Ergod. Th. & Dynam. Sys. (2013), **33**, 560–584
doi:10.1017/S0143385711000940

Ergodic properties of invariant measures for $C^{1+\alpha}$ non-uniformly hyperbolic systems

CHAO LIANG†, WENXIANG SUN‡ and XUETING TIAN§¶

† *Applied Mathematical Department, The Central University of Finance and Economics, Beijing 100081, China*
(e-mail: chaol@cufe.edu.cn)

‡ *LMAM, School of Mathematical Sciences, Peking University, Beijing 100871, China*
(e-mail: sunwx@math.pku.edu.cn)

§ *Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*

¶ *School of Mathematical Sciences, Peking University, Beijing 100871, China*
(e-mail: tianxt@amss.ac.cn, txt@pku.edu.cn)

(*Received* 20 *December* 2010 *and accepted in revised form* 16 *September* 2011)

Abstract. For every ergodic hyperbolic measure ω of a $C^{1+\alpha}$ diffeomorphism, there is an ω-full-measure set $\tilde{\Lambda}$ (the union of $\tilde{\Lambda}_l = \mathrm{supp}(\omega|_{\Lambda_l})$, the support sets of ω on each Pesin block Λ_l, $l = 1, 2, \ldots$) such that every non-empty, compact and connected subset $V \subseteq \mathrm{Closure}(\mathcal{M}_{\mathrm{inv}}(\tilde{\Lambda}))$ coincides with $V_f(x)$, where $\mathcal{M}_{\mathrm{inv}}(\tilde{\Lambda})$ denotes the space of invariant measures supported on $\tilde{\Lambda}$ and $V_f(x)$ denotes the accumulation set of time averages of Dirac measures supported at *one orbit* of some point x. For each fixed set V, the points with the above property are dense in the support $\mathrm{supp}(\omega)$. In particular, points satisfying $V_f(x) = \mathrm{Closure}(\mathcal{M}_{\mathrm{inv}}(\tilde{\Lambda}))$ are dense in $\mathrm{supp}(\omega)$. Moreover, if $\mathrm{supp}(\omega)$ is isolated, the points satisfying $V_f(x) \supseteq \mathrm{Closure}(\mathcal{M}_{\mathrm{inv}}(\tilde{\Lambda}))$ form a residual subset of $\mathrm{supp}(\omega)$. These extend results of K. Sigmund [On dynamical systems with the specification property. *Trans. Amer. Math. Soc.* **190** (1974), 285–299] (see also M. Denker, C. Grillenberger and K. Sigmund [*Ergodic Theory on Compact Spaces (Lecture Notes in Mathematics, 527)*. Springer, Berlin, Ch. 21]) from the uniformly hyperbolic case to the non-uniformly hyperbolic case. As a corollary, irregular$^+$ points form a residual set of $\mathrm{supp}(\omega)$.

1. *Introduction*

Sigmund [**11, 12**] (see also [**4**, Ch. 21]) established in the 1970s two approximation properties for C^1 uniformly hyperbolic diffeomorphisms: one is that invariant measures can be approximated by periodic measures; the other is that every non-empty, compact and connected subset of the space of invariant measures coincides with the accumulation set of time averages of Dirac measures supported at one orbit, and such orbits are dense. Similar discussions for uniformly hyperbolic flows can be found in [**3**].

The first approximation property was realized for some $C^{1+\alpha}$ non-uniformly hyperbolic diffeomorphisms in 2003, when Hirayama [**5**] proved that periodic measures are dense in the set of invariant measures supported on a full-measure set which in some sense is very close to being supp(ω), with respect to a hyperbolic mixing measure. In 2009, Liang *et al* [**7**] replaced the assumption of hyperbolic mixing measure with a more natural and weaker assumption of hyperbolic ergodic measure and generalized Hirayama's result. The proofs in [**5, 7**] are both based on Katok's closing and shadowing lemmas of the $C^{1+\alpha}$ Pesin theory. Moreover, the first approximation property is also valid in the C^1 setting with limit domination by using Liao's shadowing lemma for quasi-hyperbolic orbit segments [**13**], and furthermore is valid for the isolated transitive sets of C^1 generic diffeomorphisms [**1**] by using Mañé's closing lemma and a newly introduced notion called the barycenter property.

The specification property for Axiom A systems ensures the two approximation properties in [**11, 12**] (see also [**4**, Ch. 21]). To achieve the second approximation property, Sigmund ([**12**] or [**4**, Proposition 21.14]) uses the specification property infinitely many times to find the required orbit: he uses a periodic orbit shadowing finitely many orbit segments and new orbit segments to constitute a new periodic pseudo-orbit and obtain a new shadowing periodic orbit, and repeats this process to get a sequence of periodic orbits by induction, whose accumulation approximates all the given infinitely many orbit segments.

However, for the non-uniformly hyperbolic case, this process is not applicable: one can only get some version of specification on $\tilde{\Lambda}$ and, for a given pseudo-orbit consisting of finitely many orbit segments in $\tilde{\Lambda}$, one cannot guarantee that its shadowing periodic orbit also stays in $\tilde{\Lambda}$. Thus one cannot connect the shadowing periodic orbit with the new orbit segments to create a new periodic pseudo-orbit (the assumption of ergodicity only implies that positive measure sets can be connected by orbit segments) and then Sigmund's process stops. In other words, the specification property for *finitely* many orbit segments (for uniformly hyperbolic systems, see [**11, 12**] and [**4**, Ch. 21]; and for non-uniformly hyperbolic systems, see [**5, 7, 13**]), cannot be used *infinitely* many times (even twice).

Therefore, to deal with the non-uniformly hyperbolic case, we introduce a new specification property for *infinitely* many orbit segments (perhaps belonging to different Pesin blocks) and use it only once to find the required orbit and hence avoid induction. Remark that the specification introduced in [**5, 7, 13**] only holds for all orbit segments whose beginning and ending point are in the *same fixed* Pesin block, but the specification in the present paper allows different orbit segment corresponding to different Pesin blocks. We now begin introducing our results.

Let M be a smooth compact Riemannian manifold. Throughout this paper, we consider an $f \in \mathrm{Diff}^{1+\alpha}(M)$ and an ergodic hyperbolic measure ω for f. Let $\Lambda = \bigcup_{l=1}^{\infty} \Lambda_l$ be the Pesin set associated with ω. We denote by $\omega|_{\Lambda_l}$ the conditional measure of ω on Λ_l. Set $\tilde{\Lambda}_l = \mathrm{supp}(\omega|_{\Lambda_l})$ and $\tilde{\Lambda} = \bigcup_{l=1}^{\infty} \tilde{\Lambda}_l$. Clearly $f^{\pm 1}\tilde{\Lambda}_l \subset \tilde{\Lambda}_{l+1}$, and the sub-bundles $E^s(x)$, $E^u(x)$ depend continuously on $x \in \tilde{\Lambda}_l$. Moreover, $\tilde{\Lambda}$ is f-invariant with ω-full measure.

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Let $\mathcal{M}(M)$ be the set of all probability measures supported on M and $\mathcal{M}_{\text{inv}}(f)$ be the subset consisting of all invariant measures. Let $\mathcal{M}_{\text{inv}}(\tilde{\Lambda})$ denote the space of invariant measures supported on $\tilde{\Lambda}$. In other words, $\mathcal{M}_{\text{inv}}(\tilde{\Lambda}) = \{\nu \in \mathcal{M}_{\text{inv}}(f) \mid \nu(\tilde{\Lambda}) = 1\}$.

Remark 1.1. Note that $\mathcal{M}_{\text{inv}}(\tilde{\Lambda})$ is convex but may not be compact. For a better understanding of the following theorem and corollary, here we construct a compact connected subset of $\mathcal{M}_{\text{inv}}(\tilde{\Lambda})$. Let $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots)$ be a non-increasing sequence of positive real numbers which approach zero. Let

$$\mathcal{M}_{\bar{\varepsilon}} = \{\nu \in \mathcal{M}_{\text{inv}}(f) : \nu(\tilde{\Lambda}_l) \geq 1 - \varepsilon_l, l = 1, 2, \ldots\}.$$

Since each $\tilde{\Lambda}_l$ is compact, the map $\nu \to \nu(\tilde{\Lambda}_l)$ is upper-semicontinuous. Hence, $\mathcal{M}_{\bar{\varepsilon}}$ is a closed convex subset of $\mathcal{M}_{\text{inv}}(f)$. This implies that $\mathcal{M}_{\bar{\varepsilon}}$ is a compact connected subset of $\mathcal{M}_{\text{inv}}(f)$. Since every $\nu \in \mathcal{M}_{\bar{\varepsilon}}$ satisfies $\nu(\tilde{\Lambda}) = 1$, $\mathcal{M}_{\bar{\varepsilon}}$ is a subset of $\mathcal{M}_{\text{inv}}(\tilde{\Lambda})$. Thus, $\mathcal{M}_{\bar{\varepsilon}}$ must be a compact connected subset of $\mathcal{M}_{\text{inv}}(\tilde{\Lambda})$.

For any measure $\nu \in \mathcal{M}(M)$, we denote by $V_f(\nu)$ the set of accumulation measures of time averages

$$\nu^N = \frac{1}{N}\sum_{j=0}^{N-1} f_*^j \nu.$$

Then $V_f(\nu)$ is a non-empty, closed and connected subset of $\mathcal{M}_{\text{inv}}(f)$. And we denote by $V_f(x)$ the set of accumulation measures of time averages

$$\delta(x)^N = \frac{1}{N}\sum_{j=0}^{N-1} \delta(f^j x),$$

where $\delta(x)$ denotes the Dirac measure at x. We now state our main theorems as follows.

THEOREM 1.2. *For every non-empty connected compact set* $V \subseteq \text{Closure}(\mathcal{M}_{\text{inv}}(\tilde{\Lambda}))$, *there exists a point* $x \in M$ *such that*

$$V = V_f(x). \tag{1.1}$$

Moreover, the set of such x *is dense in* $\text{supp}(\omega)$*, that is, the closure of this set contains* $\text{supp}(\omega)$.

Remark 1.3. Pfister and Sullivan [**9**] introduced in 2007 a weak specification condition and a weak version of asymptotic h-expansivity, called the g-almost product property and uniform separation property, respectively. This is one of the recent innovations in the study of topological dynamics. They proved that if (X, f) has these two weaker properties, and if $V \subset \mathcal{M}_{\text{inv}}(X)$ is compact and connected, then

$$h_{\text{top}}\{x \in X \mid V_f(x) = V\} = \inf_{\mu \in V} h_\mu(f),$$

where h_{top} is the topological entropy in the sense of Bowen (for non-compact sets). And if only the g-almost product property is satisfied, then $h_{\text{top}}\{x \in X \mid V_f(x) = \mu\} = h_\mu(f)$. Under certain assumptions, these are stronger results than the existence of x with $V_f(x) = V$. We are not sure whether those assumptions apply in the present setting, or whether the

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

present proof can be adapted to obtain their stronger results. The main observation is the difference between their definition in the non-uniform case and our case. Their version of specification can deal with non-uniform cases and allow the transition time from one orbit segment to the next to become arbitrarily large as the orbit segments become large, provided that growth is sub-exponential. However, this is clearly a different notion of specification than the one in the present paper.

A point $x \in M$ is called *a generic point* for an f-invariant measure ν if, for any $\phi \in C^0(M, \mathbb{R})$, the limit $\lim_{n\to\infty}(1/n)\sum_{i=0}^{n-1}\phi(f^i x)$ exists and is equal to $\int \phi\, d\nu$. By Theorem 1.2, the following result holds.

COROLLARY 1.4. *Every f-invariant measure supported on $\tilde{\Lambda}$ has generic points and, for every such measure, the set of generic points is dense in* $\mathrm{supp}(\omega)$.

Remark 1.5. It is known that every ergodic measure has generic points. The heart of the matter considered in the above corollary is what happens for non-ergodic measures, which in general need not have generic points.

In a Baire space, a set is *residual* if it contains a countable intersection of dense open sets. A point $x \in M$ is said to have *maximal oscillation* if

$$V_f(x) \supseteq \mathrm{Closure}(\mathcal{M}_{\mathrm{inv}}(\tilde{\Lambda})).$$

We can deduce from Theorem 1.2 that the points having maximal oscillation are dense in $\mathrm{supp}(\omega)$. As an extension to Theorem 1.2, we go on to prove that they form a residual subset of $\mathrm{supp}(\omega)$.

In the next two theorems, we show two generic results provided that $\mathrm{supp}(\omega)$ is isolated. Λ is called *isolated* if there is some neighborhood U of Λ in M such that

$$\Lambda = \bigcap_{k\in\mathbb{Z}} f^k(U).$$

The isolated property condition is necessary even for uniformly hyperbolic systems. Note that for the non-uniformly hyperbolic systems studied here, the shadowing points need not stay in the support of ω. But they will remain in the support if the isolation property is satisfied. Hence the residual set must be contained in $\mathrm{supp}(\omega)$ (in [**4, 11, 12**] the specification is considered for a hyperbolic basic set which is naturally isolated or for a compact invariant set in which the shadowing point still remains). This is important to avoid the residual set becoming empty. For example, if $M = \mathbb{S}^n$ and $\mathrm{supp}(\omega) = \mathbb{S}^{n-1}$, it is easy to see that $B_n = \{x \in M : 0 < d(x, \mathrm{supp}(\omega)) < 1/n\}$ are open subsets of M and are dense in $\mathrm{supp}(\omega)$, but $\bigcap_{n\geq 1} B_n$ is empty.

THEOREM 1.6. *Suppose that the support set* $\mathrm{supp}(\omega)$ *is isolated (or* $\mathrm{supp}(\omega) = M$*). Then the set of points in* $\mathrm{supp}(\omega)$ *having maximal oscillation is residual in* $\mathrm{supp}(\omega)$.

Remark 1.7. Note that in the definition of maximal oscillation, we only require that $V_f(x)$ contains the whole closure of $\mathcal{M}_{\mathrm{inv}}(\tilde{\Lambda})$ rather than the equality. On the other hand, there exists at most one subset $V \subseteq \mathrm{Closure}(\mathcal{M}_{\mathrm{inv}}(\tilde{\Lambda}))$ such that points satisfying equality (1.1) form a residual set. In other words, for any two distinct non-empty connected subsets V_1

and V_2, at least one of the corresponding sets of points satisfying equality (1.1) cannot be a residual set. Otherwise, notice that the intersection of these two residual sets is also residual so that it is still non-empty. Hence, there exists some point such that the collection of weak* limits of Dirac measures along its orbit is equal to two different sets V_1 and V_2. This is a contradiction. Thus, the set in Theorem 1.6 is residual, while this is not true for the sets in Theorem 1.2. However, if $\text{Closure}(\mathcal{M}_{\text{inv}}(\tilde{\Lambda})) = \mathcal{M}_{\text{inv}}(\text{supp}(\omega))$, then points satisfying the equality $V_f(x) = \mathcal{M}_{\text{inv}}(\text{supp}(\omega))$ form a residual set in $\text{supp}(\omega)$.

A point is said to be an *irregular$^+$ point* if there is a continuous function $\phi \in C^0(M, \mathbb{R})$ such that the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x)$$

does not exist. As an application of Theorems 1.2 and 1.6, we have the following result.

THEOREM 1.8. *If* $\text{Closure}(\mathcal{M}_{\text{inv}}(\tilde{\Lambda}))$ *is non-trivial (i.e., contains at least one measure different from ω), then the set of all irregular$^+$ points is dense in* $\text{supp}(\omega)$. *Furthermore, if* $\text{supp}(\omega)$ *is isolated (or* $\text{supp}(\omega) = M$*), then irregular$^+$ points are residual in* $\text{supp}(\omega)$.

We remark that results similar to Theorems 1.6 and 1.8 can be derived for C^1 generic diffeomorphisms on isolated transitive sets by using the barycenter property introduced in [**1**] and the density of periodic measures among invariant measures proved in [**14**].

For any hyperbolic set, note that Katok's shadowing lemma (see Lemma 2.2 below) holds for all points in this set (here, all Pesin blocks are equal to this hyperbolic set). And transitivity can replace ergodicity to get the corresponding specification (see Definition 3.1). Thus, for a uniformly hyperbolic system, if it is supported on one basic set, all our results in this paper are valid, since the hyperbolic basic set is always isolated (and transitive), which implies that the shadowing point can still be in the basic set. In other words, our proofs in the present paper can be another different method to prove the related results in [**4, 11**]. Moreover, we remark that Theorem 1.2 and Corollary 1.4 are still true for any transitive (not necessarily isolated) uniformly hyperbolic sets by our proofs. However, conversely, the proofs in [**4, 11**] fail to derive these without the isolation assumption, since they require the shadowing point always to stay in the given hyperbolic set.

This paper is organized as follows. In §2, we recall the definition of the Pesin set and Katok's shadowing lemma. In §3, we develop a new specification property and verify that (f, ω) admits this property. In §4, we prove three propositions describing the feature of an invariant measure by the time average along some orbit segments. In §5 we use the results in §§3 and 4 to prove Theorem 1.2, and then in §6 we use Theorem 1.2 to prove Theorems 1.6 and 1.8.

2. *Preliminaries*

In this section we recall the definition of the Pesin set and some preliminary lemmas.

2.1. *Pesin set* [**6, 10**]. Given $\lambda, \mu \gg \varepsilon > 0$, and for all $k \in \mathbb{Z}^+$, we define $\Lambda_k = \Lambda_k(\lambda, \mu; \varepsilon)$ to be all points $x \in M$ for which there is a splitting $T_{\text{Orb}(x)}M = E^s_{\text{Orb}(x)} \oplus E^u_{\text{Orb}(x)}$ on the orbit $\text{Orb}(x)$ with the invariant property $D_x f^m(E^s_x) = E^s_{f^m x}$ and $D_x f^m(E^u_x) = E^u_{f^m x}$ satisfying:

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

(a) $\|Df^n|_{E^s_{f^m x}}\| \le e^{\varepsilon k} e^{-(\lambda-\varepsilon)n} e^{\varepsilon|m|}$, for all $m \in \mathbb{Z}$, $n \ge 1$;
(b) $\|Df^{-n}|_{E^u_{f^m x}}\| \le e^{\varepsilon k} e^{-(\mu-\varepsilon)n} e^{\varepsilon|m|}$, for all $m \in \mathbb{Z}$, $n \ge 1$;
(c) $\tan(\angle(E^s_{f^m x}, E^u_{f^m x})) \ge e^{-\varepsilon k} e^{-\varepsilon|m|}$, for all $m \in \mathbb{Z}$.

We set $\Lambda = \Lambda(\lambda, \mu; \varepsilon) = \bigcup_{k=1}^{+\infty} \Lambda_k$ and call Λ a Pesin set (see [**10**, p. 64] or [**6**, p. 146]).

The hyperbolicity requirement of (a), (b) and (c) implies that for every point in a Pesin set, its corresponding invariant splitting is unique, exactly similar to the Axiom A case. It is obvious that if $\varepsilon_1 < \varepsilon_2$ then $\Lambda(\lambda, \mu; \varepsilon_1) \subseteq \Lambda(\lambda, \mu; \varepsilon_2)$.

According to the Oseledec theorem [**8**], ω has s ($s \le d = \dim M$) non-zero Lyapunov exponents

$$\lambda_1 < \cdots < \lambda_r < 0 < \lambda_{r+1} < \cdots < \lambda_s$$

with associated Oseledec splitting

$$T_x M = E^1_x \oplus \cdots \oplus E^s_x, \quad x \in O(\omega),$$

where we recall that $O(\omega)$ denotes an Oseledec basin of ω. If we denote by λ the absolute value of the largest negative Lyapunov exponent λ_r and by μ the smallest positive Lyapunov exponent λ_{r+1} and set $E^s_x = E^1_x \oplus \cdots \oplus E^r_x$, $E^u_x = E^{r+1}_x \oplus \cdots \oplus E^s_x$, then we get a Pesin set $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$ for a small ε. We call it the Pesin set associated with ω. It follows (see, for example, [**10**, Proposition 4.2]) that $\omega(\Lambda \backslash O(\omega)) + \omega(O(\omega) \backslash \Lambda) = 0$. We remark that, by the Oseledec theorem, at each point $x \in O(\omega)$ there is a splitting $T_x M = E^s_x \oplus E^u_x$ with the invariant property $D_x f^m(E^s_x) = E^s_{f^m x}$ and $D_x f^m(E^u_x) = E^u_{f^m x}$.

The following statements are elementary:

(P1) $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3 \subseteq \cdots$;
(P2) $f(\Lambda_k) \subseteq \Lambda_{k+1}$, $f^{-1}(\Lambda_k) \subseteq \Lambda_{k+1}$;
(P3) Λ_k is compact for all $k \ge 1$;
(P4) for all $k \ge 1$ the splitting $x \to E^u_x \oplus E^s_x$ depends continuously on $x \in \Lambda_k$.

2.2. *Shadowing lemma.* Let $\{\delta_k\}_{k=1}^{+\infty}$ be a sequence of positive real numbers. Let $\{x_n\}_{n=-\infty}^{+\infty}$ be a sequence of points in $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$ for which there exists a sequence $\{s_n\}_{n=-\infty}^{+\infty}$ of positive integers satisfying:

(a) $x_n \in \Lambda_{s_n}$, for all $n \in \mathbb{Z}$;
(b) $|s_n - s_{n-1}| \le 1$, for all $n \in \mathbb{Z}$;
(c) $d(f x_n, x_{n+1}) \le \delta_{s_n}$, for all $n \in \mathbb{Z}$;

then we call $\{x_n\}_{n=-\infty}^{+\infty}$ a $\{\delta_k\}_{k=1}^{+\infty}$ pseudo-orbit. Given $\eta > 0$, a point $x \in M$ is an η-shadowing point for the $\{\delta_k\}_{k=1}^{+\infty}$ pseudo-orbit if $d(f^n x, x_n) \le \eta \varepsilon_{s_n}$, for all $n \in \mathbb{Z}$, where $\varepsilon_k = e^{-\varepsilon k}$ and ε is the number from Pesin set $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$.

Remark 2.1. Here ε_k is slightly different from the number in [**6, 10**], where $\varepsilon_k = \varepsilon_0 e^{-\varepsilon k}$, where ε_0 is a constant only dependent on the system of f. But it is easy to see that we can eliminate ε_0 by small modification. This allows our paper to be more transparent as well as easing the exposition.

LEMMA 2.2. (Shadowing lemma [**10**, Theorem 5.1]) *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism, with a non-empty Pesin set $\Lambda = \Lambda(\lambda, \mu; \varepsilon)$ and fixed parameters, $\lambda, \mu \gg \varepsilon > 0$. For any $\eta > 0$ there exists a sequence $\{\delta_k\}_{k=1}^{+\infty}$ such that for any $\{\delta_k\}_{k=1}^{+\infty}$ pseudo-orbit there exists a unique η-shadowing point.*

3. *Specification property for non-uniformly hyperbolic systems*

In this section, we formulate and prove a new version of the specification property that applies to non-uniformly hyperbolic systems as below. The classical specification property allows arbitrary orbit segments to be glued together in uniformly bounded time. In the new version, the connecting time is allowed to depend on which Pesin block Λ_k each segment begins and ends in.

Definition 3.1. Let X be a compact metric space and $f : X \to X$ be a continuous map. We say that f has specification relative to a sequence of increasing sets $\{\Delta_k\}_{k\in\mathbb{N}}$ and a number $\varepsilon > 0$ if, for every $\eta > 0$, there exist integers $\{M_{k,l} = M_{k,l}(\eta)\}_{k,l\in\mathbb{N}}$ such that the following is true.

Given any collection of finite or infinite orbit segments $\Gamma := \{\{f^i(x_s)\}_{i=0}^{n_s}\}_{s\in[a,b]\cap\mathbb{Z}}$, where $a < b \in \mathbb{Z} \cup \{\pm\infty\}$ and $n_s \in \mathbb{N}$, if $\{k_s\}_{s\in[a,b]\cap\mathbb{Z}}$ are such that $x_s, f^{n_s}(x_s) \in \Delta_{k_s}$ for every $s \in [a, b] \cap \mathbb{Z}$, then there exist a shadowing point $z \in M$ and natural numbers $\{c_s\}_{s\in[a-1,b]\cap\mathbb{Z}}$ such that:

(1) $0 \le c_{s+1} - c_s - n_{s+1} \le M_{k_s,k_{s+1}}$ for every $s \in [a - 1, b - 1] \cap \mathbb{Z}$;

(2) $d(f^{c_{s-1}+j}(z), f^j(x_s)) < \eta e^{-\varepsilon k_s}$ for all $0 \le j < n_s$ and $s \in [a, b] \cap \mathbb{Z}$; and

(3) z is periodic with period $c_b - c_{a-1}$ whenever a and b are finite.

Remark 3.2. In our use of this new version of specification, we will take the sets $\{\Delta_k\}_{k\in\mathbb{N}}$ to be the Pesin blocks $\{\tilde{\Lambda}_k\}_{k\in\mathbb{N}}$ introduced in §1 and take ε to be the parameter in the Pesin set $\Lambda(\lambda, \mu; \varepsilon)$. The notion of the Pesin set is the core of the formulation of this new version since it requires some kind of hyperbolicity to realize specification, for example, the shadowing or mixing property. A Pesin set allows certain non-uniform hyperbolicity but it depends on the point in a way that does not admit uniform errors of the pseudo-orbit in the shadowing lemma. However, this is enough to possess the above specification property which considers both the desired closeness of the shadowing orbit to the pseudo-orbit segments and the Pesin set in which the pseudo-orbit lies. In other words, this specification property allows us to glue trajectories together, in such a way that the gluing time depends only on which Λ_k the trajectories start and end in.

Remark 3.3. For any homeomorphism $f : X \to X$ on a compact metric space preserving an ergodic measure ω, if (f, ω) has the specification property defined as above, analogous arguments and results as in Theorem 1.2 and Corollary 1.4 are adaptable. However, the residual results may not be true.

In the following, we prove that the above new specification property applies to $C^{1+\alpha}$ non-uniformly hyperbolic systems, which will play a crucial role in the proof of Theorem 1.2.

THEOREM 3.4. *Let f be a $C^{1+\alpha}$ diffeomorphism of a smooth compact manifold M with an ergodic hyperbolic measure ω and let $\{\Lambda_k(\lambda, \mu; \varepsilon)\}_{k\ge1}$ be the associated Pesin blocks. Let*

$$\tilde{\Lambda}_k = \mathrm{supp}(\omega|_{\Lambda_k(\lambda,\mu;\varepsilon)}).$$

Then (f, ω) has specification relative to the sets $\{\tilde{\Lambda}_k\}_{k\ge1}$ and the number ε.

Remark 3.5. The consequence of Theorem 3.4 in **[7]** or Hirayama's definition in **[5]** for the specification property is a particular case of the above theorem. More precisely, they considered *finitely* many orbit segments and required all the beginning and ending points of these segments to be in the *same fixed* block $\tilde{\Lambda}_k$.

Proof of Theorem 3.4. For any $\eta > 0$, by Lemma 2.2 there exists a sequence $\{\delta_k\}_{k=1}^{+\infty}$ such that for any $\{\delta_k\}_{k=1}^{+\infty}$ pseudo-orbit there exists a unique η-shadowing point.

Let k_* be big enough such that $\omega(\tilde{\Lambda}_k) > 0$ for all $k \geq k_*$. For every $k \geq k_*$, take and fix for $\tilde{\Lambda}_k$ a finite cover $\alpha_k = \{V_1^k, V_2^k, \ldots, V_{r_k}^k\}$ by non-empty open balls V_i^k in M such that $\operatorname{diam}(U_i^k) < \delta_{k+1}$ and $\omega(U_i^k) > 0$ where $U_i^k = V_i^k \cap \tilde{\Lambda}_k$, $i = 1, 2, \ldots, r_k$. Since ω is f-ergodic, by the Birkhoff ergodic theorem we have

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{h=0}^{n-1} \omega(f^{-h}(U_i^l) \cap U_j^k) = \omega(U_i^l)\omega(U_j^k) > 0, \tag{3.1}$$

for all $k, l \geq k_*$, for all $1 \leq i \leq r_l$, $1 \leq j \leq r_k$. Then take

$$X_{i,j}^{k,l} = \min\{h \in \mathbb{N} \mid h \geq 1 + |k-l|, \ \omega(f^{-h}(U_i^l) \cap U_j^k) > 0\}. \tag{3.2}$$

By (3.1), $1 + |k-l| \leq X_{i,j}^{k,l} < +\infty$. Let

$$M_{k,l} = \max_{1\leq i\leq r_k, 1\leq j\leq r_l} X_{i,j}^{k,l}.$$

Remark 3.6. If $y \in \Lambda_k$, $f^n(y) \in \Lambda_l$ for some $n \geq 1 + |k-l|$, then for $s_j(y) := \min\{k + j, l + n - j\}$ $(j = 0, \ldots, n)$ we have $f^j(y) \in \Lambda_{s_j(y)}$ and $|s_j(y) - s_{j-1}(y)| \leq 1$ $(j = 1, \ldots, n)$. More precisely, the fact that $f^{\pm 1}(\Lambda_k) \subset \Lambda_{k+1}$ and $y \in \Lambda_k$, $f^n(y) \in \Lambda_l$ implies that

$$f^j(y) = f^{j-n}(f^n(y)) \in \Lambda_{k+j} \cap \Lambda_{l+n-j} = \Lambda_{\min\{k+j, l+n-j\}}, \quad 0 \leq j \leq n.$$

In particular, we remark that $s_0(y) = k$, $s_n(y) = l$.

We now consider a collection of orbit segments $\Gamma := \{\{f^i(x_s)\}_{i=0}^{n_s}\}_{s\in[a,b]\cap\mathbb{Z}}$, where $a < b \in \mathbb{Z} \cup \{\pm\infty\}$ and $n_s \in \mathbb{N}$, such that x_s, $f^{n_s}(x_s) \in \tilde{\Lambda}_{k_s}$ for every $a \leq s \leq b$. For each $s \in \mathbb{Z}$, we take and fix two integers $a(s)$ and $b(s)$ so that

$$x_s \in U_{a(s)}^{k_s}, \qquad f^{n_s} x_s \in U_{b(s)}^{k_s}.$$

Take $y_s \in U_{b(s)}^{k_s}$ by (3.2) such that $f^{X_{a(s+1),b(s)}^{k_s, k_{s+1}}} y_s \in U_{a(s+1)}^{k_{s+1}}$ for $s \in \mathbb{Z}$. Thus we get a $\{\delta_k\}_{k=1}^{+\infty}$ pseudo-orbit in the Pesin set Λ:

$$\Psi := \cdots \{f^t(x_1)\}_{t=0}^{n_1-1} \cup \{f^t(y_1)\}_{t=0}^{X_{a(2),b(1)}^{k_1,k_2}-1} \cup \{f^t(x_2)\}_{t=0}^{n_2-1} \cup \{f^t(y_2)\}_{t=0}^{X_{a(3),b(2)}^{k_2,k_3}-1} \cup \cdots.$$

More precisely, note that

$$x_s,\ f^{n_s}(x_s) \in \tilde{\Lambda}_{k_s} \subseteq \Lambda_{k_s}, \quad y_s \in \tilde{\Lambda}_{k_s} \subseteq \Lambda_{k_s} \quad \text{and} \quad f^{X_{a(s+1),b(s)}^{k_s,k_{s+1}}} y_s \in \tilde{\Lambda}_{k_{s+1}} \subseteq \Lambda_{k_{s+1}}.$$

By Remark 3.6, every pair of adjacent points in the pseudo-orbit Ψ constructed above are in the same Pesin blocks or adjacent ones (see conditions (a) and (b) in §2.2) since $1 + |k-l| \leq X_{i,j}^{k,l} < +\infty$. From the choice of y_s,

$$d(f^{n_s}(x_s), y_s) < \delta_{k_s+1}, \quad d(f^{X_{a(s+1),b(s)}^{k_s,k_{s+1}}}(y_s), x_{s+1}) < \delta_{k_{s+1}+1} \quad \text{for all } s \in \mathbb{Z}.$$

So Ψ is indeed a $\{\delta_k\}_{k=1}^{+\infty}$ pseudo-orbit in Pesin set Λ.

Hence by Lemma 2.2 there exists an η-shadowing point $z \in M$ such that

$$d(f^{c_{s-1}+j}(z), f^j(x_s)) < \eta\varepsilon_{k_s} = \eta e^{-\varepsilon k_s} \quad \text{for all } j = 0, 1, \ldots, n_s - 1, s \in \mathbb{Z},$$

where

$$c_s = \begin{cases} 0 & \text{for } s = 0, \\ \sum_{j=0}^{s-1} [n_j + X^{k_j, k_{j+1}}_{a(j+1), b(j)}] & \text{for } s > 0, \\ -\sum_{j=s}^{-1} [n_j + X^{k_j, k_{j+1}}_{a(j+1), b(j)}] & \text{for } s < 0. \end{cases}$$

This concludes the proof. □

4. *Characterizing invariant measures by (finite) orbit segments*

It is well known that for ergodic systems the time average is the same for almost all initial points and coincides with the space average due to the Birkhoff ergodic theorem. However, it is not true for general measure-preserving systems (for example, the measure supported on two periodic orbits). Inspired by the ergodic decomposition theorem, we prove in the following that the space average can be approximated by the time average along finitely many orbit segments (not a true orbit).

Given a finite subset $F \subseteq C^0(M, \mathbb{R})$, we denote

$$\|F\| = \max\{\|\xi\| : \xi \in F\}.$$

PROPOSITION 4.1. *Suppose that $f : X \to X$ is a homeomorphism on a compact metric space and ν is an f-invariant measure. Then for any number $\gamma > 0$, any finite subset $F \subseteq C^0(M, \mathbb{R})$ and any set $\Delta \subseteq X$ with $\nu(\Delta) > (1 + (\gamma/16\|F\|))^{-1}$, there is a measurable partition $\{R_j\}_{j=1}^b$ of Δ ($b \in \mathbb{Z}$), such that, for any $x_j \in R_j$, there exists a positive integer T such that, for any integers $T_j \geq T$ ($1 \leq j \leq b$) and any $\xi \in F$,*

$$\left| \int \xi(x)\, d\nu - \sum_{j=1}^b \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j - 1} \xi(f^h(x_j)) \right| < \gamma,$$

for any $\theta_j > 0$ satisfying

$$\left| \theta_j - \frac{\nu(R_j)}{\nu(\Delta)} \right| < \frac{\gamma}{2b\|F\|}, \quad 1 \leq j \leq b \text{ and } \sum_{j=1}^b \theta_j = 1.$$

Proof. Let

$$Q(f) = \left\{ x \in M \,\middle|\, \text{the limit } \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \xi(f^i x) \text{ exists}, \forall \xi \in C^0(M, \mathbb{R}) \right\}.$$

For any $\xi \in C^0(M, \mathbb{R})$ and $x \in Q(f)$, we denote the limit $\lim_{n\to+\infty}(1/n)\sum_{i=0}^{n-1} \xi(f^i x)$ by $\xi^*(x)$. From the Birkhoff ergodic theorem

$$\int \xi(x)\, d\omega = \int \xi^*(x)\, d\omega \quad \text{and} \quad \omega(Q(f)) = 1 \quad \text{for all } \omega \in \mathcal{M}_{\text{inv}}(f), \xi \in C^0(M, \mathbb{R}).$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Let

$$A = A(F) = \sup\{|\xi^*(x)| \mid x \in Q(f), \xi \in F\}.$$

Denote by $[a]$ the largest integer not exceeding a. For $j = 1, \ldots, [(32A\|F\|)/(\gamma\|\xi\|)] + 1$, $\xi \in F$, set

$$Q_j(\xi) = \left\{x \in Q(f) \,\middle|\, -A + \frac{(j-1)\gamma}{16\|F\|}\|\xi\| \le \xi^*(x) < -A + \frac{j\gamma}{16\|F\|}\|\xi\|\right\}.$$

Let

$$\mathcal{B} = \bigvee_{\xi \in F} \{Q_1(\xi), \ldots, Q_{[(32A\|F\|)/(\gamma\|\xi\|)]+1}(\xi)\},$$

where $\alpha \vee \beta = \{A_i \cap B_j \mid A_i \in \alpha, B_j \in \beta\}$, for partitions $\alpha = \{A_i\}$, $\beta = \{B_j\}$. Then $\mathcal{B} = \{R_j\}_{j=1}^b$ is a partition of $Q(f)$. Hence, the positive-measure sets in $\{R_j \cap \Delta\}_{j=1}^b$ form a partition of Δ. For simplicity, we still denote this partition by $\mathcal{B} = \{R_j\}_{j=1}^b$. Then by the definition of $\mathcal{B}$ and $Q_j(\xi)$ above, for every $\xi \in F$,

$$\begin{aligned}
&\left|\int_\Delta \xi^*(x)\,d\nu - \sum_{j=1}^b \theta_j \xi^*(x_j)\right| \\
&\le \left|\int_\Delta \xi^*(x)\,d\nu - \sum_{j=1}^b \frac{\nu(R_j)}{\nu(\Delta)}\xi^*(x_j)\right| + \left|\sum_{j=1}^b \frac{\nu(R_j)}{\nu(\Delta)}\xi^*(x_j) - \sum_{j=1}^b \theta_j \xi^*(x_j)\right| \\
&\le \left|\int_\Delta \xi^*(x)\,d\nu - \sum_{j=1}^b \nu(R_j)\xi^*(x_j)\right| + \left|\sum_{j=1}^b \nu(R_j)\xi^*(x_j) - \sum_{j=1}^b \frac{\nu(R_j)}{\nu(\Delta)}\xi^*(x_j)\right| \\
&\quad + \left|\sum_{j=1}^b \left(\frac{\nu(R_j)}{\nu(\Delta)} - \theta_j\right)\xi^*(x_j)\right| \\
&\le \sum_{j=1}^b \nu(R_j) \max_{y \in R_j} |\xi^*(y) - \xi^*(x_j)| + \left|\sum_{j=1}^b \nu(R_j)\xi^*(x_j)\right| \cdot \left(\frac{1}{\nu(\Delta)} - 1\right) \\
&\quad + \frac{\gamma}{2b\|F\|}\sum_{j=1}^b |\xi^*(x_j)| \\
&\le \frac{1}{8\|F\|} \cdot \gamma\|\xi\| \cdot \sum_{j=1}^b \nu(R_j) + A \cdot \sum_{j=1}^b \nu(R_j) \cdot \frac{\gamma}{16\|F\|} + \frac{\gamma A}{2\|F\|} \\
&\le \frac{1}{8\|F\|} \cdot \gamma\|\xi\| + \frac{\gamma A}{16\|F\|} + \frac{\gamma A}{2\|F\|} \le \frac{11\gamma}{16}.
\end{aligned}$$

For the last inequality, note that $A \le \|F\|$.

On the other hand, we shall take T large enough such that, for all $T_j \ge T$ and $\xi \in F$,

$$\left|\xi^*(x_j) - \frac{1}{T_j}\sum_{h=0}^{T_j - 1} \xi(f^h(x_j))\right| < \frac{\gamma}{16} \quad \text{for all } j = 1, 2, \ldots, b.$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Thus, for every $\xi \in F$, we can deduce that

$$\begin{aligned}
&\left|\int_\Delta \xi^*(x)\,d\nu - \sum_{j=1}^{b} \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\right| \\
&\quad\le \left|\int_\Delta \xi^*(x)\,d\nu - \sum_{j=1}^{b} \theta_j \xi^*(x_j)\right| \\
&\qquad + \left|\sum_{j=1}^{b} \theta_j \xi^*(x_j) - \sum_{j=1}^{b} \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\right| \\
&\quad\le \frac{11\gamma}{16} + \left|\sum_{j=1}^{b} \theta_j \left(\xi^*(x_j) - \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\right)\right| \\
&\quad\le \frac{11\gamma}{16} + \sum_{j=1}^{b} \theta_j \frac{\gamma}{16} = \frac{3\gamma}{4}.
\end{aligned}$$

Note that

$$\nu(\Delta) > \left(1 + \frac{\gamma}{16\|F\|}\right)^{-1} > 1 - \frac{\gamma}{16\|F\|}.$$

Hence,

$$\begin{aligned}
&\left|\int_M \xi(x)\,d\nu - \sum_{j=1}^{b} \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\right| \\
&\quad= \left|\int_M \xi^*(x)\,d\nu - \sum_{j=1}^{b} \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\right| \\
&\quad\le \left|\int_M \xi^*(x)\,d\nu - \int_\Delta \xi^*(x)\,d\nu\right| + \left|\int_\Delta \xi^*(x)\,d\nu - \sum_{j=1}^{b} \theta_j \frac{1}{T_j} \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\right| \\
&\quad\le \|\xi^*\| \cdot (1 - \nu(\Delta)) + \frac{3\gamma}{4} \\
&\quad\le A \frac{\gamma}{16\|F\|} + \frac{3\gamma}{4} < \gamma.
\end{aligned}$$

□

The following lemma is a generalization of [**7**, Lemma 3.7].

LEMMA 4.2. *Let $f : X \to X$ be a homeomorphism of a compact metric space preserving an invariant measure ω. Let $\Gamma_j \subset X$ be finitely many measurable sets with $\omega(\Gamma_j) > 0$ and, for $x \in \Gamma_j$, let*

$$S(x, \Gamma_j) := \{r \in \mathbb{N} \mid f^r x \in \Gamma_j\},$$

$j = 1, \ldots, k$. Take $0 < \gamma < 1$, $0 < \beta \le 1$ and $T \ge 1$. Then there exist sets $\tilde{\Gamma}_j \subset \Gamma_j$ with $\omega(\tilde{\Gamma}_j) = \omega(\Gamma_j)$ such that, for any $x_j \in \tilde{\Gamma}_j$ $(j = 1, \ldots, k)$, there exist $n_1, n_2, \ldots, n_k$ with $n_j \in S(x_j, \Gamma_j)$ such that $n_j \ge T$ and

$$0 < \frac{\theta_1|n_1 - n_j| + \cdots + \theta_{j-1}|n_{j-1} - n_j| + \theta_{j+1}|n_{j+1} - n_j| + \cdots + \theta_k|n_k - n_j|}{\sum_{i=1}^{k} \theta_i n_i} < \gamma$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

holds for any $j = 1, \ldots, k$ *and for all* $\theta_j > 0$ *satisfying*

$$\frac{\min_{1 \le i \le k}\{\theta_i\}}{\max_{1 \le i \le k}\{\theta_i\}} \ge \beta, \quad \sum_{i=1}^{k} \theta_i = 1.$$

Proof. The statement of [**7**, Lemma 3.7] is just the particular case of $\beta = 1$ (i.e., $\theta_j = 1/k$). Though [**7**, Lemma 3.7] assumes that ω is ergodic, from its proof in [**7**] one does not need any ergodicity since the main technique is [**2**, Lemma 3.12] whose statement just assumes that ω is invariant. Here we use this particular case and the following estimate to conclude the proof.

By the particular case of [**7**, Lemma 3.7] we can take T_j (required above) such that

$$\left|\frac{|T_1 - T_j| + \cdots + |T_{j-1} - T_j| + |T_{j+1} - T_j| + \cdots + |T_b - T_j|}{\sum_{i=1}^{b} T_i}\right| \le \beta\gamma.$$

Then for all $\theta_j > 0$ satisfying

$$\frac{\min_{1 \le i \le k}\{\theta_i\}}{\max_{1 \le i \le k}\{\theta_i\}} \ge \beta, \quad \sum_{j=1}^{k} \theta_j = 1,$$

$$\begin{aligned}
&\left|\frac{\theta_1(T_1 - T_j) + \cdots + \theta_{j-1}(T_{j-1} - T_j) + \theta_{j+1}(T_{j+1} - T_j) + \cdots + \theta_b(T_b - T_j)}{\sum_{i=1}^{b} \theta_i T_i}\right| \\
&\quad \le \frac{\max_{1 \le i \le b}\{\theta_i\}}{\min_{1 \le i \le b}\{\theta_i\}} \cdot \left|\frac{|T_1 - T_j| + \cdots + |T_{j-1} - T_j| + |T_{j+1} - T_j| + \cdots + |T_b - T_j|}{\sum_{i=1}^{b} T_i}\right| \\
&\quad < \beta^{-1}\beta\gamma = \gamma.
\end{aligned}$$

□

The Poincaré recurrence theorem states that for a set with positive measure, almost all points in this set will return to it, after a sufficiently long time. Such points are called *recurrence points* and *the Poincaré recurrence time* is the length of time elapsed until the recurrence. The set $\tilde{\Gamma}_j$ in Lemma 4.2 is in fact the subset of recurrent points in Γ_j .

PROPOSITION 4.3. *Suppose that* $f : X \to X$ *is a homeomorphism on a compact metric space and* ν *is an* f*-invariant measure. Then for any number* $\gamma > 0$*, any finite subset* $F \subseteq C^0(M, \mathbb{R})$ *and any set* $\Delta \subseteq X$ *with* $\nu(\Delta) > (1 + (\gamma/32\|F\|))^{-1}$*, there is a measurable partition* $\{R_j\}_{j=1}^{b}$ *of* Δ *(*$b \in \mathbb{Z}$*) such that, for any positive integer* T *and any recurrence points* $x_j \in R_j$*, there exist recurrence times* $T_j \ge T$ *(*$1 \le j \le b$*) satisfying, for every* $\xi \in F$*,*

$$\left|\int \xi(x)\, d\nu - \frac{1}{\sum_{j=1}^{b} \theta_j T_j} \sum_{j=1}^{b} \theta_j \sum_{h=0}^{T_j - 1} \xi(f^h(x_j))\right| < \gamma,$$

for any $\theta_j > 0$ *satisfying*

$$\left|\theta_j - \frac{\nu(R_j)}{\nu(\Delta)}\right| < \min\left\{\frac{\gamma}{4b\|F\|}, \frac{1}{2}\frac{\nu(R_j)}{\nu(\Delta)}\right\}, \quad 1 \le j \le b \text{ and } \sum_{j=1}^{b} \theta_j = 1.$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Proof. Since

$$\nu(\Delta) > \left(1+\frac{\gamma}{32\|F\|}\right)^{-1} = \left(1+\frac{(\gamma/2)}{16\|F\|}\right)^{-1},$$

we can take a partition $\{R_j\}_{j=1}^{b}$ as in Proposition 4.1, just replacing γ with $\gamma/2$. Take

$$\beta := \frac{\min_{1\le i\le b}\{(1/2)(\nu(R_j)/\nu(\Delta))\}}{\max_{1\le i\le b}\{(3/2)(\nu(R_j)/\nu(\Delta))\}} > 0.$$

By Poincaré's recurrence lemma, ν-almost every point in R_j is a recurrence point. We consider recurrence points $x_j \in R_j$ $(j = 1, \ldots, b)$. By the finiteness of F and Lemma 4.2, for any $T > 0$ we can choose integers $T_j \ge T$ such that $f^{T_j}(x_j) \in R_j$ $(j = 1, \ldots, b)$ and

$$\left|\frac{\theta_1(T_1-T_j)+\cdots+\theta_{j-1}(T_{j-1}-T_j)+\theta_{j+1}(T_{j+1}-T_j)+\cdots+\theta_b(T_b-T_j)}{\sum_{i=1}^{b}\theta_i T_i}\right| < \frac{\gamma}{32\|F\|}$$

for all $\theta_j > 0$ satisfying

$$\left|\theta_j - \frac{\nu(R_j)}{\nu(\Delta)}\right| < \min\left\{\frac{\gamma}{4b\|F\|}, \frac{1}{2}\frac{\nu(R_j)}{\nu(\Delta)}\right\}, \quad 1 \le j \le b \text{ and } \sum_{j=1}^{b}\theta_j = 1.$$

In the process we use the inequality

$$\frac{\min_{1\le i\le b}\{\theta_i\}}{\max_{1\le i\le b}\{\theta_i\}} \ge \frac{\min_{1\le i\le b}\{(1/2)(\nu(R_j)/\nu(\Delta))\}}{\max_{1\le i\le b}\{(3/2)(\nu(R_j)/\nu(\Delta))\}} = \beta.$$

Thus, for every $\xi \in F$,

$$\begin{aligned}
&\left|\sum_{j=1}^{b}\theta_j\frac{1}{T_j}\sum_{h=0}^{T_j-1}\xi(f^h(x_j)) - \frac{1}{\sum_{i=1}^{b}\theta_i T_i}\sum_{j=1}^{b}\theta_j\sum_{h=0}^{T_j-1}\xi(f^h(x_j))\right| \\
&\quad= \left|\sum_{j=1}^{b}\frac{\theta_j}{(\theta_1+\cdots+\theta_b)}\frac{1}{T_j}\sum_{h=0}^{T_j-1}\xi(f^h(x_j)) - \sum_{j=1}^{b}\frac{\theta_j}{\theta_1T_1+\cdots+\theta_bT_b}\sum_{h=0}^{T_j-1}\xi(f^h(x_j))\right| \\
&\quad= \left|\sum_{j=1}^{b}\theta_j\frac{\theta_1(T_1-T_j)+\cdots+\theta_{j-1}(T_{j-1}-T_j)+\theta_{j+1}(T_{j+1}-T_j)+\cdots+\theta_b(T_b-T_j)}{\sum_{i=1}^{b}\theta_i T_i}\right. \\
&\qquad \left.\cdot\frac{1}{T_j}\sum_{h=0}^{T_j-1}\xi(f^h(x_j))\right| \\
&\quad\le \left|\sum_{j=1}^{b}\theta_j\frac{\theta_1(T_1-T_j)+\cdots\theta_{j-1}(T_{j-1}-T_j)+\theta_{j+1}(T_{j+1}-T_j)+\cdots\theta_b(T_b-T_j)}{\sum_{i=1}^{b}\theta_i T_i}\right|\cdot\|\xi\| \\
&\quad\le \sum_{j=1}^{b}\theta_j\left|\frac{\theta_1(T_1-T_j)+\cdots\theta_{j-1}(T_{j-1}-T_j)+\theta_{j+1}(T_{j+1}-T_j)+\cdots\theta_b(T_b-T_j)}{\sum_{i=1}^{b}\theta_i T_i}\right|\cdot\|\xi\| \\
&\quad\le \sum_{j=1}^{b}\theta_j\frac{\gamma}{32\|F\|}\|\xi\| \le \frac{\gamma}{32}.
\end{aligned} \tag{4.3}$$

Note that

$$\nu(\Delta) > \left(1+\frac{\gamma}{32\|F\|}\right)^{-1} > 1-\frac{\gamma}{32\|F\|}.$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Combining with Proposition 4.1 and inequality (4.3), one deduces that, for every $\xi \in F$,

$$\begin{aligned}
&\bigg|\int_M \xi(x)\,d\nu - \frac{1}{\sum_{i=1}^b \theta_i T_i}\sum_{j=1}^b \theta_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\bigg| \\
&\le \bigg|\int_M \xi(x)\,d\nu - \sum_{j=1}^b \theta_j \frac{1}{T_j}\sum_{h=0}^{T_j-1} \xi(f^h(x_j))\bigg| \\
&\quad + \bigg|\sum_{j=1}^b \theta_j \frac{1}{T_j}\sum_{h=0}^{T_j-1} \xi(f^h(x_j)) - \frac{1}{\sum_{i=1}^b \theta_i T_i}\sum_{j=1}^b \theta_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\bigg| \\
&\le \frac{\gamma}{2} + \frac{\gamma}{32} \\
&< \gamma,
\end{aligned}$$

and the proof is complete. □

Remark 4.4. Through the proof of the previous proposition, one can obtain that the conclusion is suitable for any finer partition of $\{R_j\}_{j=1}^b$, which will be used in the proof of the next proposition.

The previous two propositions are stated in a general setting and considering homeomorphisms on a compact space with an f-invariant measure. In the following result we go back to focus on an f-invariant measure ν supported on $\tilde{\Lambda}$ with respect to an $f \in \mathrm{Diff}^{1+\alpha}(M)$ and an ergodic hyperbolic measure ω for f.

PROPOSITION 4.5. *Let ν be an f-invariant measure supported on $\tilde{\Lambda}$. Then for any number $\zeta > 0$ and any finite subset $F \subseteq C^0(M, \mathbb{R})$, there is a number $k_\nu \in \mathbb{Z}^+$ such that, for any numbers $\delta > 0$ and $T > 0$, there exist positive numbers $\{n_j\}_{j=1}^c$ with $n_j > T$ and orbit segments $\{z_j, fz_j, \ldots, f^{n_j-1}z_j\}_{j=1}^c$ with $z_j, f^{n_j}z_j \in \tilde{\Lambda}_{k_\nu}$ and $d(f^{n_j}z_j, z_{(j+1) \bmod c}) < \delta$, $j = 1, \ldots, c$, satisfying, for all $\xi \in F$,*

$$\bigg|\int \xi(x)\,d\nu - \frac{1}{\sum_{j=1}^c n_j}\sum_{j=1}^c \sum_{h=0}^{n_j-1} \xi(f^h(z_j))\bigg| < \zeta.$$

Proof. Take k_ν large such that

$$\nu(\tilde{\Lambda}_{k_\nu}) > \bigg(1 + \frac{\zeta}{32\|F\|}\bigg)^{-1}.$$

Applying Proposition 4.3 with $\Delta = \tilde{\Lambda}_{k_\nu}$ and $\gamma = \zeta$ and Remark 4.4, we can obtain a finite partition $\{R_j\}_{j=1}^b$ of $\tilde{\Lambda}_{k_\nu}$ with $\mathrm{diam} R_j < \delta$ and recurrence points $x_j \in R_j$ with large recurrence times $T_j > T$, $j = 1, \ldots, b$, satisfying, for all $\xi \in F$,

$$\bigg|\int \xi(x)\,d\nu - \frac{1}{\sum_{j=1}^b \theta_j T_j}\sum_{j=1}^b \theta_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j))\bigg| < \zeta, \tag{4.4}$$

for any $\theta_j > 0$ satisfying

$$\sum_{j=1}^k \theta_j = 1 \quad \text{and} \quad \bigg|\theta_j - \frac{\nu(R_j)}{\nu(\tilde{\Lambda}_{k_\nu})}\bigg| < \min\bigg\{\frac{\zeta}{4b\|F\|}, \frac{1}{2}\frac{\nu(R_j)}{\nu(\Delta)}\bigg\}, \quad 1 \le j \le b.$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Recall that ω is ergodic and thus, for any $1 \le j \le b$, there is an integer $X_j \ge T$ such that

$$f^{X_j} R_j \cap R_{j+1} \ne \emptyset, \quad 1 \le j < b,$$

and

$$f^{X_b} R_b \cap R_1 \ne \emptyset.$$

Take $y_j \in R_j$ so that $f^{X_j} y_j \in R_{j+1}$, $1 \le j < b$, and $f^{X_b} y_b \in R_1$.

For ζ, F and b, there exists $S \in \mathbb{N}$ such that, for any integer $s > S$, we have

$$0 < 1/s < \min\left\{\frac{\zeta}{4b\|F\|}, \frac{1}{2}\frac{\nu(R_j)}{\nu(\Delta)}\right\}.$$

Then there exist integers $\bar{s}_1, \bar{s}_2, \ldots, \bar{s}_b$ satisfying

$$\frac{\bar{s}_j}{s} \le \frac{\nu(R_j)}{\nu(\tilde{\Lambda}_{k_\nu})} \le \frac{\bar{s}_j + 1}{s}.$$

It follows from taking $s_j = \bar{s}_j$ or $\bar{s}_j + 1$ that

$$s = \sum_{j=1}^{b} s_j \quad \text{and} \quad \left|\frac{\nu(R_j)}{\nu(\tilde{\Lambda}_{k_\nu})} - \frac{s_j}{s}\right| \le \frac{1}{s} < \min\left\{\frac{\zeta}{4b\|F\|}, \frac{1}{2}\frac{\nu(R_j)}{\nu(\Delta)}\right\}.$$

Taking T_j large enough such that $\sum_{j=1}^{b} X_j \ll \sum_{j=1}^{b} s_j T_j$,

$$\left|\frac{1}{\sum_{j=1}^{b}(s_j T_j + X_j)} \sum_{j=1}^{b}\left(s_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) + \sum_{h=0}^{X_j-1} \xi(f^h(y_j))\right) - \frac{1}{\sum_{j=1}^{b} s_j T_j} \sum_{j=1}^{b} s_j \sum_{h=0}^{T_j} \xi(f^h(x_j))\right| < \zeta.$$

Let $\theta_j = s_j/s$. Observe that the definition of θ_j yields $\sum_{j=1}^{k} \theta_j = 1$. The above inequality and inequality (4.4) imply that

$$\left|\int \xi(x)\,d\nu - \frac{1}{\sum_{j=1}^{b}(s_j T_j + X_j)} \sum_{j=1}^{b}\left(s_j \sum_{h=0}^{T_j-1} \xi(f^h(x_j)) + \sum_{h=0}^{X_j-1} \xi(f^h(y_j))\right)\right| < 3\zeta. \tag{4.5}$$

Let

$$z_1 = \cdots = z_{s_1} = x_1, \quad z_{s_1+1} = y_1,$$
$$z_{s_1+2} = \cdots = z_{s_1+s_2+1} = x_2, \quad z_{s_1+s_2+2} = y_2,$$
$$\cdots$$
$$z_{\sum_{h=1}^{j} s_h + j+1} = \cdots = z_{\sum_{h=1}^{j+1} s_h + j} = x_{j+1},$$
$$z_{\sum_{h=1}^{j+1} s_h + j+1} = y_{j+1},$$
$$\cdots$$
$$z_{\sum_{h=1}^{b-1} s_h + b} = \cdots = z_{\sum_{h=1}^{b} s_h + b-1} = x_b,$$
$$z_{\sum_{h=1}^{b} s_h + b} = y_b.$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

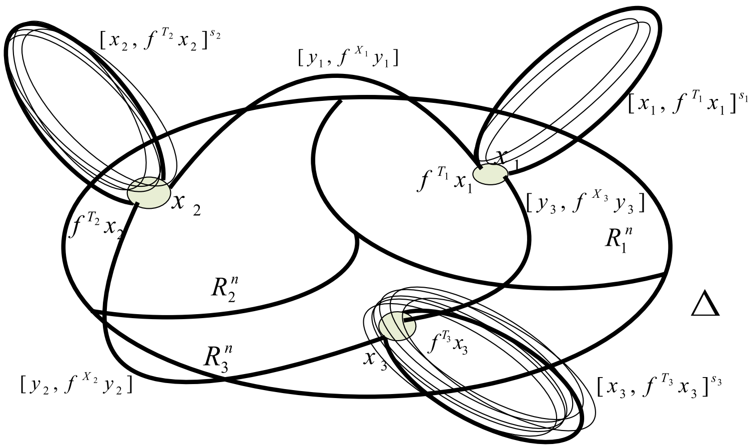

FIGURE 1. Schematic for choice of periodic pseudo-orbit (here $[x, f^T x]^s$ indicates that the number of loops on x is s).

These $\{z_j\}_{j=1}^{\sum_{h=1}^{b} s_h+b}$ are the points we want in the proposition by the previous evaluation (4.5) with

$$c = \sum_{h=1}^{b} s_h + b$$

and

$$n_{\sum_{h=1}^{l-1} s_h+t} = T_l, \quad n_{\sum_{h=1}^{l} s_h+l} = X_l,$$

for $1 \le l \le b$, $l \le t \le s_l - 1$ (see Figure 1 for more explanation). □

5. *Proof of Theorem 1.2*

In this section, we prove Theorem 1.2 by using the specification property developed in §3 and Proposition 4.5 in §4.

Proof of Theorem 1.2. If $\{\varphi_j\}_{j=1}^{\infty}$ is a dense subset of $C^0(M, \mathbb{R})$, then

$$\tilde{d}(\nu, m) = \sum_{j=1}^{\infty} \frac{|\int \varphi_j \, d\nu - \int \varphi_j \, dm|}{2^j \|\varphi_j\|}$$

is a metric on $\mathcal{M}(M)$ giving the weak* topology (see, for example, [**15**]). It is well known that $\mathcal{M}_{\text{inv}}(f)$ is a compact metric subspace of $\mathcal{M}(M)$. For any non-empty connected compact set $V \subseteq \text{Closure}(\mathcal{M}_{\text{inv}}(\tilde{\Lambda}))$, there exists a sequence of open balls B_n in $\mathcal{M}_{\text{inv}}(f)$ with radius ζ_n in the metric $\tilde{d}$ such that the following hold:

(a) $B_n \cap B_{n+1} \cap V \neq \emptyset$;

(b) $\bigcap_{N=1}^{\infty} \bigcup_{n \ge N} B_n = V$;

(c) $\lim_{n \to +\infty} \zeta_n = 0$.

Since B_n is open, by (a) we can take $Y_n \in B_n \cap \mathcal{M}_{\text{inv}}(\tilde{\Lambda})$. Note that Y_n is in $\mathcal{M}_{\text{inv}}(\tilde{\Lambda})$ and thus satisfies $Y_n(\tilde{\Lambda}) = 1$.

Remark 5.1. In [**12**] or [**4**, Proposition 21.14], Sigmund assumes that Y_n is an atomic measure and thus its information can be characterized by its support (periodic orbit). Hence it remains to deal with these periodic orbits by means of the specification property for Axiom A systems. But for our case we cannot directly take Y_n as an atomic measure. The main observation is that the support of these periodic measures may not be contained in $\tilde{\Lambda}$, and therefore the specification property as in Theorem 3.4 becomes invalid. So we just emphasize that Y_n must be in V and thus satisfies $Y_n(\tilde{\Lambda}) = 1$. This allows us to choose pseudo-orbits in $\tilde{\Lambda}$ whose information can characterize that of Y_n and for which the specification property is valid.

Take a finite set $F_n = \{\varphi_j\}_{j=1}^n \subseteq \{\varphi_j\}_{j=1}^\infty$. Let $x_* \in \tilde{\Lambda}$ be given. Fix $\delta_* > 0$ and let U_0 be the open ball of radius δ_* around x_* and let k_0 be large enough such that $x_* \in \Lambda_{k_0}$. We have to show that there exists an $x \in U_0$ such that $V = V_f(x)$. We divide the following proof into four steps.

Step 1. An estimation of Y_n $(n \geq 1)$ via a single pseudo-orbit.

Fix $0 < \eta \leq \delta_*$. Fix $n \in \mathbb{N}$. For ζ_n, F_n, by Proposition 4.5 we choose $k_n = k(Y_n)$ such that, for any $T > 0$, there exist orbit segments $\{z_j^n,\ f z_j^n, \ldots,\ f^{p_{n,j}-1} z_j^n\}_{j=1}^{c_n}$ with $p_{n,j} > T$ and z_j^n, $f^{p_{n,j}} z_j^n \in \tilde{\Lambda}_{k_n}$, satisfying

$$\bigg| \int \xi(x)\, dY_n - \frac{1}{\sum_{j=1}^{c_n} p_{n,j}} \sum_{j=1}^{c_n} \sum_{h=0}^{p_{n,j}-1} \xi(f^h(z_j^n)) \bigg| < \zeta_n \quad \text{for all } \xi \in F_n. \tag{5.6}$$

Moreover, we can take $k_n < k_{n+1}$ for all n. Let $M_n = M_{k_{n-1},k_n}(\eta)$ and $L_n = M_{k_n,k_n}(\eta)$ be numbers defined as in Theorem 3.4, and, for every n, take the above number T to be $2^n L_n$.

Remark 5.2. Note that if we use Katok's shadowing lemma (Lemma 2.2) and its corollary (Theorem 3.4) simultaneously, then we can choose $\{\delta_k\}_{k=1}^{+\infty}$ such that the shadowing property holds. This can furthermore guarantee that $d(f^{p_{n,j}} z_j^n, z_{j+1}^n) < \delta_{k_{n+1}}$ $(j = 1, \ldots, c_n - 1)$ when we use Proposition 4.5 for $\delta = \delta_{k_{n+1}}$. In other words, for any fixed n, these orbit segments $\{z_j^n,\ f z_j^n, \ldots,\ f^{p_{n,j}-1} z_j^n\}_{j=1}^{c_n}$ form a 'periodic' pseudo-orbit. This implies that we do not need the numbers L_n and the corresponding parts in the following steps (see Figure 2 for a rough idea: the thick line denotes and contains the pseudo-orbit of $\{z_j^n,\ f z_j^n, \ldots,\ f^{p_{n,j}-1} z_j^n\}$ and the thin line denotes and contains the shadowing orbit of $\hat{x}$). But here we want to give a proof just using the newly introduced specification of Theorem 3.4 (which verifies our Remark 3.3) allowing some time lag for the shadowing between $f^{p_{n,j}} z_j^n$ and z_{j+1}^n (see Figure 3).

Step 2. Finding a point $\hat{x} \in U_0$ tracing this pseudo-orbit.

Denote $p_n = \sum_{j=1}^{c_n} p_{n,j} + c_n L_n$. Define

$$\begin{aligned}
&\bar{a}_0 = \bar{b}_0 = 0, \\
\bar{a}_1 = \bar{b}_0 + M_1, \quad &\bar{b}_1 = \bar{a}_1 + 2(\bar{a}_1 + M_2 + p_2) p_1, \\
\bar{a}_2 = \bar{b}_1 + M_2, \quad &\bar{b}_1 = \bar{a}_1 + 2^2(\bar{a}_2 + M_3 + p_3) p_2, \\
&\cdots \\
\bar{a}_n = \bar{b}_{n-1} + M_n, \quad &\bar{b}_n = \bar{a}_n + 2^n(\bar{a}_n + M_{n+1} + p_{n+1}) p_n, \\
&\cdots.
\end{aligned}$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

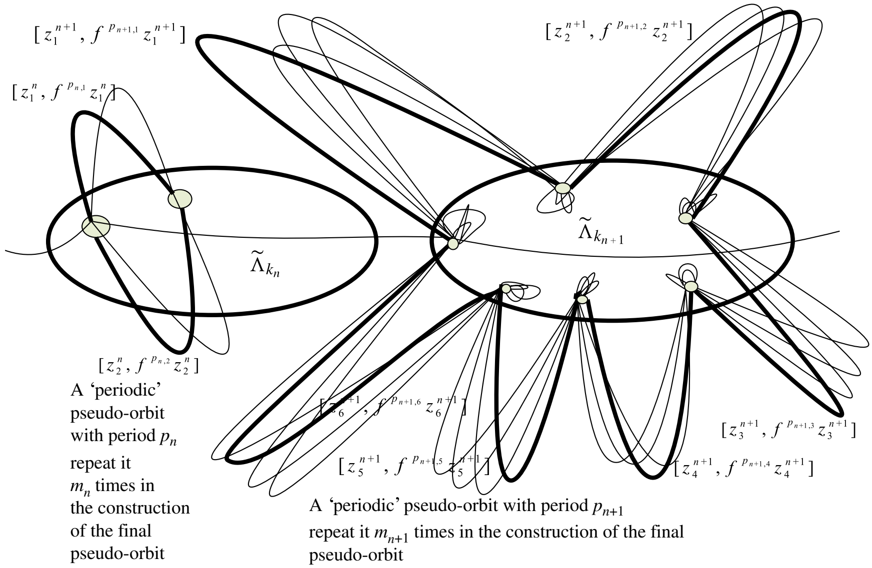

FIGURE 2. Schematic for choice of $\hat{x}$ to shadow pseudo-orbit by shadowing.

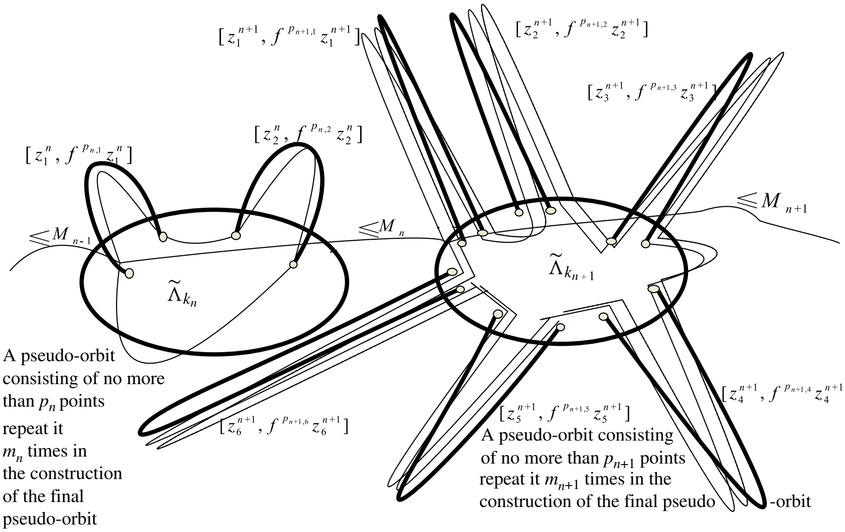

FIGURE 3. Schematic for choice of $\hat{x}$ to shadow pseudo-orbit by specification.

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Using the specification of Theorem 3.4, we can find a point $\hat{x} \in \Lambda$, δ_*-close to x_*, which η-shadows each orbit segment $\{z_j^n, \ldots, f^{p_{n,j}-1}(z_j^n)\}$, $j = 1, 2, \ldots, c_n$, $m_n = 2^n(\bar{a}_n + M_{n+1} + p_{n+1})$ times for all n and runs from a neighborhood of $f^{p_{n,j}} z_j^n$ to that of z_{j+1}^n with a time lag of no more than L_n (here for convenience we assume that the time lag from a neighborhood of $f^{p_{n,j}} z_j^n$ to that of z_{j+1}^n is exactly L_n since in this way we have enlarged the error made by the time lag to its maximum) and from a neighborhood of $f^{p_{n,c_n}} z_{c_n}^n$ to that of z_1^{n+1} with a time lag of no more than M_{n+1}. Note that if the time lag from a neighborhood of $f^{p_{n,j}} z_j^n$ to that of z_{j+1}^n is much smaller, the estimation will be much easier as can be seen in the calculation below. So for convenience and to avoid using too many new symbols, here we assume the worst case where all these time lags are the same and are equal to the maximum. Then there exist two increasing sequences of integers $\{a_n\}$, $\{b_n\}$ with

$$a_0 = b_0 = 0,$$
$$b_n = a_n + m_n p_n \quad \text{and} \quad a_n - b_{n-1} \le M_n,$$

such that

$$d(f^h \hat{x}, f^{(h-a_n) \mod p_n - \sum_{i=1}^{j-1} p_{n,i} - (j-1)L_n} z_j^n) < \eta e^{-\varepsilon k_n}, \tag{5.7}$$

for all $a_n \le h \le b_n$ satisfying

$$\sum_{i=1}^{j-1} p_{n,i} + (j-1)L_n < (h - a_n) \mod p_n \le \sum_{i=1}^{j} p_{n,i} + (j-1)L_n,$$
$$1 \le j \le c_n, n \in \mathbb{N}.$$

Remark 5.3. Note that $a_n \le \bar{a}_n$, $b_n \le \bar{b}_n$ and

$$b_n - a_n = \bar{b}_n - \bar{a}_n = m_n p_n, \quad a_n - b_{n-1} \le \bar{a}_n - \bar{b}_{n-1} = M_n.$$

So as $n \to +\infty$, b_n and a_{n+1} become much larger than a_n, M_{n+1}, p_n and p_{n+1} since $m_n \to +\infty$. And by the definition of m_n, one can obtain that

$$a_n \ll m_n \sum_{j=1}^{c_n} p_{n,j} \quad \text{and} \quad M_{n+1} \ll m_n \sum_{j=1}^{c_n} p_{n,j}.$$

The original technique for Axiom A systems in [**12**] and [**4**, Proposition 21.14] is not suitable for non-uniformly hyperbolic ones. Sigmund [**4, 12**] uses the specification property to build inductively a sequence of periodic orbits such that the nth orbit shadows both the $(n-1)$th orbit and the support of the nth center. In this process the support of the centers and these shadowing periodic orbits are always in the hyperbolic set such that the specification property can be used iteratively. Finally, these periodic orbits converge to one orbit of some point $\hat{x}$. However, for the non-uniform hyperbolic case, Sigmund's idea encounters a difficulty. That is, the specification property cannot be used more than once, since we cannot predetermine the Pesin block in which the shadowing periodic orbits stay. Therefore, to deal with non-uniformly hyperbolic cases, we introduce a new specification property, inspired by Katok's shadowing lemma. More precisely, we construct a pseudo-orbit consisting of infinitely many orbit segments and use Katok's shadowing lemma once and for all to find the required $\hat{x}$ as shown in (5.7) and hence avoid induction.

Step 3. Verifying $V \subseteq V_f(\hat{x})$.

Let $\nu \in V$ be given. By (b) and (c) there exists an increasing sequence $n_k \uparrow \infty$ such that

$$Y_{n_k} \to \nu. \tag{5.8}$$

Let $\xi \in \{\varphi_j\}_{j=1}^{\infty} = \bigcup_{n\geq 1} F_n$ be given. Then there is an integer $n_\xi > 0$ such that $\xi \in F_n$, for any $n \geq n_\xi$. Denote by $w_\xi(\gamma)$ the oscillation

$$\max\{\|\xi(y) - \xi(z)\| \mid d(y, z) \leq \gamma\}$$

and by ν_n the measure $\delta_*(\hat{x})^{b_n}$. Thus

$$\int \xi \, d\nu_n = \frac{1}{b_n} \sum_{j=0}^{b_n-1} \xi(f^j \hat{x}). \tag{5.9}$$

Remark that if A is a finite subset of $\mathbb{N}$,

$$\left| \frac{1}{\#A} \sum_{j\in A} \varphi(f^j x) - \frac{1}{\max A + 1} \sum_{j=0}^{\max A} \varphi(f^j x) \right| \leq \frac{2(\max A + 1 - \#A)}{\#A} \|\varphi\| \tag{5.10}$$

for any $x \in M$ and $\varphi \in C^0(M, \mathbb{R})$, where $\#A$ denotes the cardinality of the set A. Inequality (5.10), with

$$A_n = \bigcup_{r=1}^{m_n} \bigcup_{j=1}^{c_n} \Bigg[a_n + (r-1)p_n + \sum_{i=1}^{j-1} p_{n,i} + (j-1)L_n, a_n + (r-1)p_n + \sum_{i=1}^{j} p_{n,i} + (j-1)L_n \Bigg],$$

implies that

$$\left| \frac{1}{m_n \sum_{j=1}^{c_n} p_{n,j}} \sum_{j\in A_n} \xi(f^j \hat{x}) - \frac{1}{b_n} \sum_{j=0}^{b_n-1} \xi(f^j \hat{x}) \right| \leq \frac{2(b_n - m_n \sum_{j=1}^{c_n} p_{n,j})}{m_n \sum_{j=1}^{c_n} p_{n,j}} \|\xi\| \quad \text{for all } n \geq n_\xi. \tag{5.11}$$

On the other hand, combining inequalities (5.6) and (5.7), one can obtain that

$$\left| \int \xi \, dY_n - \frac{1}{m_n \sum_{j=1}^{c_n} p_{n,j}} \sum_{j\in A_n} \xi(f^j \hat{x}) \right| \leq \zeta_n + w_\xi(\eta e^{-\varepsilon k_n}) \quad \text{for all } n \geq n_\xi. \tag{5.12}$$

Note that

$$\frac{2(b_n - m_n \sum_{j=1}^{c_n} p_{n,j})}{m_n \sum_{j=1}^{c_n} p_{n,j}} = \frac{2(a_n + c_n m_n L_n)}{m_n \sum_{j=1}^{c_n} p_{n,j}} = \frac{2a_n}{m_n \sum_{j=1}^{c_n} p_{n,j}} + \frac{2c_n L_n}{\sum_{j=1}^{c_n} p_{n,j}} \to 0 \quad \text{as } n \to \infty \tag{5.13}$$

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

due to Remark 5.3 and the choice of the segments $\{z_j^n, \ldots, f^{p_{n,j}-1}(z_j^n)\}$ and $\zeta_n \to 0$ and $w_\xi(\eta e^{-\varepsilon k_n}) \to 0$ since $k_n \to +\infty$, as $n \to \infty$. It can be deduced by (5.9), (5.11) and (5.12) that

$$\left|\int \xi \, d\nu_n - \int \xi \, dY_n\right| \le \zeta_n + w_\xi(\eta e^{-\varepsilon k_n}) + \frac{2(b_n - m_n \sum_{j=1}^{c_n} p_{n,j})}{m_n \sum_{j=1}^{c_n} p_{n,j}} \|\xi\| \to 0 \quad \text{as } n \to \infty.$$

Hence, together with (5.8), it implies that $\nu_{n_k} \to \nu$ and thus $\nu \in V_f(\hat{x})$. Therefore, $V \subseteq V_f(\hat{x})$.

Step 4. Verifying $V_f(\hat{x}) \subseteq V$.

Let $\nu \in V_f(\hat{x})$ be given. There exists a sequence $n_k \uparrow \infty$ such that $\nu_{n_k} \to \nu$. Let $\gamma > 0$ and $\xi \in \{\varphi_j\}_{j=1}^{\infty} = \bigcup_{n \ge 1} F_n$ be given. For fixed n_k, let $i = i(n_k)$ be the largest integer such that $b_{i-1} \le n_k$. Let n_k (and hence i) be so large that

$$w_\xi(2^{-i+1}) < w_\xi(2^{-i+2}) < \frac{\gamma}{4}.$$

Let $\alpha = 1$ if $b_{i-1} \le n_k \le a_i$. Otherwise $a_i < n_k \le b_i$. Write $A = A_{i-1} \cup (A_i \cap [a_i, n_k))$ and define

$$\alpha = (\#A_{i-1})(\#A)^{-1}.$$

We divide the proof into two cases.

Case 1. $n_k = a_i + r_k p_i + \sum_{j=1}^{t_k-1} p_{i,j} + (t_k - 2)L_i + l_k$ with $0 \le l_k < L_i$, $0 \le r_k < m_i$ and $0 \le t_k < c_i$.

Recall that

$$\int \xi \, d\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} \xi(f^j \hat{x}) \quad \text{for all } n \in \mathbb{N}.$$

Note that, in this case, $\max A + 1 = n_k - l_k$. Using inequality (5.10) again, with A as above, we obtain

$$\begin{aligned}
&\left|\int \xi \, d\nu_{n_k - l_k} - \frac{1}{\#A}\left(\sum_{j \in A_{i-1}} \xi(f^j \hat{x}) + \sum_{j \in A_i \cap [a_i, n_k)} \xi(f^j \hat{x})\right)\right| \\
&\quad \le 2(n_k - l_k - \#A)(\#A)^{-1} \|\xi\| \\
&\quad \le 2\left(\frac{a_{i-1}}{m_{i-1} \sum_{j=1}^{c_{i-1}} p_{i-1,j}} + \frac{m_{i-1} c_{i-1} L_{i-1}}{m_{i-1} \sum_{j=1}^{c_{i-1}} p_{i-1,j}} + \frac{M_i}{m_{i-1} \sum_{j=1}^{c_{i-1}} p_{i-1,j}} \right. \\
&\qquad \left. + \frac{r_k c_i L_i}{r_k \sum_{j=1}^{c_i-1} p_{i,j}} + \frac{(t_k - 2)L_i}{\sum_{j=1}^{t_k-1} p_{i,j}}\right) \|\xi\| \\
&\quad \le \gamma \|\xi\|
\end{aligned} \tag{5.14}$$

provided the n_k are large enough, due to Remark 5.3.

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Note that

$$\begin{aligned}
&\left|\int \xi \, d\nu_{n_k} - \int \xi \, d\nu_{n_k-l_k}\right| \\
&= \left|\frac{1}{n_k}\sum_{j=0}^{n_k-1}\xi(f^j\hat{x}) - \frac{1}{n_k-l_k}\sum_{j=0}^{n_k-l_k-1}\xi(f^j\hat{x})\right| \\
&= \frac{1}{n_k(n_k-l_k)}\left|(-l_k)\sum_{j=0}^{n_k-l_k-1}\xi(f^j\hat{x}) + (n_k-l_k)\sum_{j=n_k-l_k}^{n_k-1}\xi(f^j\hat{x})\right| \\
&\le \frac{l_k}{n_k}\|\xi\| + \frac{l_k}{n_k}\|\xi\| \\
&\le \frac{2L_i}{n_k}\|\xi\| \\
&\le \gamma\|\xi\| \qquad (5.15)
\end{aligned}$$

provided the n_k are large enough.

Case 2. $n_k = a_i + r_k \mod p_i + \sum_{j=1}^{t_k-1} p_{i,j} + (t_k-1)L_i + l_k$ with $0 \le l_k < p_{i,t_k} < p_i$, $0 \le r_k < m_i$ and $0 \le t_k < c_i$.

In this case $\max A + 1 = n_k - l_k - L_i$. Using inequality (5.10) again, with A as above, we obtain two inequalities analogous to (5.14) and (5.15):

$$\left|\int \xi \, d\nu_{n_k-l_k-L_i} - \frac{1}{\#A}\left(\sum_{j\in A_{i-1}}\xi(f^j\hat{x}) + \sum_{j\in A_i\cap[a_i,n_k)}\xi(f^j\hat{x})\right)\right| \le \gamma\|\xi\| \qquad (5.14')$$

and

$$\left|\int \xi \, d\nu_{n_k} - \int \xi \, d\nu_{n_k-l_k-L_i}\right| \le \gamma\|\xi\| \qquad (5.15')$$

provided the n_k are large enough, due to Remark 5.3.

Then combining (5.14) with inequality (5.15) or (5.14′) with (5.15′),

$$\left|\int \xi \, d\nu_{n_k} - -\frac{1}{\#A}\left(\sum_{j\in A_{i-1}}\xi(f^j\hat{x}) + \sum_{j\in A_i\cap[a_i,n_k)}\xi(f^j\hat{x})\right)\right| \le 2\gamma\|\xi\|.$$

This implies that

$$\begin{aligned}
&\left|\int \xi \, d\nu_{n_k} - \left[\alpha\frac{1}{\#A_{i-1}}\sum_{j\in A_{i-1}}\xi(f^j\hat{x}) + (1-\alpha)\frac{1}{\#A-\#A_{i-1}}\sum_{j\in A_i\cap[a_i,n_k)}\xi(f^j\hat{x})\right]\right| \\
&\le 2\gamma\|\xi\|.
\end{aligned}$$

Remark 5.4. In [**4**, Proposition 21.14], Sigmund defined

$$a_0 = b_0 = 0$$

and

$$a_i = b_{i-1} + M_i, \quad b_i = a_i + 2^i(a_i + M_{i+1})p_i, \quad i \in \mathbb{N}.$$

It is obvious that these b_{i-1} and a_i were chosen independent of p_i. Here, in our definition (before (5.7)), b_{i-1} and a_i are chosen much larger not only than a_{i-1}, M_i, p_{i-1} but also

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

than p_i. This is one of the important differences with respect to Sigmund's proof. In fact, step 4 in Sigmund's proof only holds for some special cases since he assumes that

$$n_k - a_i = mp_i.$$

The remainder l_k is not greater than p_i. However, in his proof, the period p_i may not be small compared to the lag $b_{i-1} - a_{i-1}$, and hence l_k may not be small enough with respect to $b_{i-1} - a_{i-1}$, which is necessary to the proof as shown in the above inequality (5.15).

Set

$$\rho_{n_k} = \alpha Y_{i-1} + (1-\alpha) Y_i.$$

Using inequality (5.12),

$$\left| \int \xi \, d\nu_{n_k} - \int \xi \, d\rho_{n_k} \right| \leq 3\gamma \|\xi\|$$

for k large enough such that $n_k \gg n_\xi$ (defined below (5.8)). Thus ρ_{n_k} has the same limit as ν_{n_k}, that is, ν.

On the other hand, the limit of ρ_{n_k} has to be in V, since

$$\tilde{d}(\rho_{n_k}, V) \leq \tilde{d}(\rho_{n_k}, Y_i) \leq \tilde{d}(Y_{i-1}, Y_i) \leq 2\zeta_{i-1} + 2\zeta_i$$

and $\zeta_i \downarrow 0$. Hence, $\nu \in V$.

The arbitrariness of $x_* \in \tilde{\Lambda}$ and δ_* implies the density of $\hat{x}$ in $\tilde{\Lambda}$. Note that $\tilde{\Lambda} \subseteq \text{supp}(\omega)$ and $\omega(\tilde{\Lambda}) = 1$, and ω is an ergodic measure. All these conditions ensure that $\text{Closure}(\tilde{\Lambda}) = \text{supp}(\omega)$. Hence, such $\hat{x}$ are dense in $\text{supp}(\omega)$. This concludes the proof. □

6. *Proof of Theorem 1.6 and 1.8*

In this section, we use Theorem 1.2 (or Corollary 1.4) to prove Theorem 1.6 and then use Theorem 1.6 to prove Theorem 1.8.

Proof of Theorem 1.6. Note that if $\text{supp}(\omega)$ is isolated, then the orbit of the chosen point $\hat{x}$ in Theorem 1.2 is in $\text{supp}(\omega)$ provided that δ_* is small enough, since $\tilde{\Lambda} \subseteq \text{supp}(\omega)$ and the orbit of $\hat{x}$ is δ_* close to a $\{\delta_k\}$-pseudo-orbit contained completely in $\tilde{\Lambda}$ (containing the inverse orbit of x_*) by using Katok's shadowing lemma and the argument in the proof of Theorem 3.4. So for any $\nu \in \text{Closure}(\mathcal{M}_{\text{inv}}(\tilde{\Lambda}))$, its generic points of Corollary 1.4 can be chosen in $\text{supp}(\omega)$ and dense in $\text{supp}(\omega)$. However, we remark that if we only use the specification of Definition 3.1, we cannot guarantee that the orbit of the chosen point $\hat{x}$ can be totally in a small neighborhood of $\text{supp}(\omega)$ since we cannot control the iterates of $\hat{x}$ running from a neighborhood of $f^{p_{n,c_n}} z^n_{c_n}$ to that of z^{n+1}_1 with a time lag of no more than M_{n+1} (see Step 2).

We now show that the set of points in $\text{supp}(\omega)$ with maximal oscillation contains a dense G_δ-set. The proof is not difficult and is analogous to [**4**, Proof of Proposition 21.18].

Since $\text{supp}(\omega)$ is always compact, $\mathcal{M}(\text{supp}(\omega))$ is compact and convex. Thus we can find open balls B_n, C_n in $\mathcal{M}(\text{supp}(\omega))$ such that:

(a) $B_n \subset \text{Closure}(B_n) \subset C_n$;
(b) $\text{diam}\, C_n \to 0$;
(c) $B_n \cap \text{Closure}(\mathcal{M}_{\text{inv}}(\tilde{\Lambda})) \neq \emptyset$;
(d) each point of $\text{Closure}(\mathcal{M}_{\text{inv}}(\tilde{\Lambda}))$ lies in infinitely many B_n.

https://doi.org/10.1017/S0143385711000940 Published online by Cambridge University Press

Put

$$P(C_n) = \{x \in \mathrm{supp}(\omega) \mid V_f(x) \cap C_n \neq \emptyset\} \quad \text{for all } n \in \mathbb{Z}^+.$$

It can be verified that the set of points with maximal oscillation is just $\bigcap_{n\geq 1} P(C_n)$. Note that

$$\begin{aligned} P(C_n) &\supseteq \{x \in \mathrm{supp}(\omega) \mid \forall N_0 \in \mathbb{Z}^+, \exists N > N_0 \text{ with } \delta(x)^N \in B_n\} \\ &\supseteq \bigcap_{N_0=1}^{\infty} \bigcup_{N>N_0} \{x \in \mathrm{supp}(\omega) \mid \delta(x)^N \in B_n\}. \end{aligned}$$

Since $x \to \delta(x)^N$ is continuous (for fixed N), the sets $\bigcup_{N>N_0}\{x \in \mathrm{supp}(\omega) \mid \delta(x)^N \in B_n\}$ are open. Since $B_n \cap \mathrm{Closure}(\mathcal{M}_{\mathrm{inv}}(\tilde{\Lambda})) \neq \emptyset$, these sets are also dense, as shown in the first paragraph above. Hence, $\bigcap_{n\geq 1} P(C_n)$ contains a dense G_δ-set. □

Proof of Theorem 1.8. Let x be a point having maximal oscillation. By assumption there exist at least two invariant measures $\mu_1 \neq \mu_2 \in V_f(x)$. So there is a continuous function ϕ such that

$$\int \phi \, d\mu_1 \neq \int \phi \, d\mu_2. \tag{6.16}$$

Due to the definition of $V_f(x)$, there are two sequences of integers $n_k, m_k \to +\infty$ such that

$$\delta^{n_k}(x) \to \mu_1, \quad \delta^{m_k}(x) \to \mu_2.$$

These imply that

$$\lim_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \phi(f^j x) = \int \phi \, d\mu_1 \quad \text{and} \quad \lim_{k\to+\infty} \frac{1}{m_k} \sum_{j=0}^{m_k-1} \phi(f^j x) = \int \phi \, d\mu_2.$$

Combining these equalities with (6.16), we can deduce that

$$\lim_{k\to+\infty} \frac{1}{n_k} \sum_{j=0}^{n_k-1} \phi(f^j x) = \int \phi \, d\mu_1 \neq \int \phi \, d\mu_2 = \lim_{k\to+\infty} \frac{1}{m_k} \sum_{j=0}^{m_k-1} \phi(f^j x)$$

does not exist. □

Acknowledgements. The authors wish to thank the whole seminar of dynamical systems in Peking University and the referee for his constructive suggestions and careful reading. Liang is supported by the National Natural Science Foundation of China (# 10901167 and # 10671006). Sun is supported by the National Natural Science Foundation of China (# 10671006, # 10831003) and the National Basic Research Program of China (973 Program, # 2006CB805903). Tian is the corresponding author and supported by CAPES.

REFERENCES

[1] F. Abdenur, C. Bonatti and S. Crovisier. Non-uniform hyperbolicity of C^1-generic diffeomorphisms. *Israel J. Math.* to appear.

[2] J. Bochi. Genericity of zero Lyapunov exponents. *Ergod. Th. & Dynam. Sys.* **22** (2002), 1667–1696.

[3] R. Bowen. Periodic orbits for hyperbolic flows. *Amer. J. Math.* **94** (1972), 1–30.

[4] M. Denker, C. Grillenberger and K. Sigmund. *Ergodic Theory on Compact Spaces (Lecture Notes in Mathematics, 527)*. Springer, Berlin.

[5] M. Hirayama. Periodic probability measures are dense in the set of invariant measures. *Discrete Contin. Dyn. Syst.* **9** (2003), 1185–1192.

[6] A. Katok. Liapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **51** (1980), 137–173.

[7] C. Liang, G. Liu and W. Sun. Approximation properties on invariant measure and Oseledec splitting in non-uniformly hyperbolic systems. *Trans. Amer. Math. Soc.* **361** (2009), 1543–1579.

[8] V. I. Oseledec. Multiplicative ergodic theorem, Liapunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.* **19** (1968), 197–221, translated from Russian.

[9] C. Pfister and W. Sullivan. On the topological entropy of saturated sets. *Ergod. Th. & Dynam. Sys.* **27** (2007), 929–956.

[10] M. Pollicott. *Lectures on Ergodic Theory and Pesin Theory on Compact Manifolds*. Cambridge University Press, Cambridge, 1993.

[11] K. Sigmund. Generic properties of invariant measures for Axiom A-diffeomorphisms. *Invent. Math.* **11** (1970), 99–109.

[12] K. Sigmund. On dynamical systems with the specification property. *Trans. Amer. Math. Soc.* **190** (1974), 285–299.

[13] W. Sun and X. Tian. Pesin set, closing lemma and shadowing lemma in C^1 non-uniformly hyperbolic systems with limit domination, arXiv:1004.0486.

[14] W. Sun and X. Tian. The structure on invariant measures of C^1 generic diffeomorphisms. *Acta Math. Sinica* to appear.

[15] P. Walters. *An Introduction to Ergodic Theory*. Springer, Berlin, 2001.