Bull. Aust. Math. Soc. **106** (2022), 280–287
doi:10.1017/S0004972722000181

ON THE DIVISOR FUNCTION OVER NONHOMOGENEOUS BEATTY SEQUENCES

WEI ZHANG

(Received 12 December 2021; accepted 13 January 2022; first published online 4 March 2022)

Abstract

We consider sums involving the divisor function over nonhomogeneous ($\beta \neq 0$) Beatty sequences $\mathcal{B}_{\alpha,\beta} := \{[\alpha n + \beta]\}_{n=1}^{\infty}$ and show that

$$\sum_{n \leq N,\ n \in \mathcal{B}_{\alpha,\beta}} d(n) = \alpha^{-1} \sum_{m \leq N} d(m) + O(N^{1-1/(\tau+1)+\varepsilon}),$$

where N is a sufficiently large integer, α is of finite type τ and $\beta \neq 0$. Previously, such estimates were only obtained for homogeneous Beatty sequences or for almost all α.

2020 *Mathematics subject classification*: primary 11L20; secondary 11L07, 11B83.

Keywords and phrases: exponential sums, Beatty sequences, divisor problems.

1. Introduction

We investigate sums involving the divisor function over nonhomogeneous Beatty sequences. The *nonhomogeneous* Beatty sequences of integers are defined by

$$\mathcal{B}_{\alpha,\beta} := \{[\alpha n + \beta]\}_{n=1}^{\infty},$$

where α and β are fixed real numbers and $\beta \neq 0$. Here, $[x]$ denotes the greatest integer not larger than x. The distribution properties of such sequences are related to the type of α. For an irrational number α, we define its type τ by the relation

$$\tau := \sup\{\theta \in \mathbb{R} : \lim_{r \to +\infty} \inf_{r \in \mathbb{Z}^+} r^{\theta} \|r\alpha\| = 0\},$$

where $\|u\|$ denotes the distance of u from the nearest integer. Thus, an irrational number α is of type τ if and only if for every $\epsilon > 0$, there is a constant $c(\tau, \alpha)$ such that

$$r\|r\alpha\| \geq c(\tau, \alpha) r^{-\tau-\varepsilon+1}.$$

For properties and extensions of the type, see [2, 3].

Let $\alpha > 1$ and β be fixed real numbers with α positive, irrational and of finite type $\tau = \tau(\alpha)$. The classical divisor function $d(n)$ denotes the number of divisors of the integer n. There are precise estimates for sums of the divisor function over homogeneous Beatty sequences. Abercrombie proved in [1] that for almost all $\alpha > 1$ with respect to the Lebesgue measure,

$$\sum_{n\le x,\, n\in\mathcal{B}_{\alpha,0}} d(n) = \alpha^{-1}\sum_{n\le x} d(n) + O(x^{5/7+\varepsilon}),$$

where the implied constant may depend on α and ε. This result was subsequently improved and extended in various ways (see [2, 9, 11, 14]). Zhai [14] proved that for almost all $\alpha > 1$ with respect to the Lebesgue measure,

$$\sum_{n\le x,\, n\in\mathcal{B}_{\alpha,0}} d(n) = \alpha^{-1}\sum_{n\le x} d(n) + O(x^{1/2+\varepsilon}),$$

where the implied constant may depend on α and ε. In fact, this result can be modified to apply to an individual α.

The main aim of this paper is to generalise such sums to *nonhomogeneous* Beatty sequences and an individual number α with an error term as strong as previous results (obtained for almost all numbers). By the method of [2] or [14], it is not easy to obtain such results for *nonhomogeneous* Beatty sequences and an individual α, and we borrow some ideas from [3].

Before we focus on sums of the divisor function over Beatty sequences, we investigate a related double exponential sum, analogous to a result of Vaughan [12].

THEOREM 1.1. *Let $\alpha > 1$ be a real number. Suppose that $a, q, h \in \mathbb{N}^+$ and $H, x \ge 1$ with $H \ll x$. If*

$$|\alpha - a/q| \le 1/q^2, \quad (a, q) = 1,$$

then

$$\sum_{h\le H}\Big|\sum_{n\le x} d(n)e(\alpha hn)\Big| \ll (Hx^{1/2} + q + Hxq^{-1})x^{\varepsilon}.$$

Estimates for exponential functions twisted with divisor functions are classical problems in analytic number theory. For example, Chowla [4] proved that for almost all irrational α,

$$\sum_{1\le n\le x} d(n)e(\alpha n) = o(x\log x)$$

as $x \to \infty$. Erdös [5] improved the error term in this result to

$$\sum_{1\le n\le x} d(n)e(\alpha n) = O(x^{1/2}\log x)$$

for almost all α. However, such estimates give no idea about the numbers α to which the result applies. The estimates we obtain for such sums depend on the type of α and we show that the estimate applies to any individual α whose rational approximations satisfy certain hypotheses. In this way, we can derive estimates for specific values of α

https://doi.org/10.1017/S0004972722000181 Published online by Cambridge University Press

(or over interesting classes of α such as the class of algebraic numbers). For example, we give the following consequence of Theorem 1.1.

COROLLARY 1.2. *For all irrational $\alpha > 1$ of finite type $\tau < \infty, h \in \mathbb{N}^+$ and $H, x \geq 1$ with $H \ll x$,*

$$\sum_{h\leq H}\Big|\sum_{n\leq x} d(n)e(\alpha hn)\Big| \ll Hx^{1/2+\varepsilon} + (Hx)^{1-1/(\tau+1)+\varepsilon},$$

where the implied constant may depend on α and ε.

REMARK 1.3. Taking $\tau = 1$ and $H = 1$ gives a similar upper bound for the sum

$$\sum_{1\leq n\leq x} d(n)e(\alpha n)$$

for individual numbers α of finite type $\tau < \infty$.

By adapting the method of proving Theorem 1.1, we can obtain the following result for inhomogeneous Beatty sequences.

THEOREM 1.4. *Let $\alpha > 1$ be a fixed irrational number of finite type $\tau < \infty$ and $\beta \in \mathbb{R}$ be fixed. Then there is a constant $\varepsilon > 0$ such that*

$$\sum_{n\leq N,\ n\in\mathcal{B}_{\alpha,\beta}} d(n) = \alpha^{-1}\sum_{m\leq N} d(m) + O(N^{1-1/(\tau+1)+\varepsilon}),$$

where N is a sufficiently large integer and the implied constant depends only on α, β and ε.

REMARK 1.5. Previously, such estimates were proved only for almost all $\alpha > 1$ (not for an individual α) and for homogeneous Beatty sequences. Our result also gives almost all results for nonhomogeneous Beatty sequences because, by the theorems of Khinchin [7] and of Roth [10], almost all real numbers and all irrational algebraic numbers are of type $\tau = 1$. One can also consider generalised divisor functions, which were studied in [9, 14] only for the case of homogeneous Beatty sequences.

2. Proof of Theorem 1.1

To prove the theorem, we need the concept of discrepancy. For a sequence u_m, $m = 1, 2, \ldots, M$, of points of $\mathbb{R}/\mathbb{Z}$, the discrepancy $D(M)$ of the sequence is

$$D(M) = \sup_{\mathcal{I}\in[0,1)}\left|\frac{\mathcal{V}(\mathcal{I}, M)}{M} - |\mathcal{I}|\right|, \tag{2.1}$$

where the supremum is taken over all subintervals $\mathcal{I} = (c, d)$ of the interval $[0, 1), \mathcal{V}(\mathcal{I}, M)$ is the number of positive integers $m \leq M$ such that $u_m \in \mathcal{I}$, and $|\mathcal{I}| = d - c$ is the length of $|\mathcal{I}|$.

Let $D_{\alpha,\beta}(M)$ denote the discrepancy of the sequence $\{\alpha m + \beta\}, m = 1, 2, \ldots, M$, where $\{x\} = x - [x]$. We introduce several auxiliary lemmas.

https://doi.org/10.1017/S0004972722000181 Published online by Cambridge University Press

LEMMA 2.1 [3]. *Let $\alpha > 1$. An integer m has the form $m = [\alpha n + \beta]$ for some integer n if and only if*

$$0 < \{\alpha^{-1}(m - \beta + 1)\} \le \alpha^{-1}.$$

The value of n is determined uniquely by m.

LEMMA 2.2 [8, Ch. 2, Theorem 3.2]. *Let α be a fixed irrational number of finite type $\tau < \infty$. Then, for all $\beta \in \mathbb{R}$, we have*

$$D_{\alpha,\beta}(M) \le M^{-1/\tau + o(1)},$$

as $M \to \infty$, where the function implied by $o(1)$ depends only on α.

LEMMA 2.3 [13, page 32]. *For any $\Delta \in \mathbb{R}$ such that $0 < \Delta < 1/8$ and $\Delta \le 1/2 \min\{\gamma, 1 - \gamma\}$, there exists a periodic function $\Psi_\Delta(x)$ of period 1 such that:*

(1) $0 \le \Psi_\Delta(x) \le 1$ *for all* $x \in \mathbb{R}$;
(2) $\Psi_\Delta(x) = \Psi(x)$ *if* $\Delta \le x \le \gamma - \Delta$ *or* $\gamma + \Delta \le x \le 1 - \Delta$ *where*

$$\Psi(x) = \begin{cases} 1 & \text{if } 0 < x \le \gamma, \\ 0 & \text{if } \gamma < x \le 1; \end{cases}$$

(3) $\Psi_\Delta(x)$ *can be represented as a Fourier series*

$$\Psi_\Delta(x) = \gamma + \sum_{j=1}^{\infty} g_j e(jx) + h_j e(-jx), \tag{2.2}$$

where the coefficients g_j and h_j satisfy $\max\{|g_j|, |h_j|\} \ll \min\{j^{-1}, j^{-2}\Delta^{-1}\}$ *for* $j \ge 1$.

LEMMA 2.4. *Let α be of finite type $\tau < \infty$ and let K be sufficiently large. For an integer $w \ge 1$, there exist $a, q \in \mathbb{Z}$, $a/q \in \mathbb{Q}$ with $(a, q) = 1$ and $K^{1/\tau-\varepsilon} w^{-1} < q \le K$ such that*

$$\left|\alpha w - \frac{a}{q}\right| \le \frac{1}{qK}.$$

PROOF. By the Dirichlet approximation theorem, there is a rational number a/q with $(a, q) = 1$ and $q \le K$ such that

$$\left|\alpha w - \frac{a}{q}\right| < \frac{1}{qK},$$

that is, $\|qw\alpha\| \le 1/K$. Since α is of type $\tau < \infty$, for sufficiently large K, we have

$$\|qw\alpha\| \ge (qw)^{-\tau-\varepsilon}.$$

Thus

$$1/K \ge \|qw\alpha\| \ge (qw)^{-\tau-\varepsilon},$$

which gives

$$q \ge K^{1/\tau-\varepsilon} w^{-1}. \qquad \square$$

https://doi.org/10.1017/S0004972722000181 Published online by Cambridge University Press

LEMMA 2.5 [6, Section 13.5]. *If* $|\alpha - a/q| \le q^{-2}$, $a, q \in \mathbb{N}$ *and* $(a, q) = 1$, *then*

$$\sum_{1\le n\le M} \min\left\{\frac{x}{n}, \frac{1}{2\|n\alpha\|}\right\} \ll (M + q + xq^{-1}) \log 2qx.$$

PROOF OF THEOREM 1.1. By the Dirichlet hyperbolic method,

$$\sum_{h\le H}\Big|\sum_{1\le n\le x} d(n)e(\alpha hn)\Big| = \sum_{h\le H}\Big|\sum_{n_1n_2\le x} e(\alpha hn_1n_2)\Big| \le \sum_{h\le H} 2\Big|\sum_{\substack{n_1n_2\le x\\ n_1\le n_2}} e(\alpha hn_1n_2)\Big|. \tag{2.3}$$

However,

$$\Big|\sum_{\substack{n_1n_2\le x\\ n_1\le n_2}} e(\alpha hn_1n_2)\Big| \ll \sum_{n_1\le x^{1/2}}\Big|\sum_{n_2\le x/n_1} e(\alpha hn_1n_2)\Big|.$$

By the well-known estimate

$$\sum_{1\le n\le x} e(\alpha n) \le \min\left(x, \frac{1}{2\|\alpha\|}\right),$$

we have

$$\Big|\sum_{n_2\le x/n_1} e(\alpha hn_1n_2)\Big| \le \min\left(\frac{x}{n_1}, \frac{1}{2\|\alpha hn_1\|}\right).$$

Hence by Lemma 2.5,

$$\begin{aligned}\sum_{h\le H}\Big|\sum_{\substack{n_1n_2\le x\\ n_1\le n_2}} e(\alpha hn_1n_2)\Big| &\ll \sum_{h\le H}\sum_{n_1\le x^{1/2}} \min\left(\frac{x}{n_1}, \frac{1}{2\|\alpha hn_1\|}\right)\\ &\ll x^{\varepsilon}\sum_{n\le Hx^{1/2}} \min\left(\frac{Hx}{n}, \frac{1}{2\|\alpha n\|}\right)\\ &\ll x^{\varepsilon}(Hx^{1/2} + q + Hxq^{-1}) \log 2qx. \end{aligned}\tag{2.4}$$

Hence, by Lemma 2.4, (2.3) and (2.4), for all irrational α of finite type $\tau < \infty$,

$$\sum_{h\le H}\sum_{n\le x} d(n)e(\alpha n) \ll Hx^{1/2+\varepsilon} + (Hx)^{1-1/(\tau+1)+\varepsilon}.$$

This completes the proof of Theorem 1.1 and Corollary 1.2. □

3. Proof of Theorem 1.4

It is easy to see that

$$\sum_{n\le x,\ n\in\mathcal{B}_{\alpha,\beta}} d(n) = \sum_{n\le(x-\beta)/\alpha} d([\alpha n + \beta]).$$

https://doi.org/10.1017/S0004972722000181 Published online by Cambridge University Press

Hence, we can focus on the right-hand sum. The proof is similar to the argument of Theorem 1.1. Let $\delta = \alpha^{-1}(1-\beta)$ and $M = [\alpha N + \beta]$. Then by Lemma 2.1,

$$\sum_{n\le N} d([\alpha n+\beta]) = \sum_{\substack{m\le M\\ 0<\{\gamma m+\delta\}\le\gamma}} d(m) + O(1) = \sum_{m\le M} d(m)\Psi(\gamma m+\delta) + O(1), \tag{3.1}$$

where $\Psi(x)$ is the periodic function with period one for which

$$\Psi(x) = \begin{cases} 1 & \text{if } 0 < x \le \gamma,\\ 0 & \text{if } \gamma < x \le 1.\end{cases}$$

Let Δ and $\Psi_\Delta(x)$ satisfy the conditions of Lemma 2.3 with

$$0 < \Delta < 1/8 \quad \text{and} \quad \Delta \le \min\{\gamma, 1-\gamma\}/2.$$

From (3.1),

$$\begin{aligned}\sum_{n\le N} d([\alpha n+\beta]) &= \sum_{m\le M} d(m)\Psi(\gamma m+\delta) + O(1)\\ &= \sum_{m\le M} d(m)\Psi_\Delta(\gamma m+\delta) + O(1+\mathcal{V}(I,M)\log N),\end{aligned} \tag{3.2}$$

where $\mathcal{V}(I,M)$ denotes the number of positive integers $m \le M$ such that

$$\{\gamma m+\delta\} \in I = [0,\Delta) \cup (\gamma-\Delta, \gamma+\Delta) \cup (1-\Delta, 1).$$

Since $|I| \ll \Delta$, it follows from the definition (2.1) and Lemma 2.2 that

$$\mathcal{V}(I,M) \ll \Delta N + N^{(1-1)/\tau+\varepsilon}, \tag{3.3}$$

where the implied constant depends only on α. By (2.2),

$$\begin{aligned}&\sum_{m\le M} d(m)\Psi_\Delta(\gamma m+\delta)\\ &\quad = \gamma\sum_{m\le M} d(m) + \sum_{k=1}^{\infty} g_k e(\delta k)\sum_{m\le M} d(m)e(\gamma km) + \sum_{k=1}^{\infty} h_k e(-\delta k)\sum_{m\le M} d(m)e(-\gamma km).\end{aligned} \tag{3.4}$$

By Lemma 2.3, for

$$|\gamma - a/q| \le 1/qK, \tag{3.5}$$

we have

$$K^{1/\tau} \le q \le K. \tag{3.6}$$

https://doi.org/10.1017/S0004972722000181 Published online by Cambridge University Press

Then by Theorem 1.1, (3.5) and (3.6),

$$\begin{aligned}\sum_{k\le N^{2/(\tau+1)}} g_k e(\delta k)\sum_{m\le M} d(m)e(\gamma km) &\ll N^{\varepsilon}\sum_{k\sim H} g_k e(\delta k)\sum_{m\le M} d(m)e(\gamma km)\\ &\ll N^{\varepsilon}(N^{1/2}+q+N/q)\\ &\ll N^{1-1/(\tau+1)+\varepsilon}, \end{aligned}\tag{3.7}$$

where $1\le H\le N^{2/(\tau+1)}$ and q is determined by (3.5) and (3.6) with $K=N^{\tau/(\tau+1)}$. Similarly,

$$\sum_{k\le N^{2/(\tau+1)}} g_k e(-\delta k)\sum_{m\le M} d(dm+c)e(-\gamma km)\ll N^{1-1/(\tau+1)+\varepsilon}.\tag{3.8}$$

However, the well-known bound

$$\sum_{m\le M} d(m)e(\gamma km)\ll N(\log N)^2$$

implies that

$$\sum_{k\ge N^{2/(\tau+1)}} g_k e(\delta k)\sum_{m\le M} d(dm+c)e(\gamma km)\ll N^{1+\varepsilon}\sum_{k\ge N^{2/(\tau+1)}} k^{-2}\Delta^{-1}\ll N^{1-1/(\tau+1)+\varepsilon}\tag{3.9}$$

and

$$\sum_{k\ge N^{2/(\tau+1)}} g_k e(-\delta k)\sum_{m\le M} d(m)e(-\gamma km)\ll N^{1+\varepsilon}\sum_{k\ge N^{2/(\tau+1)}} k^{-2}\Delta^{-1}\ll N^{1-1/(\tau+1)+\varepsilon},\tag{3.10}$$

where $\Delta=N^{-1/(\tau+1)}$. Inserting the bounds (3.7)–(3.10) into (3.4),

$$\sum_{m\le M} d(m)\Psi_{\Delta}(\gamma m+\delta)=\gamma\sum_{m\le M} d(m)+O(N^{1-1/(\tau+1)+\varepsilon}),$$

where the implied constant depends on α,β and ε. Substituting this bound and (3.3) into (3.2) and recalling the choice of $\Delta=N^{-1/(\tau+1)}$ completes the proof of Theorem 1.4.

Acknowledgement

I am deeply grateful to the referee(s) for carefully reading the manuscript and making useful suggestions.

References

[1] A. G. Abercrombie, 'Beatty sequences and multiplicative number theory', *Acta Arith.* **70** (1995), 195–207.

[2] A. G. Abercrombie, W. D. Banks and I. E. Shparlinski, 'Arithmetic functions on Beatty sequences', *Acta Arith.* **136** (2009), 81–89.

[3] W. D. Banks and I. E. Shparlinski, 'Short character sums with Beatty sequences', *Math. Res. Lett.* **13** (2006), 539–547.

[4] S. Chowla, 'Some problems of Diophantine approximation I', *Math. Z.* **33** (1931), 544–563.

[5] P. Erdős, 'Some remarks on Diophantine approximations', *J. Indian Math. Soc. (N.S.)* **12** (1948), 67–74.
[6] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, AMS Colloquium Publications, 53 (American Mathematical Society, Providence, RI, 2004).
[7] A. Y. Khinchin, 'Zur metrischen Theorie der diophantischen Approximationen', *Math. Z.* **24** (1926), 706–714.
[8] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences* (Wiley-Interscience, New York–London–Sydney, 1974).
[9] G. S. Lü and W. G. Zhai, 'The divisor problem for the Beatty sequences', *Acta Math. Sinica (Chin. Ser.)* **47** (2004), 1213–1216.
[10] K. F. Roth, 'Rational approximations to algebraic numbers', *Mathematika* **2** (1955), 1–20.
[11] M. Technau and A. Zafeiropoulos, 'Metric results on summatory arithmetic functions on Beatty sets', *Acta Arith.* **197** (2021), 93–104.
[12] R. C. Vaughan, 'On the distribution of αp modulo 1', *Mathematika* **24** (1977), 135–141.
[13] I. M. Vinogradov, *The Method of Trigonometrical Sums in the Theory of Numbers* (Dover, New York, 2004).
[14] W. G. Zhai, 'Note on a result of Abercrombie', *Chinese Sci. Bull.* **42** (1997), 804–806.

WEI ZHANG, School of Mathematics and Statistics,
Henan University, Kaifeng 475004, Henan, PR China
e-mail: zhangweimath@126.com

https://doi.org/10.1017/S0004972722000181 Published online by Cambridge University Press