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Multiple positive solutions for an unbounded Dirichlet boundary problem involving sign-changing weight

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We study the multiplicity of positive solutions for the following semilinear elliptic equation:

$$\begin{aligned} -\Delta u + u &= f_\lambda(x)u^{q-1} + g_\mu(x)u^{p-1} && \text{in } \mathbb{A}, \\ u &\geqslant 0 && \text{in } \mathbb{A}, \\ u &\in H_0^1(\mathbb{A}), \end{aligned}$$

where $1 < q < 2 < p < 2^*$ ($2^* = 2N/(N-2)$ if $N \geqslant 3$, $2^* = \infty$ if $N = 2$), the parameters $\lambda, \mu \geqslant 0$, $\mathbb{A} = \Theta \times \mathbb{R}$ is an infinite strip in $\mathbb{R}^N$ and Θ is a bounded domain in $\mathbb{R}^{N-1}$. We assume that $f_\lambda(x) = \lambda f_+(x) + f_-(x)$ and $g_\mu(x) = a(x) + \mu b(x)$, where the functions $f_\pm$, a and b satisfy suitable conditions.

1. Introduction

We consider the multiplicity results of positive solutions of the following semilinear elliptic problem:

$$\left.\begin{aligned} -\Delta u + u &= f_\lambda(x)u^{q-1} + g_\mu(x)u^{p-1} && \text{in } \mathbb{A}, \\ u &\geqslant 0 && \text{in } \mathbb{A}, \\ u &\in H_0^1(\mathbb{A}), \end{aligned}\right\} \qquad (E_{f_\lambda, g_\mu})$$

where

$$1 < q < 2 < p < 2^* \quad \text{and} \quad 2^* = \begin{cases} \dfrac{2N}{N-2} & \text{if } N \geqslant 3, \\ \infty & \text{if } N = 2, \end{cases}$$

the parameters $\lambda, \mu \geqslant 0$, $\mathbb{A} = \Theta \times \mathbb{R}$ is an infinite strip in $\mathbb{R}^N$ and Θ is a bounded domain in $\mathbb{R}^{N-1}$. Let $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. We assume that $f_\lambda(x) = \lambda f_+(x) + f_-(x), g_\mu(x) = a(x) + \mu b(x)$, where the functions $f_\pm$, a and b satisfy the following conditions:

(D1) $f_\pm(x) = \pm\max\{\pm f, 0\}$, $f_+ \not\equiv 0$ and $f \in L^{q^*}(\mathbb{A})$, where $q^* = 2/(2-q)$;

(D2) $a(x) \in C(\bar{\mathbb{A}})$ and there exist $\delta > \theta_1$ and $0 < C_0 < 1$ such that

$$1 \geqslant a(x) \geqslant 1 - C_0 \exp(-2\sqrt{1+\delta}|x_N|) \text{ for all } x = (x', x_N) \in \mathbb{A},$$

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where θ_1 is the first eigenvalue of the Dirichlet problem $-\Delta\phi = \theta\phi$ in Θ, $\phi = 0$ on $\partial\Theta$;

(D3) $b \in C(\bar{\mathbb{A}})$ with $b(x) \geqslant 0$ for all $x \in \mathbb{A}$ and $b(x) \to 0$ as $|x_N| \to \infty$;

(D4) there exists $d_0 > 0$ such that $b(x) \geqslant d_0 \exp(-r|x_N|)$ for some $0 < r < q$.

Under the assumption that $f_\lambda(x) \not\equiv 0$, equation (E_{f_λ,g_μ}) can be regarded as a perturbation problem of the following semilinear elliptic equation:

$$\left.\begin{aligned} -\Delta u + u &= g_\mu(x)u^{p-1} && \text{in } \mathbb{A},\\ u &> 0 && \text{in } \mathbb{A},\\ u &\in H_0^1(\mathbb{A}). \end{aligned}\right\} \qquad (E_{0,g_\mu})$$

When $\mu \geqslant 0$ and $a(x) \equiv 1$, it is known that equation (E_{0,g_μ}) has a ground-state positive solution w_μ [26]. When $\mu < 0$, the function $b(x)$ satisfies the condition (D3) and $b(x) \geqslant 0$ on $\mathbb{A}$ with a strict inequality on a set of positive measure. Then, for equation (E_{0,g_μ}), we can see that the mountain-pass value is equal to the first level of breakdown of the Palais–Smale condition [15, p. 38] and we cannot get a positive solution through the mountain-pass theorem (i.e. equation (E_{0,g_μ}) does not admit any ground-state solution for all $\mu < 0$). See also the existence of ground-state solutions of equation (E_{0,g_μ}) under $\mathbb{A}$ replaced by $\mathbb{R}^N$ and various conditions (cf. [6–8, 13, 25, 27, 28], etc.).

For the above situation, several authors have made some progress on the multiplicity of positive solutions for the following non-homogeneous semilinear elliptic equation:

$$\left.\begin{aligned} -\Delta u + u &= \tilde{g}(x)u^{p-1} + h(x) && \text{in } \mathbb{R}^N,\\ u &> 0 && \text{in } \mathbb{R}^N,\\ u &\in H^1(\mathbb{R}^N), \end{aligned}\right\} \qquad (\tilde{E}_h)$$

where $h(x) \in H^{-1}(\mathbb{R}^N) \setminus \{0\}$ is non-negative and $\tilde{g} \in C(\mathbb{R}^N)$. When the homogeneous equation $(\tilde{E}_0)$ has a ground-state solution, Cao and Zhou [14], Hirano [23], Jeanjean [24] and Zhu [38] proved that equation $(\tilde{E}_h)$ has at least two positive solutions under the assumption that $\|h\|_{H^{-1}}$ is sufficiently small. When the homogeneous equation $(\tilde{E}_0)$ does not admit any ground-state solution, Adachi and Tanaka [1,2] proved that equation $(\tilde{E}_h)$ has at least four positive solutions under the assumptions $\tilde{g}(x) \leqslant 1 = \lim_{|x|\to\infty} \tilde{g}(x)$, $\tilde{g}(x) \geqslant 1 - C(-(2+\delta)|x|)$ for all $x \in \mathbb{R}^N$, for some $\delta > 0$, $C > 0$ and $\|h\|_{H^{-1}}$ is sufficiently small.

Similar problems have been the focus of a great deal of research in recent years. Chabrowski and Bezzera do Ó [16] and Goncalves and Miyagaki [22] have investigated the following equation:

$$\left.\begin{aligned} -\Delta u + V(x)u &= \lambda h(x)u^{q-1} + \bar{g}(x)u^{p-1} && \text{in } \mathbb{R}^N,\\ u &> 0 && \text{in } \mathbb{R}^N,\\ u &\in H^1(\mathbb{R}^N), \end{aligned}\right\} \qquad (\bar{E}_\lambda)$$

where $1 < q < 2 < p < 2^*$ and $\bar{g} \in C(\mathbb{R}^N)$. They found some existence and multiplicity results, which can be summarized as follows. In [22], the following conditions were assumed:

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(a_0) $V(x) \geqslant a_0 > 0$, $x \in \mathbb{R}^N$;

(a_∞) $V(x) \to \infty$ as $|x| \to \infty$;

(f_0) $h \geqslant 0$ and $h \in L^{2^*/(2^*-q)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Then it was proved that there exists $\lambda' > 0$ such that equation $(\bar{E}_\lambda)$ has at least two positive solutions for all $\lambda \in (0, \lambda')$.

In [16], the following conditions were assumed:

(a_1) $V(x)$ is positive, locally Hölder continuous and bounded in $\mathbb{R}^N$;

(f_c) h is a positive constant.

Then it was proved that there exists $\bar{\lambda} > 0$ such that equation $(\bar{E}_\lambda)$ has at least one positive solution for all $\lambda \in (0, \bar{\lambda})$.

Furthermore, Wu [35] has investigated the following equation:

$$\left.\begin{aligned} -\Delta u + u = h(x)u^{q-1} + \hat{g}(x)u^{p-1} \quad & \text{in } \mathbb{A}, \\ u > 0 \quad & \text{in } \mathbb{A}, \\ u \in H_0^1(\mathbb{A}), \quad & \end{aligned}\right\} \tag{$\bar{E}_{h,g}$}$$

where $1 \leqslant q < 2 < p < 2N/(N-2)$, $h(x) \in L^{q^*}(\mathbb{A}) \setminus \{0\}$ is non-negative, $\hat{g}(x) \leqslant 1 = \lim_{|x|\to\infty} \hat{g}(x)$ on $\mathbb{A}$ with a strict inequality on a set of positive measure and there exist $\delta > \theta_1$ and $0 < C_0 < 1$ such that

$$\hat{g}(x) - 1 \geqslant -C_0 \exp(-2\sqrt{1+\delta}|x_N|) \quad \text{for all } x = (x', x_N) \in \mathbb{A},$$

and θ_1 is the first eigenvalue of the Dirichlet problem $-\Delta\phi = \theta\phi$ in Θ, $\phi = 0$ on $\partial\Theta$. It was proved that equation $(\bar{E}_{h,g})$ has at least three positive solutions under the assumption that $\|h\|_{L^{q^*}}$ is sufficiently small.

From the above results, we know that the existence of a ground-state solution of the homogeneous equation affects the number of positive solutions of the perturbation problem. Actually, if the homogeneous equation has a ground-state solution, then the perturbation problem is presently only able to prove the existence of at least two positive solutions. The main purpose of this paper is to consider the possible existence of more than two positive solutions of (E_{f_λ,g_μ}), even if the homogeneous equation (E_{0,g_μ}) has a ground-state positive solution. Let

$$\Lambda_0 = (2-q)^{2-q}\left(\frac{p-2}{\|f_+\|_{L^{q^*}}}\right)^{p-2}\left(\frac{S_p}{p-q}\right)^{p-q},$$

where S_p is a best Sobolev constant for the embedding of $H_0^1(\mathbb{A})$ in $L^p(\mathbb{A})$. Then our main result is the following.

Theorem 1.1. *If, in addition to the conditions (D1)–(D4), we have*

(D5) $f_- \not\equiv 0$,

then

(i) *for each $\lambda > 0$ and $\mu > 0$ with $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$, equation (E_{f_λ,g_μ}) has at least two positive solutions,*

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(ii) *there exist positive numbers* λ_0, μ_0 *with* $\lambda_0^{p-2}(1+\mu_0\|b\|_\infty)^{2-q} < \frac{1}{2}q\Lambda_0$ *such that, for each* $\lambda \in (0, \lambda_0)$ *and* $\mu \in (0, \mu_0)$*, equation* (E_{f_λ, g_μ}) *has at least three positive solutions.*

Our analysis also makes use of the following result.

THEOREM 1.2. *If in addition to the conditions (D1)–(D4), we have*

(D6) $1 \leqslant q < 2 < p < 2^*$,

(D7) $f_- \equiv 0$ *and* $a(x) \leqslant 1$ *on* $\mathbb{A}$ *with a strict inequality on a set of positive measure,*

then

(i) *for each* $\lambda > 0$ *and* $\mu > 0$ *with* $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$*, equation* (E_{f_λ, g_μ}) *has at least two positive solutions,*

(ii) *there exist positive numbers* $\tilde{\lambda}_0$, $\tilde{\mu}_0$ *with* $\tilde{\lambda}_0^{p-2}(1+\tilde{\mu}_0\|b\|_\infty)^{2-q} < \frac{1}{2}q\Lambda_0$ *such that, for each* $\lambda \in (0, \tilde{\lambda}_0)$ *and* $\mu \in (0, \tilde{\mu}_0)$*, equation* (E_{f_λ, g_μ}) *has at least three positive solutions.*

Proof. The proofs of the multiplicity results are similar to those of theorem 1.1 (see § 6), so we leave the details to the reader. □

Among other interesting results, Ambrosetti *et al*. [4] investigated the following equation:

$$\left.\begin{aligned} -\Delta u &= \lambda u^{q-1} + u^{p-1} && \text{in } \Omega, \\ u &> 0 && \text{in } \Omega, \\ u &\in H_0^1(\Omega), \end{aligned}\right\} \tag{E_λ}$$

where $1 < q < 2 < p \leqslant 2^*$ ($2^* = 2N/(N-2)$ if $N \geqslant 3$; $2^* = \infty$ if $N = 1, 2$), $\lambda > 0$ and Ω is a bounded domain in $\mathbb{R}^N$. They found that there exists $\lambda_0 > 0$ such that equation (E_λ) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, a positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Actually, Adimurthy *et al*. [3], Damascelli *et al*. [18], Ouyang and Shi [29] and Tang [31] proved that there exists $\lambda_0 > 0$ such that there are exactly two positive solutions of equation (E_λ) in the unit ball $B^N(0; 1)$ for $\lambda \in (0, \lambda_0)$, exactly one positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$. Generalizations of the result of equation (E_λ) were given in [5, 10, 11, 19, 37].

In the following sections, we proceed to prove theorem 1.1. We use variational methods to find positive solutions of equation (E_{f_λ, g_μ}). Associated with equation (E_{f_λ, g_μ}), we consider the energy functional J_{f_λ, g_μ} in $H_0^1(\mathbb{A})$ for given $\lambda, \mu \geqslant 0$, $f(x)$, $a(x)$ and $b(x)$:

$$J_{f_\lambda, g_\mu}(u) = \tfrac{1}{2}\|u\|_{H^1}^2 - \frac{1}{q}\int_{\mathbb{A}} f_\lambda(x)|u|^q \, \mathrm{d}x - \frac{1}{p}\int_{\mathbb{A}} g_\mu(x)|u|^p \, \mathrm{d}x,$$

where

$$\|u\|_{H^1} = \left(\int_{\mathbb{A}} |\nabla u|^2 + u^2 \, \mathrm{d}x\right)^{1/2}$$

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is the standard norm in $H_0^1(\mathbb{A})$. It is well known that the solutions of equation (E_{f_λ,g_μ}) are the critical points of the energy functional J_{f_λ,g_μ} in $H_0^1(\mathbb{A})$ [30].

This paper is organized as follows. In § 2, we give some notation and preliminaries. In § 3, we establish the existence of a local minimum for J_{f_λ,g_μ}. In § 4, we give an estimate of energy. In § 5, we discussion some concentration behaviour in the Nehari manifold. In § 6, we prove theorem 1.1.

2. Notation and preliminaries

Throughout this section, we denote by S_p the best Sobolev constant for the embedding of $H_0^1(\mathbb{A})$ in $L^p(\mathbb{A})$. In particular,

$$\|u\|_{L^p} \leqslant S_p^{-1/2}\|u\|_{H^1} \quad \text{for all } u \in H_0^1(\mathbb{A}) \setminus \{0\}. \tag{2.1}$$

First, we define the Palais–Smale (PS) sequences, and (PS)-values and (PS)-conditions in $H_0^1(\mathbb{A})$ for J_{f_λ,g_μ} as follows.

Definition 2.1.

(i) For $\beta \in \mathbb{R}$, a sequence $\{u_n\}$ is a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\mathbb{A})$ for J_{f_λ,g_μ} if $J_{f_\lambda,g_\mu}(u_n) = \beta + o(1)$ and $J'_{f_\lambda,g_\mu}(u_n) = o(1)$ strongly in $H^{-1}(\mathbb{A})$ as $n \to \infty$.

(ii) $\beta \in \mathbb{R}$ is a (PS)-value in $H_0^1(\mathbb{A})$ for J_{f_λ,g_μ} if there exists a $(\mathrm{PS})_\beta$-sequence in $H_0^1(\mathbb{A})$ for J_{f_λ,g_μ}.

(iii) J_{f_λ,g_μ} satisfies the $(\mathrm{PS})_\beta$-condition in $H_0^1(\mathbb{A})$ if every $(\mathrm{PS})_\beta$-sequence in $H_0^1(\mathbb{A})$ for J_{f_λ,g_μ} contains a convergent subsequence.

As the energy functional J_{f_λ,g_μ} is not bounded below on $H_0^1(\mathbb{A})$, it is useful to consider the functional on the Nehari manifold

$$\boldsymbol{N}_{f_\lambda,g_\mu} = \{u \in H_0^1(\mathbb{A}) \setminus \{0\} \mid \langle J'_{f_\lambda,g_\mu}(u), u\rangle = 0\}.$$

Thus, $u \in \boldsymbol{N}_{f_\lambda,g_\mu}$ if and only if

$$\|u\|_{H^1}^2 - \int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x - \int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x = 0.$$

Note that $\boldsymbol{N}_{f_\lambda,g_\mu}$ contains every non-zero solution of equation (E_{f_λ,g_μ}). Furthermore, we have the following results.

Lemma 2.2. *The energy functional J_{f_λ,g_μ} is coercive and bounded below on $\boldsymbol{N}_{f_\lambda,g_\mu}$.*

Proof. If $u \in \boldsymbol{N}_{f_\lambda,g_\mu}$, then, by the Hölder and Sobolev inequalities,

$$\begin{aligned} J_{f_\lambda,g_\mu}(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{H^1}^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{A}} (\lambda f_+(x) + f_-(x))|u|^q \,\mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{H^1}^2 - \left(\frac{1}{q} - \frac{1}{p}\right)\int_{\mathbb{A}} \lambda f_+(x)|u|^q \,\mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{p}\right)\|u\|_{H^1}^2 - \lambda\left(\frac{p-q}{pq}\right)\|f_+\|_{L^{q^*}} S^{-q/2}\|u\|_{H^1}^q. \end{aligned} \tag{2.2}$$

Thus, J_{f_λ,g_μ} is coercive and bounded below on $\boldsymbol{N}_{f_\lambda,g_\mu}$. □

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The Nehari manifold $\boldsymbol{N}_{f_\lambda,g_\mu}$ is closely linked to the behaviour of the function of the form $h_u : t \to J_{f_\lambda,g_\mu}(tu)$ for $t > 0$. Such maps are known as fibering maps and were introduced by Dradek and Pohozaev [20] (they are also discussed in [10–12]). If $u \in H_0^1(\mathbb{A})$, we have

$$h_u(t) = \frac{t^2}{2}\|u\|_{H^1}^2 - \frac{t^q}{q}\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x - \frac{t^p}{p}\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x,$$

$$h_u'(t) = t\|u\|_{H^1}^2 - t^{q-1}\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x - t^{p-1}\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x,$$

$$h_u''(t) = \|u\|_{H^1}^2 - (q-1)t^{q-2}\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x - (p-1)t^{p-2}\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x.$$

It is easy to see that

$$th_u'(t) = \|tu\|_{H^1}^2 - \int_{\mathbb{A}} f_\lambda(x)|tu|^q \,\mathrm{d}x - \int_{\mathbb{A}} g_\mu(x)|tu|^p \,\mathrm{d}x$$

and so, for $u \in H_0^1(\mathbb{A}) \setminus \{0\}$ and $t > 0$, $h_u'(t) = 0$ if and only if $tu \in \boldsymbol{N}_{f_\lambda,g_\mu}$, i.e. positive critical points of h_u correspond to points on the Nehari manifold. In particular, $h_u'(1) = 0$ if and only if $u \in \boldsymbol{N}_{f_\lambda,g_\mu}$. Thus, it is natural to split $\boldsymbol{N}_{f_\lambda,g_\mu}$ into three parts corresponding to local minima, local maxima and points of inflection. Accordingly, we define

$$\begin{aligned}
\boldsymbol{N}_{f_\lambda,g_\mu}^{+} &= \{u \in \boldsymbol{N}_{f_\lambda,g_\mu} \mid h_u''(1) > 0\},\\
\boldsymbol{N}_{f_\lambda,g_\mu}^{0} &= \{u \in \boldsymbol{N}_{f_\lambda,g_\mu} \mid h_u''(1) = 0\},\\
\boldsymbol{N}_{f_\lambda,g_\mu}^{-} &= \{u \in \boldsymbol{N}_{f_\lambda,g_\mu} \mid h_u''(1) < 0\}.
\end{aligned}$$

We now derive some basic properties of $\boldsymbol{N}_{f_\lambda,g_\mu}^{+}$, $\boldsymbol{N}_{f_\lambda,g_\mu}^{0}$ and $\boldsymbol{N}_{f_\lambda,g_\mu}^{-}$.

Lemma 2.3. *Suppose that u_0 is a local minimizer for J_{f_λ,g_μ} on $\boldsymbol{N}_{f_\lambda,g_\mu}$ and that $u_0 \notin \boldsymbol{N}_{f_\lambda,g_\mu}^{0}$. Then $J_{f_\lambda,g_\mu}'(u_0) = 0$ in $H^{-1}(\mathbb{A})$.*

Proof. Our proof is almost the same as that in [12, theorem 2.3] (or see [9]). □

For each $u \in \boldsymbol{N}_{f_\lambda,g_\mu}$ we have

$$\begin{aligned}
h_u''(1) &= \|u\|_{H^1}^2 - (q-1)\int_{\mathbb{A}} f_\lambda|u|^q \,\mathrm{d}x - (p-1)\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x\\
&= (2-p)\|u\|_{H^1}^2 - (q-p)\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x \qquad (2.3\,a)\\
&= (2-q)\|u\|_{H^1}^2 - (p-q)\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x. \qquad (2.3\,b)
\end{aligned}$$

Then we have the following result.

Lemma 2.4.

(i) *For any $u \in \boldsymbol{N}_{f_\lambda,g_\mu}^{+} \cup \boldsymbol{N}_{f_\lambda,g_\mu}^{0}$, we have*

$$\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x > 0.$$

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(ii) *For any* $u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$, *we have*

$$\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x > 0.$$

Proof. The results now follow immediately from (2.3 *a*) and (2.3 *b*). □

Let

$$\Lambda_0 = (2-q)^{2-q}\left(\frac{p-2}{\|f_+\|_{L^{q^*}}}\right)^{p-2}\left(\frac{S_p}{p-q}\right)^{p-q}.$$

Then we have the following results.

LEMMA 2.5. *For each* $\lambda > 0$ *and* $\mu \geqslant 0$ *with* $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$, *we have* $\boldsymbol{N}^0_{f_\lambda,g_\mu} = \emptyset$.

Proof. Suppose the contrary. Then there exist $\lambda > 0$ and $\mu \geqslant 0$ with $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$ such that $\boldsymbol{N}^0_{f_\lambda,g_\mu} \neq \emptyset$. Then, for $u \in \boldsymbol{N}^0_{f_\lambda,g_\mu}$, by (2.3 *a*) and the Hölder and Sobolev inequalities we have

$$\|u\|^2_{H^1} = \frac{p-q}{p-2}\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x \leqslant \lambda S_p^{-q/2}\frac{p-q}{p-2}\|f_+\|_{L^{q^*}}\|u\|^q_{H^1}$$

and so

$$\|u\|^2_{H^1} \leqslant S_p^{q/(q-2)}\left[\lambda\|f_+\|_{L^{q^*}}\frac{p-q}{p-2}\right]^{2/(2-q)}.$$

Similarly, using (2.3 *b*) and the Sobolev inequality we have

$$\frac{2-q}{p-q}\|u\|^2_{H^1} = \int_{\mathbb{A}}[a(x)+\mu b(x)]|u|^p \,\mathrm{d}x \leqslant (1+\mu\|b\|_\infty)S_p^{-p/2}\|u\|^p_{H^1},$$

which implies

$$\|u\|^2_{H^1} \geqslant S_p^{p/(p-2)}\left[\frac{2-q}{(1+\mu\|b\|_\infty)(p-q)}\right]^{2/(p-2)} \quad \text{for all } \mu \geqslant 0.$$

Hence, we must have

$$\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} \geqslant (2-q)^{2-q}\left(\frac{p-2}{\|f_+\|_{L^{q^*}}}\right)^{p-2}\left(\frac{S_p}{p-q}\right)^{p-q} = \Lambda_0,$$

which is a contradiction. This completes the proof. □

In order to get a better understanding of the Nehari manifold and fibering maps, we consider the function $m_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$m_u(t) = t^{2-q}\|u\|^2_{H^1} - t^{p-q}\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x \quad \text{for } t > 0. \tag{2.4}$$

Clearly, $tu \in \boldsymbol{N}_{f_\lambda,g_\mu}$ if and only if

$$m_u(t) = \int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x.$$

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Moreover,

$$m_u'(t) = (2-q)t^{1-q}\|u\|_{H^1}^2 - (p-q)t^{p-q-1}\int_{\mathbb{A}} g_\mu(x)|u|^p\,\mathrm{d}x \tag{2.5}$$

and so it is easy to see that if $tu \in \boldsymbol{N}_{f_\lambda,g_\mu}$, then $t^{q-1}m_u'(t) = h_u''(t)$. Hence, $tu \in \boldsymbol{N}^+_{f_\lambda,g_\mu}$ (or $\boldsymbol{N}^-_{f_\lambda,g_\mu}$) if and only if $m_u'(t) > 0$ (or $m_u'(t) < 0$).

Suppose that $u \in H_0^1(\mathbb{A}) \setminus \{0\}$. Then, by (2.5), m_u has a unique critical point at $t = t_{\max,\mu}(u)$, where

$$t_{\max,\mu}(u) = \left(\frac{(2-q)\|u\|_{H^1}^2}{(p-q)\int_{\mathbb{A}} g_\mu(x)|u|^p\,\mathrm{d}x}\right)^{1/(p-2)} > 0 \tag{2.6}$$

and clearly m_u is strictly increasing on $(0, t_{\max,\mu}(u))$ and strictly decreasing on $(t_{\max,\mu}(u), \infty)$ with $\lim_{t\to\infty} m_u(t) = -\infty$. Moreover, if $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \varLambda_0$, then

$$\begin{aligned}
&m_u(t_{\max,\mu}(u)) \\
&\quad= \left[\left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)} - \left(\frac{2-q}{p-q}\right)^{(p-q)/(p-2)}\right]\frac{\|u\|_{H^1}^{2(p-q)/(p-2)}}{(\int_{\mathbb{A}} g_\mu(x)|u|^p\,\mathrm{d}x)^{(2-q)/(p-2)}} \\
&\quad= \|u\|_{H^1}^q\left(\frac{p-2}{p-q}\right)\left(\frac{2-q}{p-q}\right)^{(2-q)/(p-2)}\left(\frac{\|u\|_{H^1}^p}{\int_{\mathbb{A}} g_\mu(x)|u|^p\,\mathrm{d}x}\right)^{(2-q)/(p-2)} \\
&\quad\geqslant \left(\frac{2-q}{1+\mu\|b\|_\infty}\right)^{2-q}\left(\frac{p-2}{\lambda\|f_+\|_{L^{q^*}}}\right)^{p-2}\left(\frac{S_p}{p-q}\right)^{p-q}\int_{\mathbb{A}} f_\lambda(x)|u|^q\,\mathrm{d}x \\
&\quad> \int_{\mathbb{A}} f_\lambda(x)|u|^q\,\mathrm{d}x.
\end{aligned}$$

Thus, we have the following lemma.

Lemma 2.6. *For each $u \in H_0^1(\mathbb{A}) \setminus \{0\}$ we have the following.*

(i) *If*

$$\int_{\mathbb{A}} f_\lambda(x)|u|^q\,\mathrm{d}x \leqslant 0,$$

then there is a unique $t^- = t^-(u) > t_{\max,\mu}(u)$ such that $t^-u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ and h_u is increasing on $(0, t^-)$ and decreasing on (t^-, ∞). Moreover,

$$J_{f_\lambda,g_\mu}(t^-u) = \sup_{t\geqslant 0} J_{f_\lambda,g_\mu}(tu). \tag{2.7}$$

(ii) *If*

$$\int_{\mathbb{A}} f_\lambda(x)|u|^q\,\mathrm{d}x > 0,$$

then there is a unique $0 < t^+ = t^+(u) < t_{\max,\mu}(u) < t^-$ such that $t^+u \in \boldsymbol{N}^+_{f_\lambda,g_\mu}$, $t^-u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$, m_u is decreasing on $(0, t^+)$, increasing on (t^+, t^-) and decreasing on (t^-, ∞). Moreover,

$$J_{f_\lambda,g_\mu}(t^+u) = \inf_{0\leqslant t\leqslant t_{\max,\mu}(u)} J_{f_\lambda,g_\mu}(tu) \quad \text{and} \quad J_{f_\lambda,g_\mu}(t^-u) = \sup_{t\geqslant t^+} J_{f_\lambda,g_\mu}(tu). \tag{2.8}$$

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(iii) *$t^-(u)$ is a continuous function for $u \in H_0^1(\mathbb{A})$;*

(iv) $\boldsymbol{N}^-_{f_\lambda,g_\mu} = \left\{ u \in H_0^1(\mathbb{A}) \;\middle|\; \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) = 1 \right\}.$

Proof. The proofs are almost the same as in [36, lemma 2.5] and are left to the reader. □

REMARK 2.7.

(i) If $\lambda = 0$, then by lemma 2.6(i) $\boldsymbol{N}^+_{f_0,g_\mu} = \emptyset$, and so $\boldsymbol{N}_{f_0,g_\mu} = \boldsymbol{N}^-_{f_0,g_\mu}$ for all $\mu \geqslant 0$.

(ii) If $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$, then, by (2.3 *a*), for each $u \in \boldsymbol{N}^+_{f_\lambda,g_\mu}$ we have

$$\begin{aligned} \|u\|^2_{H^1} &< \frac{p-q}{p-2} \int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x \\ &\leqslant \lambda \frac{p-q}{p-2} \int_{\mathbb{A}} f^+(x)|u|^q \,\mathrm{d}x \\ &\leqslant \Lambda_0^{1/(p-2)} S_p^{-q/2} \frac{p-q}{p-2} \|f_+\|_{L^{q^*}} \|u\|^q_{H^1}, \end{aligned}$$

and so

$$\|u\|_{H^1} \leqslant \left(\Lambda_0^{1/(p-2)} S_p^{-q/2} \frac{p-q}{p-2} \|f_+\|_{L^{q^*}}\right)^{1/(2-q)} \quad \text{for all } u \in \boldsymbol{N}^+_{f_\lambda,g_\mu}. \tag{2.9}$$

3. Existence of the first solution

First, we remark that it follows from lemma 2.5 that

$$\boldsymbol{N}_{f_\lambda,g_\mu} = \boldsymbol{N}^+_{f_\lambda,g_\mu} \cup \boldsymbol{N}^-_{f_\lambda,g_\mu}$$

for all $\lambda > 0$ and $\mu \geqslant 0$ with $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$. Furthermore, by lemma 2.6 it follows that $\boldsymbol{N}^+_{f_\lambda,g_\mu}$ and $\boldsymbol{N}^-_{f_\lambda,g_\mu}$ are non-empty and, by lemma 2.2, we may define

$$\alpha^+_{f_\lambda,g_\mu} = \inf_{u \in \boldsymbol{N}^+_{f_\lambda,g_\mu}} J_{f_\lambda,g_\mu}(u) \quad \text{and} \quad \alpha^-_{f_\lambda,g_\mu} = \inf_{u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}} J_{f_\lambda,g_\mu}(u).$$

Then we have the following result.

THEOREM 3.1. *We have the following:*

(i) *$\alpha^+_{f_\lambda,g_\mu} < 0$ for all $\lambda > 0$ and $\mu \geqslant 0$ with $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$;*

(ii) *if $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \frac{1}{2}q\Lambda_0$, then $\alpha^-_{f_\lambda,g_\mu} > c_0$ for some $c_0 > 0$.*

In particular, for each $\lambda > 0$ and $\mu \geqslant 0$ with $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \frac{1}{2}q\Lambda_0$, we have $\alpha^+_{f_\lambda,g_\mu} = \inf_{u \in \boldsymbol{N}_{f_\lambda,g_\mu}} J_{f_\lambda,g_\mu}(u)$.

Proof. (i) Let $u \in \boldsymbol{N}^{+}_{f_\lambda,g_\mu}$, Then, by (2.3 a),

$$\|u\|^2_{H^1} < \frac{p-q}{p-2}\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x$$

and so

$$\begin{aligned} J_{f_\lambda,g_\mu}(u) &= \frac{p-2}{2p}\|u\|^2_{H^1} - \frac{p-q}{pq}\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x \\ &< -\frac{(p-q)(2-q)}{2pq}\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x < 0. \end{aligned}$$

Thus, $\alpha^{+}_{f_\lambda,g_\mu} < 0$.

(ii) Let $u \in \boldsymbol{N}^{-}_{f_\lambda,g_\mu}$. Since

$$\frac{2}{p}\|u\|^2_{H^1} < \int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x \quad \text{if } \int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x \leqslant 0$$

and

$$\frac{2-q}{p-q}\|u\|^2_{H^1} < \int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x \quad \text{if } \int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x > 0,$$

we have

$$\frac{2-q}{p-q}\|u\|^2_{H^1} < \int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x \quad \text{for all } u \in \boldsymbol{N}^{-}_{f_\lambda,g_\mu}.$$

Then, by (2.1),

$$\frac{2-q}{p-q}\|u\|^2_{H^1} < \int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x \leqslant (1+\mu\|b\|_\infty)S_p^{-p/2}\|u\|^p_{H^1},$$

which implies

$$\|u\|_{H^1} > S^{p/2(p-2)}\left(\frac{2-q}{(1+\mu\|b\|_\infty)(p-q)}\right)^{1/(p-2)} \quad \text{for all } u \in \boldsymbol{N}^{-}_{f_\lambda,g_\mu}.$$

By (2.2), we have

$$\begin{aligned} J_{f_\lambda,g_\mu}(u) &\geqslant \|u\|^q_{H^1}\left(\frac{p-2}{2p}\|u\|^{2-q}_{H^1} - \lambda\|f_+\|_{L^{q^*}}S^{-q/2}\left(\frac{p-q}{pq}\right)\right) \\ &> S^{pq/2(p-2)}\left(\frac{2-q}{(1+\mu\|b\|_\infty)(p-q)}\right)^{q/(p-2)} \\ &\quad \times \left(\frac{p-2}{2p}S^{p(2-q)/2(p-2)}\right. \\ &\quad\quad \left.\times\left(\frac{2-q}{(1+\mu\|b\|_\infty)(p-q)}\right)^{(2-q)/(p-2)} - \lambda\|f_+\|_{L^{q^*}}S^{-q/2}\left(\frac{p-q}{pq}\right)\right). \end{aligned}$$

Thus, if $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \frac{1}{2}q\Lambda_0$, then

$$\alpha^{-}_{f_\lambda,g_\mu} > c_0 \quad \text{for some } c_0 > 0.$$

This completes the proof. □

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Now, we consider the following elliptic problem:

$$-\Delta u + u = |u|^{p-2}u \text{ in } \Omega, \quad u \in H_0^1(\Omega), \tag{E_0}$$

where Ω is a domain in $\mathbb{R}^N$. Associated with equation (E_0), we consider the energy functional J_0 in $H_0^1(\Omega)$,

$$J_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + u^2 - \frac{1}{p}\int_\Omega |u|^p \, dx.$$

Consider a minimizing problem

$$\inf_{u \in \boldsymbol{N}_0(\Omega)} J_0(u) = \alpha_0(\Omega),$$

where

$$\boldsymbol{N}_0(\Omega) = \{u \in H_0^1(\Omega) \setminus \{0\} \mid \langle J_0'(u), u\rangle = 0\}$$

is the Nehari manifold. It is known that if $\Omega = \mathbb{A}$, equation (E_0) has a positive solution $w_0(x)$ such that $J_0(w_0) = \alpha_0(\mathbb{A})$. The following proposition then provides a precise description for the (PS)-sequence of J_{f_λ,g_μ}.

Proposition 3.2. *Each sequence $\{u_n\} \subset H_0^1(\mathbb{A})$ satisfying the following has a convergent subsequence:*

$J_{f_\lambda,g_\mu}(u_n) = \beta + o(1)$ *with* $\beta < \alpha^+_{f_\lambda,g_\mu} + \alpha_0(\mathbb{A})$;

$J_{f_\lambda,g_\mu}(u_n) = o(1)$ *in* $H^{-1}(\mathbb{A})$.

Proof. The proof is almost the same as [35, proposition 2.9]. □

Theorem 3.3. *For each $\lambda > 0$ and $\mu \geqslant 0$ with $\lambda^{p-2}(1 + \mu\|b\|_\infty)^{2-q} < \Lambda_0$, the functional J_{f_λ,g_μ} has a minimizer $u^+_{\lambda,\mu}$ in $\boldsymbol{N}^+_{f_\lambda,g_\mu}$ and it satisfies*

(i) $J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) = \alpha^+_{f_\lambda,g_\mu}$,

(ii) $u^+_{\lambda,\mu}$ *is a positive solution of equation* (E_{f_λ,g_μ}),

(iii) $\|u^+_{\lambda,\mu}\|_{H^1} \to 0$ *as* $\lambda \to 0$.

Proof. By the Ekeland variational principle [21] (or [36, proposition 1]), there is $\{u_n\} \subset \boldsymbol{N}^+_{f_\lambda,g_\mu}$ such that it is a $(\mathrm{PS})_{\alpha^+_{f_\lambda,g_\mu}}$-sequence for J_{f_λ,g_μ}. Then, by proposition 3.2, there is a subsequence $\{u_n\}$ and $u^+_{\lambda,\mu} \in \boldsymbol{N}^+_{f_\lambda,g_\mu}$ is a solution of equation (E_{f_λ,g_μ}) such that $u_n \to u^+_{\lambda,\mu}$ strongly in $H_0^1(\mathbb{A})$ and $J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) = \alpha^+_{f_\lambda,g_\mu}$. Since

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) = J_{f_\lambda,g_\mu}(|u^+_{\lambda,\mu}|) \quad \text{and} \quad |u^+_{\lambda,\mu}| \in \boldsymbol{N}^+_{f_\lambda,g_\mu},$$

by lemma 2.3 we may assume that $u^+_{\lambda,\mu}$ is a positive solution of equation (E_{f_λ,g_μ}). Finally, by (2.3 *a*),

$$\|u^+_{\lambda,\mu}\|^{2-q}_{H^1} < \lambda\frac{p-q}{p-2}\|f_+\|_{L^{q^*}} S_p^{-q/2} \tag{3.1}$$

and so $\|u^+_{\lambda,\mu}\|_{H^1} \to 0$ as $\lambda \to 0$. □

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4. The estimate of energy

First, we let $w_0(x)$ be a x_N-symmetric positive solution of equation (E_0) in $\Omega = \mathbb{A}$ such that $J_0(w_0) = \alpha_0(\mathbb{A})$. Then, by the result in [17], for any $0 < \varepsilon < 1 + \theta_1$, there exist $A_\varepsilon > 0$ and $B_\varepsilon > 0$ such that

$$\begin{aligned} A_\varepsilon \phi_1(x') \exp\{-\sqrt{1+\theta_1+\varepsilon}|x_N|\} &\leqslant w_0(x', x_N) \\ &\leqslant B_\varepsilon \phi_1(x') \exp\{-\sqrt{1+\theta_1-\varepsilon}|x_N|\} \end{aligned} \tag{4.1}$$

for all $(x', x_N) \in \mathbb{A}$, where θ_1 is the first eigenvalue and ϕ_1 is the corresponding first positive eigenfunction of the Dirichlet problem $-\Delta\phi = \theta\phi$ in Θ, $\phi = 0$ on $\partial\Theta$. Let

$$w_l(x) = w_0(x', x_N + l), \quad l \in \mathbb{R}. \tag{4.2}$$

Clearly,

$$\int_{\mathbb{A}} f_\lambda(x)|w_l|^q \, dx = 0 \quad \text{as } |l| \to \infty.$$

Then we have the following results.

Proposition 4.1. *For each $\lambda > 0$ and $\mu > 0$ with $\lambda^{p-2}(1 + \mu\|b\|_\infty)^{2-q} < \Lambda_0$ we have*

$$\alpha^-_{f_\lambda,g_\mu} < \alpha^+_{f_\lambda,g_\mu} + \alpha_0(\mathbb{A}).$$

Proof. Let $u^+_{\lambda,\mu}$ be a positive solution of equation (E_{f_λ,g_μ}) as in theorem 3.3. Then

$$\begin{aligned} &J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + tw_l) \\ &\quad = \tfrac{1}{2}\|u^+_{\lambda,\mu} + tw_l\|^2_{H^1} - \frac{1}{q}\int_{\mathbb{A}} f_\lambda |u^+_{\lambda,\mu} + tw_l|^q \, dx - \frac{1}{p}\int_{\mathbb{A}} g_\mu |u^+_{\lambda,\mu} + tw_l|^p \, dx \\ &\quad \leqslant J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) + J_0(tw_0) \\ &\qquad + \frac{1}{p}\int_{\mathbb{A}} (1-a)t^p w_l^p \, dx - \frac{\mu}{p}\int_{\mathbb{A}} bt^p w_l^p \, dx + \int_{\mathbb{A}} |f_\lambda| \biggl\{\int_0^{tw_l} \eta^{q-1} d\eta\biggr\} \, dx \\ &\qquad - \frac{a_{\min}}{p}\int_{\mathbb{A}} [(u^+_{\lambda,\mu} + tw_y)^p - (u^+_{\lambda,\mu})^p - t^p w_l^p - p(u^+_{\lambda,\mu})^{p-1} tw_l] \, dx \\ &\quad = J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) + J_0(tw_0) \\ &\qquad + \frac{1}{p}\int_{\mathbb{A}} (1-a)t^p w_l^p \, dx - \frac{\mu}{p}\int_{\mathbb{A}} gt^p w_l^p \, dx + \frac{t^q}{q}\int_{\mathbb{A}} |f_\lambda| w_l^q \, dx \\ &\qquad - \frac{a_{\min}}{p}\int_{\mathbb{A}} [(u^+_{\lambda,\mu} + tw_y)^p - (u^+_{\lambda,\mu})^p - t^p w_l^p - p(u^+_{\lambda,\mu})^{p-1} tw_l] \, dx, \end{aligned} \tag{4.3}$$

where $a_{\min} = \inf\{a(x) \mid x \in \mathbb{A}\} > 0$. By [12, 34], we know that

$$J_0(tw_0) \leqslant \alpha_0(\mathbb{A}) \quad \text{for all } l \in \mathbb{R}.$$

Thus, by (4.3) and $\lambda^{p-2}(1+\mu\|b\|_\infty)^{2-q} < \Lambda_0$ there exists $c_0 > 0$ such that

$$\begin{aligned} J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + tw_l) &\leqslant J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) + \alpha_0(\mathbb{A}) \\ &\quad + \frac{t^p}{p}\int_{\mathbb{A}} (1-a) w_l^p \,\mathrm{d}x - \frac{\mu t^p}{p}\int_{\mathbb{A}} b w_l^p \,\mathrm{d}x + \frac{c_0 t^q}{q}\int_{\mathbb{A}} |f| w_l^q \,\mathrm{d}x \\ &\quad - \frac{a_{\min}}{p}\int_{\mathbb{A}} [(u^+_{\lambda,\mu} + tw_y)^p - (u^+_{\lambda,\mu})^p - t^p w_l^p - p(u^+_{\lambda,\mu})^{p-1} t w_l] \,\mathrm{d}x. \end{aligned} \tag{4.4}$$

Since

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + tw_l) \to J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu}) = \alpha^+_{f_\lambda,g_\mu} < 0 \quad \text{as } t \to 0$$

and

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + tw_l) \to -\infty \quad \text{as } t \to \infty,$$

we can easily find $0 < t_1 < t_2$ such that

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + tw_l) < \alpha^+_{f_\lambda,g_\mu} + \alpha_0(\mathbb{A}) \quad \text{for all } t \in [0, t_1] \cup [t_2, \infty). \tag{4.5}$$

Thus, we only need to show that there exists $l_0 > 0$ such that, for $|l| > l_0$,

$$\sup_{t_1 \leqslant t \leqslant t_2} J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + tw_l) < \alpha^+_{f_\lambda,g_\mu} + \alpha_0(\mathbb{A}). \tag{4.6}$$

By (4.1), there exists $B_0 > 0$ such that

$$w_0(x', x_N) \leqslant B_0 \phi_1(x') \exp\{-|x_N|\} \quad \text{for all } (x', x_N) \in \mathbb{A}.$$

We also remark that

(i) $(u+v)^p - u^p - v^p - pu^{p-1}v \geqslant 0$ for all $(u, v) \in [0,\infty) \times [0,\infty)$,

(ii) for any $r > 0$ we can find a constant $C(r) > 0$ such that

$$(u+v)^p - u^p - v^p - pu^{p-1}v \geqslant C(r) v^2$$

for all $(u, v) \in [r,\infty) \times [0,\infty)$.

Thus, if $\mathbb{A}_{-1,1} = \{(x', x_N) \in \mathbb{A} \mid -1 < x_N < 1\}$ is a finite strip, setting

$$C_{\lambda,\mu} = C\Big(\min_{x\in\mathbb{A}_{-1,1}} u^+_{\lambda,\mu}(x)\Big) > 0 \quad \text{and} \quad C_0 = C\Big(\min_{x\in\mathbb{A}_{-1,1}} w_0^p(x)\Big) > 0,$$

we have

$$\begin{aligned} &\int_{\mathbb{A}} [(u^+_{\lambda,\mu} + tw_y)^p - (u^+_{\lambda,\mu})^p - t^p w_l^p - p(u^+_{\lambda,\mu})^{p-1} t w_l] \,\mathrm{d}x \\ &\qquad \geqslant \int_{\mathbb{A}_{-1,1}} [(u^+_{\lambda,\mu} + tw_y)^p - (u^+_{\lambda,\mu})^p - t^p w_l^p - p(u^+_{\lambda,\mu})^{p-1} t w_l] \,\mathrm{d}x \\ &\qquad \geqslant C_{\lambda,\mu} \int_{\mathbb{A}_{-1,1}} w_0^2(x', x_N + l) \,\mathrm{d}x \\ &\qquad > C_{\lambda,\mu} A_\varepsilon \exp(-2\sqrt{1+\theta_1+\varepsilon}|l|) \end{aligned} \tag{4.7}$$

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and, by condition (D4),

$$\begin{aligned}\int_{\mathbb{A}} bw_l^p \,dx &= \int_{\mathbb{A}} b(x', x_N - l) w_0^p(x) \,dx \\ &\geqslant C_0 \int_{\mathbb{A}_{-1,1}} b(x', x_N - l) \,dx \\ &\geqslant C_0 d_0 \exp(-r|l|). \end{aligned} \tag{4.8}$$

From conditions (D1) and (D2), we also have

$$\begin{aligned}\int_{\mathbb{A}} |f| w_l^q \,dx &\leqslant \bar{C} \int_{\mathbb{A}} B_0^q \phi_1^q(x') \exp(-q|x_N + l|) \,dx \\ &\leqslant \bar{C} \exp(-q|l|) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\int_{\mathbb{A}} (1-a) w_l^p \,dx &\leqslant \int_{\mathbb{A}} C_0 \exp(-2\sqrt{1+\delta}|l|) B_0^p \phi_1(x') \exp(-p|x_N + l|) \\ &\leqslant \tilde{C} \exp(-\min\{p, 2\sqrt{1+\delta}\}|l|). \end{aligned} \tag{4.10}$$

Since (4.7) holds for any $0 < \varepsilon < 1 + \theta_1$, choosing $\varepsilon < \delta - \theta_1$, we can find $l_1 > 0$ such that

$$\frac{t^p}{p} \int_{\mathbb{A}} (1-a) w_l^p \,dx < \frac{a_{\min}}{p} \int_{\mathbb{A}} [(u_{\lambda,\mu}^+ + tw_y)^p - (u_{\lambda,\mu}^+)^p - t^p w_l^p - p(u_{\lambda,\mu}^+)^{p-1} tw_l] \,dx \tag{4.11}$$

for all $|l| \geqslant l_1$. Moreover, since $r < q$ and $t_1 \leqslant t \leqslant t_2$, by (4.8) and (4.9), we can find $l_2 > 0$ such that

$$\frac{c_0 t^q}{q} \int_{\mathbb{A}} |f| w_l^q \,dx < \frac{\mu t^p}{p} \int_{\mathbb{A}} g w_l^p \,dx \quad \text{for all } |l| \geqslant l_2. \tag{4.12}$$

Thus, by (4.4), (4.5), (4.11) and (4.12), we obtain

$$\sup_{t \geqslant 0} J_{f_\lambda, g_\mu}(u_{f_\lambda, g_\mu}^+ + tw_l) < \alpha_{f_\lambda, g_\mu}^+ + \alpha_0(\mathbb{A}) \quad \text{for all } |l| \geqslant l_0 = \max\{l_1, l_2\}.$$

To complete the proof of proposition 4.1, it remains to show that there exists a positive number t_* such that $u_{f_\lambda, g_\mu}^+ + t_* w_l \in \boldsymbol{N}_{f_\lambda, g_\mu}^-$. Let

$$U_1 = \left\{ u \in H_0^1(\mathbb{A}) \;\middle|\; \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) > 1 \right\} \cup \{0\};$$
$$U_2 = \left\{ u \in H_0^1(\mathbb{A}) \;\middle|\; \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right) < 1 \right\}.$$

Then $\boldsymbol{N}_{f_\lambda, g_\mu}^-$ separates $H_0^1(\mathbb{A})$ into two connected components U_1 and U_2, and

$$H_0^1(\mathbb{A}) \setminus \boldsymbol{N}_{f_\lambda, g_\mu}^- = U_1 \cup U_2.$$

For each $u \in \boldsymbol{N}_{f_\lambda, g_\mu}^+$, we have

$$1 < t_{\max,\mu}(u) < t^-(u).$$

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Since

$$t^-(u) = \frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right),$$

then $\boldsymbol{N}^+_{f_\lambda,g_\mu} \subset U_1$. In particular, $u^+_{f_\lambda,g_\mu} \in U_1$. We claim that there exists $t_0 > 0$ such that $u^+_{f_\lambda,g_\mu} + t_0 w_l \in U_2$. First, we find a constant $c > 0$ such that

$$0 < t^-\left(\frac{u^+_{f_\lambda,g_\mu} + t w_l}{\|u^+_{f_\lambda,g_\mu} + t w_l\|_{H^1}}\right) < c \quad \text{for each } t \geqslant 0.$$

Otherwise, there exists a sequence $\{t_n\}$ such that $t_n \to \infty$ and

$$t^-\left(\frac{u^+_{f_\lambda,g_\mu} + t_n w_l}{\|u^+_{f_\lambda,g_\mu} + t_n w_l\|_{H^1}}\right) \to \infty \quad \text{as } n \to \infty.$$

Let

$$v_n = \frac{u^+_{f_\lambda,g_\mu} + t_n w_l}{\|u^+_{f_\lambda,g_\mu} + t_n w_l\|_{H^1}}.$$

Since $t^-(v_n)v_n \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ and, by the Lebesgue dominated convergence theorem,

$$\begin{aligned}\int_{\mathbb{A}} g_\mu(x) v_n^p \,\mathrm{d}x &= \frac{1}{\|u^+_{\lambda,\mu} + t_n w_l\|^p_{H^1}} \int_{\mathbb{A}} g_\mu(x)(u^+_{\lambda,\mu} + t_n w_l)^p \,\mathrm{d}x \\ &= \frac{1}{\|(u^+_{\lambda,\mu}/t_n) + w_l\|^p_{H^1}} \int_{\mathbb{A}} g_\mu(x)\left(\frac{u^+_{\lambda,\mu}}{t_n} + w_l\right)^p \mathrm{d}x \\ &\to \frac{\int_{\mathbb{A}} g_\mu(x)^p w_l^p \,\mathrm{d}x}{\|w_l\|^p_{H^1}} \quad \text{as } n \to \infty,\end{aligned}$$

we have

$$\begin{aligned}&J_{f_\lambda,g_\mu}(t^-(v_n)v_n) \\ &\quad = \tfrac{1}{2}[t^-(v_n)]^2 - \frac{[t^-(v_n)]^q}{q} \int_{\mathbb{A}} f_\lambda(x) v_n^q \,\mathrm{d}x - \frac{[t^-(v_n)]^p}{p} \int_{\mathbb{A}} g_\mu(x) v_n^p \,\mathrm{d}x \\ &\quad \to -\infty \quad \text{as } n \to \infty.\end{aligned}$$

This contradicts the statement that J_{f_λ,g_μ} is bounded below on $\boldsymbol{N}_{f_\lambda,g_\mu}$. Let

$$t_0 = \left(\frac{p-2}{2p\alpha_0(\mathbb{A})}|c^2 - \|u^+_{\lambda,\mu}\|^2_{H^1}|\right)^{1/2} + 1.$$

Then

$$\begin{aligned}\|u^+_{\lambda,\mu} + t_0 w_l\|^2_{H^1} &= \|u^+_{\lambda,\mu}\|^2_{H^1} + t_0^2 \|w_l\|^2_{H^1} + o(1) \\ &> \|u^+_{\lambda,\mu}\|^2_{H^1} + |c^2 - \|u^+_{\lambda,\mu}\|^2_{H^1}| + o(1) \\ &> c^2 + o(1) > \left[t^-\left(\frac{u^+_{\lambda,\mu} + t_0 w_l}{\|u^+_{\lambda,\mu} + t_0 w_l\|_{H^1}}\right)\right]^2 + o(1) \quad \text{as } l \to \infty.\end{aligned}$$

Thus, there exists $l_3 \geqslant l_0$ such that for, $|l| \geqslant l_3$,

$$\frac{1}{\|u^+_{\lambda,\mu} + t_0 w_l\|_{H^1}} t^-\left(\frac{u^+_{\lambda,\mu} + t_0 w_l}{\|u^+_{\lambda,\mu} + t_0 w_l\|_{H^1}}\right) < 1$$

or $u^+_{\lambda,\mu} + t_0 w_l \in A_2$. Define a path $\gamma_l(s) = u^+_{\lambda,\mu} + s t_0 w_l$ for $s \in [0,1]$. Then

$$\gamma_l(0) = u^+_{\lambda,\mu} \in A_1, \qquad \gamma_l(1) = u^+_{\lambda,\mu} + t_0 w_l \in A_2.$$

Since

$$\frac{1}{\|u\|_{H^1}} t^-\left(\frac{u}{\|u\|_{H^1}}\right)$$

is a continuous function for non-zero u and $\gamma_l([0,1])$ is connected, there exists $s_l \in (0,1)$ such that $u^+_{\lambda,\mu} + s_l t_0 w_l \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$. This completes the proof. □

THEOREM 4.2. *For each $\lambda > 0$ and $\mu > 0$ with $\lambda^{p-2}(1 + \mu\|b\|_\infty)^{2-q} < \Lambda_0$, equation (E_{f_λ,g_μ}) has a positive solution $u^-_{\lambda,\mu} \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ such that $J_{f_\lambda,g_\mu}(u^-_{\lambda,\mu}) = \alpha^-_{f_\lambda,g_\mu}$.*

Proof. Analogously to the proof of [37, proposition 9], one can show that for the Ekeland variational principle [21] there exist minimizing sequences $\{u_n\} \subset \boldsymbol{N}^-_{f_\lambda,g_\mu}$ such that

$$J_{f_\lambda,g_\mu}(u_n) = \alpha^-_{f_\lambda,g_\mu} + o(1) \quad \text{and} \quad J'_{f_\lambda,g_\mu}(u_n) = o(1) \text{ in } H^{-1}(\mathbb{A}).$$

Since $\alpha^-_{f_\lambda,g_\mu} < \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})$, by proposition 3.2 there exists a subsequence $\{u_n\}$ and $u^-_{\lambda,\mu} \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ is a non-zero solution of equation (E_{f_λ,g_μ}) such that

$$u_n \to u^-_{\lambda,\mu} \quad \text{strongly in } H^1_0(\mathbb{A}).$$

Since

$$J_{f_\lambda,g_\mu}(u^-_{\lambda,\mu}) = J_{f_\lambda,g_\mu}(|u^-_{\lambda,\mu}|) \quad \text{and} \quad |u^-_{\lambda,\mu}| \in \boldsymbol{N}^-_{f_\lambda,g_\mu},$$

by lemma 2.3, it may be assumed that $u^-_{\lambda,\mu}$ is a positive solution of (E_{f_λ,g_μ}). □

5. Concentration behaviour

It is known that the equation (E_{0,g_0}) does not admit any solution u_0 such that

$$J_{0,g_0}(u_0) = \inf_{u \in \boldsymbol{N}_{0,g_0}} J_{0,g_0}(u)$$

and

$$\inf_{u \in \boldsymbol{N}_{0,g_0}} J_{0,g_0}(u) = \inf_{u \in \boldsymbol{N}_0(\mathbb{A})} J_0(u) = \alpha_0(\mathbb{A})$$

(see [15, p. 38]). Furthermore, we have the following lemmas.

LEMMA 5.1. *We have*

$$\inf_{u \in \boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u) = \inf_{u \in \boldsymbol{N}_{0,g_0}} J_{0,g_0}(u) = \alpha_0(\mathbb{A}).$$

Furthermore, equation (E_{f_0,g_0}) does not admit any solution u_0 such that

$$J_{f_0,g_0}(u_0) = \inf_{u \in \boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u).$$

Proof. For w_l as in (4.2). By lemma 2.6, there is a unique $t^-(w_l) > 1$ such that $t^-(w_l)w_l \in \boldsymbol{N}_{f_0,g_0}$ for all $l > 0$, that is

$$\|t^-(w_l)w_l\|_{H^1}^2 = \int_{\mathbb{A}} f_-(x)|t^-(w_l)w_l|^q \,\mathrm{d}x + \int_{\mathbb{A}} a|t^-(w_l)w_l|^p \,\mathrm{d}x.$$

Since

$$\|w_l\|_{H^1}^2 = \int_{\mathbb{A}} |w_l|^p \,\mathrm{d}x = \frac{2p}{p-2}\alpha_0(\mathbb{A}) \quad \text{for all } l \in \mathbb{R},$$

$$\int_{\mathbb{A}} f_-(x)|w_l|^q \,\mathrm{d}x = o(1) \quad \text{and} \quad \int_{\mathbb{A}} a|w_l|^p \,\mathrm{d}x = \int_{\mathbb{A}} |w_l|^p \,\mathrm{d}x + o(1) \quad \text{as } |l| \to \infty,$$

we have $t^-(w_l) \to 1$ as $|l| \to \infty$. Thus,

$$\lim_{|l|\to\infty} J_{f_0,g_0}(t^-(w_l)w_l) = \lim_{|l|\to\infty} J_{0,g_0}(t^-(w_l)w_l) = \alpha_0(\mathbb{A}) \quad \text{as } |l| \to \infty.$$

Then

$$\inf_{u\in\boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u) \leqslant \inf_{u\in\boldsymbol{N}_{0,g_0}} J_{0,g_0}(u) = \alpha_0(\mathbb{A}).$$

Let $u \in \boldsymbol{N}_{f_0,g_0}$. Then, by lemma 2.6(i), $J_{f_0,g_0}(u) = \sup_{t\geqslant 0} J_{f_0,g_0}(tu)$. Moreover, there is a unique $s_u > 0$ such that $s_u u \in \boldsymbol{N}_{0,g_0}$. Thus,

$$J_{f_0,g_0}(u) \geqslant J_{f_0,g_0}(s_u u) \geqslant J_{0,g_0}(s_u u) \geqslant \alpha_0(\mathbb{A})$$

and so $\inf_{u\in\boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u) \geqslant \alpha_0(\mathbb{A})$. Therefore,

$$\inf_{u\in\boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u) = \inf_{u\in\boldsymbol{N}_{0,g_0}} J_{0,g_0}(u) = \alpha_0(\mathbb{A}).$$

Next, we will show that equation (E_{f_0,g_0}) does not admit any solution u_0 such that

$$J_{f_0,g_0}(u_0) = \inf_{u\in\boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u).$$

Suppose the contrary. Then we can assume that there exists $u_0 \in \boldsymbol{N}_{f_0,g_0}$ such that

$$J_{f_0,g_0}(u_0) = \inf_{u\in\boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u).$$

Then, by lemma 2.6(i), $J_{f_0,g_0}(u_0) = \sup_{t\geqslant 0} J_{f_0,g_0}(tu_0)$. Moreover, there is a unique $s_{u_0} > 0$ such that $s_{u_0}u_0 \in \boldsymbol{N}_{0,g_0}$. Thus,

$$\begin{aligned}\alpha_0(\mathbb{A}) &= \inf_{u\in\boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u) = J_{f_0,g_0}(u_0) \geqslant J_{f_0,g_0}(s_{u_0}u_0) \\ &\geqslant J_{0,g_0}(s_{u_0}u_0) + s_{u_0}^q \int_{\mathbb{A}} |f_-(x)||u_0|^q \,\mathrm{d}x \geqslant \alpha_0(\mathbb{A}) + s_{u_0}^q \int_{\mathbb{A}} |f_-(x)||u_0|^q \,\mathrm{d}x.\end{aligned}$$

This implies

$$\int_{\mathbb{A}} |f_-(x)||u_0|^q \,\mathrm{d}x = 0$$

and so $u_0 \in \boldsymbol{N}_{0,g_0}$ and $u_0 \equiv 0$ in $\{x \in \mathbb{A} \mid f_-(x) \neq 0\}$ form condition (D5). Therefore,

$$\alpha_0(\mathbb{A}) = \inf_{u\in\boldsymbol{N}_{0,g_0}} J_{0,g_0}(u) = J_{0,g_0}(u_0).$$

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By the Lagrange multiplier and the maximum principle, we can assume that u_0 is a positive solution of (E_{0,g_0}). This contradicts $u_0 \equiv 0$ in $\{x \in \mathbb{A} \mid f_-(x) \neq 0\}$ and completes the proof. □

LEMMA 5.2. *Suppose that $\{u_n\}$ is a minimizing sequence for J_{f_0,g_0} in $\boldsymbol{N}_{f_0,g_0}$. Then*

$$\int_{\mathbb{A}} f_-(x)|u_n|^q \, dx = o(1).$$

Furthermore, $\{u_n\}$ is a (PS)$_{\alpha_0(\mathbb{A})}$-sequence for J_{0,g_0} in $H_0^1(\mathbb{A})$.

Proof. For each n, there is a unique $t_n > 0$ such that $t_n u_n \in \boldsymbol{N}_{0,g_0}$, that is

$$t_n^2 \|u_n\|_{H^1}^2 = t_n^p \int_{\mathbb{A}} a(x)|u_n|^p \, dx.$$

Then, by lemma 2.6(i),

$$\begin{aligned} J_{f_0,g_0}(u_n) \geqslant J_{f_0,g_0}(t_n u_n) &= J_{0,g_0}(t_n u_n) + \frac{t_n^q}{q} \int_{\mathbb{A}} f_-(x)|u_n|^q \, dx \\ &\geqslant \alpha_0(\mathbb{A}) + \frac{t_n^q}{q} \int_{\mathbb{A}} f_-(x)|u_n|^q \, dx. \end{aligned}$$

Since $J_{f_0,g_0}(u_n) = \alpha_0(\mathbb{A}) + o(1)$ from lemma 5.1, we have

$$\frac{t_n^q}{q} \int_{\mathbb{A}} f_-(x)|u_n|^q \, dx = o(1).$$

We will show that there exists $c_0 > 0$ such that $t > c_0$ for all n. Suppose the contrary. Then we may assume $t_n \to 0$ as $n \to \infty$. Since $J_{f_0,g_0}(u_n) = \alpha_0(\mathbb{A}) + o(1)$, by lemma 2.2 we have that $\|u_n\|$ is uniformly bounded and so $\|t_n u_n\|_{H^1} \to 0$ or $J_{0,g_0}(t_n u_n) \to 0$. This contradicts the statement that $J_{0,g_0}(t_n u_n) \geqslant \alpha_0(\mathbb{A}) > 0$. Thus,

$$\int_{\mathbb{A}} f_-(x)|u_n|^q \, dx = o(1),$$

which implies that

$$\|u_n\|_{H^1}^2 = \int_{\mathbb{A}} a(x)|u_n|^p \, dx + o(1)$$

and

$$J_{0,g_0}(u_n) = \alpha_0(\mathbb{A}) + o(1).$$

Moreover, by [33, lemma 7], we have that $\{u_n\}$ is a (PS)$_{\alpha_0(\mathbb{A})}$-sequence for J_{0,g_0} in $H_0^1(\mathbb{A})$. □

The following lemma is a key lemma for proving our main results. Define the upper infinite strip $\mathbb{A}_r^+$ and the lower infinite strip $\mathbb{A}_r^-$ as follows:

$$\mathbb{A}_r^+ = \{(x', x_N) \in \mathbb{A} \mid x_N > r\} \quad \text{and} \quad \mathbb{A}_r^- = \{(x', x_N) \in \mathbb{A} \mid x_N < r\}.$$

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For a positive number δ, we consider the filtration of the manifold $\boldsymbol{N}_{f_0,g_0}$ as follows:

$$N_{f_0,g_0}(\delta,\mathbb{A}) = \{u \in \boldsymbol{N}_{f_0,g_0} \mid J_{f_0,g_0}(u) \leqslant \alpha_0(\mathbb{A}) + \delta\};$$

$$N^+_{f_0,g_0}(\delta,\mathbb{A}) = \left\{u \in N_{f_0,g_0}(\delta,\mathbb{A}) \;\middle|\; \int_{\mathbb{A}_0^+} |u|^p \,\mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A})\right\};$$

$$N^-_{f_0,g_0}(\delta,\mathbb{A}) = \left\{u \in N_{f_0,g_0}(\delta,\mathbb{A}) \;\middle|\; \int_{\mathbb{A}_0^-} |u|^p \,\mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A})\right\}.$$

Then we have the following results.

LEMMA 5.3. *There exists $\delta_0 > 0$ such that if $u \in N_{f_0,g_0}(\delta_0,\mathbb{A})$, then either*

$$\int_{\mathbb{A}_0^+} |u|^p \,\mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}) \quad \textit{or} \quad \int_{\mathbb{A}_0^-} |u|^p \,\mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}).$$

Proof. We divide the proof into two steps.

STEP 1 (existence). Suppose that there exists a sequence $\{u_n\} \subset \boldsymbol{N}_{f_0,g_0}$ such that $J_{f_0,g_0}(u_n) = \alpha_0(\mathbb{A}) + o(1)$,

$$\int_{\mathbb{A}_0^+} |u_n|^p \,\mathrm{d}x \geqslant \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}) \quad \text{and} \quad \int_{\mathbb{A}_0^-} |u_n|^p \,\mathrm{d}x \geqslant \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}). \tag{5.1}$$

By lemmas 5.1 and 5.2, equation (E_{f_0,g_0}) does not admit any solution u_0 such that

$$J_{f_0,g_0}(u_0) = \inf_{u \in \boldsymbol{N}_{f_0,g_0}} J_{f_0,g_0}(u)$$

and $\{u_n\}$ is a $(\mathrm{PS})_{\alpha_0(\mathbb{A})}$-sequence in $H^1_0(\mathbb{A})$ for J_{0,g_0}. Analogously to the proof in [35, lemma 2.8], there exists a subsequence $\{u_n\}$ such that $\{\xi_n u_n\}$ is a $(\mathrm{PS})_{\alpha_0(\mathbb{A})}$-sequence in $H^1_0(\mathbb{A})$ for J_0,

$$\|u_n - \xi_n u_n\|_{H^1} = o(1) \tag{5.2}$$

and

$$\int_{\mathbb{A}} a(x)|\xi_n u_n|^p \,\mathrm{d}x = \int_{\mathbb{A}} |\xi_n u_n|^p \,\mathrm{d}x + o(1) = \int_{\mathbb{A}} |u_n|^p \,\mathrm{d}x + o(1), \tag{5.3}$$

where $\xi_n(x) = \xi(2|x|/n)$ and $\xi \in C^\infty([0,\infty))$ such that $0 \leqslant \xi \leqslant 1$ and

$$\xi(t) = \begin{cases} 0 & \text{for } t \in [0,1], \\ 1 & \text{for } t \in [2,\infty). \end{cases}$$

Let $v_n = \xi_n u_n$. Then, by (5.2), (5.3) and [33, lemma 7], we obtain

$$J_0(v_n) = \alpha_0(\mathbb{A}) + o(1) \text{ and } J_0'(v_n) = o(1) \text{ in } H^{-1}(\mathbb{A}) \quad \text{as } n \to \infty \tag{5.4}$$

and $v_n = 0$ in $\bar{\mathbb{A}}_{-1,1}$ for $n > 2$, where $\mathbb{A}_{-1,1} = \{(x', x_N) \in \mathbb{A} \mid |x_N| < 1\}$. Moreover, $v_n = v_n^+ + v_n^-$ and

$$v_n^\pm(z) = \begin{cases} v_n(z) & \text{for } z \in \mathbb{A}_0^\pm, \\ 0 & \text{for } z \notin \mathbb{A}_0^\pm. \end{cases} \tag{5.5}$$

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Then, by (5.4) and (5.5), $v_n^{\pm} \in H_0^1(\mathbb{A}_0^{\pm})$ and $\{v_n^{\pm}\}$ are bounded sequences. This implies that

$$\langle J_0'(v_n), v_n^{\pm} \rangle = \|v_n^{\pm}\|_{H^1}^2 - \int_{\mathbb{A}_0^{\pm}} |v_n^{\pm}|^p \, \mathrm{d}x = o(1).$$

Again using (5.4), we obtain

$$J_0'(v_n^{\pm}) = o(1) \quad \text{strongly in } H^{-1}(\mathbb{A}_0^{\pm})$$

and

$$J_0(v_n) = J_0(v_n^+) + J_0(v_n^-) = \alpha_0(\mathbb{A}) + o(1).$$

Assume that $J_0(v_n^{\pm}) = c^{\pm} + o(1)$. Then

$$c^+ + c^- = \alpha_0(\mathbb{A}). \tag{5.6}$$

Since $c^{\pm}$ are (PS)-values in $H_0^1(\mathbb{A}_0^{\pm})$ for J_0, by [32, lemma 2.38], they are non-negative. Moreover, by [26, lemma 2.6], $\alpha_0(\mathbb{A}) = \alpha_0(\mathbb{A}_0^{\pm}) > 0$. Thus, by (5.6) and the definition of the Nehari minimization problem, we may assume that $c^+ = \alpha_0(\mathbb{A}_0^+) = \alpha_0(\mathbb{A})$ and $c^- = 0$. Next, for $n > 2$,

$$\begin{aligned}\int_{\mathbb{A}} |u_n|^p \, \mathrm{d}x = \int_{\mathbb{A}} |v_n|^p \, \mathrm{d}x + o(1) &= \int_{\mathbb{A}_0^+} |v_n^+|^p \, \mathrm{d}x + \int_{\mathbb{A}_0^-} |v_n^-|^p \, \mathrm{d}x + o(1) \\ &= \int_{\mathbb{A}_0^+} |v_n^+|^p \, \mathrm{d}x + \int_{\mathbb{A}_0^-} |u_n|^p \, \mathrm{d}x + o(1).\end{aligned}$$

Thus,

$$\int_{\mathbb{A}_0^-} |u_n|^p \, \mathrm{d}x = \int_{\mathbb{A}} |u_n|^p \, \mathrm{d}x - \int_{\mathbb{A}_0^+} \left|v_n^+\right|^p \, \mathrm{d}x = o(1),$$

which contradicts (5.1).

Step 2 (uniqueness). By lemma 5.2 and $1 < q < 2 < p$, we can find $\delta_0 > 0$ such that

$$\inf\left\{ \int_{\mathbb{A}} a(x)|u|^p \, \mathrm{d}x \;\middle|\; u \in N_{f_0,g_0}(\delta_0, \mathbb{A}) \right\} \geqslant \frac{qp}{p-q}\alpha_0(\mathbb{A}).$$

Claim. If $u \in N_{f_0,g_0}(\delta_0, \mathbb{A})$, then either

$$\int_{\mathbb{A}_0^+} |u|^p \, \mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}) \quad \text{or} \quad \int_{\mathbb{A}_0^-} |u|^p \, \mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}).$$

Suppose the contrary. Then there exists $u_0 \in N_{f_0,g_0}(\delta_0, \mathbb{A})$ such that

$$\int_{\mathbb{A}_0^+} |u_0|^p \, \mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}) \quad \text{and} \quad \int_{\mathbb{A}_0^-} |u_0|^p \, \mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}).$$

Then

$$\frac{qp}{p-q}\alpha_0(\mathbb{A}) \leqslant \int_{\mathbb{A}} a(x)|u_0|^p \, \mathrm{d}x \leqslant \int_{\mathbb{A}_0^+} |u_0|^p \, \mathrm{d}x + \int_{\mathbb{A}_0^-} |u_0|^p \, \mathrm{d}x < \frac{qp}{p-q}\alpha_0(\mathbb{A}).$$

This is a contradiction. We thus complete the proof of lemma 5.3. □

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By the consequence of lemma 5.3, it is easy to prove the following result.

LEMMA 5.4. *There exists $\delta_0 > 0$ such that*

(i) $N^{\pm}_{f_0,g_0}(\delta_0, \mathbb{A}) \neq \emptyset$,

(ii) $N^{+}_{f_0,g_0}(\delta_0, \mathbb{A}) \cap N^{-}_{f_0,g_0}(\delta_0, \mathbb{A}) = \emptyset$,

(iii) $N_{f_0,g_0}(\delta_0, \mathbb{A}) = N^{+}_{f_0,g_0}(\delta_0, \mathbb{A}) \cup N^{-}_{f_0,g_0}(\delta_0, \mathbb{A})$.

By (2.3 b), (2.6) and lemma 2.6(i), for each $u \in \boldsymbol{N}^{-}_{f_\lambda,g_\mu}$ there is a unique $t_0^-(u) > 0$ such that $t_0^-(u)u \in \boldsymbol{N}_{f_0,g_0}$ and

$$t_0^-(u) > t_{\max,0}(u) = \left(\frac{(2-q)\|u\|^2_{H^1}}{(p-q)\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x}\right)^{1/(p-2)} > 0.$$

Then we have the following results.

LEMMA 5.5. *There exist $\lambda_1, \tilde{\mu} > 0$ with $\lambda_1^{p-2}(1+\tilde{\mu}\|b\|_\infty)^{2-q} < \frac{1}{2}q\Lambda_0$ such that for every $\lambda \in (0,\lambda_1)$ and $\mu \in (0,\tilde{\mu})$ we have*

(i) $\alpha^+_{f_\lambda,g_\mu} \leqslant \alpha^+_{f_\lambda,g_0}$,

(ii) $\displaystyle\int_{\mathbb{A}} f_\lambda(x)|u|^q\,\mathrm{d}x > 0$ *for all* $u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ *with* $J_{f_\lambda,g_\mu}(u) \leqslant \alpha^+_{f_\lambda,g_\mu} + \alpha_0(\mathbb{A})$.

Proof. (i) By lemma 2.6 and theorem 3.3, for each $\lambda^{p-2} < \frac{1}{2}q\Lambda_0$ there exists $u^+_{\lambda,0} \in \boldsymbol{N}^+_{f_\lambda,g_0}$, a positive solution of (E_{f_λ,g_0}) such that $J_{f_\lambda,g_0}(u^+_{\lambda,0}) = \alpha^+_{f_\lambda,g_0}$,

$$t_{\max,0}(u^+_{\lambda,0}) = \left(\frac{(2-q)\|u^+_{\lambda,0}\|^2_{H^1}}{(p-q)\int_{\mathbb{A}}|u^+_{\lambda,0}|^p\,\mathrm{d}x}\right)^{1/(p-2)} > 1 \tag{5.7}$$

and

$$t_{\max,0}(u^+_{\lambda,0}) \to \infty \quad \text{as } \lambda \to 0. \tag{5.8}$$

Moreover, by (2.9) there exists a positive constant c_0 independent of λ such that

$$\mu\int_{\mathbb{A}} b(x)|u^+_{\lambda,0}|^p\,\mathrm{d}x \leqslant \mu c_0\|b\|_\infty. \tag{5.9}$$

Then, by (5.7)–(5.9), there exists $\lambda_1, \tilde{\mu} > 0$ with

$$\lambda_1^{p-2}(1+\tilde{\mu}\|b\|_\infty)^{2-q} < \tfrac{1}{2}q\Lambda_0$$

such that, for every $\lambda \in (0,\lambda_1)$ and $\mu \in (0,\tilde{\mu})$,

$$t_{\max,0}(u^+_{\lambda,0}) \geqslant t_{\max,\mu}(u^+_{\lambda,0}) = \left(\frac{(2-q)\|u^+_{\lambda,0}\|^2_{H^1}}{(p-q)\int_{\mathbb{A}} g_\mu(x)|u^+_{\lambda,0}|^p\,\mathrm{d}x}\right)^{1/(p-2)} > 1.$$

Thus, by lemma 2.6,

$$J_{f_\lambda,g_0}(u) \geqslant J_{f_\lambda,g_\mu}(u) \geqslant \inf_{0\leqslant t\leqslant t_{\max,\mu}(u)} J_{f_\lambda,g_\mu}(tu) \geqslant \alpha^+_{f_\lambda,g_\mu}$$

and so

$$\alpha^+_{f_\lambda,g_\mu} \leqslant \alpha^+_{f_\lambda,g_0}$$

for all $\lambda \in (0,\lambda_1)$ and $\mu \in (0,\tilde{\mu})$.

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(ii) First, we will show that there exists $d_0 > 0$ independent of μ such that $\|u\|_{H^1} \leqslant d_0$ for all $\lambda \in (0, \lambda_0), \mu \in (0, \tilde{\mu})$ and for all $u \in \boldsymbol{N}^-_{f_\lambda, g_\mu}$ with $J_{f_\lambda, g_\mu}(u) \leqslant \alpha^+_{f_\lambda, g_\mu} + \alpha_0(\mathbb{A})$. By (2.2),

$$\begin{aligned}\alpha_0(\mathbb{A}) > \alpha^+_{f_\lambda, g_\mu} + \alpha_0(\mathbb{A}) &\geqslant J_{f_\lambda, g_\mu}(u) \\ &= \frac{p-2}{2p}\|u\|^2_{H^1} - \frac{p-q}{pq}\int_{\mathbb{A}}[\lambda f^+(x) + f^-(x)]|u|^q \,\mathrm{d}x \\ &\geqslant \frac{p-2}{2p}\|u\|^2_{H^1} - \frac{p-q}{pq}\lambda\|f^+\|_{L^{q^*}} S^{-q/2}\|u\|^q_{H^1} \\ &\geqslant \frac{p-2}{2p}\|u\|^2_{H^1} - \frac{p-q}{pq}\lambda_0\|f^+\|_{L^{q^*}} S^{-q/2}\|u\|^q_{H^1}.\end{aligned}$$

Then there exists $d_0 > 0$ independent of λ, μ such that $\|u\|_{H^1} \leqslant d_0$ for all $\lambda \in (0, \lambda_1), \mu \in (0, \tilde{\mu})$ and for all $u \in \boldsymbol{N}^-_{f_\lambda, g_\mu}$ with $J_{f_\lambda, g_\mu}(u) \leqslant \alpha^+_{f_\lambda, g_\mu} + \alpha_0(\mathbb{A})$. Moreover, there is a unique $t^-(u) > 0$ such that $t^-(u)u \in \boldsymbol{N}_{0, g_0}$, where

$$t^-(u) = \left(\frac{\|u\|^2_{H^1}}{\int_{\mathbb{R}^N}|u|^p \,\mathrm{d}x}\right)^{1/(p-2)} < \left[\frac{(1 + \mu\|b\|_\infty)(p-q)}{(2-q)}\right]^{1/(p-2)}.$$

Then, by lemma 2.6 and theorem 3.1,

$$\begin{aligned}J_{f_\lambda, g_\mu}(u) &= \sup_{t \geqslant 0} J_{f_\lambda, g_\mu}(tu) \geqslant J_{f_\lambda, g_\mu}(t^-(u)u) \\ &\geqslant J_{0, g_0}(t^-(u)u) - \frac{[t^-(u)]^q}{q}\int_{\mathbb{R}^N} f_\lambda(x)|u|^q \,\mathrm{d}x - \frac{\mu[t^-(u)]^p}{p}\int_{\mathbb{R}^N} b(x)|u|^p \,\mathrm{d}x \\ &\geqslant \alpha_0(\mathbb{A}) - \frac{[t^-(u)]^q}{q}\int_{\mathbb{R}^N} f_\lambda(x)|u|^q \,\mathrm{d}x - \frac{\mu[t^-(u)]^p}{p}\int_{\mathbb{R}^N} b(x)|u|^p \,\mathrm{d}x.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{[t^-(u)]^q}{q}\int_{\mathbb{R}^N} f_\lambda(x)|u|^q \,\mathrm{d}x \\ &\qquad \geqslant -\alpha^+_{f_\lambda, g_\mu} - \frac{\mu[t^-(u)]^p}{p}\int_{\mathbb{R}^N} b(x)|u|^p \,\mathrm{d}x \\ &\qquad > -\alpha^+_{f_\lambda, g_\mu} - \frac{\mu}{p}\left[\frac{(1 + \mu\|b\|_\infty)(p-q)}{(2-q)}\right]^{p/(p-2)}\|b\|_\infty S^{-2/p} d_0^p.\end{aligned}$$

Since $\alpha^+_{f_\lambda, g_\mu} \leqslant \alpha^+_{f_\lambda, g_0}$ for all $\lambda \in (0, \lambda_1)$ and $\mu \in (0, \tilde{\mu})$, we have

$$\frac{[t^-(u)]^q}{q}\int_{\mathbb{R}^N} f_\lambda(x)|u|^q \,\mathrm{d}x > -\alpha^+_{f_\lambda, g_0} - \frac{\mu}{p}\left[\frac{(1 + \mu\|b\|_\infty)(p-q)}{(2-q)}\right]^{p/(p-2)}\|b\|_\infty S^{-2/p} d_0^p.$$

Thus, we can conclude that for every $\lambda \in (0, \lambda_1)$ and $\mu \in (0, \tilde{\mu})$ we have

$$\int_{\mathbb{R}^N} f_\lambda(x)|u|^q \,\mathrm{d}x > 0$$

for all $u \in \boldsymbol{N}^-_{f_\lambda, g_\mu}$ with $J_{f_\lambda, g_\mu}(u) \leqslant \alpha^+_{f_\lambda, g_\mu} + \alpha_0(\mathbb{A})$. □

Let $\lambda_1, \tilde{\mu} > 0$, as in lemma 5.5, and let

$$\theta_0 = \biggl[\frac{(1+\tilde{\mu}\|b/a\|_\infty)(p-q)}{2-q} \times \biggl(1 + \|f_-\|_{L^{q^*}}\biggl(\frac{(1+\tilde{\mu}\|b/a\|_\infty)(p-q)}{S_p^{(p-q)/(2-q)}(2-q)}\biggr)^{(2-q)/(p-2)}\biggr)\biggr]^{p/(p-2)}.$$

Then we have the following results.

Lemma 5.6. *There exists a positive number $\lambda_2 \leqslant \lambda_1$ such that, for every $\lambda \in (0, \lambda_2)$ and $\mu \in (0, \tilde{\mu})$, we have*

(i) $1 < [t_0^-(u)]^p < \theta_0$,

(ii) $\displaystyle\int_{\mathbb{A}} |u|^p \,\mathrm{d}x \geqslant \frac{qp}{\theta_0(p-q)}\alpha_0(\mathbb{A})$ *for all* $u \in \boldsymbol{N}^-_{f_\lambda, g_\mu}$ *with*

$$J_{f_\lambda, g_\mu}(u) < \alpha^+_{f_\lambda, g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A}).$$

Proof. (i) For $u \in \boldsymbol{N}^-_{f_\lambda, g_\mu}$ with $J_{f_\lambda, g_\mu}(u) < \alpha^+_{f_\lambda, g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})$, we have

$$\|u\|^2_{H^1} - \int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x - \int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x = 0$$

and

$$(2-q)\|u\|^2_{H^1} < (p-q)\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x.$$

By lemma 2.6(i) there is a unique $t_0^-(u) > 0$ such that $t_0^-(u)u \in \boldsymbol{N}_{f_0, g_0}$ and so

$$\begin{aligned}[t_0^-(u)]^2\|u\|^2_{H^1} &= [t_0^-(u)]^q \int_{\mathbb{A}} f_-(x)|u|^q \,\mathrm{d}x + [t_0^-(u)]^p \int_{\mathbb{A}} a|u|^p \,\mathrm{d}x \\ &\leqslant [t_0^-(u)]^p \int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x.\end{aligned}$$

This implies

$$[t_0^-(u)]^{p-2} > \frac{\|u\|^2_{H^1}}{\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x} = 1 + \frac{\int_{\mathbb{A}} f_\lambda(x)|u|^q \,\mathrm{d}x}{\int_{\mathbb{A}} g_\mu(x)|u|^p \,\mathrm{d}x}$$

and so, by lemma 5.5(ii), $t_0^-(u) > 1$. Moreover,

$$\begin{aligned}[t_0^-(u)]^p \int_{\mathbb{A}} a|u|^p \,\mathrm{d}x &= [t_0^-(u)]^2\|u\|^2_{H^1} - [t_0^-(u)]^q \int_{\mathbb{A}} f_-(x)|u|^q \,\mathrm{d}x \\ &< [t_0^-(u)]^2\biggl(\|u\|^2_{H^1} - \int_{\mathbb{A}} f_-(x)|u|^q \,\mathrm{d}x\biggr);\end{aligned}$$

thus, we have

$$[t_0^-(u)]^{p-2} \leqslant \frac{\|u\|^2_{H^1} + \int_{\mathbb{A}} f_-(x)|u|^q \,\mathrm{d}x}{\int_{\mathbb{A}} a|u|^p \,\mathrm{d}x}. \tag{5.10}$$

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By $u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ and (2.3 b),

$$\|u\|^2_{H^1} < \frac{p-q}{2-q}\int_{\mathbb{A}} g_\mu(x)|u|^p\,\mathrm{d}x \leqslant \frac{p-q}{2-q}(1+\mu\|b/a\|_\infty)\int_{\mathbb{A}} a(x)|u|^p\,\mathrm{d}x$$
$$\leqslant (1+\mu\|b/a\|_\infty)S_p^{-p/2}\frac{p-q}{2-q}\|u\|^p_{H^1} \tag{5.11}$$

and so

$$\|u\|_{H^1} \geqslant \left(\frac{2-q}{(1+\mu\|b/a\|_\infty)(p-q)}\right)^{1/(p-2)} S_p^{p/2(p-2)}. \tag{5.12}$$

Thus, by (5.10)–(5.12),

$$\begin{aligned}
[t_0^-(u)]^{p-2} &\leqslant (1+\mu\|b/a\|_\infty)\left(\frac{p-q}{2-q}\right)\left(1+\frac{\int_{\mathbb{A}} f_-(x)|u|^q\,\mathrm{d}x}{\|u\|^2_{H^1}}\right)\\
&\leqslant (1+\mu\|b/a\|_\infty)\left(\frac{p-q}{2-q}\right)\left(1+\frac{\|f_-\|_{L^{q^*}}}{S_p^{-q/2}\|u\|^{2-q}_{H^1}}\right)\\
&\leqslant (1+\mu\|b/a\|_\infty)\left(\frac{p-q}{2-q}\right)\\
&\quad\times\left(1+S_p^{(q-p)/(p-2)}\left(\frac{(1+\mu\|b/a\|_\infty)(p-q)}{2-q}\right)^{(2-q)/(p-2)}\|f_-\|_{L^{q^*}}\right)\\
&< (1+\mu_0\|b/a\|_\infty)\left(\frac{p-q}{2-q}\right)\\
&\quad\times\left(1+S_p^{(q-p)/(p-2)}\left(\frac{(1+\mu_0\|b/a\|_\infty)(p-q)}{2-q}\right)^{(2-q)/(p-2)}\|f_-\|_{L^{q^*}}\right)
\end{aligned}$$

or $[t_0^-(u)]^p \leqslant \theta_0$.

(ii) By lemma 5.1 and $t_0^-(u)u \in \boldsymbol{N}_{f_0,g_0}$,

$$\begin{aligned}
\alpha_0(\mathbb{A}) &\leqslant J_{f_0,g_0}(t_0^-(u)u)\\
&= \left(\frac{1}{2}-\frac{1}{q}\right)[t_0^-(u)]^2\|u\|^2_{H^1} + \left(\frac{1}{q}-\frac{1}{p}\right)[t_0^-(u)]^p\int_{\mathbb{A}}|u|^p\,\mathrm{d}x\\
&< \left(\frac{1}{q}-\frac{1}{p}\right)[t_0^-(u)]^p\int_{\mathbb{A}}|u|^p\,\mathrm{d}x.
\end{aligned}$$

This implies

$$\int_{\mathbb{A}}|u|^p\,\mathrm{d}x \geqslant \frac{1}{[t_0^-(u)]^p}\left(\frac{pq}{p-q}\right)\alpha_0(\mathbb{A}).$$

By part (i), we can conclude that

$$\int_{\mathbb{A}}|u|^p\,\mathrm{d}x \geqslant \frac{pq}{\theta_0(p-q)}\alpha_0(\mathbb{A})$$

for all $u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})$. This completes the proof. □

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Now, we consider the filtration of the submanifold $\boldsymbol{N}^-_{f_\lambda,g_\mu}$ as follows:

$$N_{\lambda,\mu}(\mathbb{A}) = \{u \in \boldsymbol{N}^-_{f_\lambda,g_\mu} \mid J_{f_\lambda,g_\mu}(u) \leqslant \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})\};$$

$$N^+_{\lambda,\mu}(\mathbb{A}) = \left\{u \in N_{\lambda,\mu}(\mathbb{A}) \;\middle|\; \int_{\mathbb{A}^+_0} |u|^p < \frac{pq}{2\theta_0(p-q)}\alpha_0(\mathbb{A})\right\};$$

$$N^-_{\lambda,\mu}(\mathbb{A}) = \left\{u \in N_{\lambda,\mu}(\mathbb{A}) \;\middle|\; \int_{\mathbb{A}^-_0} |u|^p < \frac{pq}{2\theta_0(p-q)}\alpha_0(\mathbb{A})\right\}.$$

By the proof of proposition 4.1, for each $l \in \mathbb{R}$ with $|l| \geqslant l_1$ there exists a positive number $t_*(l)$ such that $u^+_{\lambda,\mu} + t_* w_l \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ and

$$J_{f_\lambda,g_\mu}(u^+_{\lambda,\mu} + t_*(l)w_l) < \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A}).$$

This implies $N^\pm_{\lambda,\mu}(\mathbb{A}) \neq \emptyset$. Then we have the following result.

Lemma 5.7. *There exist positive numbers $\lambda_0 \leqslant \lambda_2$ and $\mu_0 \leqslant \tilde{\mu}$ such that, for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, we have*

(i) $N^\pm_{\lambda,\mu}(\mathbb{A}) \neq \emptyset$,

(ii) $N^+_{\lambda,\mu}(\mathbb{A}) \cap N^-_{\lambda,\mu}(\mathbb{A}) = \emptyset$,

(iii) $N_{\lambda,\mu}(\mathbb{A}) = N^+_{\lambda,\mu}(\mathbb{A}) \cup N^-_{\lambda,\mu}(\mathbb{A})$.

Proof. For $u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) \leqslant \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})$, by lemma 2.6(i) there exists $t^-_0(u) > 0$ such that $t^-_0(u)u \in \boldsymbol{N}_{f_0,g_0}$. Moreover,

$$\begin{aligned} J_{f_\lambda,g_\mu}(u) &= \sup_{t\geqslant 0} J_{f_\lambda,g_\mu}(tu) \geqslant J_{f_\lambda,g_\mu}(t^-_0(u)u) \\ &= J_{f_0,g_0}(t^-_0(u)u) - \frac{\lambda[t^-_0(u)]^q}{q}\int_{\mathbb{A}} f_+(x)|u|^q \,\mathrm{d}x - \frac{\mu[t^-_0(u)]^p}{p}\int_{\mathbb{A}} g(x)|u|^p \,\mathrm{d}x. \end{aligned}$$

Thus, by lemma 5.6 and the Hölder and Sobolev inequalities

$$\begin{aligned} J_{f_0,g_0}(t^-_0(u)u) &\leqslant J_{f_\lambda,g_\mu}(u) + \frac{\lambda[t^-_0(u)]^q}{q}\int_{\mathbb{R}^N} f_+(x)|u|^q \,\mathrm{d}x \\ &\qquad + \frac{\mu[t^-_0(u)]^p}{p}\int_{\mathbb{R}^N} g(x)|u|^p \,\mathrm{d}x \\ &< \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A}) + \frac{\lambda\theta_0^{q/p}}{q}\|f_+\|_{L^{q^*}} S_p^{-q/2}\|u\|^q_{H^1} \\ &\qquad + \frac{\mu\theta_0\|b\|_\infty}{p} S_p^{-p/2}\|u\|^p_{H^1}. \end{aligned}$$

Since $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A}) < \alpha_0(\mathbb{A})$. By (2.2), there exists a positive number $\tilde{c}$ such that $\|u\|_{H^1} \leqslant \tilde{c}$ for all $\lambda \in (0,\lambda_2)$, $\mu \in [0,\tilde{\mu})$ and for all $u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}(\mathbb{A})$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})$. Therefore,

$$J_{f_0,g_0}(t^-_0(u)u) < \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A}) + \frac{\lambda\theta_0^{q/p}}{q}\|f_+\|_{L^{q^*}} S_p^{-q/2}\tilde{c}^q + \frac{\mu\theta_0\|b\|_\infty}{p} S_p^{-p/2}\tilde{c}^p.$$

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Let $\delta_0 > 0$ as in lemma 5.3. Then there exist positive numbers $\lambda_0 \leqslant \lambda_2$ and $\mu_0 \leqslant \tilde{\mu}$ such that for $\lambda \in (0, \lambda_0)$ and $\mu \in [0, \mu_0)$,

$$J_{f_0,g_0}(t^-(u)u) < \alpha_0(\mathbb{A}) + \delta_0. \tag{5.13}$$

Since $t_0^-(u)u \in \boldsymbol{N}_{f_0,g_0}$, by lemma 5.3 and (5.13) either

$$\int_{\mathbb{A}_0^+} |t_0^-(u)u|^p \,\mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}) \quad \text{or} \quad \int_{\mathbb{A}_0^-} |t_0^-(u)u|^p \,\mathrm{d}x < \frac{pq}{2(p-q)}\alpha_0(\mathbb{A}).$$

Then, by lemma 5.6(i), either

$$\int_{\mathbb{A}_0^+} |u|^p \,\mathrm{d}x < \frac{pq}{2\theta_0(p-q)}\alpha_0(\mathbb{A}) \quad \text{or} \quad \int_{\mathbb{A}_0^-} |u|^p \,\mathrm{d}x < \frac{pq}{2\theta_0(p-q)}\alpha_0(\mathbb{A})$$

for all $u \in \boldsymbol{N}^-_{f_\lambda,g_\mu}$ with $J_{f_\lambda,g_\mu}(u) < \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})$. To complete the proof of lemma 5.7, it remains to show that

$$N^+_{\lambda,\mu}(\mathbb{A}) \cap N^-_{\lambda,\mu}(\mathbb{A}) = \emptyset.$$

Suppose the contrary. Then there exists $u_0 \in N_{\lambda,\mu}(\mathbb{A})$ such that

$$\int_{\mathbb{A}_0^+} |u|^p \,\mathrm{d}x < \frac{pq}{2\theta_0(p-q)}\alpha_0(\mathbb{A}) \quad \text{and} \quad \int_{\mathbb{A}_0^-} |u|^p \,\mathrm{d}x < \frac{pq}{2\theta_0(p-q)}\alpha_0(\mathbb{A}).$$

By lemma 5.6(ii),

$$\frac{qp}{\theta_0(p-q)}\alpha_0(\mathbb{A}) \leqslant \int_{\mathbb{A}} |u_0|^p \,\mathrm{d}x \leqslant \int_{\mathbb{A}_0^+} |u_0|^p \,\mathrm{d}x + \int_{\mathbb{A}_0^-} |u_0|^p \,\mathrm{d}x < \frac{qp}{\theta_0(p-q)}\alpha_0(\mathbb{A}),$$

which is a contradiction. This completes the proof. □

Let $\overline{N^{\pm}_{\lambda,\mu}(\mathbb{A})}$ denote the closure of $N^{\pm}_{\lambda,\mu}(\mathbb{A})$. Then we have the following result.

Lemma 5.8. $\overline{N^{\pm}_{\lambda,\mu}(\mathbb{A})} = N^{\pm}_{\lambda,\mu}(\mathbb{A})$.

Proof. The proofs of the '+' and '−' cases are similar. Therefore, we only need to prove the '+' case. Suppose that u_0 is a limit point of $N^+_{\lambda,\mu}(\mathbb{A})$. Then $J_{f_\lambda,g_\mu}(u_0) \leqslant \alpha^+_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})$ and

$$\int_{\mathbb{A}_0^+} |u_0|^p \,\mathrm{d}x \leqslant \frac{qp}{\theta_0(p-q)}\alpha_0(\mathbb{A}).$$

This implies $u_0 \in N_{\lambda,\mu}(\mathbb{A})$. If

$$\int_{\mathbb{A}_0^+} |u_0|^p \,\mathrm{d}x = \frac{qp}{2\theta_0(p-q)}\alpha_0(\mathbb{A}),$$

then by lemma 5.7 $u_0 \in N^-_{\lambda,\mu}(\mathbb{A})$. Thus, by lemma 5.6(ii),

$$\frac{qp}{\theta_0(p-q)}\alpha_0(\mathbb{A}) \leqslant \int_{\mathbb{A}} |u_0|^p \,\mathrm{d}x \leqslant \int_{\mathbb{A}_0^+} |u_0|^p \,\mathrm{d}x + \int_{\mathbb{A}_0^-} |u_0|^p \,\mathrm{d}x < \frac{qp}{\theta_0(p-q)}\alpha_0(\mathbb{A}),$$

which is a contradiction. Thus, $u_0 \in N^+_{\lambda,\mu}(\mathbb{A})$ and so $\overline{N^+_{\lambda,\mu}(\mathbb{A})} = N^+_{\lambda,\mu}(\mathbb{A})$. □

6. Proof of theorem 1.1

First, we establish the existence of local minima for J_{f_λ,g_μ} on $N^{\pm}_{\lambda,\mu}(\mathbb{A})$. We need the following result.

PROPOSITION 6.1. *Let $\lambda_0, \mu_0 > 0$, as in lemma 5.7. Then, for each $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, there exist minimizing sequences $\{u_n^{\pm}\} \subset N^{\pm}_{\lambda,\mu}(\mathbb{A})$ such that*

$$J_{f_\lambda,g_\mu}(u_n^{\pm}) = \sigma^{\pm}_{\lambda,\mu} + o(1) \quad \textit{and} \quad J'_{f_\lambda,g_\mu}(u_n^{\pm}) = o(1) \textit{ in } H^{-1}(\mathbb{A}),$$

where $\sigma^{\pm}_{\lambda,\mu} = \inf\{J_{f_\lambda,g_\mu}(u) \mid u \in N^{\pm}_{\lambda,\mu}(\mathbb{A})\}$.

Proof. Analogously to the proof of [37, proposition 9], one can show that by the Ekeland variational principle [21] there exist minimizing sequences $\{u_n^{\pm}\} \subset N^{\pm}_{\lambda,\mu}(\mathbb{A})$ such that

$$J_{f_\lambda,g_\mu}(u_n^{\pm}) = \sigma^{\pm}_{\lambda,\mu} + o(1) \quad \text{and} \quad J'_{f_\lambda,g_\mu}(u_n^{\pm}) = o(1) \text{ in } H^{-1}(\mathbb{A}).$$

We will omit a more detailed proof here. □

THEOREM 6.2. *Let $\lambda_0, \mu_0 > 0$, as in lemma 5.7. Then for each, $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, equation (E_{f_λ,g_μ}) has positive solutions $u_0^{\pm} \in N^{\pm}_{\lambda,\mu}(\mathbb{A})$ such that $J_{f_\lambda,g_\mu}(u_0^{\pm}) = \sigma^{\pm}_{\lambda,\mu}$.*

Proof. By proposition 6.1, there exist sequences $\{u_n^{\pm}\} \subset N^{\pm}_{\lambda,\mu}(\mathbb{A})$ such that

$$J_{f_\lambda,g_\mu}(u_n^{\pm}) = \sigma^{\pm}_{\lambda,\mu} + o(1) \text{ and } J'_{f_\lambda,g_\mu}(u_n^{\pm}) = o(1) \text{ in } H^{-1}(\mathbb{A}).$$

Since $\sigma^{\pm}(\bar{\delta}) < \alpha^{+}_{f_\lambda,g_\mu}(\mathbb{A}) + \alpha_0(\mathbb{A})$, by proposition 3.2 and lemma 5.8 there exist subsequences $\{u_n^{\pm}\}$ and $u_0^{\pm} \in N^{\pm}_{\lambda,\mu}(\mathbb{A})$ which are non-zero solutions of equation (E_{f_λ,g_μ}) such that

$$u_n^{\pm} \to u_0^{\pm} \quad \text{strongly in } H_0^1(\mathbb{A}).$$

Since $J_{f_\lambda,g_\mu}(u_0^{\pm}) = J_{f_\lambda,g_\mu}(|u_0^{\pm}|)$ and $|u_0^{\pm}| \in N^{\pm}_{\lambda,\mu}(\mathbb{A})$, by lemma 2.3, we may assume that $u_0^{\pm}$ are positive solutions of equation (E_{f_λ,g_μ}). □

Sketch of the proof of theorem 1.1.
(i) Combining the results of theorems 3.3 and 4.2, equation (E_{f_λ,g_μ}) has two positive solutions $u^{+}_{\lambda,\mu}$ and $u^{-}_{\lambda,\mu}$ such that $u^{+}_{\lambda,\mu} \in \boldsymbol{N}^{+}_{f_\lambda,g_\mu}$, $u^{-}_{\lambda,\mu} \in \boldsymbol{N}^{-}_{f_\lambda,g_\mu}$. Since

$$\boldsymbol{N}^{+}_{f_\lambda,g_\mu} \cap \boldsymbol{N}^{-}_{f_\lambda,g_\mu} = \emptyset,$$

this implies that $u^{+}_{\lambda,\mu}$ and $u^{-}_{\lambda,\mu}$ are different.

(ii) Combining the results of theorems 3.3 and 6.2, equation (E_{f_λ,g_μ}) has three positive solutions $u^{+}_{\lambda,\mu}$, u_0^{+} and u_0^{-} such that $u^{+}_{\lambda,\mu} \in \boldsymbol{N}^{+}_{f_\lambda,g_\mu}$ and $u_0^{\pm} \in N^{\pm}_{\lambda,\mu}(\mathbb{A})$. Since

$$\boldsymbol{N}^{+}_{f_\lambda,g_\mu} \cap \boldsymbol{N}^{-}_{f_\lambda,g_\mu} = \emptyset \quad \text{and} \quad N^{+}_{\lambda,\mu}(\mathbb{A}) \cap N^{-}_{\lambda,\mu}(\mathbb{A}) = \emptyset,$$

this implies that $u^{+}_{\lambda,\mu}$, u_0^{+} and u_0^{-} are different. □

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