Probability in the Engineering and Informational Sciences, **33**, 2019, 367–386.
doi:10.1017/S0269964818000268

ON THE TIME-DEPENDENT BEHAVIOR OF A MARKOVIAN REENTRANT-LINE MODEL

Brian Fralix
Department of Mathematical Sciences
Clemson University, Clemson, SC, USA
E-mail: bfralix@clemson.edu

We use the random-product technique from [5] to study both the steady-state and time-dependent behavior of a Markovian reentrant-line model, which is a generalization of the preemptive reentrant-line model studied in the work of Adan and Weiss [2]. Our results/observations yield additional insight into why the stationary distribution of the reentrant-line model from [2] exhibits an almost-geometric product-form structure: indeed, our generalized reentrant-line model, when stable, admits a stationary distribution with a similar product-form representation as well. Not only that, the Laplace transforms of the transition functions of our reentrant-line model also have a product-form structure if it is further assumed that both Buffers 2 and 3 are empty at time zero.

Keywords: random-product representation, reentrant-line model, time-dependent behavior

1. INTRODUCTION

We consider a reentrant-line model consisting of three buffers and two servers. The buffers are labeled as Buffer 1, Buffer 2, and Buffer 3, and the servers are labeled as Server 1 and Server 2. We assume there is an infinite supply of jobs present at Buffer 1, while additional jobs arrive from outside of the system to Buffer 3 in accordance to a homogeneous Poisson process with rate μ_4. Server 2 always stays at Buffer 2 and processes jobs there in a First-Come-First-Served (FCFS) manner, while Server 1 moves between Buffers 1 and 3, giving preemptive priority to jobs at Buffer 3: this means that Server 1 always devotes its full attention to jobs at Buffer 3, in an FCFS manner, whenever any jobs are present there. Once Buffer 3 empties, Server 1 then devotes its full attention to the infinite pile of jobs at Buffer 1 until a job arrives to Buffer 3, at which time it immediately stops serving at Buffer 1 to work at Buffer 3, only to resume processing at Buffer 1 whenever Buffer 3 is again empty. Server 2 serves jobs at Buffer 2 at a rate μ_2, while server 1 serves jobs at Buffers 1 and 3 with rates μ_1 and μ_3, respectively.

Each job found among the infinite pile of jobs at Buffer 1 possesses three i.i.d. unit exponentially distributed amounts of work to be processed at the three buffers, independent of all other jobs present, and each job that arrives from outside of the system to Buffer 3 possesses a single amount of work that is unit exponentially distributed, and independent of everything else. Observe, however, that not all work possessed by a job needs to be processed

 0269-9648/18 $25.00

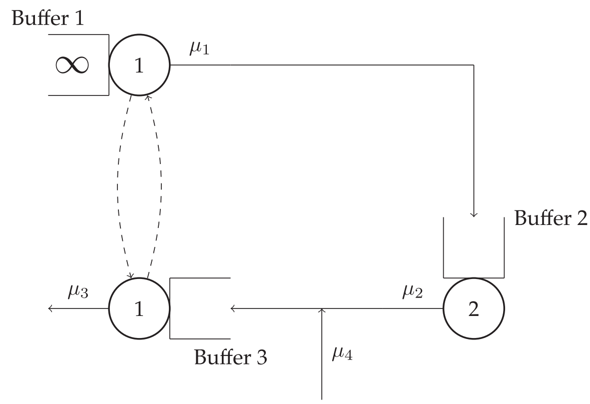

FIGURE 1. An illustration of our three-buffer, two-server reentrant-line model. Readers should note that this diagram does not display that jobs leaving Buffer 2 could further leave the system, and some of these jobs may also take a customer—waiting or in service—at Buffer 3. The dashed lines represent the ability of Server 1 to move between Buffers 1 and 3, where the time it takes to move from one buffer to another is assumed to be instantaneous.

in order for it to leave the system. More specifically, whenever a job finishes processing at Buffer 1, it moves on to Buffer 2, but when a job finishes processing at Buffer 2, it will either (i) leave the system *and* take a single job currently either waiting or in service at Buffer 3 (if such a job is present at Buffer 3) with probability α_1, (ii) leave the system without attempting to take a job from Buffer 3 with probability α_2, or (iii) move on to Buffer 3 with probability α_3: here $\alpha_1 + \alpha_2 + \alpha_3 = 1$. As soon as a job at Buffer 3 is processed by Server 1, it leaves the queueing system. This deletion skill possessed by jobs leaving Buffer 2 makes this process somewhat reminiscent of the negative customer models studied in Gelenbe [9]. Readers should note that when $\alpha_1 = \alpha_2 = \mu_4 = 0$, our model becomes the reentrant-line model studied in Adan and Weiss [2]: the reentrant-line model from [2] is also covered in the recent text of Adan et al. [3]. An illustration of our three-buffer, two-server reentrant-line model can be found in Figure 1.

The dynamics of this modified reentrant-line model can be fully described by a continuous-time Markov chain (CTMC) $X := \{X(t); t \geq 0\}$ having state space

$$\mathbb{Z}_+^2 := \{(i,j) : i,j \in \{0,1,2,3,\ldots\}\}$$

where for each real number $t \geq 0$, we write $X(t) := (X_2(t), X_3(t))$ with $X_i(t)$ representing the number of jobs present (either waiting or in service) at Buffer i at time t, $i = 2, 3$. Figure 2 illustrates, through a transition rate diagram, the structure of the generator (i.e. transition rate matrix) $\mathbf{Q} := [q(x,y)]_{(x,y)\in E}$ associated with X.

Our primary goal is to derive computable expressions for the Laplace transform of each transition function of X, whenever $X(0) = (0,0)$. More specifically, for each state $(i,j) \in E$, we seek to derive $\pi_{(0,0),(i,j)}$, which is well-defined on the set $\mathbb{C}_+ := \{\alpha \in \mathbb{C} : Re(\alpha) > 0\}$—the set of complex numbers having positive real part—as

$$\pi_{(0,0),(i,j)}(\alpha) := \int_0^\infty e^{-\alpha t} p_{(0,0),(i,j)}(t)dt, \quad \alpha \in \mathbb{C}_+$$

where for each real number $t \geq 0$, the transition function $p_{(0,0),(i,j)} : [0,\infty) \to [0,1]$ is defined for each real $t \geq 0$ as

$$p_{(0,0),(i,j)}(t) = \mathbb{P}(X(t) = (i,j) \mid X(0) = (0,0)).$$

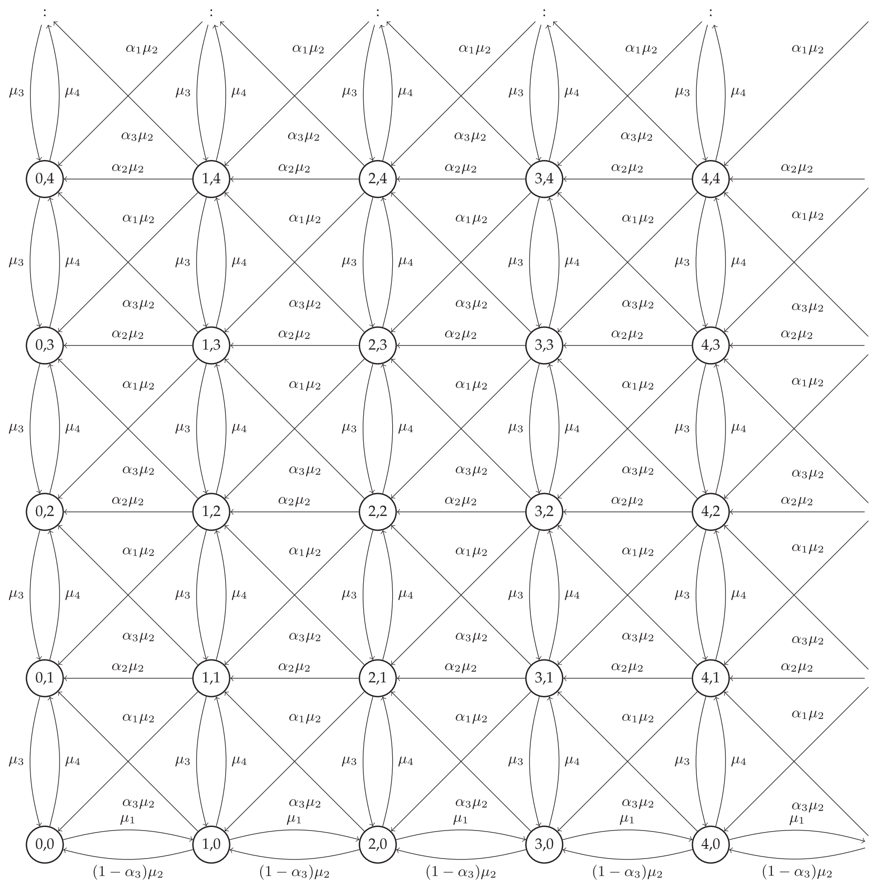

FIGURE **2.** The rate diagram associated with the generator $\mathbf{Q}$ of X.

Since throughout this paper our focus will always be on the case where $X(0) = (0,0)$, we denote each Laplace transform $\pi_{(0,0),(i,j)}$ simply as $\pi_{(i,j)}$. Once these $\pi_{(i,j)}$ transforms are known, numerical transform inversion algorithms can be used to numerically compute each of the corresponding transition functions, and this can give some insight into how fast the reentrant-line model converges in distribution to its stationary distribution $\mathbf{p} := [p_{(i,j)}]_{(i,j)\in E}$, whenever $\mathbf{p}$ exists. Observe that $p_{(0,0),(i,j)}$ is a transition function, whereas $p_{(i,j)}$ is an element of $\mathbf{p}$: clearly, since X is irreducible, when $\mathbf{p}$ exists we have

$$p_{(i,j)} = \lim_{t\to\infty} p_{(0,0),(i,j)}(t).$$

Our results contribute to the applied probability literature in the following ways. First, our analysis of this generalized reentrant-line model provides a new understanding of why the stationary distribution of the reentrant-line model considered in [2] admits a modified-geometric product-form structure having computable parameters, and it gives us an idea of how much we can generalize the transition dynamics of the reentrant-line model without changing the overall tractability of the stationary distribution $\mathbf{p}$. Second, our techniques allow us to show that the Laplace transforms of the transition functions of our generalized reentrant-line model also have a modified-geometric product-form structure that is computable, when we further assume Buffers 2 and 3 are empty at time zero. Finally, this paper illustrates another application of the random-product technique introduced in [5] to the study of the steady-state and/or the time-dependent behavior of continuous-time Markov chains. This appears to be a new technique, and in future studies of Markov chains, it may prove to be a key tool in ways that as of now are unknown.

2. MAIN RESULTS

Our approach towards deriving a computable expression for each Laplace transform $\pi_{(i,j)}$ consists of making use of the random-product technique. In order to use this technique, we must select another CTMC $\{\tilde{X}(t); t \geq 0\}$ having the same state space E as $\{X(t); t \geq 0\}$, but with a generator $\tilde{\mathbf{Q}}$ that satisfies two properties: (i) $\tilde{q}(x,x) = q(x,x)$ for each $x \in E$, and (ii) for any pair of distinct states $x, y \in E$, $\tilde{q}(x,y) > 0$ if and only if $q(y,x) > 0$. Further associated with $\{\tilde{X}(t); t \geq 0\}$ is its embedded DTMC $\{\tilde{X}_n\}_{n \geq 0}$, where $\tilde{X}_0 := \tilde{X}(0)$, and for each integer $n \geq 1$, $\tilde{X}_n$ represents the state of $\tilde{X}$ immediately after its nth state transition.

Given both $\{\tilde{X}(t); t \geq 0\}$ and $\{\tilde{X}_n\}_{n \geq 0}$, we introduce a collection of functions $\{w_{i,j}\}_{(i,j) \in E}$ defined on $\mathbb{C}_+ \cup \{0\}$, where $w_{(0,0)}(\alpha) := 1$ for $\alpha \in \mathbb{C}_+ \cup \{0\}$, and for each state $(i,j) \neq (0,0)$,

$$w_{(i,j)}(\alpha) := \mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{(0,0)} < \infty) e^{-\alpha \tilde{\tau}_{(0,0)}} \prod_{\ell=1}^{\tilde{\eta}_{(0,0)}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right], \quad \alpha \in \mathbb{C}_+ \cup \{0\} \tag{1}$$

where the hitting times $\tilde{\eta}_{(0,0)}$ and $\tilde{\tau}_{(0,0)}$ are defined as

$$\tilde{\eta}_{(0,0)} := \inf\{n \geq 1 : \tilde{X}_n = (0,0)\}, \quad \tilde{\tau}_{(0,0)} := \inf\{t \geq 0 : \tilde{X}(t-) \neq \tilde{X}(t) = (0,0)\}$$

and where in (1), $\mathbb{E}_{(i,j)}[\cdot]$ represents a conditional expectation, conditional on the event $\tilde{X}_0 = (i,j)$, and $\mathbb{P}_{(i,j)}(\cdot)$ is its corresponding conditional probability. We could have omitted the indicator function $\mathbf{1}(\tilde{\eta}_{(0,0)} < \infty)$ from our definition of (1) when $\alpha \in \mathbb{C}_+$, due to the fact that under $\mathbb{P}_{(i,j)}$, we have $e^{-\alpha \tilde{\tau}_{(0,0)}} = 0$ on the event $\{\tilde{\eta}_{(0,0)} = \infty\}$ when $\alpha \in \mathbb{C}_+$.

It has been shown in [5]—see also [7] for a proper extension to the complex plane—that for each state $(i,j) \in E$,

$$\pi_{(i,j)}(\alpha) = \pi_{(0,0)}(\alpha) w_{(i,j)}(\alpha), \quad \alpha \in \mathbb{C}_+. \tag{2}$$

In particular, we may also say that when the stationary distribution $\mathbf{p}$ of X exists, its elements can be expressed as follows: given $\tilde{\mathbf{Q}}$, we have for each state $(i,j) \in E$ that

$$p_{(i,j)} = p_{(0,0)} w_{(i,j)}(0) \tag{3}$$

Note that (3) is a consequence of (2), since

$$p_{(i,j)} = \lim_{\alpha \downarrow 0} \alpha \pi_{(i,j)}(\alpha) = \lim_{\alpha \downarrow 0} \alpha \pi_{(0,0)}(\alpha) w_{(i,j)}(\alpha) = p_{(0,0)} w_{(i,j)}(0).$$

We focus primarily on deriving the Laplace transforms $\pi_{(i,j)}$, for $i, j \geq 0$, but a quick comparison between (2) and (3) reveals that similar expressions can be derived for each element of $\mathbf{p}$ as well. In fact, the $w_{(i,j)}(0)$ terms themselves can reveal whether or not state $(0,0)$ is positive recurrent, due to (see [5])

$$\sum_{(i,j)\in E} w_{(i,j)}(0) = q((0,0))\mathbb{E}_{(0,0)}\left[\tau_{(0,0)}\right], \tag{4}$$

where $\tau_{(0,0)} := \inf\{t \geq 0 : X(t-) \neq X(t) = (0,0)\}$ denotes the first time $\{X(t); t \geq 0\}$ returns to state $(0,0)$. The expectation found on the right-hand side of (4) denotes a conditional expectation, conditional on $X(0) = (0,0)$. We will always use the notation $\mathbb{E}_x$ to denote conditional expectation, when we condition on $X(0) = x$ or $\tilde{X}(0) = x$: while this is a somewhat sloppy practice to follow, readers should be able to infer from the form of the random element within such a given conditional expectation precisely which type of conditional expectation is being used. Readers interested in seeing other recent applications of the random product technique are referred to [8,13].

We are now ready to state the main result of this paper, but before doing so, we first need to introduce a few additional sets and functions, as well as additional notation. Given integers $i, j, k \in \{1, 2, 3, 4\}$, let $\mu_{ij} = \mu_i + \mu_j$, and let $\mu_{ijk} := \mu_i + \mu_j + \mu_k$. Let $\overline{D}(0,1) := \{z \in \mathbb{C} : |z| \leq 1\}$ denote the closed unit disk in $\mathbb{C}$ centered at the origin, and define $\mathbb{C}_+^0 := \mathbb{C}_+ \cup \{0\}$. Next, define the functions $\beta : \mathbb{C}_+^0 \to \mathbb{C}$, $\phi : \mathbb{C}_+^0 \times \overline{D}(0,1) \to \mathbb{C}$, and $\psi : \mathbb{C}_+^0 \to \mathbb{C}$ as

$$\beta(\alpha) := w_{(1,0)}(\alpha) = \mathbb{E}_{(1,0)}\left[\mathbf{1}(\tilde{\eta}_{(0,0)} < \infty)e^{-\alpha\tilde{\tau}_{(0,0)}} \prod_{\ell=1}^{\tilde{\eta}_{(0,0)}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}\right], \tag{5}$$

$$\phi(\alpha, z) := \frac{(\mu_{234} + \alpha - \alpha_2\mu_2 z) \pm \sqrt{(\mu_{234} + \alpha - \alpha_2\mu_2 z)^2 - 4(\mu_3 + \alpha_1\mu_2 z)(\mu_4 + \alpha_3\mu_2 z)}}{2(\mu_3 + \alpha_1\mu_2 z)}, \tag{6}$$

where the $\pm$ symbol is replaced with either plus or minus in order to ensure that $|\phi(\alpha, z)| < 1$—we will see in the Appendix that there is only one way to choose the sign when $\alpha \in \mathbb{C}_+$ and $z \in \mathbb{C}$ satisfies $|z| < 1$—and

$$\psi(\alpha) := \frac{(\mu_3 + \mu_4 + \alpha) - \sqrt{(\mu_3 + \mu_4 + \alpha)^2 - 4\mu_3\mu_4}}{2\mu_4}. \tag{7}$$

Notice that ψ is only well-defined for the case where $\mu_4 > 0$.

Both β and ϕ will be described further later in this paper, but readers should recognize that ψ is just the Laplace–Stieltjes transform of the busy period of an $M/M/1$ queue having arrival rate μ_4 and service rate μ_3. From these functions, we construct the functions $\Gamma_0 : \mathbb{C}_+^0 \to \mathbb{C}$, $\Gamma_1 : \mathbb{C}_+^0 \to \mathbb{C}$, and $\Gamma_2 : \mathbb{C}_+^0 \to \mathbb{C}$, defined as

$$\Gamma_0(\alpha) := \frac{\mu_2}{\mu_3}\frac{\psi(\alpha)\beta(\alpha)(\alpha_3 + \alpha_2\phi(\alpha, \beta(\alpha)) + \alpha_1\phi(\alpha, \beta(\alpha))^2)}{(\rho\psi(\alpha) - \phi(\alpha, \beta(\alpha)))(1 - \psi(\alpha)\phi(\alpha, \beta(\alpha)))},$$

$$\Gamma_1(\alpha) := \frac{\mu_2\beta(\alpha)}{\mu_4(1 - \rho\psi(\alpha)^2)}[\alpha_3 + \alpha_2\phi(\alpha, \beta(\alpha)) + \alpha_1\phi(\alpha, \beta(\alpha))^2],$$

and

$$\Gamma_2(\alpha) := \frac{\mu_2\beta(\alpha)[\alpha_3 + \alpha_2\phi(\alpha, \beta(\alpha)) + \alpha_1\phi(\alpha, \beta(\alpha))^2]}{\mu_3(1 - \phi(\alpha, \beta(\alpha))) + \alpha}.$$

Our first result shows how to write each Laplace transform $\pi_{(i,j)}$—for $i \geq 1$, $j \geq 0$—in terms of the Laplace transform $\pi_{(0,0)}$ times a product of geometric terms.

THEOREM 2.1: *For each integer $i \geq 1$, and each integer $j \geq 0$, we have*

$$\pi_{(i,j)}(\alpha) = \pi_{(0,0)}(\alpha)\beta(\alpha)^i \phi(\alpha, \beta(\alpha))^j, \quad \alpha \in \mathbb{C}_+^0$$

where $\beta(\alpha)$ is a root of the polynomial $g : \mathbb{C} \to \mathbb{C}$, defined as

$$\begin{aligned} g(x) = {} & \alpha_1\mu_2 x^4 \\ & + [-2(1-\alpha_3)\mu_2(\mu_{124}+\alpha) + (\mu_{124}+\alpha)\alpha_2\mu_2 \\ & + (1-\alpha_3)\mu_2(\mu_{234}+\alpha) + \alpha_3\mu_2\mu_3 + \alpha_1\mu_2\mu_4]\, x^3 \\ & + \left[(\mu_{124}+\alpha)^2 + 2\mu_1\mu_2(1-\alpha_3) - (\mu_{124}+\alpha)(\mu_{234}+\alpha) - \alpha_2\mu_1\mu_2 + \mu_3\mu_4\right] x^2 \\ & + [-2\mu_1(\mu_{124}+\alpha) + \mu_1(\mu_{234}+\alpha)]\, x + \mu_1^2. \end{aligned} \tag{8}$$

The next three results address how to compute $\pi_{(0,j)}(\alpha)$ for each integer $j \geq 0$, but three different cases need to be considered separately in order to carry out the calculations. Theorem 2.2 considers the case where $\mu_4 > 0$, and where $\alpha \in \mathbb{C}_+^0$ satisfies $\phi(\alpha, \beta(\alpha)) \neq \rho\psi(\alpha)$.

THEOREM 2.2: *Suppose $\mu_4 > 0$, and $\alpha \in \mathbb{C}_+$ satisfies $\phi(\alpha, \beta(\alpha)) \neq \rho\psi(\alpha)$. Then the Laplace transforms $\pi_{(0,j)}$, for $j \geq 0$ are as follows. First, for $\alpha \in \mathbb{C}_+$,*

$$\pi_{(0,0)}(\alpha) = \frac{1}{\alpha C(\alpha)}, \tag{9}$$

where

$$\begin{aligned} C(\alpha) = {} & 1 + (1+\Gamma_0(\alpha))\left[\frac{\rho\psi(\alpha)}{1-\rho\psi(\alpha)}\right] - \Gamma_0(\alpha)\left[\frac{\phi(\alpha,\beta(\alpha))}{1-\phi(\alpha,\beta(\alpha))}\right] \\ & + \frac{\beta(\alpha)}{(1-\beta(\alpha))(1-\phi(\alpha,\beta(\alpha)))}. \end{aligned} \tag{10}$$

Furthermore, for each integer $j \geq 1$, we have for $\alpha \in \mathbb{C}_+$ that

$$\pi_{(0,j)}(\alpha) = \frac{1}{\alpha C(\alpha)}\left[(1+\Gamma_0(\alpha))(\rho\psi(\alpha))^j - \Gamma_0(\alpha)\phi(\alpha,\beta(\alpha))^j\right]. \tag{11}$$

Our next result considers the case where $\mu_4 > 0$, and where α satisfies $\phi(\alpha, \beta(\alpha)) = \rho\psi(\alpha)$.

THEOREM 2.3: *Suppose $\mu_4 > 0$, and $\phi(\alpha, \beta(\alpha)) = \rho\psi(\alpha)$. Then the Laplace transforms $\pi_{(0,j)}$, $j \geq 0$ are as follows. First, for $\alpha \in \mathbb{C}_+$*

$$\pi_{(0,0)}(\alpha) = \frac{1}{\alpha C(\alpha)}, \tag{12}$$

where

$$C(\alpha) = 1 + \left[\frac{\rho\psi(\alpha)}{1-\rho\psi(\alpha)}\right] + \Gamma_1(\alpha)\left(\frac{\phi(\alpha,\beta(\alpha))}{1-\phi(\alpha,\beta(\alpha))}\right)^2 + \frac{\beta(\alpha)}{(1-\beta(\alpha))(1-\phi(\alpha,\beta(\alpha)))}. \tag{13}$$

Furthermore, for each integer $j \geq 1$, we have for $\alpha \in \mathbb{C}_+$ that

$$\pi_{(0,j)}(\alpha) = \frac{1}{\alpha C(\alpha)}\left[(\rho\psi(\alpha))^j + \Gamma_1(\alpha)\binom{j-1}{1}\phi(\alpha,\beta(\alpha))^j\right]. \tag{14}$$

The Laplace transforms $\pi_{(0,j)}$, for $j \geq 0$, change in form slightly when $\mu_4 = 0$.

THEOREM 2.4: *Suppose now that $\mu_4 = 0$. Then the Laplace transforms $\pi_{(0,j)}$, for $j \geq 0$, are as follows: for $\alpha \in \mathbb{C}_+$,*

$$\pi_{(0,0)}(\alpha) = \frac{1}{\alpha C(\alpha)}, \tag{15}$$

where

$$C(\alpha) = 1 + \Gamma_2(\alpha)\frac{1}{1-\phi(\alpha,\beta(\alpha))} + \frac{\beta(\alpha)}{(1-\beta(\alpha))(1-\phi(\alpha,\beta(\alpha)))}. \tag{16}$$

Furthermore, for each integer $j \geq 1$, we have for $\alpha \in \mathbb{C}_+$ that

$$\pi_{(0,j)}(\alpha) = \frac{1}{\alpha C(\alpha)}\Gamma_2(\alpha)\phi(\alpha,\beta(\alpha))^{j-1}. \tag{17}$$

In the next section, we will describe a procedure for determining which root of g corresponds to $\beta(\alpha)$, as this is a somewhat nontrivial problem. Fortunately, numerical experiments indicate that this procedure appears to always yield a unique root, as well as correct Laplace transform values: we verified this numerically in R for various cases using both simulation, and the numerical transform inversion algorithm of Abate and Whitt [1].

3. PROOF OF THEOREMS 2.1–2.4

Our ability to simplify each Laplace transform $\pi_{(i,j)}$ depends heavily on our ability to simplify each function $w_{(i,j)}$, and these functions are expressed in terms of expected values of random functions of the CTMC $\{\tilde{X}(t); t \geq 0\}$ having generator $\mathbf{Q}$. In order to simplify these $w_{(i,j)}$ terms, we must first choose a proper generator matrix $\tilde{\mathbf{Q}}$. First, we define the interior transitions of $\tilde{\mathbf{Q}}$ as follows: for $i, j \geq 1$, define

$$\begin{aligned}\tilde{q}((i,j),(i,j-1)) &= q((i,j-1),(i,j)) = \mu_4,\\ \tilde{q}((i,j),(i,j+1)) &= q((i,j+1),(i,j)) = \mu_3\tilde{q}((i,j),(i+1,j-1))\\ &= q((i+1,j-1),(i,j)) = \alpha_3\mu_2,\\ \tilde{q}((i,j),(i+1,j)) &= q((i+1,j),(i,j)) = \alpha_2\mu_2\end{aligned}$$

and

$$\tilde{q}((i,j),(i+1,j+1)) = q((i+1,j+1),(i,j)) = \alpha_1\mu_2.$$

We chose to define these transition rates from interior states in this way so that each transition made by $\tilde{X}$ from an interior state contributes the value of one to the random product, which makes the random products found within each $w_{(i,j)}$ term easier to study: this will be clearer in our derivations.

The next step involves defining the transition rates made by $\tilde{X}$ from the set $\{(1,0),(2,0),(3,0),\ldots\}$. For each integer $i \geq 0$, we set

$$\tilde{q}((i,0),(i-1,0)) = \tilde{q}((i,0),(i+1,0)) = \tilde{q}((i,0),(i,1)) = \tilde{q}((i,0),(i+1,1)) = \mu_{124}/4.$$

Choosing this uniform transition structure is really not a crucial step, we mainly need to choose these transition rates so that $\tilde{X}$ has a homogeneous transition structure among the transition rates corresponding to transitions from states in the set $\{(i,j) : i \geq 1, j \geq 0\}$: more specifically, these transition rates do not depend on i, for $i \geq 1$. We refrain from choosing specific transition rates within $\tilde{\mathbf{Q}}$ from states of the form $(0,j)$, $j \geq 0$, as these will not be needed in our derivation: the reader will understand why this is the case as he/she reads through the arguments in this section.

Our justification of Theorems 2.1–2.4 proceeds in the following manner. Step one is to show that for each integer $i \geq 0$, and each $\alpha \in \mathbb{C}_+$, that

$$\pi_{(i,0)}(\alpha) = \pi_{(0,0)}(\alpha)\beta(\alpha)^i.$$

In Step two, we show that given any integers $i, j \geq 1$, we have that for each $\alpha \in \mathbb{C}_+$,

$$\pi_{(i,j)}(\alpha) = \pi_{(0,0)}(\alpha)\beta(\alpha)^i \phi(\alpha, \beta(\alpha))^j.$$

Step three consists of showing that $\beta(\alpha)$ is a root of the polynomial defined in (8): these three steps establish the validity of Theorem 2.1.

The next step is to express, for each integer $j \geq 1$ and each $\alpha \in \mathbb{C}_+$, the Laplace transform values $\pi_{(0,j)}(\alpha)$ in terms of $\beta(\alpha)$, for each of the three cases addressed by Theorems 2.2–2.4. Once all of these expressions have been calculated, we can, for each $\alpha \in \mathbb{C}_+$, compute the normalization terms $C(\alpha)$ appearing in Theorems 2.2–2.4, and we conclude by explaining how to compute the remaining unknown $\beta(\alpha)$ term numerically.

3.1. Derivation of $\pi_{(i,0)}(\alpha)$, for $i \geq 0$

Our first task is, for each integer $i \geq 1$ and each $\alpha \in \mathbb{C}_+$, to express $\pi_{(i,0)}(\alpha)$ in terms of $\pi_{(0,0)}(\alpha)$ and $\beta(\alpha)$, and this can be achieved by writing $w_{(i,0)}(\alpha)$ in terms of $\beta(\alpha)$ when $i \geq 1$. Due to the spatially homogeneous structure possessed by both $\mathbf{Q}$ and $\tilde{\mathbf{Q}}$ among transitions from any states within the set $\{(i,j); i \geq 1, j \geq 0\}$, combined with the fact that $q((i,0),(i+1,0)) = q((0,0),(1,0))$ for each integer $i \geq 0$, observe that for each integer $i \geq 1$ and each $\alpha \in \mathbb{C}_+$, the law (i.e. the distribution) of the random variable

$$\mathbf{1}(\tilde{\eta}_{(i-1,0)} < \infty) e^{-\alpha \tilde{\tau}_{(i-1,0)}} \prod_{\ell=1}^{\tilde{\eta}_{(i-1,0)}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}$$

under the probability measure $\mathbb{P}_{(i,0)}$—which denotes a conditional probability, conditional on $\tilde{X}_0 = (i,0)$—is the same as the law of the random variable

$$\mathbf{1}(\tilde{\eta}_{(0,0)} < \infty) e^{-\alpha \tilde{\tau}_{(0,0)}} \prod_{\ell=1}^{\tilde{\eta}_{(0,0)}} \frac{q(\tilde{X}_\ell, \tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1}, \tilde{X}_\ell)}$$

under the probability measure $\mathbb{P}_{(1,0)}$. Furthermore, any feasible path $\tilde{X}$ takes from state $(i,0)$ to state $(0,0)$ must first eventually reach state $(i-1,0)$, then eventually $(i-2,0)$, and so on, before reaching state $(1,0)$, then eventually state $(0,0)$. By the strong Markov property applied to the stopping time $\tilde{\tau}_{(i-1,0)}$, we find

$$\begin{aligned} w_{(i,0)}(\alpha) &= \mathbb{E}_{(i,0)}\left[\left[\mathbf{1}(\tilde{\eta}_{(0,0)}<\infty)e^{-\alpha(\tau_{(0,0)}-\tau_{(i-1,0)})}\prod_{\ell=\tilde{\eta}_{(i-1,0)+1}}^{\tilde{\eta}_{(0,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]\right. \\ &\qquad \left.\times\,\mathbf{1}(\tilde{\eta}_{(i-1,0)}<\infty)e^{-\alpha\tilde{\tau}_{(i-1,0)}}\prod_{\ell=1}^{\tilde{\eta}_{(i-1,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right] \\ &= \beta(\alpha)\mathbb{E}_{(i-1,0)}\left[\mathbf{1}(\tilde{\eta}_{(0,0)}<\infty)e^{-\alpha\tilde{\tau}_{(0,0)}}\prod_{\ell=1}^{\tilde{\eta}_{(0,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right] \\ &= w_{(i-1,0)}(\alpha)\beta(\alpha). \end{aligned}$$

Repeated applications of this formula show, since $w_{(0,0)}(\alpha)=1$, that for each integer $i \geq 1$,

$$w_{(i,0)}(\alpha) = \beta(\alpha)^i,$$

which implies $\pi_{(i,0)}(\alpha) = \pi_{(0,0)}(\alpha)\beta(\alpha)^i$.

3.2. Derivation of $\pi_{(i,j)}(\alpha)$, for $i, j \geq 1$

Fix two integers $i, j \in \{1,2,3,\ldots\}$, where possibly $i=j$, and define the hitting-time random variables

$$\begin{aligned} \tilde{\eta}_{(\cdot,0)} &:= \inf\{n \geq 1 : \tilde{X}_n \in \{(1,0),(2,0),(3,0),\ldots\}\}, \\ \tilde{\tau}_{(\cdot,0)} &:= \inf\{t \geq 0 : \tilde{X}(t) \in \{(1,0),(2,0),(3,0),\ldots\}\}. \end{aligned}$$

Then for each $\alpha \in \mathbb{C}_+$, $w_{(i,j)}(\alpha)$ can be expressed as

$$\begin{aligned} &\mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{(\cdot,0)}<\infty)\left[e^{-\alpha\tilde{\tau}_{(\cdot,0)}}\prod_{\ell=1}^{\tilde{\eta}_{(\cdot,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]\mathbf{1}(\tilde{\eta}_{(0,0)}-\tilde{\eta}_{(\cdot,0)}<\infty)\right. \\ &\qquad \left.\times\left[e^{-\alpha(\tilde{\tau}_{(0,0)}-\tilde{\tau}_{(\cdot,0)})}\prod_{\ell=\tilde{\eta}_{(\cdot,0)}+1}^{\tilde{\eta}_{(0,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]\right]. \end{aligned}$$

Next, observe that due to the way we chose transition rates of $\tilde{\mathbf{Q}}$ corresponding to transitions from states within the set $\{(i,j); i \geq 1, j \geq 1\}$, on the set $\{\tilde{\eta}_{(\cdot,0)} < \infty\}$, we have

$$\prod_{\ell=1}^{\tilde{\eta}_{(\cdot,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)} = 1$$

with probability one under the measure $\mathbb{P}_{(i,j)}$. This then gives

$$\begin{aligned} w_{(i,j)}(\alpha) &= \mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{(\cdot,0)}<\infty)e^{-\alpha\tilde{\tau}_{(\cdot,0)}}\right. \\ &\qquad \left.\left[\mathbf{1}(\tilde{\eta}_{(0,0)}-\tilde{\eta}_{(\cdot,0)}<\infty)e^{-\alpha(\tilde{\tau}_{(0,0)}-\tilde{\tau}_{(\cdot,0)})}\prod_{\ell=\tilde{\eta}_{(\cdot,0)}+1}^{\tilde{\eta}_{(0,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]\right]. \end{aligned} \tag{18}$$

https://doi.org/10.1017/S0269964818000268 Published online by Cambridge University Press

The next step is to keep track of how $\tilde{X}$ first reaches the set $\{(1,0),(2,0),(3,0),(4,0),\ldots\}$, given $\tilde{X}(0)=(i,j)$. Define the random variable $\tilde{N}_{(\cdot,0)}$ so that it satisfies

$$\tilde{X}(\tilde{\tau}_{(\cdot,0)}) = (\tilde{N}_{(\cdot,0)}, 0).$$

Then the right-hand side of (18) can be further simplified as follows:

$$\begin{aligned}
&\mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{(\cdot,0)}<\infty)e^{-\alpha\tilde{\tau}_{(\cdot,0)}}\left[\mathbf{1}(\tilde{\eta}_{(0,0)}-\tilde{\eta}_{(\cdot,0)}<\infty)e^{-\alpha(\tilde{\tau}_{(0,0)}-\tilde{\tau}_{(\cdot,0)})}\prod_{\ell=\tilde{\eta}_{(\cdot,0)}+1}^{\tilde{\eta}_{(0,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]\right]\\
&\quad=\mathbb{E}_{(i,j)}\left[\mathbf{1}(\tilde{\eta}_{(\cdot,0)}<\infty)e^{-\alpha\tilde{\tau}_{(\cdot,0)}}\beta(\alpha)^{\tilde{N}_{(\cdot,0)}}\right]\\
&\quad=\beta(\alpha)^i\phi(\alpha,\beta(\alpha))^j,
\end{aligned}$$

where the first equality follows from conditioning on the history of $\tilde{X}$ up to the stopping time $\tilde{\tau}_{(\cdot,0)}$ and applying the strong Markov property: more particularly,

$$\mathbb{E}\left[\mathbf{1}(\tilde{\eta}_{(0,0)}-\tilde{\eta}_{(\cdot,0)}<\infty)e^{-\alpha(\tau_{(0,0)}-\tau_{(\cdot,0)})}\prod_{\ell=\tilde{\eta}_{(\cdot,0)+1}}^{\tilde{\eta}_{(0,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\mid\tilde{\mathcal{F}}_{\tilde{\tau}_{(\cdot,0)}}\right]=\beta(\alpha)^{\tilde{N}_{(\cdot,0)}}$$

where $\{\tilde{\mathcal{F}}_t\}_{t\geq 0}$ is the filtration induced by $\{\tilde{X}(t);t\geq 0\}$, $\tilde{\mathcal{F}}_\infty := \bigvee_{t\geq 0}\tilde{\mathcal{F}}_t$, and

$$\tilde{\mathcal{F}}_{\tilde{\tau}_{(\cdot,0)}} := \{A\in\mathcal{F}_\infty : A\cap\{\tilde{\tau}_{(\cdot,0)}\leq t\}\in\mathcal{F}_t \quad \forall \quad t\geq 0\}.$$

Readers should recall that $\tilde{\mathcal{F}}_{\tilde{\tau}_{(\cdot,0)}}$ is the σ-field that represents the history of $\{\tilde{X}(t);t\geq 0\}$ up to the stopping time $\tilde{\tau}_{(\cdot,0)}$. Finally, the second equality stems from $(\tilde{\tau}_{(\cdot,0)},\tilde{N}_{(\cdot,0)})$ having a tractable joint Laplace–Stieltjes transform ϕ defined in (6), which we derive in the Appendix.

3.3. Showing $\beta(\alpha)$ is a root of g

It remains to compute $\beta(\alpha)$, for $\alpha\in\mathbb{C}_+^0$. Starting with the definition of $\beta(\alpha)$, and applying the first-step analysis gives

$$\begin{aligned}
\beta(\alpha)&=\mathbb{E}_{(1,0)}\left[\mathbf{1}(\tilde{\eta}_{(0,0)}<\infty)e^{-\alpha\tilde{\tau}_{(0,0)}}\prod_{\ell=1}^{\tilde{\eta}_{(0,0)}}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]\\
&=\frac{\mu_1}{\mu_1+\mu_2+\mu_4+\alpha}+\frac{(1-\alpha_3)\mu_2}{\mu_1+\mu_2+\mu_4+\alpha}\beta(\alpha)^2+\frac{\mu_3}{\mu_1+\mu_2+\mu_4+\alpha}\beta(\alpha)\phi(\alpha,\beta(\alpha))\\
&\quad+\frac{\alpha_1\mu_2}{\mu_1+\mu_2+\mu_4+\alpha}\beta(\alpha)^2\phi(\alpha,\beta(\alpha)).
\end{aligned}$$

Further multiplying both sides by $\mu_{124}+\alpha$ gives

$$(\mu_{124}+\alpha)\beta(\alpha)=\mu_1+(1-\alpha_3)\mu_2\beta(\alpha)^2+\mu_3\beta(\alpha)\phi(\alpha,\beta(\alpha))+\alpha_1\mu_2\beta(\alpha)^2\phi(\alpha,\beta(\alpha))$$

or, equivalently,

$$(\mu_{124}+\alpha)\beta(\alpha)-\mu_1-(1-\alpha_3)\mu_2\beta(\alpha)^2=[\mu_3\beta(\alpha)+\alpha_1\mu_2\beta(\alpha)^2]\phi(\alpha,\beta(\alpha)). \qquad \mathbf{(19)}$$

To simplify (19) further, we need to plug in our derived expression for $\phi(\alpha,\beta(\alpha))$, and then isolate the square root term. Doing so yields

$$\begin{aligned}&2\left[(\mu_{124}+\alpha)\beta(\alpha)-(1-\alpha_3)\mu_2\beta(\alpha)^2-\mu_1\right]-\beta(\alpha)(\mu_{234}+\alpha-\alpha_2\mu_2\beta(\alpha))\\&\quad=\pm\beta(\alpha)\sqrt{(\mu_{234}+\alpha-\alpha_2\mu_2\beta(\alpha))^2-4(\mu_3+\alpha_1\mu_2\beta(\alpha))(\mu_4+\alpha_3\mu_2\beta(\alpha))}\end{aligned}\tag{20}$$

and after squaring both sides of (20) to eliminate the square root term and simplifying further, we find

$$\begin{aligned}&4\left[(\mu_{124}+\alpha)\beta(\alpha)-(1-\alpha_3)\mu_2\beta(\alpha)^2-\mu_1\right]^2\\&\quad-4\left[(\mu_{124}+\alpha)\beta(\alpha)-(1-\alpha_3)\mu_2\beta(\alpha)^2-\mu_1\right]\beta(\alpha)\left[\mu_{234}+\alpha-\alpha_2\mu_2\beta(\alpha)\right]\\&\qquad=-4\beta(\alpha)^2\left[\mu_3+\alpha_1\mu_2\beta(\alpha)\right]\left[\mu_4+\alpha_3\mu_2\beta(\alpha)\right],\end{aligned}\tag{21}$$

which establishes that $\beta(\alpha)$ is the root of an at-most fourth-degree polynomial. Finally, after applying some tedious algebra to further simplify (21), we conclude that $\beta(\alpha)$ is a root of the fourth-degree polynomial g defined in (8). These observations prove Theorem 2.1.

3.4. Derivation of $\pi_{(0,j)}$, for $j \geq 1$

The next step is, for each $j \geq 1$ and each $\alpha \in \mathbb{C}_+$, to express $\pi_{(0,j)}(\alpha)$ in terms of $\pi_{(0,0)}(\alpha)$ and $\beta(\alpha)$, meaning we must write $w_{(0,j)}(\alpha)$ in terms of $\beta(\alpha)$. The approach we use to simplify $w_{(0,j)}(\alpha)$ is a slight extension of the CAP method recently introduced in [6]. This slight extension of the CAP method was also recently used in [14], within the context of analyzing the time-dependent behavior of the 2-class $M/M/c$ preemptive priority queueing system.

Fix a subset A of E, and define the hitting times $\tilde{\eta}_A$ and $\tilde{\tau}_A$, where

$$\tilde{\eta}_A := \inf\{n \geq 1 : X_n \in A\}$$

represents the index corresponding to the first transition time into the set A, and

$$\tilde{\tau}_A := \inf\{t \geq 0 : X(t-) \neq X(t) \in A\}$$

represents the actual time at which this transition takes place.

Our method for computing each $w_{(0,j)}(\alpha)$ term requires us to choose the set A as $A := \{(0,0)\} \cup \{(i,j) : i \geq 1, j \geq 0\}$. Given a fixed integer $j \geq 1$, observe that under $\mathbb{P}_{(0,j)}$, summing over all ways at which the process $\tilde{X}$ can first reach the set A from state $(0,j)$ yields, after repeatedly applying the Strong Markov property at the stopping time $\tilde{\tau}_A$, the equality

$$\begin{aligned}w_{(0,j)}(\alpha)&=\mathbb{E}_{(0,j)}\left[\mathbf{1}(\tilde{X}_{\tilde{\eta}_A}=(0,0))e^{-\alpha\tilde{\tau}_A}\prod_{\ell=1}^{\tilde{\eta}_A}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]\\&\quad+\sum_{k=0}^{\infty}\mathbb{E}_{(0,j)}\left[\mathbf{1}(\tilde{X}_{\tilde{\eta}_A}=(1,k))e^{-\alpha\tilde{\tau}_A}\prod_{\ell=1}^{\tilde{\eta}_A}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]w_{(1,k)}(\alpha).\end{aligned}\tag{22}$$

In order to simplify the right-hand side of (22) further, it will be helpful to have a general result that allows us to pass back and forth between X and $\tilde{X}$. This result is not formally stated in any previously written papers that feature the random-product technique,

https://doi.org/10.1017/S0269964818000268 Published online by Cambridge University Press

although it is really not a new result since it follows from a straightforward use of ideas found in the papers [5,7,10,11]. We omit its proof.

Theorem 3.1: *Let A be a subset of E. Then for each state $x \in A$, and each state $y \in A^c$, we have for each $\alpha \in \mathbb{C}_+$ that*

$$\begin{aligned}(q(x)+\alpha)\mathbb{E}_x\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=y)dt\right] \\ = \mathbb{E}_y\left[\mathbf{1}(\tilde{\eta}_A<\infty)\mathbf{1}(\tilde{X}_{\tilde{\eta}_A}=x)e^{-\alpha\tilde{\tau}_A}\prod_{\ell=1}^{\tilde{\eta}_A}\frac{q(\tilde{X}_\ell,\tilde{X}_{\ell-1})}{\tilde{q}(\tilde{X}_{\ell-1},\tilde{X}_\ell)}\right]\end{aligned} \quad \mathbf{(23)}$$

where $\tau_A := \inf\{t \geq 0 : X(t-) \neq X(t) \in A\}$.

Applying now Theorem 3.1 to each expectation within (22) yields, for each $\alpha \in \mathbb{C}_+$,

$$\begin{aligned}w_{(0,j)}(\alpha) &= (q((0,0))+\alpha)\mathbb{E}_{(0,0)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\ &+ \sum_{k=0}^{\infty} w_{(1,k)}(\alpha)(q((1,k))+\alpha)\mathbb{E}_{(1,k)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right].\end{aligned} \quad \mathbf{(24)}$$

Readers should note that (24) could have also been derived from a formula found at the top of page 124 of Latouche and Ramaswami [12].

Each of the unknown expectations found in (24) can be further simplified by applying, to each term, a first-step analysis argument, a change-of-variable argument, and the strong Markov property, here

$$\begin{aligned}&(q((0,0))+\alpha)\mathbb{E}_{(0,0)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] = q((0,0),(0,1))\mathbb{E}_{(0,1)} \\ &\quad\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\ &(q((1,0))+\alpha)\mathbb{E}_{(1,0)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\ &\quad = q((1,0),(0,1))\mathbb{E}_{(0,1)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\ &q((1,1))+\alpha)\mathbb{E}_{(1,1)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\ &\quad = q((1,1),(0,1))\mathbb{E}_{(0,1)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\ &\quad\quad + q((1,1),(0,2))\mathbb{E}_{(0,2)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right]\end{aligned}$$

and for each $k \geq 2$,

$$\begin{aligned}
&(q((1,k))+\alpha)\mathbb{E}_{(1,k)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\
&\quad = q((1,k),(0,k-1))\mathbb{E}_{(0,k-1)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\
&\qquad + q((1,k),(0,k))\mathbb{E}_{(0,k)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\
&\qquad + q((1,k),(0,k+1))\mathbb{E}_{(0,k+1)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right].
\end{aligned}$$

Plugging these observations into (24) and further simplifying gives

$$\begin{aligned}
w_{(0,j)}(\alpha) &= \mu_4\mathbb{E}_{(0,1)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\
&\quad + \sum_{\ell=1}^{\infty} w_{(1,\ell-1)}(\alpha)\alpha_3\mu_2\mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\
&\quad + \sum_{\ell=1}^{\infty} w_{(1,\ell)}(\alpha)\alpha_2\mu_2\mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] \\
&\quad + \sum_{\ell=1}^{\infty} w_{(1,\ell+1)}(\alpha)\alpha_1\mu_2\mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right]. \qquad \textbf{(25)}
\end{aligned}$$

It remains to simplify the expectations found in (25).

Proposition 3.1: *Let ℓ, j be two positive integers, and suppose $\alpha \in \mathbb{C}_+$. When $\mu_4 > 0$, the following statements are true.*

(i) For ℓ, j satisfying $1 \leq \ell < j$,

$$\mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] = \Omega(\alpha)(\rho\psi(\alpha))^{j-\ell}(1-(\rho\psi(\alpha)^2)^{\ell}) \qquad \textbf{(26)}$$

where

$$\Omega(\alpha) := \frac{\rho\psi(\alpha)}{\mu_4(1-\rho\psi(\alpha)^2)}.$$

Similarly, (ii) for ℓ, j satisfying $\ell \geq j$,

$$\mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] = \Omega(\alpha)\psi(\alpha)^{\ell-j}(1-(\rho\psi(\alpha)^2)^{j}). \qquad \textbf{(27)}$$

If instead $\mu_4 = 0$, the following statements are true: (i) for $1 \leq \ell < j$,

$$\mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] = 0. \qquad \textbf{(28)}$$

Similarly, (ii) for $\ell \geq j$,

$$\mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right] = \frac{1}{\mu_3+\alpha}\left(\frac{\mu_3}{\mu_3+\alpha}\right)^{\ell-j}. \qquad \textbf{(29)}$$

https://doi.org/10.1017/S0269964818000268 Published online by Cambridge University Press

PROOF: Both statements (26) and (27) have been established in [14]. Statements (28) and (29) have been proven in the Appendix of [6] for the case where $\alpha \geq 0$, but similar expressions still hold for $\alpha \in \mathbb{C}_+^0$ as well: we leave it to the reader to fill in the details. ■

We are now ready to calculate each $w_{(0,j)}(\alpha)$ term. Suppose first that $\mu_4 > 0$: applying both (26) and (27) to (25) gives

$$\begin{aligned} w_{(0,j)}(\alpha) = (\rho\psi(\alpha))^j &+ \Omega(\alpha)\mu_2\beta(\alpha)\left(\alpha_3\phi(\alpha,\beta(\alpha))^{-1} + \alpha_2 + \alpha_1\phi(\alpha,\beta(\alpha))\right) \\ &\times \left[\sum_{\ell=1}^{j-1}\phi(\alpha,\beta(\alpha))^\ell(\rho\psi(\alpha))^{j-\ell}(1-(\rho\psi(\alpha)^2)^\ell)\right. \\ &\left. + \sum_{\ell=j}^{\infty}\phi(\alpha,\beta(\alpha))^\ell\psi(\alpha)^{\ell-j}(1-(\rho\psi(\alpha)^2)^j)\right]. \end{aligned} \quad \textbf{(30)}$$

The key to simplifying (30) further is to consider separately the case where $\alpha \in \mathbb{C}_+^0$ satisfies $\rho\psi(\alpha) \neq \phi(\alpha,\beta(\alpha))$, as well as the case where $\rho\psi(\alpha) = \phi(\alpha,\beta(\alpha))$. Assume first that $\phi(\alpha,\beta(\alpha)) \neq \rho\psi(\alpha)$, then

$$\begin{aligned} &\sum_{\ell=1}^{j-1}\phi(\alpha,\beta(\alpha))^\ell(\rho\psi(\alpha))^{j-\ell}(1-(\rho\psi(\alpha)^2)^\ell) + \sum_{\ell=j}^{\infty}\phi(\alpha,\beta(\alpha))^\ell\psi(\alpha)^{\ell-j}(1-(\rho\psi(\alpha)^2)^j) \\ &\quad = \frac{(\rho\psi(\alpha))^j\phi(\alpha,\beta(\alpha)) - \rho\psi(\alpha)\phi(\alpha,\beta(\alpha))^j}{\rho\psi(\alpha)-\phi(\alpha,\beta_(\alpha))} \\ &\qquad - \left[\frac{(\rho\psi(\alpha))^j\rho\psi(\alpha)^2\phi(\alpha,\beta(\alpha)) - \rho\psi(\alpha)(\rho\psi(\alpha)^2\phi(\alpha,\beta(\alpha)))^j}{\rho\psi(\alpha)-\rho\psi(\alpha)^2\phi(\alpha,\beta(\alpha))}\right] \\ &\qquad + \frac{\phi(\alpha,\beta(\alpha))^j(1-(\rho\psi(\alpha)^2)^j)}{1-\psi(\alpha)\phi(\alpha,\beta(\alpha))} \\ &\quad = \left[\frac{\phi(\alpha,\beta(\alpha))(1-\rho\psi(\alpha)^2)}{(\rho\psi(\alpha)-\phi(\alpha,\beta(\alpha)))(1-\psi(\alpha)\phi(\alpha,\beta(\alpha)))}\right](\rho\psi(\alpha))^j \\ &\qquad - \left[\frac{\phi(\alpha,\beta(\alpha))(1-\rho\psi(\alpha)^2)}{(\rho\psi(\alpha)-\phi(\alpha,\beta(\alpha)))(1-\psi(\alpha)\phi(\alpha,\beta(\alpha)))}\right]\phi(\alpha,\beta(\alpha))^j \end{aligned}$$

Plugging this expression into (30) and performing some algebra finally yields

$$w_{(0,j)}(\alpha) = (\rho\psi(\alpha))^j + \Gamma_0(\alpha)(\rho\psi(\alpha))^j - \Gamma_0(\alpha)\phi(\alpha,\beta(\alpha))^j$$

or, equivalently,

$$\pi_{(0,j)}(\alpha) = \pi_{(0,0)}(\alpha)\left[(\rho\psi(\alpha))^j + \Gamma_0(\alpha)(\rho\psi(\alpha))^j - \Gamma_0(\alpha)\phi(\alpha,\beta(\alpha))^j\right]$$

which establishes (11).

In the case where $\rho\psi(\alpha) = \phi(\alpha, \beta(\alpha))$, we instead find that

$$\sum_{\ell=1}^{j-1} \phi(\alpha,\beta(\alpha))^{\ell} (\rho\psi(\alpha))^{j-\ell} (1-(\rho\psi(\alpha)^2)^{\ell}) + \sum_{\ell=j}^{\infty} \phi(\alpha,\beta(\alpha))^{\ell} \psi(\alpha)^{\ell-j} (1-(\rho\psi(\alpha)^2)^{j})$$
$$= \phi(\alpha,\beta(\alpha))^{j} \sum_{\ell=1}^{j-1} (1-(\rho\psi(\alpha)^2)^{\ell}) + \phi(\alpha,\beta(\alpha))^{j} (1-(\rho\psi(\alpha)^2)^{j}) \sum_{\ell=j}^{\infty} (\phi(\alpha,\beta(\alpha))\psi(\alpha))^{\ell-j}$$
$$= \binom{j-1}{1} \phi(\alpha,\beta(\alpha))^{j} - \left[\frac{1-(\rho\psi(\alpha)^2)^{j}}{1-\rho\psi(\alpha)^2}\right] \phi(\alpha,\beta(\alpha))^{j}$$
$$+ \left[\frac{1}{1-\phi(\alpha,\beta(\alpha))\psi(\alpha)}\right] \phi(\alpha,\beta(\alpha))^{j} - \left[\frac{1}{1-\phi(\alpha,\beta(\alpha))\psi(\alpha)}\right] (\rho\psi(\alpha)^2 \phi(\alpha,\beta(\alpha)))^{j}$$
$$= \binom{j-1}{1} \phi(\alpha,\beta(\alpha))^{j}$$

since, under the assumption $\phi(\alpha, \beta(\alpha)) = \rho\psi(\alpha)$,

$$\frac{1}{1-\rho\psi(\alpha)^2} - \frac{1}{1-\psi(\alpha)\phi(\alpha,\beta(\alpha))} = 0.$$

This in turn yields

$$w_{(0,j)}(\alpha) = (\rho\psi(\alpha))^{j} + \big[\Omega(\alpha)\mu_2\beta(\alpha)\left(\alpha_3\phi(\alpha,\beta(\alpha))^{-1} + \alpha_2 \right.$$
$$\left. +\ \alpha_1\phi(\alpha,\beta(\alpha))\right)\big] \binom{j-1}{1} \phi(\alpha,\beta(\alpha))^{j}$$

or

$$w_{(0,j)}(\alpha) = (\rho\psi(\alpha))^{j} + \Gamma_1(\alpha) \binom{j-1}{1} \phi(\alpha,\beta(\alpha))^{j}$$

proving (14).

It remains to determine $w_{(0,j)}(\alpha)$ for the case where $\mu_4 = 0$. Plugging both (29) and (28) into (30) give

$$w_{(0,j)}(\alpha) = \sum_{\ell=1}^{\infty} w_{(1,\ell-1)}(\alpha)\alpha_3\mu_2 \mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right]$$
$$+ \sum_{\ell=1}^{\infty} w_{(1,\ell)}(\alpha)\alpha_2\mu_2 \mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right]$$
$$+ \sum_{\ell=1}^{\infty} w_{(1,\ell+1)}(\alpha)\alpha_1\mu_2 \mathbb{E}_{(0,\ell)}\left[\int_0^{\tau_A} e^{-\alpha t}\mathbf{1}(X(t)=(0,j))dt\right]$$
$$= \frac{\mu_2\beta(\alpha)}{\mu_3+\alpha}\left[\alpha_3 + \alpha_2\phi(\alpha,\beta(\alpha)) + \alpha_1\phi(\alpha,\beta(\alpha))^2\right]\phi(\alpha,\beta(\alpha))^{j-1}$$
$$\times \sum_{\ell=j}^{\infty}\left(\frac{\mu_3\phi(\alpha,\beta(\alpha))}{\mu_3+\alpha}\right)^{\ell-j}$$

https://doi.org/10.1017/S0269964818000268 Published online by Cambridge University Press

$$= \left[\frac{\mu_2\beta(\alpha)[\alpha_3 + \alpha_2\phi(\alpha,\beta(\alpha)) + \alpha_1\phi(\alpha,\beta(\alpha))^2]}{\mu_3(1-\phi(\alpha,\beta(\alpha))) + \alpha}\right]\phi(\alpha,\beta(\alpha))^{j-1}$$
$$= \Gamma_2(\alpha)\phi(\alpha,\beta(\alpha))^{j-1}.$$

This establishes (17).

3.5. Computing $\pi_{(0,0)}(\alpha)$

It is now easy, for each $\alpha \in \mathbb{C}_+$, to express each Laplace transform value $\pi_{(0,0)}(\alpha)$ in terms of $\beta(\alpha)$. Observe that $\{X(t); t \geq 0\}$ is a regular CTMC, meaning that with probability one, the number of state transitions it can take in the interval $[0,t]$ must be finite. This of course implies that $X(t) \in E$ with probability one, and as a consequence, for each $\alpha \in \mathbb{C}_+$,

$$\sum_{(i,j)\in E} \pi_{(i,j)}(\alpha) = \sum_{(i,j)\in E} \int_0^\infty e^{-\alpha t}\mathbb{P}(X(t) = (i,j) \mid X(0) = 0,0)dt = \int_0^\infty e^{-\alpha t}dt = \frac{1}{\alpha}.$$

This in turn implies

$$\pi_{(0,0)}(\alpha) = \frac{1}{\alpha C(\alpha)},$$

where

$$C(\alpha) := \sum_{(i,j)\in E} w_{(i,j)}(\alpha). \tag{31}$$

Expression (31) makes it easy to calculate $C(\alpha)$, but the form of $C(\alpha)$ will change depending on both μ_4 and α. In the case, where $\mu_4 > 0$ and $\alpha \in \mathbb{C}_+$ satisfies $\phi(\alpha,\beta(\alpha)) \neq \rho\psi(\alpha)$, we get

$$C(\alpha) = 1 + \sum_{j=1}^{\infty}(\rho\psi(\alpha))^j + \Gamma_0(\alpha)\sum_{j=1}^{\infty}(\rho\psi(\alpha))^j - \Gamma_0(\alpha)\sum_{j=1}^{\infty}\phi(\alpha,\beta(\alpha))^j$$
$$= + \sum_{i=1}^{\infty}\sum_{j=0}^{\infty}\beta(\alpha)^i\phi(\alpha,\beta(\alpha))^j$$
$$= 1 + (1+\Gamma_0(\alpha))\frac{\rho\psi(\alpha)}{1-\rho\psi(\alpha)} - \Gamma_0(\alpha)\frac{\phi(\alpha,\beta(\alpha))}{1-\phi(\alpha,\beta(\alpha))} + \frac{\beta(\alpha)}{(1-\beta(\alpha))(1-\phi(\alpha,\beta(\alpha)))}$$

which establishes (10), and furthermore Theorem 2.2. Similar arguments can be used to prove statements (13) and (16), and furthermore Theorems 2.3 and 2.4.

3.6. Calculating $\beta(\alpha)$ numerically

The last step in computing $\beta(\alpha)$ numerically involves correctly selecting the root of g that corresponds to $\beta(\alpha)$, when $\alpha \in \mathbb{C}_+$. One way to select this root is to make use of the fact that $\beta(\alpha)$ must satisfy

$$|\beta(\alpha)| \leq \beta(\mathrm{Re}(\alpha)) < 1$$

so if g, with α replaced by $\mathrm{Re}(\alpha)$, has only one real root r_0 in $(0,1)$—which must be $\beta(Re(\alpha))$—and only one root r of g satisfies $|r| \leq r_0$, then r must be $\beta(\alpha)$. Numerical

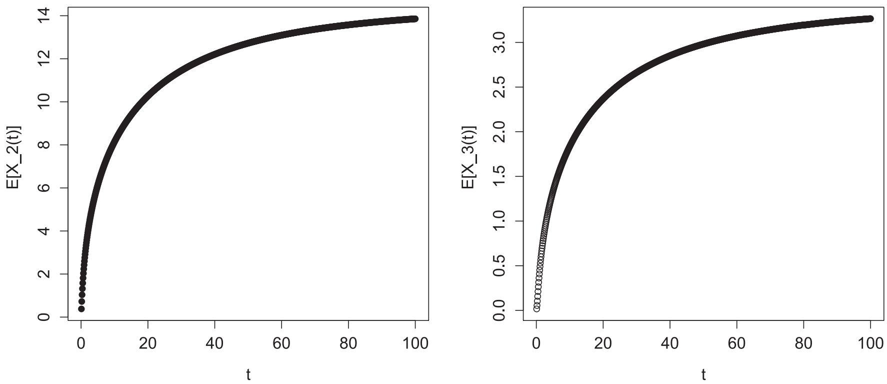

FIGURE 3. The graph on the left shows how the mean number of customers present at Buffer 2 grows as t increases. Similarly, the graph on the right indicates how the mean number of customers present at Buffer 3 grows as t increases. Here $\mu_1 = 4$, $\mu_2 = \mu_3 = 1$, and $\alpha_3 = 1$.

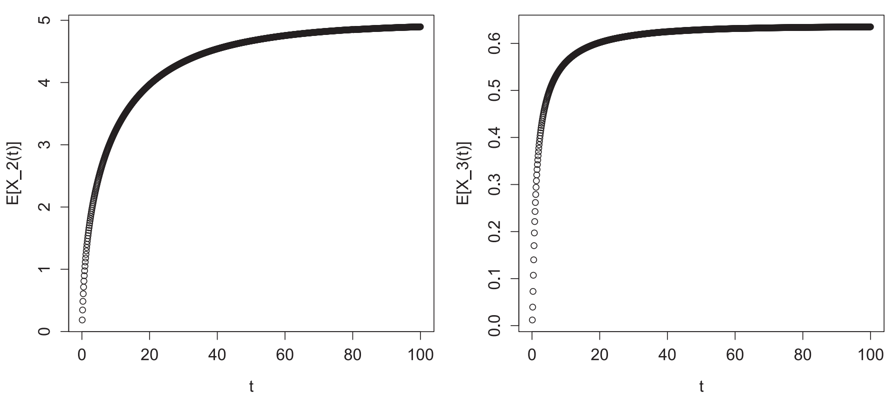

FIGURE 4. Graph of the mean number of customers at Buffers 2 and 3, respectively, as a function of t. Here $\mu_1 = 2$, $\mu_2 = 1.5$, $\mu_3 = 3$, and $\alpha_3 = 1$.

experiments seem to indicate that this procedure always correctly points out a single candidate root for $\beta(\alpha)$, but we do not have a proof that shows there is always exactly one root satisfying this criterion.

Our transform expressions can be used, in conjunction with the numerical transform inversion algorithm of [1] to numerically compute many time-dependent performance measures associated with our reentrant-line model. These graphs were created using R: readers interested in running further numerical experiments may download from http://bfralix.people.clemson.edu/preprints.htm the R code used and written by the author.

The parameter choices we used to create Figures 3 and 4 were made so that comparisons could be made to the values found in Tables 2 and 3 on page 182 of [2]. In both cases, we found that the time-dependent means converge to their corresponding steady-state mean values, which were also calculated in [2]. Similar graphs may be generated for each transition function as well.

The polynomial g simplifies in interesting ways whenever $\alpha = 0$, as the next corollary illustrates. Keep in mind that $\beta(0)$ must be a root in the interval $[0, 1]$, since $\beta(\alpha) \in (0, 1)$ whenever $\alpha > 0$, and $\beta(0) = \lim_{\alpha \downarrow 0} \beta(\alpha)$ by the monotone convergence theorem.

https://doi.org/10.1017/S0269964818000268 Published online by Cambridge University Press

Corollary 3.1: *When $\alpha = 0$, the polynomial g has a root at 1, and g can be factored as*

$$g(x) = (x-1)p(x),$$

where

$$p(x) = \alpha_1 \mu_2^2 x^3 + \mu_2[\mu_3 - \mu_1(2\alpha_1 + \alpha_2)]x^2 + \mu_1(\mu_{124} - \mu_3)x - \mu_1^2.$$

Proof: This is an immediate consequence of the form of the polynomial g under these additional assumptions and follows from simple algebra. ■

We conclude this section by further studying the polynomial p for the case where $\alpha_1 = \alpha_2 = \mu_4 = 0$. In this case, p is simply the quadratic polynomial

$$p(x) = \mu_2\mu_3 x^2 + \mu_1(\mu_1 + \mu_2 - \mu_3)x - \mu_1^2,$$

which must have two real roots, one positive and one negative. The only non-negative root of this polynomial is

$$\frac{-\mu_1(\mu_1 + \mu_2 - \mu_3) + \mu_1\sqrt{(\mu_1 + \mu_2 - \mu_3)^2 + 4\mu_2\mu_3}}{2\mu_2\mu_3}$$

and this is the only possible value for $\beta(0)$. A small amount of algebra shows that indeed,

$$\begin{aligned}\frac{-\mu_1(\mu_1 + \mu_2 - \mu_3) + \mu_1\sqrt{(\mu_1 + \mu_2 - \mu_3)^2 + 4\mu_2\mu_3}}{2\mu_2\mu_3} \\ = \frac{\mu_1}{\mu_2}\left[\frac{-\mu_1 - \mu_2 + \mu_3 + \sqrt{(\mu_1 + \mu_2 + \mu_3)^2 - 4\mu_1\mu_3}}{2\mu_3}\right],\end{aligned}$$

which is the corresponding geometric parameter found in Theorem 3.1 of [2].

Acknowledgement

The author first learned about the reentrant-line model of [2] from Johan van Leeuwaarden during a visit—funded by a travel grant from the Dutch stochastics cluster STAR—to EURANDOM (Eindhoven University of Technology) in May 2013. The author thanks EURANDOM for its gracious hospitality during his visit, as well as STAR for making this visit possible. BF also gratefully acknowledges receiving support from the National Science Foundation, via grant NSF-CMMI-1435261.

The author also thanks the Associate Editor, as well as two anonymous referees for providing useful comments that significantly improved both the style and content of this paper.

References

1. Abate, J. & Whitt, W. (1995). Numerical inversion of Laplace transforms of probability distributions. *ORSA Journal of Computing* 7: 36–43.
2. Adan, I. & Weiss, G. (2006). Analysis of a simple Markovian re-entrant line with infinite supply of work under the LBFS policy. *Queueing Systems* 54: 169–183.
3. Adan, I., van Leeuwaarden, J., & Selen, J. (2017). Analysis of Structured Markov Chains. Draft available at https://arxiv.org/abs/1709.09060.
4. Brown, J.W. & Churchill, R.V. (2009). *Complex variables and applications*, 8th ed. New York: McGraw-Hill Companies.
5. Buckingham, P. & Fralix, B. (2015). Some new insights into Kolmogorov's criterion, with applications to hysteretic queues. *Markov Processes and Related Fields* 21: 339–368.
6. Doroudi, S., Fralix, B., & Harchol-Balter, M. (2016). Clearing Analysis on Phases: exact limiting probabilities for skip-free, unidirectional, quasi-birth-death processees. *Stochastic Systems* 6: 420–458.

7. Fralix, B. (2015). When are two Markov chains similar? *Statistics and Probability Letters* 107: 199–203.
8. Fralix, B. (2018). A new look at a smart polling model. *Mathematical Methods of Operations Research*, to appear.
9. Gelenbe, E. (1991). Product-form queueing networks with negative and positive customers. *Journal of Applied Probability* 28: 656–663.
10. Joyner, J. & Fralix, B. (2015). A new look at Markov processes of $G/M/1$-type. *Stochastic Models* 32: 253–274.
11. Joyner, J. & Fralix, B. (2017). A new look at block-structured Markov processes. Under revision: a draft of this paper can be accessed at http://bfralix.people.clemson.edu/preprints.htm.
12. Latouche, G. & Ramaswami, V. (1999). *Introduction to matrix-analytic methods in stochastic modeling.* Philadelphia: ASA-SIAM Publications.
13. Liu, X. & Fralix, B. (2017). New applications of lattice-path counting to Markovian queues. Submitted for publication: a draft of this work can be downloaded from http://bfralix.people.clemson.edu/preprints.htm.
14. Selen, J. & Fralix, B. (2017). Time-dependent analysis of a multi-server priority system. *Queueing Systems* 87: 379–415.

APPENDIX A

Here we state and prove a result that is needed in our derivations and calculations. Suppose $\{Y(t); t \geq 0\}$ is a CTMC on the state space $E = \{(i,j) : i,j = 0,1,2,\ldots\}$ having generator $\mathbf{Q}_Y := [q_Y(x,y)]_{x,y\in E}$. Here the states $(0,0),(1,0),(2,0),\ldots$ are all absorbing states, with the remaining states having the following transition structure: for $i,j = 1,2,3,\ldots$,

$$q_Y((i,j),(i,j+1)) = \mu_3, \quad q_Y((i,j),(i+1,j-1)) = \alpha_3\mu_2, \quad q_Y((i,j),(i+1,j)) = \alpha_2\mu_2$$

$$q_Y((i,j),(i+1,j+1)) = \alpha_1\mu_2, \quad q_Y((i,j),(i,j-1)) = \mu_4.$$

Our goal is to derive the joint Laplace–Stieltjes transform of τ_Y and N_Y, where

$$\tau_Y := \inf\{t \geq 0 : Y(t-) \neq Y(t) \in \{(0,0),(1,0),(2,0),\ldots\}$$

is the amount of time it takes Y to reach an absorbing state, and N_Y is a random variable satisfying $X(\tau_Y) = (N_Y, 0)$, in that $(N_Y, 0)$ denotes the absorbing state reached by Y when τ_Y is finite. Here the random vector (τ_Y, N_Y) will play the role of the random vector $(\tilde{\tau}_{(\cdot,0)}, \tilde{N}_{(\cdot,0)})$ in the study of our reentrant-line model.

Our first lemma reveals a fairly obvious property of the joint LST of τ_Y and N_Y, given $Y(0) = (i,j)$, for $i \geq 0$, $j \geq 1$.

LEMMA A.1: *For $\alpha \in \mathbb{C}_+$, $z \in \overline{D}(0,1)$, we have*

$$\mathbb{E}_{(0,j)}\left[e^{-\alpha\tau_Y} z^{N_Y}\right] = \phi(\alpha,z)^j$$

where

$$\phi(\alpha,z) := \mathbb{E}_{(0,1)}\left[e^{-\alpha\tau_Y} z^{N_Y}\right].$$

Moreover, we also have

$$\mathbb{E}_{(i,j)}\left[e^{-\alpha\tau_Y} z^{N_Y}\right] = z^i \mathbb{E}_{(0,j)}\left[e^{-\alpha\tau_Y} z^{N_Y}\right] = z^i \mathbb{E}_{(0,1)}\left[e^{-\alpha\tau_Y} z^{N_Y}\right]^j.$$

PROOF: The first statement can be verified through repeated use of the strong Markov property. The second statement follows as a consequence of the homogeneous structure of $\mathbf{Q}_Y$ on the set of all nonabsorbing states. ■

Our next result addresses the problem of computing $\phi(\alpha, z)$, when $\alpha \in \mathbb{C}_+^0$ and $z \in \mathbb{C}$ satisfies $|z| < 1$.

THEOREM A.1: *The quantity $\phi(\alpha, z)$ is the unique solution of the equation*

$$(\mu_3 + \alpha_1\mu_2 z)\phi(\alpha, z)^2 - (\mu_{234} + \alpha - \alpha_2\mu_2 z)\phi(\alpha, z) + (\mu_4 + \alpha_3\mu_2 z) = 0$$

that is contained in the open disk $\{z \in \mathbb{C} : |z| < 1\}$. Furthermore, in the case where $z \in (0, 1]$, and $\alpha \geq 0$, we have

$$\phi(\alpha, z) = \frac{(\mu_{234} + \alpha - \alpha_2\mu_2 z) - \sqrt{(\mu_{234} + \alpha - \alpha_2\mu_2 z)^2 - 4(\mu_3 + \alpha_1\mu_2 z)(\mu_4 + \alpha_3\mu_2 z)}}{2(\mu_3 + \alpha_1\mu_2 z)}.$$

PROOF: Using the first-step analysis, we have

$$\phi(\alpha, z) = \frac{\mu_4}{\mu_{234} + \alpha} + \frac{\mu_3}{\mu_{234} + \alpha}\phi(\alpha, z)^2 + \frac{\alpha_3\mu_2}{\mu_{234} + \alpha}z + \frac{\alpha_2\mu_2}{\mu_{234} + \alpha}z\phi(\alpha, z) + \frac{\alpha_1\mu_2}{\mu_{234} + \alpha}z\phi(\alpha, z)^2.$$

Multiplying both sides of this equality by $\mu_{234} + \alpha$ and collecting terms reveals that

$$(\mu_3 + \alpha_1\mu_2 z)\phi(\alpha, z)^2 - (\mu_{234} + \alpha - \alpha_2\mu_2 z)\phi(\alpha, z) + (\mu_4 + \alpha_3\mu_2 z) = 0,$$

thus implying that $\phi(\alpha, z)$ is a root of a quadratic polynomial.

We next show that this quadratic polynomial has exactly one root within the unit circle whenever $\alpha \in \mathbb{C}_+$, and $z \in \mathbb{C}$ satisfies $|z| < 1$. Fix such an α and z, and define the polynomials f and g as

$$f(x) = -(\mu_2 + \mu_3 + \mu_4 + \alpha - \alpha_2\mu_2 z)x, \quad g(x) = (\mu_3 + \alpha_1\mu_2 z)x^2 + (\mu_4 + \alpha_3\mu_2 z).$$

One can show that when $x \in \mathbb{C}$ satisfies $|x| = 1$, then $|g(x)| < |f(x)|$. Then since f has exactly one root inside the unit circle, we can conclude through an application of Rouché's Theorem (see e.g. page 294 of [4]) that the polynomial $f + g$ has exactly one root inside the unit circle as well.

Furthermore, since for each fixed $z \in (0, 1]$, $\phi(\cdot, z)$ is both continuous and decreasing on $[0, \infty)$, it must be the case that when $\alpha \geq 0$, and $z \in (0, 1]$,

$$\phi(\alpha, z) = \frac{(\mu_{234} + \alpha - \alpha_2\mu_2 z) - \sqrt{(\mu_{234} + \alpha - \alpha_2\mu_2 z)^2 - 4(\mu_3 + \alpha_1\mu_2 z)(\mu_4 + \alpha_3\mu_2 z)}}{2(\mu_3 + \alpha_1\mu_2 z)}.$$

■