorem of this section. We use the same notation as in Definition 3.1.

THEOREM 5.2. *Let G be a countable amenable group and Γ an infinite subgroup of G. Suppose that Γ has a free action on the Lebesgue space (X, μ). Then the co-induced action of G on (Y, ν) has cpe if and only if the action of Γ on (X, μ) has cpe. This action of G on (Y, ν) is Bernoulli if and only if the action of Γ on (X, μ) is Bernoulli.*

The proof of this theorem will be preceded by a definition and two lemmas.

Definition 5.3. Let $P \subset \Gamma\backslash G$. Then $X^P = \prod_P (X, \mathcal{B}, \mu)$ may be considered as a quotient space of $Y = \prod_{\Gamma\backslash G} (X, \mathcal{B}, \mu)$ with quotient map $\tau : Y \to X^P$ given by

$$\tau(x)_\gamma = x_\gamma \quad \text{for } x \in Y,\ \gamma \in P.$$

We shall say that a partition η of Y is *subjugated to P* if η reduces to a partition of the quotient space X^P in the sense that there exists a partition η' of X^P such that $\eta = \tau^{-1}(\eta')$.

Suppose we are given a finite partition ξ of Y. Choose a section s and a finite subset $K \subset G$ so that $K^{-1} \subset s(\Gamma\backslash G)$ (and hence K^{-1} meets each right coset at most once).

Further, let d be the Rokhlin metric for partitions with a finite entropy [**28,** §6]. (It is denoted by d_{ent} by Glasner [**10**].) Then we may choose, for any $\varepsilon > 0$, a finite subset K as above and a finite partition $\widetilde{\eta}$ of Y such that $\widetilde{\eta}$ is subjugated to $s^{-1}(K^{-1})$ and also $d(\xi, \widetilde{\eta}) < \varepsilon/6$.

Next, let η_0 be a finite partition of Y subjugated to $\{s^{-1}(e)\}$ and chosen so that for some partition $\eta \preceq \overline{\eta} = \bigvee_{k\in K} k\eta_0$ one has $d(\widetilde{\eta}, \eta) < \varepsilon/6$. Hence

$$d(\xi, \eta) < \varepsilon/3. \tag{5.1}$$

Since the Γ-action on X has cpe, we can find a finite subset $J \subset \Gamma$ such that $e \in J$ and for any finite J-spread subset $S \subset \Gamma$.

$$H(\eta_0) - \frac{1}{|S|} H\bigg(\bigvee_{g\in S} g\eta_0\bigg) < \frac{\varepsilon}{3|K|}. \tag{5.2}$$

This follows immediately from Corollary 4.3; see also Rudolph and Weiss [**37,** Theorem 2.3].

We will need the following simple result.

LEMMA 5.4. *Let Q be a finite KJK^{-1}-spread subset of G. Then $|QK| = |Q| \cdot |K|$ and QK is J-spread.*

Proof. It is obvious that QK is J-spread. Let $q_i \in Q$ and $k_i \in K$, $i = 1, 2$, then $q_1k_1 = q_2k_2$ if and only if $q_1 = q_2$ and $k_1 = k_2$ because $e \in J$. But this implies $|QK| = |Q| \cdot |K|$. □

LEMMA 5.5. *Let J be as above. Then for any finite KJK^{-1}-spread subset $Q \subset G$ one has*

$$H(\eta) - \frac{1}{|Q|} H\left(\bigvee_{g\in Q} g\eta\right) < \frac{\varepsilon}{3}. \tag{5.3}$$

Proof. By Lemma 5.1, it suffices to verify equation (5.3) with η replaced by $\overline{\eta}$.

Let $\pi : G \to G/\Gamma$ be the natural projection. It follows from our choice of K that it meets each left coset at most once, so we can choose a section $s' : G/\Gamma \to G$ for π in such a way that $s'(k\Gamma) = k$ for all $k \in K$. Now, with $Q \subset G$ finite and KJK^{-1}-spread we can use equation (5.2) and independence of the components corresponding to different elements of G/Γ to obtain

$$\begin{aligned}
|Q| \cdot H(\overline{\eta}) - H\left(\bigvee_{g\in Q} g\overline{\eta}\right) &= |Q| \cdot |K| \cdot H(\eta_0) - H\left(\bigvee_{g\in Q}\bigvee_{k\in K} gk\eta_0\right) \\
&= |QK| \cdot H(\eta_0) - H\left(\bigvee_{\theta\in G/\Gamma} \bigvee_{\substack{(g,k)\in Q\times K\\ \pi(gk)=\theta}} gk\eta_0\right) \\
&= |QK| \cdot H(\eta_0) - \sum_{\theta\in G/\Gamma} H\left(\bigvee_{\substack{(g,k)\in Q\times K\\ \pi(gk)=\theta}} s'(\theta)^{-1}gk\eta_0\right) \\
&< |QK| \cdot H(\eta_0) - \sum_{\theta\in G/\Gamma} |QK \cap \pi^{-1}(\theta)| H(\eta_0) - \frac{\varepsilon}{3|K|} \\
&= |QK| \frac{\varepsilon}{3|K|} = |Q| \cdot |K| \cdot \frac{\varepsilon}{3|K|} = \frac{|Q| \cdot \varepsilon}{3}.
\end{aligned}$$

Hence, we have

$$H(\overline{\eta}) - \frac{1}{|Q|} H\left(\bigvee_{g\in Q} g\overline{\eta}\right) < \frac{\varepsilon}{3}, \tag{5.4}$$

and hence equation (5.3) follows from Lemma 5.1. □

Proof of Theorem 5.2. It follows from Proposition 3.2 and Corollary 3.3 that the co-induced action of G is Bernoulli if and only if the action of Γ is Bernoulli.

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Suppose that the co-induced action of G has cpe on (Y, ν); then the action of Γ on (X, μ) also has cpe by Corollary 3.5.

Suppose now that the action of Γ has cpe on (X, μ). We will prove that the action of G on (Y, ν) has cpe. By Theorem 4.2 it is enough to show that the co-induced action of G on Y is uniform mixing (see Definition 4.1).

Recall that for any finite partition ξ of Y and any $\varepsilon > 0$ there is a finite partition η of Y with $d(\xi, \eta) < \varepsilon/3$. Hence, the standard properties of the Rokhlin metric yield the following estimates:

$$|H(\xi) - H(\eta)| < d(\xi, \eta) < \frac{\varepsilon}{3},$$

$$\left|\frac{1}{|Q|} H\left(\bigvee_{g \in Q} g\xi\right) - \frac{1}{|Q|} H\left(\bigvee_{g \in Q} g\eta\right)\right| < d(\xi, \eta) < \frac{\varepsilon}{3}.$$

Here Q is a finite KJK^{-1}-spread subset of G as in the statement of Lemma 5.5. It follows from equation (5.3) and from these two estimates that

$$\left|H(\xi) - \frac{1}{|Q|} H\left(\bigvee_{g \in Q} g\xi\right)\right| < \varepsilon. \tag{5.5}$$

Thus we have proved that for any finite partition ξ of Y and any $\varepsilon > 0$ there exists a finite subset KJK^{-1} of G such that for any finite KJK^{-1}-spread subset Q the inequality (5.5) holds. This means that the co-induced action of G on Y is uniform mixing. □

COROLLARY 5.6. *Suppose that a countable amenable group G contains an element of infinite order. Then G has a non-Bernoulli action with cpe.*

Proof. This follows from Theorem 5.2 and the fact [**7, 15, 16, 24–26**] that $\mathbb{Z}$ has a non-Bernoulli action with cpe. □

COROLLARY 5.7. *Suppose that G is as in the statement of Corollary 5.6. Given $t \in (0, \infty]$, there is a non-Bernoulli cpe action α_t of G such that $h(\alpha_t) = t$.*

Proof. Indeed, for each t, there is a non-Bernoulli K-automorphism S_t of $\mathbb{Z}$ such that $h(S_t) = t$. This follows from Corollary 3 of Feldman [**7**]. Let α_t be the action of G co-induced from S_t. Then α_t is a cpe action of G by Theorem 5.2 and $h(\alpha_t) = h(S_t) = t$ (see Proposition 3.4).

Another way to see this assertion is to use the weak Pinsker property (see the remarks before Proposition 3.7 above). If T is a K-automorphism with the weak Pinsker property then one can easily construct a family $\{S_t\}_{t \in (0,\infty]}$ of K-automorphisms with this property and such that $h(S_t) = t$. Examples of these automorphisms will be demonstrated in §7. □

Kalikow proved [**17**] that there exist non-Bernoulli K-automorphisms S such that S and S^{-1} are isomorphic. We can use co-induction and its properties to generalize this result to any abelian group containing $\mathbb{Z}$ as a subgroup.

COROLLARY 5.8. *Let G be a countable abelian group containing $\mathbb{Z}$ as a subgroup. Then there is a non-Bernoulli cpe action U of G such that the actions $h \mapsto U_h$ and $h \mapsto U_{h^{-1}}$ are isomorphic.*

Proof. Suppose that S acts on a Lebesgue space (X, μ). It is not difficult to verify that there exists $V \in \mathrm{Aut}(X, \mu)$ such that $VSV^{-1} = S^{-1}$. We shall give a sketch of the proof of this proposition for the case $G = \mathbb{Z}^2$. The general case can be treated similarly. By Definition 3.1, the Lebesgue space (Y, ν) has the form $(Y, \nu) = \prod_{\mathbb{Z}}(X, \mu)$, and if $y \in Y$ then $y = (y_i)$, $i \in \mathbb{Z}$, $y_i \in X$. We have

$$(T_1 y)_i = y_{i+1}$$
$$(T_2 y)_i = S y_i, \quad i \in \mathbb{Z}.$$

Setting $(Wy)_i = Vy_i$, $i \in \mathbb{Z}$, we have $WT_2W^{-1} = T_2^{-1}$ and $WT_1W^{-1} = T_1$. Now it is easy to see that there is a transformation J of (Y, ν) such that $JT_1J = T_1^{-1}$, $J^2 = I$, $JW = WJ$ and $JT_2 = T_2J$. □

Thus we have shown that any countable amenable group G, containing $\mathbb{Z}$ as a subgroup, has non-Bernoulli cpe actions (see Corollary 5.6).

In the case of integer actions, Ornstein and Shields [**26**] showed that there exists an uncountable family of cpe actions of G with the same entropy which are pairwise non-isomorphic. We will generalize their result below. However, we need to overcome some complications.

The obvious first approach is to apply Corollary 5.6 to the uncountable family of non-Bernoulli pairwise non-isomorphic K-automorphisms with the same entropy defined by Ornstein and Shields [**26**]. We obtain a family of cpe co-induced actions of G with the same entropy and if G is abelian we can relatively easily show, using Corollary 4.5 and methods of Ornstein and Shields [**26**], that the corresponding co-induced actions of G are pairwise non-isomorphic.

However, in Example 7.3.4 below, we show that there is a countable solvable group with a subgroup Γ isomorphic to $\mathbb{Z}$, for which one cannot apply the methods of Ornstein and Shields [**26**] to distinguish the co-induced actions of G. Thus we need a more subtle approach.

Probably the most general approach to the Ornstein–Shields problem for integer actions is that of Hoffman [**15**]. Unfortunately, direct application of the methods of Hoffman [**15**] do not allow us to construct non-isomorphic co-induced actions.

Nevertheless, the restriction of the co-induced action of G on a subgroup $\Gamma \simeq \mathbb{Z}$ yields an action of $\mathbb{Z}$ which is of the general form studied by Rudolph [**33**] and Hoffman [**15**]. We develop this observation in the next section, in combination with methods of Ornstein and Shields [**26**], to prove the existence of a uncountable family of K-automorphisms of $\mathbb{Z}$ with some extra properties. This will allow us to solve the problem for any countable amenable group G containing $\mathbb{Z}$ as a subgroup.

6. *An uncountable family of K-systems*

6.1. *Uniform somewhat disjointness.* We construct here a family of K-automorphisms T_α where α is any infinite string of 0's and 1's, $\alpha \in \{0, 1\}^{\mathbb{N}}$. Such families have been constructed elsewhere but none have quite the properties we seek. This collection mirrors much of the examples of Ornstein and Shields [**26**] and Hoffman's K-counterexample machine [**15**]. In particular, $T_\alpha \simeq T_\beta$ if and only if α and β are asymptotically equal. More precisely, our construction is of the same form and in many ways simpler than that of Hoffman.

Definition 6.1. Suppose that $(X, \mathcal{F}, \mu)$ is a standard probability space. We say a finite partition P is ε-contained in a sub-σ-algebra $\mathcal{H}$ (written $P \overset{\varepsilon}{\subseteq} \mathcal{H}$) if there is a $P' \subseteq \mathcal{H}$ and

$$\mu(\{x : P(x) \neq P'(x)\}) \leq \varepsilon.$$

The maps T_α are constructed as transformations on a common sequence space $\{0, e, f, s\}^{\mathbb{Z}}$ and hence possess a canonical generating partition P_α given by the time-zero value of these sequences. We will now state the central result we expect this family to possess and deduce some of its properties. Then we will describe the construction. We will always assume the maps T and S discussed below are of finite entropy. The following somewhat complex definition is the central idea of our work.

Definition 6.2. Suppose that $(T, X, \mathcal{F}, \mu)$ and $(S, Y, \mathcal{G}, \nu)$ are two measure-preserving and ergodic dynamical systems and P is a partition of X. We say (T, P) is *somewhat disjoint* from S if there is a value $a > 0$ for any joining $\hat{\mu}$ of T and S and for any partition $P' \in \mathcal{G}$ we have

$$\hat{\mu}(P \triangle P') \geq a.$$

That is to say, if $P \overset{\varepsilon}{\subseteq} \mathcal{G}$ then $\varepsilon \geq a$. We say T is somewhat disjoint from S if for some partition P, (T, P) is somewhat disjoint from S.

As usual, by $P \triangle P'$ we mean $\{z : P(z) \neq P'(z)\}$. Notice that if for some P we know that (T, P) is somewhat disjoint from S then the same will be true for any partition Q which generates P under the action of T. In particular, this will be true for any generating partition.

Definition 6.3. We say that T is *uniformly somewhat disjoint* (which we abbreviate u.s.d.) from S if for some partition P of X there is a value $a > 0$ so that for any $m \in \mathbb{N}$ and values $j_0, j_1, \ldots, j_m \in \mathbb{Z}\backslash 0$ and any ergodic joining $\hat{\mu}$ of T^{j_0} and $S^{j_1} \times S^{j_2} \times \cdots \times S^{j_m}$, (that is to say, a $T^{j_0} \times (S^{j_1} \times \cdots \times S^{j_m})$ invariant measure with marginals μ and ν^m) which for all $0 \leq j < j_0$ satisfies $T^j(P) \overset{\varepsilon_i}{\subseteq} (\mathcal{G})^m$, then $(1/j_0) \sum_{j=0}^{j_0-1} \varepsilon_j \geq a$.

As before, it is not difficult to see that if a partition is u.s.d. from one partition P it is also u.s.d. from any other partition Q that generates P under the action of T. It is also clear that to be uniformly somewhat disjoint implies that no power of T is a factor of some product of powers of S. The converse is probably not true as we require the value a to be uniform over all joinings. Our goal is to construct an uncountable family of K-systems of the same entropy, any pair of which is u.s.d. Notice that somewhat disjointness is not a symmetric property. We begin with an easy consequence of somewhat disjointness.

COROLLARY 6.4. *If $(T, X, \mathcal{F}, \mu)$ and $(S, Y, \mathcal{G}, \nu)$ are u.s.d. K-systems, then for any $j_0 \neq 0$, T^{j_0} cannot arise as a factor of any action of the form $B \times \bigotimes_{k=1}^{\infty} S^{j_k}$ where all $j_k \neq 0$ and B is a Bernoulli shift of finite or infinite entropy.*

Proof. Without loss of generality we can assume B is of infinite entropy. As S is of positive entropy, by Sinai's theorem B arises as a factor of $\bigotimes_{k=1}^{\infty} S$. Hence, without loss

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of generality, we need only show that T^{j_0} cannot arise as a factor of $\bigotimes_{k=1}^{\infty} S^{j_k}$. We argue by contradiction. If it is a factor, then for any partition $P \subseteq \mathcal{F}$ and $\varepsilon > 0$ there is a value of K and an ergodic joining of T^{j_0} and $\bigotimes_{k=1}^{K} S^{j_k}$ with

$$\bigvee_{j=0}^{|j_0|-1} T^j(P) \overset{\varepsilon}{\subseteq} \mathcal{G}^K.$$

This contradicts the hypothesis of u.s.d. □

Definition 6.5. For a K-system $(S, Y, \mathcal{G}, \nu)$, by a *permuted power* of S we mean any map $\hat{S}$ of the following form. $\hat{S}$ acts on a finite or countably infinite product of copies of Y written $\otimes_{k\in\mathcal{K}} Y$ and has the form

$$\hat{S}\{x_k\}_{k\in\mathcal{K}} = \{S^{j_k}(x_{\pi^{-1}(k)})\}_{k\in\mathcal{K}} \tag{6.1}$$

where $\pi : \mathcal{K} \to \mathcal{K}$ is a bijection and if $L \subseteq \mathcal{K}$ is any finite cycle of π, then $\sum_{k\in L} j_k \neq 0$.

Special cases of permuted powers have been considered [**15, 28**]. The last condition on sums over finite cycles guarantees that a permuted power is always ergodic. Also note that if you construct two permuted powers by using a common π, using perhaps different choices for the j_k but with the sums over finite cycles agreeing, then the two permuted powers will be isomorphic. We now prove the result which gives us what we need.

PROPOSITION 6.6. *Suppose $(T, X, \mathcal{F}, \mu)$ and $(S, Y, \mathcal{G}, \mu)$ are two K-systems. If T is uniformly somewhat disjoint from S, then T cannot arise as a factor of a permuted power of S.*

Proof. We argue by contradiction. Therefore, suppose that T does arise as a factor of some permuted power of S. Break $\mathcal{K}$ into cycles of π and let P be a generator for T. For all $\varepsilon > 0$ we can select a subset $\mathcal{K}' \subseteq \mathcal{K}$ consisting of finitely many cycles of π so that relative to the joining $\hat{\mu}$ given by the factor map,

$$P \overset{\varepsilon}{\subseteq} \bigotimes_{k\in\mathcal{K}'} \mathcal{G}.$$

Since this algebra is invariant under the permuted power, we see that for all $t \in \mathbb{Z}$ we have

$$T^t(P) \overset{\varepsilon}{\subseteq} \bigotimes_{k\in\mathcal{K}'} \mathcal{G}.$$

Let $\hat{S}'$ be the restriction of $\hat{S}$ to this invariant sub-σ-algebra. The cycles of π in $\mathcal{K}'$ are either finite or infinite and we write $\mathcal{K}' = \mathcal{K}'_1 \cup \mathcal{K}'_2$ where $\mathcal{K}'_1$ is a finite set consisting of the finite cycles of π and $\mathcal{K}'_2$ is a collection of infinite cycles. If $\mathcal{K}'_2$ is not empty, then $\hat{S}'$ acting on these coordinates is an infinite entropy Bernoulli shift. For some sufficiently high power t_0 of $\hat{S}'$, it follows that $\hat{S}'^{t_0}$ is of the form $\bigotimes_{k\in\mathcal{K}'_1} S^{t_k} \times B$ where B is a possibly trivial Bernoulli shift. Hence $\hat{\mu}$ is a joining of this action and T^{t_0}. As in the previous corollary, we can replace the Bernoulli action B with a product of powers of S. Hence for all $\varepsilon > 0$ we have an ergodic joining of some T^{t_0} and some $\bigotimes_{k\in\mathcal{K}''} S^{t_k}$ with $T^t(P)$ being ε-contained in $\bigotimes_{k\in\mathcal{K}''} \mathcal{G}$. This contradicts somewhat disjointness. □

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6.2. *Building windows, wiggles and block names.* We have now explained the implications of u.s.d. which we will use, and we begin our construction. We borrow many ideas from the history of such counterexamples. In particular, we use what Hoffman refers to as pseudorandom sequences, chosen generically from independent identically distributed (i.i.d.) processes. As is usual, we will be constructing a series of collections of names inductively,

$$\mathcal{B}_n^\alpha \subseteq \{0, e, f, s\}^{h(n)}.$$

Here $h(n)$ is the common length of all the names. This set will actually depend only on the first n terms of α, with the next term determining how the construction continues to the next value $n+1$. Names will be built by concatenation, which we will write as a product. In this context, $\prod_{i=1}^{K} a_k$ will represent the name $a_1 a_2 \ldots a_k$, i.e. the 'product' is from left to right. We set

$$\mathcal{B}_0^\alpha = 0^{N(0)}$$

and so $h(0) = N(0)$.

It will then be convenient to have $B_{-1}^\alpha = 0$ and $h(-1) = 1$. We will use the parameter $N(0)$ to push down the entropy of the maps T^α.

The construction is governed by a set of inductively defined parameters.

(1) $N(n)$ is the number of $(n-1)$-block names that occur across an n-block name.

(2) $c(n)$ is the smallest value $> n$ such that $h(n-1) + c(n)$ is divisible by $n!$. This is the independent 'wiggle', which we will allow each $(n-1)$-block in the n-block.

(3) $\mathbf{A}^0(n) = \{a_1^0(n), \ldots, a_{N(n)}^0(n)\}$ and $\mathbf{A}^1(n) = \{a_1^1(n), \ldots, a_{N(n)}^1(n)\}$ are two 'pseudorandom' sequences of values in $\{1, \ldots, 2^n\}$.

(4) Each $(n-1)$-block sits inside a window in the n-block of length $w(n)$ where $w(0) = 1$.

(5) $d(n)$ is the least multiple of $n!$ larger than the value $2w(n-1) + n!$.

To begin the inductive description, we describe a general 'window' framing a word $b \in \mathcal{B}_{n-1}^\alpha$. The window depends on parameters $1 \le a \le 2^n$ and $0 \le j \le c(n)$ and is given by

$$W_n(b, a, j) = e^{ad(n)} s^j b s^{c(n)-j} f^{(2^n+1-a)d(n)}.$$

The e, s and f strings form the 'frame' of the window. Notice that the number of s's is smaller than the value $d(n)$. The e and f sections will be deterministic in that they will be set by the pseudorandom sequences. The s sections will be random, in that all values j will be allowed. All of this follows the general theme of such constructions.

To define the words allowed in $\mathcal{B}_n^\alpha$ let $b_1, \ldots, b_{N(n)}$ be any sequence of values from $\mathcal{B}_{n-1}^\alpha$ and $j_1, \ldots, j_{N(n)}$ be any sequence of values from $\{1, \ldots, c(n)\}$. The names in $\mathcal{B}_n^\alpha$ will all be possible names of the form

$$\prod_{k=1}^{N(n)} W_n(b_k, a_k^{\alpha_n}, j_k).$$

That is to say, we concatenate $N(n)$ windows with the e and f sections determined by $\mathbf{A}^0(n)$ or $\mathbf{A}^1(n)$ depending on whether $\alpha_n = 0$ or 1. The $\mathcal{B}_{n-1}^\alpha$ names in the windows are arbitrary as are the 'wiggles' produced by the s's.

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We calculate that $\#(\mathcal{B}_0^\alpha) = 1$, and letting $\eta(n)$ be the number of n-block names, we have inductively

$$\#(\mathcal{B}_n^\alpha) = \eta(n-1) = \#(\mathcal{B}_{n-1}^\alpha) \cdot c(n)^{N(n)}.$$

Notice this value does not depend on α. We set $w(n)$ to be the length of a window in the n-block name and observe that

$$w(n) = h(n-1) + (2^n + 1)d(n) + c(n).$$

Notice this length is divisible by $n!$. Furthermore, we have

$$h(n) = N(n)w(n).$$

Notice that the frame of a window has two components, a 'deterministic' part consisting of the e and f sections whose lengths are multiples of $d(n) > 2w(n-1) + n!$, and a 'random' part consisting of the s's. As $n! \geq c(n) > n$ and $h(n)$ will grow extremely fast, the random wiggle will be very small in relation to the deterministic part of the frame. This is of course the standard trick for making non-Bernoulli K-systems, referring to Ornstein and Shields [**26**].

Having set the value $c(n)$, the value $w(n)$ is determined. What remains so far unspecified is the size of $N(n)$ and the pseudorandom sequences $\mathbf{A}^0(n)$ and $\mathbf{A}^1(n)$.

One calculates that the fraction of each name in $\mathcal{B}_n^\alpha$ occupied by frames around $(n-1)$-block names is

$$F(n) = \frac{(2^n+1)d(n) + c(n)}{h(n-1) + (2^n+1)d(n) + c(n)} \leq \frac{2^{n+1}n!}{N(n-1)} \quad \text{for } n \geq 1.$$

We will define the conditions on $N(n)$ inductively. Our conditions will set lower limits for how large $N(n)$ must be, given the construction through stage $n-1$. As a first condition we ask that

$$N(n) > 10 \cdot 2^n \cdot (2^{n+2}+1) \cdot 2 \cdot n(n+1)!.$$

This guarantees that $F(n) < 1/(10n2^n)$, a bound we will use later. It follows that $\sum_{n=k}^{\infty} F(n) < 1(10 \cdot 2^{k-1})$ and the fraction of names occupied by frames around windows at level k or beyond decays exponentially in k. Hence the block names $\mathcal{B}_n^\alpha$ can be used to construct measure-preserving actions on probability spaces $(X^\alpha, \mathcal{F}^\alpha, \mu^\alpha)$. For our purposes, the best way to think of this is to take for X_α all words in $\{0, e, f, s\}^{\mathbb{Z}}$ for which every finite substring appears as a string in some $\mathcal{B}_n^\alpha$. This is a closed and shift-invariant set. As the e and f strings in any frame are non-empty, any word in X^α can be parsed uniquely into copies of words in $\mathcal{B}_n^\alpha$ and frames of stage n and higher. These topological systems are not uniquely ergodic. In fact, we want a measure of maximal entropy and the natural way to define it is to build an i.i.d. process on $\eta(n)$ symbols, each equally likely. Now build a tower over this system of height $h(n)$ and paint over each symbol one of the $\eta(n)$ names occurring in $\mathcal{B}_n^\alpha$. This produces a shift-invariant measure μ_α^n on $\{0, e, f, s\}^{\mathbb{Z}}$. One moves across the n-block names and, at the end of each one, decides independently which name to move to next. It is not difficult to check that if you take μ_α^{n+1} and induce on the occurrences of n-block names, i.e. erase the frames from the $(n+1)$-blocks, one obtains the measure μ_α^n. Since the density of the frames in the $(n+1)$-block names is constant and decaying in n exponentially, these measures μ_α^n must converge to a shift-invariant measure μ_α. More precisely, one can obtain μ_α by successively ramifying the σ-algebra and exducing to move from μ_α^n to μ_α^{n+1}. This later description is tantamount to cutting and stacking. For further details, see Ornstein and Shields [**26**].

Each of these actions has a canonical time-zero partition P^α whose sets we label 0, e, f, and s. As stated earlier, a name in X^α parses uniquely into copies of names in $\mathcal{B}_n^\alpha$ and frames around them. This means that there are well-defined sets B_n^α of points whose names from index 0 through index $h(n)-1$ are in $\mathcal{B}_n^\alpha$. Thus, for any $x \in X^\alpha$ there are values $d_1, d_2, \ldots \geq 0$ such that $T_\alpha^{-d_n}(x) \in B_n^\alpha$ and d_n is the minimal such value. If $d_n < h(n)$ then we refer to d_n as the *index* of x in its n-block. Otherwise, we say that x lies in a frame of level n or higher. This is consistent with saying x belongs to some set of names if one of these names is a subname of x containing the origin.

PROPOSITION 6.7. *For any particular choice of parameters as described above, the collection of maps* $(T_\alpha, X_\alpha, \mathcal{F}_\alpha, \mu_\alpha)$ *are all K-systems of equal entropy and all have the weak Pinsker property.*

Proof. Consider the factor algebra obtained by replacing all symbols in an occurrence of a block $b \in \mathcal{B}_n^\alpha$ by the single symbol 1. Call this list of factor algebras $\mathcal{F}_\alpha^n$. As n grows, these algebras decrease to the trivial algebra and in particular their entropies go to zero. The names from $\mathcal{B}_n^\alpha$ that occur in the intervals (now labelled with 1's) become independent and all equally likely. This implies three facts. First, one sees that all cylinder sets have constant densities of occurrence μ_α almost surely on the orbits of points $x \in X_\alpha$ and hence this is an ergodic measure. Second, the conditional entropy of T_α relative to the factor algebra $\mathcal{F}_\alpha^n$ is independent of α; hence all T_α have the same entropy. Third, the factor $\mathcal{F}_\alpha^n$ splits off with a Bernoulli complementary algebra, and hence all T_α have the weak Pinsker property. The length of the block of s's that precede the $(n-1)$-block in each window are independent, independent of all choices of $(n-1)$-blocks in any window and have range growing in n. This is sufficient, by standard arguments, to show that T_α is a K-system [**15**]. □

6.3. *Proper arrays and pseudorandom sequences.* We now begin the most technical part of our work, setting up the properties we require of $N(n)$ and the pseudorandom sequences $\mathbf{A}^0(n)$ and $\mathbf{A}^1(n)$. We need to consider the structure of overlaps of windows in n-blocks amongst a finite collection of points x_k, as each is moved by a perhaps distinct power j_k of T_α. Fix a list of $m+1$ values $j_0, j_1, \ldots, j_m$ with $-n \leq j_k \leq n$ and $m \leq n$. Let

$$\bar{j} = \operatorname{lcm}\left(\frac{w(n)}{|j_k|}\right) = \frac{w(n)}{\gcd(|j_k|)} \quad \text{and so } \bar{j} \leq w(n).$$

where, as usual, gcd denotes the greatest common divisor and lcm the least common multiple. Fix choices $1 \leq t_0^k \leq N(n)$, $k = 0, \ldots, m$ and consider the array of values

$$t_i^k = t_0^k + \frac{i\bar{j}j_k}{w(n)} \quad \text{where } i \in \mathbb{Z}.$$

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Hence we construct a series of m arithmetic progressions which begin at the values t_0^k and have transitions $(\bar{j}\, j_k)/w(n)$. We truncate this list to values $i_0 \le i < r_0$ so that all t_i^k satisfy $1 \le t_i^k \le N(n)$.

Definition 6.8. For a choice of $m \le n$, values of $j_0, \ldots, j_m$ and $t_0^1, \ldots, t_0^m$ as above we refer to the $n \times (r_0 - i_0)$ array of values $\{t_i^k\}$ as an n-array. Its dimensions are $r_0 - i_0 \times m$.

What is the significance of n-arrays? Suppose $x_0 \in X_\alpha$ and $(x_1, \ldots, x_m)$ are m points in some X_β with x_k lying in the t_0^kth window in its n-block and in this window, at index $0 \le w_n(x) < w(n)$ counting from the left end. Act on this $m+1$-tuple of points by powers of the map

$$\hat{T} = T_\alpha^{j_0} \times \bigotimes_{k=1}^{m} T_\beta^{j_k}.$$

The points $\hat{T}^{\bar{j}i}(x_0, x_1, \ldots, x_m)$ for $i_0 \le i < r_0$ all still lie in these same n-block names, but in different windows. The kth term of the list of points lies in the t_i^kth window and is again at precisely index $w_n(x_k)$ in that window. That is to say, the points recur at these times to exactly the same indices in their windows where they began. Our coding argument to show the examples are u.s.d. relies fundamentally on this recurrence of the local geometry of the overlap of n-block windows.

Definition 6.9. An n-array as defined above is *proper* if it satisfies two properties.

(1) The length of the n-array satisfies $r_0 - i_0 \ge N(n)/(10 \cdot 2^n \cdot w(n)^n)$.

(2) If for some $k_1 \ne k_2$ both ≥ 1 we have $j_{k_1} = j_{k_2}$ then $t_0^{k_1} \ne t_0^{k_2}$. That is to say, no two rows $k \ge 1$ of a proper array are allowed to be identical. We need not be concerned about row 0.

Definition 6.10. Fixing values $-n \le j_0, j_1, \ldots, j_m \le n$ we define *the associated n-array of the list of points* $\{x_0, x_1, \ldots, x_m\} \in X^\alpha \times \bigotimes_{k=1}^m X^\beta$ to be the perhaps empty n-array of indices

$$t_i^k = t_0^k + \frac{i\bar{j}\, j_k}{w(n)}$$

where for some choice i_1, $i_0 \le i_1 \le r_0$, all x_k are in the $t_{i_1}^k$th window of an n-block. This n-array will be empty if and only if some x_k is not in an n-block name.

PROPOSITION 6.11. *For each $m \le n$ and choice of values $-n \le j_0, j_1, \ldots, j_m \le n$ and $\hat{\mu}$ a joining of μ_α and μ_β^m, the $\hat{\mu}$ measure of those points $\{x_0, x_1, \ldots, x_m\}$ whose n-array is proper is at least* $1 - 1/(2 \cdot 2^n)$.

Proof. Let $0 \le t(x) < N(n)$ be the index of the window containing x in its n-block. The value $t(x)$ is undefined if x is not in an n-block. If for all x_k we have

$$\frac{N(n)}{10 \cdot 2^n} \le t(x) \le N(n)\left(1 - \frac{1}{10 \cdot 2^n}\right)$$

then the n-array of $\{x_0, \ldots, x_m\}$ will be proper. The $\hat{\mu}$ measure of this set is at least

$$1 - \left(\sum_{k=n+1}^{\infty} F(k) + \frac{1}{10 \cdot 2^n}\right) \ge 1 - \left(\frac{4}{10 \cdot 2^n}\right).$$

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The μ_β^m-measure of those points $\{x_1, \ldots, x_m\}$, such that at least two elements have the same value $t(x)$, is at most

$$\frac{n^2}{N(n)} < \frac{1}{10 \cdot 2^n}$$

and hence the $\hat{\mu}$ measure of those strings with proper n-arrays is at least $1 - 1/(2 \cdot 2^n)$. □

Simple counting gives the following lemma.

LEMMA 6.12. *The number of distinct proper n-arrays as $m \le n$ and the values of $j_0, \ldots, j_m$ are allowed to vary is bounded by*

$$n[(2n+1)N(n)]^{n+1}.$$

Definition 6.13. We say a pair of *pseudorandom sequences* $(\mathbf{A}^0, \mathbf{A}^1) \subseteq (\{1, \ldots, 2^n\}^{N(n)})^2$ *has Property 1* if for all proper n-arrays $\{t_i^k\}$ and all lists $\{a^0, a^1, \ldots, a^m\} \in \{1, 2, \ldots, 2^n\}$ we have

$$\left| \frac{\#\{i | a^0_{t_i^0} = a^0, a^1_{t_i^k} = a^k, \text{ for } k \ge 1\}}{(r_0 - i_0)} - \frac{1}{(2^n)^{m+1}} \right| < \frac{1}{10 \cdot 2^n (2^n)^{n+1}}.$$

In other words, along any proper n-array, all strings $\{a^0, \ldots, a^1\}$ occur approximately equally often. Notice the error here from uniformity is small compared to the density of occurrence.

PROPOSITION 6.14. *For all $\varepsilon > 0$, if $N(n)$ is sufficiently large, the fraction in $(\{1, \ldots, 2^n\}^{N(n)})^2$ of pairs of sequences $(\mathbf{A}^0, \mathbf{A}^1)$ which have Property 1 will be at least $1 - \varepsilon$.*

Proof. It is convenient to argue this probabilistically. Take $(\{1, \ldots, 2^n\}^2)^{\mathbb{Z}}$ with uniform Bernoulli measure as a sequence of random variables. Suppose $\{t_i^k\}_{i \in \mathcal{I}}$ is some subarray of a proper n-array with the property that no two values t_i^k, $i \in \mathcal{I}$ and $k = 1, \ldots, m$ agree. Then $\{(a^0_{t_i^0}, a^1_{t_i^1}, \ldots, a^1_{t_i^m})\}_{i \in \mathcal{I}}$ will be an i.i.d. sequence of random variables. The central limit theorem tells us that for some $\{a^0, \ldots, a^M\}$, the probability of

$$\left| \frac{\#\{i \in \mathcal{I} : a^0_{t_i^0} = a^0, a^1_{t_i^k} = a^k, k \ge 1\}}{\#\mathcal{I}} - \frac{1}{(2^n)^{m+1}} \right| \ge \frac{1}{20} \frac{1}{2^n (2^n)^{n+1}}$$

decays exponentially in $\#\mathcal{I}$. That is, for some values $C, h > 0$ (depending on n but not $N(n)$) this probability is at most $C \exp(-h\#\mathcal{I})$.

In any proper array, two rows $k \ne k' \ge 1$ can have $t_i^k = t_i^{k'}$ for at most one value of i. Hence, we can delete at most $n(n-1)$ values i and we know that on the remaining indices $\mathcal{I} \subseteq \{i_0, \ldots, r_0 - 1\}$, for each $i \in \mathcal{I}$ the indices $t_i^1, \ldots, t_i^m$ are distinct. Any particular index t can occur at most m times in the array on rows $k \ge 1$. Hence each row $t_i^1, \ldots, t_i^m$ can have a value in common with at most n^2 other rows $t_{i'}^1, \ldots, t_{i'}^n$. We claim that this means $\mathcal{I}$ can be partitioned into sets $\mathcal{I}_1, \ldots, \mathcal{I}_{m^2+1}$ where for each $\mathcal{I}_u$, all indices t_i^k, $i \in \mathcal{I}_u$ and $k = 1, \ldots, m$ are distinct and moreover

$$\#\mathcal{I}_u \ge \frac{\#\mathcal{I}}{2(n^2+1)^3}.$$

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To see this, first note that one can recursively create a partition of $\mathcal{I}$ into at most $m^2 + 1$ sets where each contains distinct elements. A new column $\{t_i^1, \ldots, t_i^m\}$ can share terms with at most m^2 others and hence can be added to one of the partition elements, while still maintaining distinctness.

Now suppose that some $\mathcal{I}_k$'s have cardinality less than $\#\mathcal{I}/2(n^2+1)^3$. These are referred to as the *small* terms. The union of all such small terms has cardinality less than $\#\mathcal{I}/2(n^2+1)^2$. Hence some other (large) $\mathcal{I}_j$ has

$$\#\mathcal{I}_j \geq \#\mathcal{I}\left(1 - \frac{1}{1(n^2+1)^2}\right)\frac{1}{m^2} > \frac{\#\mathcal{I}}{2n^2}.$$

The small terms can share elements with at most $(n^2\#\mathcal{I})/(2(n^2+1)^3) < \#\mathcal{I}/(2(n^2+1)^2)$ of the columns of $\mathcal{I}_j$. Hence, some column in $\mathcal{I}_j$ can be moved to a small term without making $\mathcal{I}_j$ small. This can be continued until no small terms remain and we have the partition we seek.

Thus, for each $\mathcal{I}_u$ for some $a^0, \ldots, a^m$, the probability that

$$\left|\frac{\#\{i \in \mathcal{I}_u | a^0_{t_i^0} = a^0, a^1_{t_i^k} = a^k, k \geq 1\}}{\#\mathcal{I}_u} - \frac{1}{(2^n)^{m+1}}\right| \geq \frac{1}{20}\left(\frac{1}{2^n}\right)^{n+2}$$

is at most

$$C\exp(-h\#\mathcal{I}_u) \leq C\exp\left(-\#\mathcal{I}\frac{h}{2(n^2+1)^3}\right)$$

and hence for some $a^0, \ldots, a^m$ the probability that

$$\left|\frac{\#\{i \in \mathcal{I} | a^0_{t_i^0} = a^0, a^1_{t_i^k} = a^k, k \geq 1\}}{\#\mathcal{I}} - \frac{1}{(2^n)^{m+1}}\right| \geq \frac{1}{20}\left(\frac{1}{2^n}\right)^{n+2}$$

is at most

$$n^2C\exp\left(-\#\mathcal{I}\frac{h}{2(n^2+1)^3}\right).$$

Recalling that $\mathcal{I}$ omits at most $n(n-1)$ of the values $i_0, \ldots, r$, and a proper array has length at least $N(n)(10 \cdot 2^n \cdot w(n)^n)^{-1}$ if we choose $N(n)$ large enough, we conclude that for each proper array for all $a^0, \ldots, a^m$, the probability that

$$V = \left|\frac{\#\{i_0 \leq i < r_0 | a^0_{t_i^0} - a^0, a^1_{t_i^k} = a^k, k \geq 1\}}{r_0 - i_0} - \frac{1}{(2^n)^{m+1}}\right| < \frac{1}{10(2^n)^{n+2}}$$

is at most

$$n^2C\exp\left(\frac{-(r_0 - i_0)h}{4(n^2+1)^3}\right) \leq n^2C\exp\left(\frac{-N(n)h}{40 \cdot 2^n \cdot w(n)^n \cdot (n^2+1)^3}\right).$$

Hence, the probability that Property 1 is not satisfied is at most

$$n[(2n+1)N(n)]^{n+1}n^2C\exp\left(\frac{-N(n)h}{40 \cdot 2^n \cdot w(n)^n \cdot (n^2+1)^3}\right).$$

This tends to zero as the choice for $N(n)$ increases to infinity. □

6.4. *Separating names in $\overline{d}$.* Property 1 is the only fact we need about the pseudorandom sequences. Henceforth, we shall require that both $(\mathbf{A}^0, \mathbf{A}^1)$ and $(\mathbf{A}^1, \mathbf{A}^0)$ satisfy Property 1. For convenience we also ask that $N(n) = k \cdot 10 \cdot 2^n \cdot w(n)^n$ where $k \geq 10$. We shall also make one further requirement on the size of $N(n)$; we establish some basic facts first. A *word*, as is standard, will be a map from a finite or infinite subinterval $i_0 \leq i < r_0$ to a symbol set. As we will work with various powers of T_α, if we are considering T_α^j then we will consider words in the symbols $\{0, e, f, s\}^j$. This is equivalent to taking a word in the symbols $\{0, e, f, s\}$ and walking across with steps of length j. Hence, we actually need only consider words in the symbols $\{0, e, f, s\}$, but will need to specify the step size with which we walk across the word. If B is a finite word of length $t = jk$ we will write $B(j)$ for the word of length k in symbols $\{0, e, f, s\}^j$ obtained by walking across B in steps of length j. This will only make sense when the size of the domain of a word B is divisible by j. The domain of $B(j)$ is not made explicit from the domain of B. In our usage these names will always be parts of a doubly infinite sequence. The arithmetic progression we walk on will start at 0 and then this will fix the domain of $B(j)$.

We will consider two words as *equivalent* if they differ only by a translation of their domains. When we say a word B sits at index i we are perhaps translating the domain of the word B to begin at index i and extend to $i +$ (the length of B) $- 1$. By the *overlap* of two words we mean the interval that is the intersection of their domains. Here, the stepsize with which we are walking across the two words is significant. Also, following standard practice we will use B to represent the set of points $x \in \{0, e, f, s\}^{\mathbb{Z}}$ for which $x_i = B(i)$, $i_0 \leq i < r_0$. Moreover, we will let $x_{i_0}^{r_0-1}$ be the word obtained by restricting x to this domain of indices. The words in $\mathcal{B}_n^\alpha$ have not been given specific domains and in this context, we regard them as sets of equivalence classes of names. When we speak of a word $b \in \mathcal{B}_n^\alpha$ we mean any word with a fixed domain in some equivalence class in $\mathcal{B}_n^\alpha$.

Definition 6.15. The *mean Hamming* or *d-bar* distance between two words B and B' with the same domain $i_0 \leq i < r_0$ is given by

$$\overline{d}(B, B') = \frac{\#\{i \mid B(i) \neq B'(i)\}}{r_0 - i_0}.$$

If $x, x' \in \{0, e, f, s\}^{\mathbb{N}}$ it is standard to write

$$\overline{d}_n(x, x') = \overline{d}(x_0^{n-1}, x'^{n-1}_0).$$

Definition 6.16. Two words $b, b' \in \mathcal{B}_n^\alpha$ are said to have *good overlap* if their overlap has length at least $h(n)/[10 \cdot 2^n \cdot w(n)^n]$ and at most $h(n) - w(n)$. That is to say, the overlap is neither too long nor too short. For $j \leq n$ we say $b(j)$ and $b'(j)$ have *good overlap* if b and b' do.

We want to show that if an overlap of two n-block names is good, then they are some fixed distance in $\overline{d}$ apart. This fact is a standard piece of all such constructions and comes from Property 1. One need not work terribly carefully to get this fact and our estimates are very sloppy.

LEMMA 6.17. *If $b, b' \in \mathcal{B}_n^\alpha$, $n \geq 1$ have good overlap then a fraction of at least $1 - 3/2^n$ of the overlap is occupied by good overlaps of pairs of names from B_{n-1}^α.*

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Proof. A good overlap of words $b, b' \in \mathcal{B}_n^\alpha$ must contain at least $N(n)/(10 \cdot 2^n \cdot w(n)^n)$ windows: note this value is at least 10. It follows that a fraction of less than $3F(n)$ of the overlap is occupied by frames in the windows, hence a fraction of at most $1 - 3F(n)$ is occupied by overlaps of pairs of words from $\mathcal{B}_{n-1}^\alpha$. Among these overlaps, a fraction of at most $w(n-1)/h(n-1) = 1/N(n-1) < 1/(10 \cdot 2^n)$ can be too short to be good overlaps. We now want to estimate the fraction of overlaps that might be too long. Suppose the $(n-1)$-block names in windows at indices t_1 in b and t_2 in b' overlap in too long a segment to be good. This means these two windows themselves overlap in at least $h(n-1) - w(n-1) > w(n)[1 - 1/(5 \cdot 2^n)]$ places. It follows that the same holds true for pairs of windows at indices $t_1 + k$ and $t_2 + k$, provided that these values stay between 1 and $N(n)$. In order for some other pair of windows $t_1 + k$ and $t_2 + k$ not to yield a good overlap of $(n-1)$-blocks, we must have

$$a_{t_1}^{\alpha_n} - a_{t_2}^{\alpha_n} = a_{t_1+k}^{\alpha_n} - a_{t_2+k}^{\alpha_n}.$$

Otherwise, the e sections in the windows will force the overlap at these new indices to change by at least $2w(n-2)$ from those at indices t_1 and t_2 and hence become good. Property 1 tells us that the pairs $a_{t_1+k}^{\alpha_n}, a_{t_2+k}^{\alpha_n}$ are essentially uniformly distributed over all possible pairs. This means their difference can be a constant on a set of density at most $1/2^n + 1/(10 \cdot 2^{2n})$ and hence that at most this fraction of the overlaps can possibly fail to be good because they are too long. Not all these overlaps of $(n-1)$-block names have the same length but their lengths differ certainly by less than a factor of two. Hence, if any overlap fails to be good because it is too long, then amongst all overlaps, the density of those which fail to be good for this reason is at most $22/10 \cdot 2^n$. Overall, the density of the overlap of b and b' which must be occupied by good overlaps of names from $\mathcal{B}_n^\alpha$ is at least

$$1 - 3F(n) - \frac{1}{10 \cdot 2^n} - \frac{22}{10 \cdot 2^n} \geq 1 - \frac{26}{10 \cdot 2^n}. \qquad \square$$

LEMMA 6.18. *For two names $b, b' \in \mathcal{B}_4^\alpha$ with good overlap and overlapping on indices $i_0 \leq i < r_0$,*

$$\overline{d}(b_{i_0}^{r_0}, b'^{r_0}_{i_0}) \geq \frac{3}{4w(4)}.$$

Proof. This is a very sloppy estimate, although it is good enough for our purposes. Suppose that the name $b_{i_0}^{r_0}$ is broken into windows of length $w(4)$. Each such window provides one index where $b_i \neq b'_i$, unless the window overlaps a window in b' whose e and s sequences are identical to those in b. But this can only happen if one name is translated by precisely a multiple of $w(4)$ with respect to the other and then only in windows where the values from $\mathbf{A}^{\alpha_4}(4)$ agree. As the pair have a good overlap, this occurs for less than a fraction $1/2^4 + 1/(10 \cdot 2^8) < 1/4$ of the windows. $\square$

COROLLARY 6.19. *For all $n \geq 4$ and all pairs of words $b, b' \in \mathcal{B}_n^\alpha$ with a good overlap, if the overlap is $i_0 \leq i < r_0$ then*

$$\overline{d}(b_{i_0}^{r_0}, b'^{r_0}_{i_0}) > \frac{3}{4w(4)} \prod_{k=5}^{n} \left(1 - \frac{3}{2^k}\right) > \frac{3}{4w(4)} \prod_{k=5}^{\infty} \left(1 - \frac{3}{2^k}\right) \stackrel{\text{def}}{=} d > 0.$$

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Proof. One verifies the first inequality inductively using the previous two lemmas. The rest follows easily. □

COROLLARY 6.20. *For all $n \geq 4$ and all pairs of words b, $b' \in \mathcal{B}_n^\alpha$ with a good overlap, if the overlap is $i_0 \leq i < r_0$ and $i_0 \leq i_1 < j_1 \leq r_0$ with $r_1 - i_1 \geq h(n)/[10 \cdot 2^n \cdot w(n)^n]$ then*

$$\overline{d}(b_{i_1}^{r_1}, b'^{r_1}_{i_1}) > d > 0.$$

Proof. The interval $[i_1, j_1)$ is large enough to contain enough $(n-1)$-blocks. We apply Lemma 6.17 to claim that at least $1 - 3/2^n$ of it is occupied by good overlaps of names from $\mathcal{B}_{n-1}^\alpha$. We can now apply the same inductive procedure of the previous corollary. □

6.5. *The nitty gritty.* We are now almost prepared to show that any T_α and T_β with $\alpha_n \neq \beta_n$ infinitely often are somewhat disjoint with value $a = d/5$. Towards that end, our work from now on will be premised on the existence of two such maps T_α and T_β. Moreover, we will assume that non-zero values $j_0, j_1, \ldots, j_m$ have been fixed as well as an ergodic joining $\hat{\mu}$ of $T_\alpha^{j_0}$ and $\bigotimes_{i=1}^m T_\beta^{j_i}$. We set $\hat{X} = X_\alpha \times \bigotimes_{i=1}^m X_\beta$ and $\hat{T} = T_\alpha^{j_0} \times \bigotimes_{i=1}^m T_\beta^{j_i}$. Given these constructions, we will consider values $n \geq \max\{|j_0|, \ldots, |j_m|, m\}$ where $\alpha_n \neq \beta_n$. We will refer to this collection of choices as a set of *basic material*. Given this basic material, by *the n-block containing a point x_i* we will mean the name $b(j_i)$ where x_i, under that action of T_α sits in the n-block name b. That is to say, we will walk across the name in steps of size j_i. By the *n-block window containing x_i* we mean that window in this n-block which contains the origin. Once more we walk across in step size j_i. By the index in the n-block or n-block window at which x_i sits we mean the position in the block, counting from the left end in units, not steps of size j_i. Thus, if $T_\alpha^{j_i}(x_i)$ still lies in the same n-block window as x_i did, then its index in this window will be j_i larger than that of x_i. Let Q_n^α be the partition of X^α labelled by $\{1, e, f, s\}$ and obtained by putting any point that lies inside an n-block name $b \in \mathcal{B}_n^\alpha$ into the single set 1 and leaving all the remaining points, those in frames around n-blocks or higher, in the sets as originally labelled. Given a set of basic material, set

$$\hat{Q}_n = Q_n^\alpha(j_0) \times \bigotimes_{i=1}^m Q_n^\beta(j_i).$$

Note that the $\hat{T}$, $\hat{Q}_n$-name of a point determines whether or not it is in a good n-overlap in the sense of Definition 6.27 below as it provides enough information to specify the positions of the n-block windows in an n-block. This name also determines the size of the deterministic e and f parts of the spacers in these windows, since these are defined by the position of the n-block window in the n-block. The remaining part of the $P^\alpha(j_0)$ name and the $P^\beta(j_k)$-names are completely arbitrary. More precisely, having used the Q_n^α name to fill in the position of the n-block windows and then the e and f sections of the frame in each window, we are left with an infinite sequence of gaps in name of width $h(n-1) + c(n)$. The number of possible names that can be placed in each such window is $c(n)\eta(n-1)$ (remember that $\eta(n)$ is the number of n-block names). The conditional expectation of the sequence of names we can see in these windows given the Q_n^α name is i.i.d. and uniformly distributed over this set of names.

LEMMA 6.21. *For any choice of $\varepsilon > 0$, if we fix the construction through that of $(n-1)$-blocks and then if $N(n)$ is chosen sufficiently large, then for all α,*

$$h(T_\alpha, Q_n^\alpha) < \varepsilon.$$

Proof. Simply note that the measure of the set labelled 1 increases to 1 as the size of $N(n)$ grows. □

Our goal now is a 'conditional $\bar{d}$' calculation and, to achieve that, we will need a definition. $\bar{d}$ is defined in many different contexts. We will need most of them, and so include here are the definitions for completeness.

Definition 6.22. The definition of $\bar{d}$ for finite names in symbols from a finite set Σ has already been given. If μ_1 and μ_2 are two measures on the space Σ^n of names in Σ of length n then we set

$$\bar{d}(\mu_1, \mu_2) = \inf_{\hat{\mu}} \bigg(\int \bar{d}(s^1, s^2)\, d\hat{\mu} \bigg)$$

where the infimum is over all couplings $\hat{\mu}$ of the two measures. For $s^1, s^2 \in \Sigma^{\mathbb{Z}}$ we set

$$\bar{d}(s^1, s^2) = \limsup_{N\to\infty} \bar{d}((s^1)_{-N}^N, (s^2)_{-N}^N).$$

If μ_1 and μ_2 are two shift-invariant measures on $\Sigma^{\mathbb{Z}}$ we set

$$\bar{d}(\mu_1, \mu_2) = \inf_{\hat{\mu}} \bigg(\int \bar{d}(s^1, s^2)\, d\hat{\mu} = \hat{\mu}(\{(s^1, s^2) : s_0^1 \neq s_0^2\}) \bigg)$$

where the infimum is over all ergodic joinings $\hat{\mu}$ of μ_1 and μ_2. If (T_1, P_1) and (T_2, P_2) are two ergodic processes with P and P' using the same label space Σ, then the map to T, P-names and T', P'-names give two ergodic measures μ_1 and μ_2 on $\Sigma^{\mathbb{Z}}$ and one sets

$$\bar{d}(T_1, P_1; T_2, P_2) = \bar{d}(\mu_1, \mu_2).$$

As a last refinement, suppose $(X_1, \mathcal{F}_1, \mu_1, T_1)$ and $(X_2, \mathcal{F}_2, \mu_2, T_2)$ are ergodic systems which possess a common invariant factor action $\mathcal{H}$. In the space of joinings of these two actions let $J_{\mathcal{H}}$ be those joinings of μ_1 and μ_2 supported on the graph of the identity on $\mathcal{H} \times \mathcal{H}$ [**31,** 6.2]. We now define

$$\bar{d}_{\mathcal{H}}(T_1, P_1; T_2, P_2) = \inf_{\hat{\mu} \in J_{\mathcal{H}}} \hat{\mu}(\{(x_1, x_2) | P_1(x_1) \neq P_2(x_2)\}).$$

Suppose we have a set of basic material. We now describe a modification of $\hat{\mu}$ which we call $\overline{\mu} = \overline{\mu}_n$. This construction is an essential ingredient of our argument.

To do this, let $\mathcal{Q}_n^\alpha = \bigvee_{i=-\infty}^{\infty} T_\alpha^{-i}(Q_n^\alpha)$ and $\mathcal{P}^\alpha = \bigvee_{i=-\infty}^{\infty} T_\alpha^{i}(P^\alpha)$. To start, we define the measure $\overline{\mu}$ on the sub-σ-algebra $\mathcal{Q}_n^\alpha \vee \bigvee_{k=1}^m \mathcal{P}^\beta$. Notice that $\bigvee_{k=1}^m \mathcal{Q}_n^\beta$ is a sub-σ-algebra of both $\bigvee_{k=1}^m \mathcal{P}^\beta$ and $\mathcal{Q}_n^\alpha \vee \bigvee_{k=1}^m \mathcal{Q}_n^\beta$. Define $\overline{\mu}$ on $\mathcal{Q}_n^\alpha \vee \bigvee_{k=1}^m \mathcal{P}^\beta$ to be the relatively independent joining of these two systems over this common factor [**31,** 6.2]. What this means is that in $\mathcal{Q}_n^\alpha \vee \bigvee_{k=1}^m \mathcal{Q}_n^\beta$, we fill in the names across the windows of 1's with spacers and n-block names not only independent of the Q_n^β-names but also independent of the Q_n^α name as well. Hence $\overline{\mu}$ is a shift-invariant measure on doubly infinite $Q_n^\alpha(j_0) \times \bigotimes_{k=1}^m P^\beta(j_k)$ names. It is not difficult to see that as $\hat{\mu}$ is ergodic, so is $\overline{\mu}$.

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PROPOSITION 6.23. *Fixing the construction through stage $n-1$, for any $\varepsilon > 0$ if $N(n)$ is chosen large enough, for any choice of a set of basic material, set $\mathcal{H} = \mathcal{Q}_n^\alpha \vee \bigvee_{k=1}^m \mathcal{Q}_n^\beta$ and then as measures on doubly infinite $Q_n^\alpha(j_0) \times \bigotimes_{k=1}^m P^\beta(j_k)$ names we will have*

$$\overline{d}_{\mathcal{H}}(\hat{\mu}, \overline{\mu}) < \varepsilon.$$

Proof. Fixing the construction up to stage n fixes the number of possible names to be filled in across an n-block window at $c(n) \cdot \#\mathcal{B}_{n-1}^\beta$. We show the $\bar{d}$-closeness for each of the $P^\beta(j_k)$ names separately. Fix the value k, and let A be the subset of points which, in the kth copy of X^β is the leftmost point in an n-block window. Partition the set A into subsets according to the $\{0, e, f, s\}$-name across this window and call $\tilde{P}_k$ this partition of A. Now induce on A, i.e. consider the action $\hat{T}_A$. The process $(\hat{T}_A, \tilde{P}_n)$ is an i.i.d. process on $c(n) \cdot \#\mathcal{B}_{n-1}^\beta$ symbols, independent of the choice of $N(n)$. This process is extremal [**27,** III §4]. To see this, use the weak Pinsker property of $\hat{T}$. Set $\mathcal{H}_k = \mathcal{Q}_n^\alpha \vee \bigvee_{k'<k} \mathcal{P}^\beta \vee \bigvee_{k' \geq k} \mathcal{Q}_n^\beta$. That is to say, we inductively add on the $\mathcal{P}^\beta$ algebras to $\mathcal{H}$. Extremality tells us there is a δ so that if we knew

$$h(\hat{T}_A, \tilde{P}_k | \mathcal{H}_k) > h(\hat{T}_A, \tilde{P}_k) - \delta = \log[c(n) \cdot \#\mathcal{B}_{n-1}^\beta] - \delta$$

relative to both $\hat{\mu}$ and $\overline{\mu}$, then we could conclude

$$\overline{d}_{\mathcal{H}_k}((\hat{T}_A, \tilde{P}_k)_{\hat{\mu}}, (\hat{T}_A, \tilde{P}_k)_{\overline{\mu}}) < \varepsilon'$$

and the proposition would follow by setting $\varepsilon' = \varepsilon/m$ and working through the m terms inductively.

With respect to $\overline{\mu}$, $(\hat{T}_A, \tilde{P}_k)$ is independent of $\mathcal{H}_k$ so the entropy bound above follows immediately. With respect to $\hat{\mu}$, $(\hat{T}_A, \tilde{P}_k)$ is independent of $\bigvee_{k'<k} \mathcal{P}^\beta \vee \bigvee_{k' \geq k} \mathcal{Q}_n^\beta$ as the m X^β coordinates are independent and the windows are filled in independently. So, by the Pinsker formula [**10**], the problem reduces to showing that once $N(n)$ is large enough, we have

$$h(\hat{T}_A, Q_n^\alpha) < \delta.$$

But by Kac's formula [**3,** Kac's Lemma] this is

$$\frac{h(T^\alpha, Q_n^\alpha)}{\mu^\beta(A)}.$$

As $N(n) \nearrow \infty$, $\mu^\beta(A) \to w(n)/j_k \geq w(n)/n > 0$ and by Lemma 6.21 we can choose $N(n)$ large enough to give the entropy bound and apply extremality. □

This last proposition gives our *final requirement on the size of* $N(n)$. We ask that it be sufficiently large that for any set of basic material from stage n we will have

$$\overline{d}_{\mathcal{H}}(\hat{\mu}, \overline{\mu}) < \frac{1}{10 \cdot 2^n}.$$

The conditions on our choices of parameters are now completely specified.

For a basic set of material, we now extend the construction of $\overline{\mu}$ to all of $\mathcal{P}^\alpha \vee \bigvee_{k=1}^m \mathcal{P}^\beta$ as follows. Take an ergodic joining $\tilde{\mu}$ in $J_{\mathcal{H}}$ which achieves the $\overline{d}_{\mathcal{H}}$ distance between $\hat{\mu}$ and $\overline{\mu}$. This is a measure on the direct product of two copies of $\mathcal{Q}_n^\alpha \vee \bigvee_{k=1}^m \mathcal{P}^\beta$.

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The $\mathcal{H}$ factors of these two systems are identified relative to $\tilde{\mu}$ and these contain $\mathcal{Q}_n^\alpha$. Now $\hat{\mu}$ extends to include the factor $\mathcal{P}^\alpha$. In this way, one can extend $\tilde{\mu}$ to include $\mathcal{P}^\alpha$, taking the relatively independent extension of $\tilde{\mu}$ over this common $\mathcal{H}$ factor with respect to the second copy (the $\bar{\mu}$ copy) of $\bigvee_{k=1}^m \mathcal{P}^\beta$. We then restrict $\tilde{\mu}$ to span $\mathcal{P}^\alpha$ and the second copy of $\bigvee_{k=1}^m \mathcal{P}^\beta$. This extends $\overline{\mu}$ to all of $\mathcal{P}^\alpha \vee \bigvee_{k=1}^m \mathcal{P}^\beta$. From its definition as the restriction of an $\mathcal{H}$-relative joining, we still have

$$\overline{d}_{\mathcal{H}}(\hat{\mu}, \overline{\mu}) < \frac{1}{10 \cdot 2^n}.$$

A set of basic material gives a lower bound for the value of n but does not specify the value. The construction of $\overline{\mu}$ depends on the set of basic material and on the value n. Hence we will now write it as $\overline{\mu}_n$. Notice that the joining $\tilde{\mu}$, which we now call $\tilde{\mu}_n$, possesses one copy of the $(T_\alpha^{j_0}, P^\alpha(j_0))$ process but two copies of the $\big(\bigotimes_{k=1}^m T_\beta^{j_k}, \bigvee_{k=1}^m P^\beta(j_k)\big)$ process which, as n grows, agree with higher and higher probability.

LEMMA 6.24. *Given a set of basic material, the measure $\overline{\mu}_n$ converges in $\overline{d}$ to $\hat{\mu}$. Moreover, the measures $\tilde{\mu}_n$ which join $\overline{\mu}_n$ and $\hat{\mu}$, converge to the diagonal two-fold self-joining of $\hat{\mu}$. In particular, for any sets $A \in \mathcal{P}^\alpha$ and $B \in \bigvee_{k=1}^m \mathcal{P}^\beta$, we have $\overline{\mu}_n(A \cap B)$ converges in n to $\hat{\mu}(A \cap B)$.*

Proof. We only need to check this property for a dense class of sets A and B. Hence we can assume they are finitely coded from the processes $(T_\alpha^{j_0}, P^\alpha(j_0))$ and $\big(\bigotimes_{k=1}^m T_\beta^{j_k}, \bigvee_{k=1}^m P^\beta(j_k)\big)$. One calculates

$$|\overline{\mu}_n(A \cap B) - \hat{\mu}(A \cap B)| \le \tilde{\mu}_n(B_1 \triangle B_2),$$

where B_1 and B_2 are the two copies of B in the joining $\tilde{\mu}_n$. The latter tends to zero as $\tilde{\mu}_n$ achieves the $\overline{d}$ distance between $\overline{\mu}_n$ and $\hat{\mu}$. □

COROLLARY 6.25. *Given a set of basic material, suppose that for $0 \le j < j_0$ we have, relative to the joining $\hat{\mu}$, $T_\alpha^j(P^\alpha) \overset{\varepsilon_j}{\subseteq} \bigvee_{k=1}^m \mathcal{P}^\beta$. It follows that there are partitions $P^0, \ldots, P^{j_0-1} \subseteq \bigvee_{k=1}^m \mathcal{P}^\beta$ so that*

$$\lim_{n\to\infty} \overline{\mu}_n(T_\alpha^j(P^\alpha) \triangle P^j) < \varepsilon_j.$$

This corollary tells us that if T_α and T_β fail to be u.s.d. the failure is already evident on the joinings $\overline{\mu}$. We now formulate a result which will be our principal tool in proving that T_α and T_β are somewhat disjoint.

PROPOSITION 6.26. *Suppose there is a value $a > 0$ so that for all choices of $j_0, \ldots, j_m$ and partitions $P^0, \ldots, P^{j_0-1}$ that are finitely coded from $\bigvee_{k=1}^m P^\beta(j_k)$ and all ergodic joinings $\hat{\mu}$ of $T_\alpha^{j_0}$ and $\otimes T_\beta^{j_k}$ there are arbitrarily large values of n so that if we construct $\overline{\mu}_n$ we find*

$$\frac{1}{j_0} \sum_{i=0}^{j_0-1} \overline{\mu}(T_\alpha^i(P^\alpha) \triangle P^i) \ge a.$$

Then T_α and T_β are u.s.d.

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Proof. Under the hypotheses of the proposition, we may conclude that for all sets of basic material and for all finitely coded sets $P^0, \ldots, P^{j_0}$ we have

$$\frac{1}{j_0}\sum_{i=0}^{j_0-1}\hat{\mu}(T^i_\alpha(P^\alpha)\triangle P^i) \geq a.$$

This is a closed property on the choice of partitions P^i and hence holds on all partitions. This gives the result. □

We now verify Proposition 6.26 for $a = d/5$.

Definition 6.27. Starting from a set of basic material as described above, we say the list of points $\{x_0, x_1, \ldots, x_m\} \in X_\alpha \times \bigotimes_{i=1}^m X_\beta$ has a *good n-overlap* if

(1) the associated n-array of this list of points is proper in the sense of Definition 6.9; and

(2) the overlap of the n-block windows containing the list has length at least $F(n) + w(n)/(10n2^n)$.

The set of such points in $\hat{X}$ is measurable and we call it G_n.

Notice that having good n-overlap is determined completely by the relative overlaps of the n-blocks and hence G_n is $\mathcal{H}_n = \mathcal{Q}^\alpha_n \vee \bigvee_{k=1}^m \mathcal{Q}^\beta_n$ measurable.

COROLLARY 6.28. *For any set of basic material, we have*

$$\hat{\mu}(G_n) \geq 1 - \frac{1}{2\cdot 2^n}.$$

Proof. By Proposition 6.11, condition (1) excludes at most $1/(10\cdot 2^n)$ of the measure space and condition (2) excludes at most $nF(n) + 1/(10\cdot 2^n)$. □

COROLLARY 6.29. *Given a set of basic material, if $x_0, \ldots, x_m$ have good n-overlap then so do all points $\hat{T}^j(x_0, \ldots, x_m)$ for all j small enough to keep these points in the same n-block windows as $\{x_0, \ldots, x_m\}$. Moreover, if $\{t^0_i, t^1_i, \ldots, t^m_i\}$ is the n-array of this list, then $\{T_\alpha^{t^0_i}(x_0), T_\beta^{t^1_i}(x_1), \ldots, T_\beta^{t^m_i}(x_m)\}$ all lie on this $\hat{T}$ orbit and not only still sit inside this n-block overlap, but sit at the same indices in their n-block windows as do $x_0, \ldots, x_m$. Hence they also have a good n-overlap.*

Partition the set G_n according to the relative translates of the n-block windows which contain the points, the relative positions of the n-block windows forming the proper n-array and the indices modulo j_k of the points in their n-blocks. This is a finite partition of G_n which we call H_n. Now take a set $h \in H_n$ and partition it according to the indices in which the points lie in their n-blocks. This is a finite partition of h. If we order these sets of indices according to the order in which $\hat{T}$ hits them, by moving in j_k-increments on the kth there will be a least set in this partition, referred to as h_0, and a set of powers of $\hat{T}$ we call $\mathcal{I}(h)$, so that

$$h = \bigcup_{i\in\mathcal{I}(h)} \hat{T}^i(h_0)$$

and such that this is a disjoint union. That is, h_0 forms the base of what is often called a *funny tower*, as the levels do not come in arithmetic progression. Our final computations will be made on these funny towers. To help set the scene, notice that $\mathcal{I}(h)$ looks like a block of consecutive integers, enough to get one across the first overlap of n-block windows. Then we jump to the second term in the proper array to begin a similar walk across this overlap of n-block windows etc. until the n-block overlap is exhausted. Notice that $\mathcal{I}(h)$ breaks up into a sequence of translates of a common n-block window overlap. Once again, this partition of the elements h is $\mathcal{H}_n$ measurable as it only depends on the geometry of the n-block overlaps of the points.

Given a proper n-array $\{t_i^k\}_{i=i_0}^{r_0-1}$, we now construct an involution of the values $i_0 \le i < r_0$. We want this involution to depend only on the geometry of the overlap.

Since the sequences $\mathbf{A}^{\alpha_n}$, $\mathbf{A}^{\beta_n}$ have Property 1, we know that for each particular choice of values $\{a^0, \ldots, a^m\}$, this vector appears as the deterministic part of the frame for almost precisely the same number of values i. For an initial array, let us choose an involution $I = I_{\{t_i^k\}}$ acting on $i_0 \le i < r_0$ so that

$$\frac{\#\{i \,|\, a^{\alpha_n}_{t_i^k} \neq a^{\alpha_n}_{t^k_{\pi(i)}} \text{ and } a^{\beta_n}_{t_i^k} = a^{\beta_n}_{t^k_{\pi(i)}}\}}{r_0 - i_0} > 1 - \frac{1}{10 \cdot 2^n},$$
where $\pi : [i_0, r_0] \to [i_0, r_0]$ is a bijection.

To do this, we first construct an involution ι of the names $a^0, \ldots, a^m$ with the property that it preserves the values of $a^1, \ldots, a^m$ but changes that of a^0. Now we attempt to define I so that it takes indices where the string $a^0, \ldots, a^m$ occurs, to indices where $\iota(a^0, \ldots, a^m)$ occurs. Property 1 tells us that we can carry this out for all but a fraction $1/(10 \cdot 2^n)$ of the indices i. We use the identity on the remaining indices. We can consider I to be determined by a list of points $x_0, \ldots, x_m$ with good n-overlap rather than the proper n-array they determine.

Notice that the choice of I is $\mathcal{H}_n$ measurable as this subalgebra determines the geometry of the n-overlap and hence the choice of I. We now use I to define an involution f on each of the sets $h \in H_n$. Begin by defining it only on the sub-σ-algebra $\mathcal{Q}_n^\alpha \vee \bigvee_{k=1}^n \mathcal{P}^\beta$. To do this, partition h into fibers η according to this partition, i.e. specify the $P^\beta(j_k)$ names across the windows. We first define f on h_0. For each n-block window overlap in the index set $\mathcal{I}(h)$, take that window overlap and the names across its overlapping n-block windows, and permute them according to the involution I associated with the n-array of the point. Fix the names across all the other n-block windows. Since we are working relative to $\overline{\mu}_n$, such a permutation of the $P^\beta(j_k)$ names is measure preserving. This defines f on h_0 but not on the full σ-algebra. To extend to the full algebra, note that any measure-preserving map defined on a subalgebra with non-atomic fibers can always be extended as a measure-preserving map on the full algebra. Our sub-σ-algebra has non-atomic fibers, so we can fill in the action on the $P^\alpha(j_0)$-names in some measure-preserving way to define f. Now, to extend f to all of h. For $i \in \mathcal{I}(h)$, set f on $\hat{T}^i(h_0)$ to be $\hat{T}^i f \hat{T}^{-i}$, extending the definition of f to all of h (see Corollary 6.29). Notice that on the funny tower over h_0 this amounts to painting the $P^\alpha(j_0) \vee \bigvee_{k=1}^m P^\beta(j_k)$ names constructed on the base h_0 onto the tower. To go along with this involution f, we define a second involution g which acts on the funny tower by permuting the levels by powers of $\hat{T}$. We move those levels at indices from a single n-block window overlap, by the translate taking them to the levels in I of this

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overlap. As the maps f and g are defined on disjoint sets of h, we can combine them into a single map, taking it to be the identity not yet defined.

We now prepare the final steps towards showing that T_α and T_β are u.s.d.

COROLLARY 6.30. *For all $h \in H_n$ and $x_0, \ldots, x_m \in h$, the two points $f(x_0, \ldots, x_m)$ and $g(x_0, \ldots, x_m)$ have identical $\bigvee_{k=1}^m P^\beta(j_k)$ names across the overlaps of their n-block windows. On the other hand, setting $P^{\alpha,i} = T_\alpha^i(P^\alpha)$ for $i = 0, \ldots, j_0 - 1$, we have*

$$\frac{1}{j_0}\sum_{i=0}^{j_0-1} \overline{\mu}_n(f(P^{\alpha,i}) \triangle g(P^{\alpha,i})) > \overline{\mu}_n(h)\left(\frac{d}{2} - \frac{1}{2 \cdot 2^n}\right).$$

Proof. The first part follows from the definitions of f and g. The second is a consequence of Corollary 6.20 as follows. For each $(x_0, x_1, \ldots, x_m) \in h_0$, if we calculate the $\overline{d}$ distance between the partitions $f(P^{\alpha,i})$ and $g(P^{\alpha,i})$ over the indices in $\mathcal{I}(h)$ then average over the values i, integrate over h_0 and multiply by $\#\mathcal{I}(h)$, we obtain the given integral. Observe that if $(x_0, \ldots, x_n) \in h_0$, then the $P^\alpha(j_0)$-names of the points $f(x_0)$ and $g(x_0)$ have good overlap in the index set $I(h)$ in the sense of Definition 6.16. Furthermore, $\#I(h) \geq h(n-1)/(10n2^n)$. Hence, in the calculation of the first $\overline{d}$ distance, we can apply Corollary 6.20 to conclude that for those n-window overlaps actually moved by the involution, there is a $\bar{d}$ error of $d/2$ and these are all but a fraction $12 \cdot 2^n$ of the indices. □

LEMMA 6.31. *Given a set of basic material, suppose that $P^0, P^1, \ldots, P^{j_0-1}$ are finitely coded from $\bigvee_{k=1}^m P^\beta(j_k)$. Then for $\delta > 0$, if n is large enough we have*

$$\frac{1}{j_0}\overline{\mu}_n(f(P^j) \triangle g(P^j)) < \delta.$$

Proof. Consider the measure of $f(P^j) \triangle g(P^j)$ on each set h separately. The names across the n-block window overlaps of $f(x_0, \ldots, x_m)$ and $g(x_0, \ldots, x_m)$ have identical $\bigvee_{k=1}^m P^\beta(j_k)$ names across them and their length grows uniformly in n. Hence the finite codes agree over an increasing fraction of the funny tower as n grows. □

THEOREM 6.32. *For T_α as constructed, if $\alpha_n \neq \beta_n$ infinitely often, then T_α and T_β are u.s.d.*

Proof. Given a basic set of material and using Proposition 6.26 we assume $P^0, \ldots, P^{j_0-1}$ are finitely coded. We can calculate that

$$\begin{aligned}
\frac{1}{j_0}\sum_{i=0}^{j_0-1} \overline{\mu}_n(\hat{T}^i(P^\alpha) \triangle P^i) &= \frac{1}{2}\frac{1}{j_0}\sum_{i=0}^{j_0-1} \overline{\mu}[f(P^{\alpha,i}) \triangle f(P^i)] + \overline{\mu}[g(P^{\alpha,i}) \triangle g(P^i)] \\
&\geq \frac{1}{2}\frac{1}{j_0}\sum_{i=0}^{j_0-1} \overline{\mu}[f(P^{\alpha,i}) \triangle f(P^i) \cup g(P^{\alpha,i}) \triangle g(P^i)] \\
&\geq \frac{1}{2}\frac{1}{j_0}\sum_{i=0}^{j_0-1} \overline{\mu}[f(P^{\alpha,i}) \triangle g(P^{\alpha,i})] - \overline{\mu}[f(P^i) \triangle g(P^i)].
\end{aligned}$$

Choosing n large enough, the previous two lemmas now give the conclusion with $a = d/5$. □

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COROLLARY 6.33. *All elements in the collection T_α satisfy the weak Pinsker property, have the same entropy, and can be constructed with any entropy $0 < h \leq \infty$ so that they still possess the u.s.d property between any pair T_α and T_β for which α and β differ infinitely often.*

Proof. That all the T_α as constructed have the same entropy follows from the fact that the number of n-block names is independent of α. To see how to obtain all entropies, first note that to verify u.s.d. one needs only to check it on one partition. We can increase the entropy, even to infinity, by taking the direct product with a Bernoulli action. The arguments in the first section show why this still yields u.s.d. for pairs. Hence we only need to see how to obtain small entropy. Notice we did not formally set the value of $N(0)$, the length of a 0-block. The larger it is chosen, the smaller the entropy of the T_α will be. □

7. *An uncountable family of non-Bernoulli cpe actions*

Let G be a countable discrete amenable group containing an element of infinite order (i.e. containing $\mathbb{Z}$ as a subgroup), and let $0 < h \leq \infty$. In Theorems 7.2 and 7.5 below, we construct an uncountable family of cpe actions of G all having entropy h.

Definition 7.1. For $i = 1, 2$, let U^i be an action of a countable discrete group G on a Lebesgue space (Y_i, ν_i). The actions U_h^1 and U_h^2, $h \in G$, of the group G are *isomorphic* if there is an isomorphism $S : (Y_1, \nu_1) \to (Y_2, \nu_2)$ such that $U_h^2 S y = S U_h^1 y$, where $h \in G$, $y \in Y_1$.

Now assume that G is countable amenable, that γ is an element of infinite order in G, and that Γ is the subgroup of G generated by γ. Let T_α be a K-automorphism of the Lebesgue space (X_α, μ_α), where $\alpha = (\alpha_i) \in \{0, 1\}^{\mathbb{N}}$ and $h(T_\alpha) = h$, as described in §6 above. Then we may consider (X_α, μ_α) as a Γ-space and define the co-induced action U_α of G on the space (Y_α, ν_α) (cf. Definition 3.1). As we saw in §3, the action U_α has the same entropy as that of T_α, that is $h(U_\alpha) = h$, for all α. Moreover, the U_α are non-Bernoulli and have the cpe property, by Theorem 5.2. Thus we must show that α's which are not asymptotically equal give non-isomorphic U_α's.

THEOREM 7.2. *Let G be a countable discrete amenable group and Γ a subgroup of G generated by an element γ of infinite order. Let $U_1 = U_{\alpha_1}$ and $U_2 = U_{\alpha_2}$ be cpe actions of G co-induced from actions T_{α_1} and T_{α_2} respectively. If α_1 and α_2 are not asymptotically equal, then U_{α_1} and U_{α_2} are not isomorphic.*

Proof. It suffices to show that the transformations $U_1(\gamma)$ and $U_2(\gamma)$ are not isomorphic. Recall that by equation (3.1), $U_i(\gamma)$ has the form

$$(U_i(\gamma)y)_\theta = T_{\alpha_i}^{n(\theta,\gamma)} y_{\theta\gamma}, \tag{7.1}$$

where $y = (y_\theta) \in Y$ and $s(\theta)\gamma s(\theta\gamma)^{-1} = \gamma^{n(\theta,\gamma)}$, for some $n(\theta, \gamma) \in \mathbb{Z}$.

This equation is related to Definition 6.5 of permuted powers. To see the relationship, set $\mathcal{K} = \Gamma\backslash G$, and note that $\pi : \theta \mapsto \theta\gamma$ is a bijection of $\mathcal{K}$. We further let $j_k = n(\theta, \gamma)$, where θ corresponds to k. With this notation, equation (7.1) coincides with equation (6.1), which defines the permuted powers $\hat{S}(T_{\alpha_i})$. Furthermore, a cycle of π on $\mathcal{K}$ is a Γ orbit in $\Gamma\backslash G$.

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We need to check the condition on sums over finite cycles $L \subseteq \mathcal{K}$ in Definition 6.5, namely $\sum_{k\in L} j_k \neq 0$. Suppose that $g \in G$ has the property that the cosets $\Gamma g, \Gamma g\gamma, \ldots, \Gamma g\gamma^{t-1}$ are distinct, and $\Gamma g\gamma^t = \Gamma g$, so we have a cycle of order $t \in \mathbb{N}$. It follows that there exists $p \in \mathbb{Z}$ such that $g\gamma^t = \gamma^p g$. Now p must be non-zero since γ has infinite order. Suppose that $\theta = \Gamma g$ with $s(\theta) = g$. Then by definition, we have

$$\gamma^{n(\theta,\gamma^t)} = g\gamma^t s(\Gamma g\gamma^t)^{-1} = g\gamma^t g^{-1}$$

and thus $n(\theta, \gamma^t) = p$. Since n is a cocycle, it follows that

$$\sum_{i=0}^{t-1} n(\theta\gamma^i, \gamma) = n(\theta, \gamma^t) = p \neq 0.$$

As T_{α_1} is uniformly somewhat disjoint from T_{α_2} (see Definition 6.3) it follows from Proposition 6.6 that T_{α_1} cannot arise as a factor of $U_2(\gamma)$. This immediately implies that the transformations $U_1(\gamma)$ and $U_2(\gamma)$ cannot be isomorphic, and hence that U_1 and U_2 are non-isomorphic cpe actions of G. □

We give some examples of the use of the above theorem.

Examples 7.3.

(1) Suppose that γ belongs to the centre of G; this holds for all γ if G is abelian. In this situation, (7.1) has the form

$$(U_i(\gamma)y)_\theta = T_{\alpha_i} y_\theta.$$

(2) Let G be the semidirect product of a direct sum of countably many copies $\mathbb{Z}_2$ with $\mathbb{Z}$, the action on the direct sum being permutation of the summands. In this example, the Γ action on $\Gamma\backslash G$ has a single one-point orbit (namely the coset $[e]$), and all the other orbits are infinite.

(3) Consider the matrix group

$$G = \left\{ \begin{pmatrix} q & r \\ 0 & 1 \end{pmatrix} : q \in \mathbb{Q}_+^*,\ r \in \mathbb{Q} \right\},$$

and let $\gamma = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ for $q \in \mathbb{Q}_+$. Then the Γ-orbits on $\Gamma\backslash G$ are either single points or are infinite.

(4) Let G be as above, but take $\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The Γ-orbits on $\Gamma\backslash G$ are all finite. This is clear if $g = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}$, $q \in \mathbb{Q}$, then for $\Gamma g\gamma^i = \Gamma g$ for all $i \in \mathbb{Z}$. On the other hand, let $g = \begin{pmatrix} 1/t & 0 \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{N}$, then

$$g\gamma^i g^{-1} = \gamma^{i/t}$$

and in particular $g\gamma^t = \gamma g$. It follows that $\{\Gamma g\gamma^i\}$, $0 \leq i \leq t-1$, is a cycle, and $n(g, \gamma) = n(g\gamma, \gamma) = \cdots = n(g\gamma^{t-2}, \gamma) = 0$. But $n([g\gamma^{t-1}], \gamma) = 1$, where we write $\Gamma g\gamma^i = [g\gamma^i]$, and hence $n([g], \gamma^t) = \sum_{i=0}^{t-1} n([g\gamma^i], \gamma) = 1$. In general, if $g_a = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where $a = s/t$, $s, t \in \mathbb{N}$, $(s, t) = 1$, then $\{\Gamma g_a \gamma^i\}$ $0 \le i \le t-1$ is a cycle, and $n([g_a], \gamma^t) = s$.

(5) Let γ_p be an automorphism of $\mathbb{Z}_p$, for each prime number p. It is known that $\gamma_p^{p-1} =$ id. Let H be the direct sum over all primes p of $\mathbb{Z}_p$ and γ be an automorphism of H such that the restriction of γ onto the summand $\mathbb{Z}_p$ is just γ_p. Define G as the semidirect product of H by Γ, where $\Gamma = \{\gamma^n, n \in \mathbb{Z}\}$. It is not hard to check that all the Γ-orbits on $\Gamma\backslash G$ are finite.

(6) Let G be the Grigorchuk finitely-generated non-elementary countable amenable group [**14**]. Let G_f be the subgroup of G containing all the elements of finite order. Then G_f is normal in G and $G/G_f \simeq \mathbb{Z}$. Let γ be an element of infinite order in G, and consider the action of ad(γ) on G_f. This action has a finite number of infinite orbits, together with the one-point orbit corresponding to the identity.

We now pass to the main result of this section. First, we need a definition.

Definition 7.4. Two actions U_1 and U_2 of a discrete countable group G on Lebesgue spaces (X_1, μ_1) and (X_2, μ_2), respectively, are called *automorphically isomorphic* if there is an isomorphism $V : (X_1, \mu_1) \to (X_2, \mu_2)$ and an (outer) automorphism β of G such that

$$U_2(\beta(g))Vx = VU_1(g)x, \quad x \in X_1, \ g \in G. \tag{7.2}$$

THEOREM 7.5. *Let G be a countable discrete abelian group containing an element of infinite order γ, which generates Γ. Let U_1 and U_2 be as in the statement of Theorem 7.2. Then the cpe action U_1 and U_2 are automorphically non-isomorphic if α_1 and α_2 are not asymptotically equal.*

Proof. Suppose that U_1 and U_2 are automorphically isomorphic. It follows from equation (7.2) that $U_2(\beta(\gamma))$ is not Bernoulli. By Definition 3.1, $\beta(\gamma)$ must have the form

$$\beta(\gamma) = hgh^{-1}$$

where $h \in G$ and $g^{m_1} = \gamma^{m_2}$ for some $m_1 \in \mathbb{N}$ and $m_2 \in \mathbb{Z} \setminus 0$. It follows, again from equation (7.2), that

$$V_1 U_2(\gamma^{m_2}) V_1^{-1} = U_1(\gamma^{m_1}),$$

where $V_1 = V^{-1}U_2(h)$. But this is impossible, because T_{α_1} is u.s.d. from T_{α_2}. Hence, U_1 and U_2 are not automorphically isomorphic. □

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