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The Group Aut(μ) is Roelcke Precompact

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Abstract. Following a similar result of Uspenskij on the unitary group of a separable Hilbert space, we show that, with respect to the lower (or Roelcke) uniform structure, the Polish group $G = \mathrm{Aut}(\mu)$ of automorphisms of an atomless standard Borel probability space (X, μ) is precompact. We identify the corresponding compactification as the space of Markov operators on $L_2(\mu)$ and deduce that the algebra of right and left uniformly continuous functions, the algebra of weakly almost periodic functions, and the algebra of Hilbert functions on G, *i.e.*, functions on G arising from unitary representations, all coincide. Again following Uspenskij, we also conclude that G is totally minimal.

Let (X, μ) be an atomless standard Borel probability space.[1] We denote by $\mathrm{Aut}(\mu)$ the Polish group of measure preserving automorphisms of (X, μ) equipped with the weak topology. If for $T \in G$ we let $U_T \colon L_2(\mu) \to L_2(\mu)$ be the corresponding unitary operator (defined by $U_T f(x) = f(T^{-1}x)$), then the map $T \mapsto U_T$ (the *Koopman map*) is a topological isomorphic embedding of the topological group G into the Polish topological group $\mathcal{U}(H)$ of unitary operators on the Hilbert space $H = L_2(\mu)$ equipped with the strong operator topology. The image of G in $\mathcal{U}(H)$ under the Koopman map is characterized as the collection of unitary operators $U \in \mathcal{U}(H)$ for which $U(\mathbf{1}) = \mathbf{1}$ and $Uf \geq 0$ whenever $f \geq 0$; see [5, Theorem A.11].

It is well known (and not hard to see) that the strong and weak operator topologies coincide on $\mathcal{U}(H)$ and that with respect to the weak operator topology, the group $\mathcal{U}(H)$ is dense in the unit ball Θ of the space $\mathcal{B}(H)$ of bounded linear operators on H. Now Θ is a compact space and as such it admits a unique uniform structure. The trace of the latter on $\mathcal{U}(H)$ defines a uniform structure on $\mathcal{U}(H)$. We denote by $\mathcal{J}$ the collection of Markov operators in Θ, where $K \in \Theta$ is *Markov* if $K(\mathbf{1}) = K^*(\mathbf{1}) = \mathbf{1}$ and $Kf \geq 0$ whenever $f \geq 0$. It is easy to see that $\mathcal{J}$ is a closed subset of Θ. Clearly the image of G in $\mathcal{U}(H)$ is contained in $\mathcal{J}$, and it is well known that this image is actually dense in $\mathcal{J}$ (see [6, 7]). Thus, via the embedding of G into $\mathcal{J}$ we obtain also a uniform structure on G. We will denote this uniform space by $(G, \mathcal{J})$.

On every topological group G there are two naturally defined uniform structures $\mathcal{L}(G)$ and $\mathcal{R}(G)$. The *lower* or the *Roelcke* uniform structure on G is defined as $\mathcal{U} = \mathcal{L} \wedge \mathcal{R}$, the greatest lower bound of the left and right uniform structures on G. If $\mathcal{N}$ is a base for the topology of G at the neutral element e, then with

$$U_L = \{(x, y) : x^{-1}y \in U\} \qquad U_R = \{(x, y) : xy^{-1} \in U\},$$

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[1] This assumption may not be an essential one. It seems that the proof of Theorem 1.1 below may hold for a general probability space, but I have not checked the details.

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the collections $\{U_L : U \in \mathcal{N}\}$ and $\{U_R : U \in \mathcal{N}\}$ constitute bases for $\mathcal{L}(G)$ and $\mathcal{R}(G)$, respectively. A base for the Roelcke uniform structure is obtained by the collection $\{V_U : U \in \mathcal{N}\}$ where

$$V_U = \{(x, y) : \exists\, z, w \in U,\ y = zxw\}.$$

The reader should be warned that whereas the left and right uniform structures on a subgroup are induced by the corresponding uniform structures of an ambient group, the Roelcke uniform structure does not have a similar property. We refer to the monograph by Roelcke and Dierolf [10] for information about uniform structures on topological groups.

In [12] Uspenskij showed that the uniform structure induced from Θ on $\mathcal{U}(H)$ coincides with the Roelcke structure of this group. In this note we show that the same is true for $G = \mathrm{Aut}(\mu)$ and then, as in [12], deduce that G is totally minimal (see the definition below).

The subject of Roelcke precompact (RPC) groups was thoroughly studied by Uspenskij. In addition to the paper [12] the interested reader can find more information about RPC groups in [13–15]. In [13] the author showed that the group $\mathrm{Homeo}(C)$ of self-homeomorphisms of the Cantor set C with the compact-open topology is RPC. See also [8, Sections§12, 13] where a new proof is given for the fact that the Polish group $S_\infty(\mathbb{N})$ of permutations of the natural numbers is RPC and where an alternative proof for the RPC property of $\mathrm{Homeo}(C)$ is indicated.

1 $\mathrm{Aut}(\mu)$ is Roelcke Precompact

Theorem 1.1 *The uniform structure induced from $\mathcal{J}$ on G coincides with the Roelcke uniform structure $\mathcal{L} \wedge \mathcal{R}$. Thus the Roelcke uniform structure on G is precompact and the natural embedding $G \to \mathcal{J}$ is a realization of the Roelcke compactification of G.*

Proof Given $\epsilon > 0$ and a finite measurable partition $\alpha = \{A_1, \dots, A_n\}$ of X we set

$$U_{\alpha,\epsilon} = \{T \in G : \mu(A_i \triangle T^{-1}A_i) < \epsilon,\ \forall\, 1 \le i \le n\},$$

$$W_{\alpha,\epsilon} = \{(S, T) \in G \times G : |\mu(A_i \cap S^{-1}A_j) - \mu(A_i \cap T^{-1}A_j)| < \epsilon, \forall\, 1 \le i, j \le n\},$$

$$\widetilde{W}_{\alpha,\epsilon} = \{(S, T) \in G \times G : \exists\, P, Q \in U_{\alpha,\epsilon},\ T = PSQ\}.$$

Note that sets of the form $W_{\alpha,\epsilon}$ constitute a base for the uniform structure on G induced from $\mathcal{J}$, while the $\widetilde{W}_{\alpha,\epsilon}$ form a base for the Roelcke uniform structure on G.

The fact that the identity map $(G, \mathcal{L} \wedge \mathcal{R}) \to (G, \mathcal{J})$ is uniformly continuous actually follows from Uspenskij's result that the map $(\mathcal{U}(H), \mathcal{L} \wedge \mathcal{R}) \to (\mathcal{U}(H), \Theta)$ is uniformly continuous. However, a direct proof is easy.

If $T = PSQ$ with $P, Q \in U_{\alpha,\epsilon}$, then

$$\begin{aligned}
|\mu(A_i \cap T^{-1}A_j) - \mu(A_i \cap S^{-1}A_j)| &= |\mu(A_i \cap Q^{-1}S^{-1}P^{-1}A_j) - \mu(A_i \cap S^{-1}A_j)| \\
&\le |\mu(A_i \cap Q^{-1}S^{-1}P^{-1}A_j) - \mu(Q^{-1}A_i \cap Q^{-1}S^{-1}P^{-1}A_j)| \\
&\quad + |\mu(A_i \cap S^{-1}P^{-1}A_j) - \mu(A_i \cap S^{-1}A_j)| \\
&< 2\epsilon.
\end{aligned}$$

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Thus $\widetilde{W}_{\alpha,\epsilon} \subset W_{\alpha,2\epsilon}$. This means that the identity map $(G, \mathcal{L} \wedge \mathcal{R}) \to (G, \mathcal{J})$ is uniformly continuous.

For the other direction we start with a given $\widetilde{W}_{\alpha,\epsilon}$. Suppose $(S, T) \in W_{\alpha,\epsilon/n^2}$. Set

$$A_{ij} = A_i \cap T^{-1} A_j, \quad A'_{ij} = A_i \cap S^{-1} A_j.$$

We have $|\mu(A_{ij}) - \mu(A'_{ij})| < \epsilon/n^2$ for every i and j. Define a measure preserving $R \in G$ as follows. For each pair i, j, let $B_{ij} \subset A_{ij}$ with $\sum_{i,j} \mu(A_{ij} \setminus B_{ij}) < \epsilon$ and $\mu(B_{ij}) \le \mu(A'_{ij})$. Next choose a measure preserving isomorphism ϕ_{ij} from B_{ij} onto a subset $B'_{ij} \subset A'_{ij}$ and let $\phi\colon \bigcup B_{ij} \to X$ be the map whose restriction to B_{ij} is ϕ_{ij}. Finally, extend ϕ to an element $R \in G$ by defining R on $X \setminus \bigcup B_{ij}$ to be any measure preserving isomorphism $X \setminus \bigcup B_{ij} \to X \setminus \bigcup B'_{ij}$.

It is easy to check that R is in $U_{\alpha,\epsilon}$, and for $TR^{-1}S^{-1}$ we have, up to sets of small measure,

$$TR^{-1}S^{-1}(A_i) = TR^{-1}S^{-1}\Big(\bigcup_{j=1}^{n} A_i \cap SA_j\Big) = TR^{-1}\Big(\bigcup_{j=1}^{n} S^{-1}A_i \cap A_j\Big)$$

$$= T\Big(\bigcup_{j=1}^{n} T^{-1}A_i \cap A_j\Big) = A_i.$$

Thus also $TR^{-1}S^{-1}$ is in $U_{\alpha,\epsilon}$, whence the equation $T = (TR^{-1}S^{-1})SR$ implies $(S, T) \in \widetilde{W}_{\alpha,\epsilon}$. We have shown that $W_{\alpha,\epsilon/n^2} \subset \widetilde{W}_{\alpha,\epsilon}$ and it follows that the identity map $(G, \mathcal{J}) \to (G, \mathcal{L} \wedge \mathcal{R})$ is also uniformly continuous. ∎

Corollary 1.2 *The Roelcke and the WAP compactifications of G coincide. Moreover, every bounded right and left uniformly continuous function — and hence also every WAP function — on G can be uniformly approximated by linear combinations of positive definite functions. In other words, every right and left uniformly continuous function arises from a Hilbert representation.*

Proof Let $\mathcal{R}o(G)$, denote the algebra of bounded right and left uniformly continuous complex-valued functions on G. We write $\mathrm{WAP}(G)$ for the algebra of weakly-almost-periodic complex-valued functions on G and finally we let $\mathcal{H}(G)$ be the algebra of Hilbert complex-valued functions on G, *i.e.*, the uniform closure of the algebra of all linear combinations of positive definite functions; the latter is also called the *Fourier–Stieltjes* algebra. We then have $\mathcal{R}o(G) \supseteq \mathrm{WAP}(G) \supseteq \mathcal{H}(G)$. By Theorem 1.1 these three algebras coincide for the topological group $G = \mathrm{Aut}(\mu)$. In fact, the functions of the form $F_f(T) = \langle Tf, f\rangle$ with $f \in L_2(\mu)$ and $T \in \mathcal{J}$, when restricted to G, are clearly positive definite. Since these functions generate the algebra $C(\mathcal{J})$, which by Theorem 1.1 is canonically isomorphic to $\mathcal{R}o(G)$, this shows that indeed $\mathcal{R}o(G) = \mathcal{H}(G)$. ∎

Remark 1.3 By [12] the same is true for the group $\mathcal{U}(H)$. For more details see [9]. In the literature a topological group G for which $\mathrm{WAP}(G) = \mathcal{H}(G)$ is called *Eberlein*. Thus both $\mathcal{U}(H)$ and $\mathrm{Aut}(\mu)$ are Eberlein groups and moreover for these groups

$$\mathcal{R}o(G) = \mathrm{WAP}(G) = \mathcal{H}(G).$$

This fact for $\mathcal{U}(H)$ was first shown in [9].

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2 Aut(μ) Is Totally Minimal

A topological group is called *minimal* if it does not admit a strictly coarser Hausdorff group topology. It is *totally minimal* if all its Hausdorff quotient groups are minimal. Stoyanov proved that the unitary group $\mathcal{U}(H)$ is totally minimal [11], [1, Theorem 7.6.18], and Uspenskij provided [12] an alternative proof based on his identification of Θ as the Roelcke compactification of this group. Using Theorem 1.1 we have the following result.

Theorem 2.1 *The topological group $G = \mathrm{Aut}(\mu)$ is totally minimal.*

For completeness we provide a proof of this theorem. It follows Uspenskij's proof with some simplifications. We will use the following theorem of Uspenskij [12, Theorem 3.2].

Theorem 2.2 *Let S be a compact Hausdorff semitopological semigroup which satisfies the following assumption:*

> *For every pair of idempotents $p, q \in S$ the conditions $pq = p$ and $qp = p$ are equivalent. (We write $p \le q$ when $p, q \in S$ satisfy these conditions.)*

Then every nonempty closed subsemigroup K of S contains a least idempotent, i.e.*, an idempotent p such that $p \le q$ for every idempotent q in K.*

It is not hard to check that Θ (and therefore also $\mathcal{J}$) satisfies the assumption of this theorem.

Proof of Theorem 2.1 Let τ denote the topology of a Hausdorff topological group G, and suppose that τ' is a coarser Hausdorff group topology. Then the identity map $(G, \tau) \to (G, \tau')$ is continuous and $\tau = \tau'$ if and only if this map is open. A moment's reflection now shows that in order to prove that G is totally minimal, it suffices to check that every surjective homomorphism of Hausdorff topological groups $f\colon G \to G'$ is an open map.

So let $f\colon G \to G'$ be such a homomorphism and observe that G' is then Roelcke precompact and satisfies $\mathcal{R}o(G') = \mathrm{WAP}(G')$ as well. We denote by $\mathcal{J}'$ the corresponding (Roelcke and WAP) compactification of G' and observe that the dynamical systems $(\mathcal{J}, G)$ and $(\mathcal{J}', G')$ are their own enveloping semigroups (see [5]). Now $(\mathcal{J}, G)$ and $(\mathcal{J}', G')$, being WAP systems, are compact semitopological semigroups. Moreover, the map $f\colon G \to G'$ naturally extends to a continuous homomorphism $F\colon \mathcal{J} \to \mathcal{J}'$. (This fact frees us from the need to use Proposition 2.1 from [12].)

Let $K = F^{-1}(e')$, where e' is the neutral element of G'. Clearly then K is a closed subsemigroup of $\mathcal{J}$. Moreover, we have $gK = Kg = F^{-1}(g')$ whenever $g' = f(g) \in G'$. In fact, clearly $gK \subset F^{-1}(g')$ and if $F(q) = g'$ for some $q \in \mathcal{J}$, then

$$F(g^{-1}q) = f(g^{-1})F(q) = g'^{-1}g' = e',$$

hence $g^{-1}q \in K$ and $q \in gK$. Thus $gK = F^{-1}(g')$ and symmetrically also $Kg = F^{-1}(g')$.

Next observe that $gKg^{-1} = K$ for every $g \in G$. Thus if p is the least idempotent in K, provided by Theorem 2.2, then $gpg^{-1} = p$ for all $g \in G$ and we conclude (an

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easy exercise) that either $p = I$, the identity element of G, or p is the projection on the space of constant functions, *i.e.*, the operator of integration on $L_2(X, \mu)$. In the second case we have $e' = F(p) = F(gp) = f(g)F(p) = f(g)e' = f(g)$ for every $g \in G$ and we conclude that $G' = \{e'\}$.

Suppose then that $p = I$. In that case we have $qK \subset K$ for every $q \in K$ and qK being a closed subsemigroup, we conclude that $I \in qK$. Similarly, we get $I \in Kq$, whence q is an invertible element of $\mathcal{J}$, *i.e.*, an element of G. Thus when $p = I$, we have $K \subset G$.

Now let g be an arbitrary element of G and let $g' = f(g) \in G'$. Suppose $G' \ni f(g_i) = g_i' \to g'$ is a convergent sequence in G'. With no loss in generality we assume that g_i converges to an element $q \in \mathcal{J}$ and it then follows that $F(q) = g'$. As $F^{-1}(g') = gK$, we conclude that $q \in gK \subset G$. Now $f(g_i q^{-1} g) = f(g_i) = g_i'$ and $g_i q^{-1} g \to g$. This shows that f is an open map, and the proof is complete. ■

Remark 2.3 The group $G = \text{Aut}(\mu)$ is in fact algebraically simple [2]. Thus minimality of G implies total minimality. Note that with only slight changes the same proof applies to $\mathcal{U}(H)$, and thus we have here a simplified version of Uspenskij's proof.

Remark 2.4 It is perhaps worthwhile mentioning here two other outstanding properties of the Polish group $G = \text{Aut}(\mu)$. The first, due to Giordano and Pestov [3, 4], is that this group has the *fixed point on compacta property*, *i.e.*, whenever G acts on a compact space it admits a fixed point (this property is also called *extreme amenability*). The second is the fact that the natural unitary representation of G on $L_2^0(X, \mu)$ (the subspace of functions with zero mean) is irreducible [5, Theorem 5.14].

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