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EXTREMA OF MULTI-DIMENSIONAL GAUSSIAN PROCESSES OVER RANDOM INTERVALS

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Abstract

This paper studies the joint tail asymptotics of extrema of the multi-dimensional Gaussian process over random intervals defined as $P(u) := \mathbb{P}\{\cap_{i=1}^{n}(\sup_{t\in[0,\mathcal{T}_i]} (X_i(t) + c_i t) > a_i u)\}$, $u \to \infty$, where $X_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, n$, are independent centered Gaussian processes with stationary increments, $\boldsymbol{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ is a regularly varying random vector with positive components, which is independent of the Gaussian processes, and $c_i \in \mathbb{R}$, $a_i > 0$, $i = 1, 2, \ldots, n$. Our result shows that the structure of the asymptotics of $P(u)$ is determined by the signs of the drifts c_i. We also discuss a relevant multi-dimensional regenerative model and derive the corresponding ruin probability.

Keywords: Joint tail asymptotic; Gaussian process; perturbed random walk; ruin probability; fluid model; fractional Brownian motion; regenerative model.

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1. Introduction

Let $X(t)$, $t \geq 0$, be an almost surely (a.s.) continuous centered Gaussian process with stationary increments and $X(0) = 0$. Motivated by its applications to the hybrid fluid and ruin models, the seminal paper [18] derived the exact tail asymptotics of

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} X(t) > u\right\}, \quad u \to \infty, \tag{1.1}$$

with $\mathcal{T}$ being a regularly varying random variable independent of the Gaussian process X. Since then, the study of the tail asymptotics of supremum on random interval has attracted substantial interest in the literature. We refer to [1], [2], [3], [10], [11], and [36] for various extensions to general (non-centered) Gaussian or Gaussian-related processes. In these contributions, various different tail distributions for $\mathcal{T}$ have been discussed, and it has been shown that the variability of $\mathcal{T}$ influences the form of the asymptotics of (1.1), leading to qualitatively different structures.

The primary aim of this paper is to analyse the asymptotics of a multi-dimensional counterpart of (1.1). More precisely, consider a multi-dimensional centered Gaussian process

$$\boldsymbol{X}(t) = (X_1(t), X_2(t), \ldots, X_n(t)), \quad t \geq 0, \tag{1.2}$$

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with independent coordinates, each $X_i(t)$, $t \ge 0$, has stationary increments, a.s. continuous sample paths and $X_i(0) = 0$, and let $\boldsymbol{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_n)$ be a regularly varying random vector with positive components, which is independent of the multi-dimensional Gaussian process $\boldsymbol{X}$ in (1.2) (we use $\boldsymbol{X}$ for short). We are interested in the exact asymptotics of

$$P(u) := \mathbb{P}\left\{\cap_{i=1}^n \left(\sup_{t\in[0,\mathcal{T}_i]} (X_i(t) + c_i t) > a_i u\right)\right\}, \quad u \to \infty, \tag{1.3}$$

where $c_i \in \mathbb{R}$, $a_i > 0$, $i = 1, 2, \ldots, n$.

Extremal analysis of multi-dimensional Gaussian processes has been an active research area in recent years; see [5], [12], [13], [15], [17], and [29], and references therein. Most of these contributions discuss the asymptotic behaviour of the probability that $\boldsymbol{X}$ (possibly with trend) enters an upper orthant over a finite-time or infinite-time interval; this problem is also connected with the conjunction problem for Gaussian processes first studied by Worsley and Friston [38]. Investigations of the joint tail asymptotics of multiple extrema as defined in (1.3) are known to be more challenging. The current literature has only focused on the case with deterministic times $\mathcal{T}_1 = \cdots = \mathcal{T}_n$ and some additional assumptions on the correlation structure of the X_i. In [17] and [31] large deviation type results are obtained, and more recently in [14] and [16] exact asymptotics are obtained for correlated two-dimensional Brownian motion. It is worth mentioning that a large deviation result for the multivariate maxima of a discrete Gaussian model has been discussed recently in [37].

In order to avoid more technical difficulties, the coordinates of the multi-dimensional Gaussian process $\boldsymbol{X}$ in (1.2) are assumed to be independent. The dependence among the extrema in (1.3) is driven by the structure of the multivariate regularly varying $\boldsymbol{\mathcal{T}}$. Interestingly, we observe in Theorem 3.1 that the form of the asymptotics of (1.3) is determined by the signs of the drifts c_i.

Apart from its theoretical interest, the motivation to analyse the asymptotic properties of $P(u)$ is related to numerous applications in modern multi-dimensional risk theory, financial mathematics, or fluid queueing networks. For example, we consider an insurance company that runs n lines of business. The surplus process of the ith business line can be modelled by a time-changed Gaussian process

$$R_i(t) = a_i u + c_i Y_i(t) - X_i(Y_i(t)), \quad t \ge 0,$$

where $a_i u > 0$ is the initial capital (considered as a proportion of u allocated to the ith business line, with $\sum_{i=1}^n a_i = 1$), $c_i > 0$ is the net premium rate, $X_i(t)$, $t \ge 0$, is the net loss process, and $Y_i(t)$, $t \ge 0$, is a positive increasing function modelling the so-called 'operational time' for the ith business line. We refer to [4, 23] and [11], respectively, for detailed discussions of multi-dimensional risk models and time-changed risk models. Of interest in risk theory is the study of the probability of ruin of all the business lines within some finite (deterministic) time $T > 0$, defined by

$$\varphi(u) := \mathbb{P}\left\{\cap_{i=1}^n \left(\inf_{t\in[0,T]} R_i(t) < 0\right)\right\} = \mathbb{P}\left\{\cap_{i=1}^n \left(\sup_{t\in[0,T]} (X_i(Y_i(t)) + c_i Y_i(t)) > a_i u\right)\right\}.$$

If additionally all the operational time processes $Y_i(t)$, $t \ge 0$, have a.s. continuous sample paths, then we have $\varphi(u) = P(u)$ with $\boldsymbol{\mathcal{T}} = \boldsymbol{Y}(T)$, and thus the derived result can be used to estimate this ruin probability. Note that the dependence among different business lines is introduced by the dependence among the operational time processes Y_i. As a simple example we can consider $Y_i(t) = \Theta_i t$, $t \ge 0$, with $\boldsymbol{\Theta} = (\Theta_1, \ldots, \Theta_n)$ being a multivariate regularly varying random vector. Additionally, multi-dimensional time-changed (or subordinate) Gaussian

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processes have recently been proved to be good candidates for modelling the log-return processes of multiple assets; see e.g. [6], [25], and [26]. As the joint distribution of extrema of asset returns is important in finance problems (e.g. [20]), we expect the results obtained for (1.3) might also be interesting in financial mathematics.

As a relevant application, we shall discuss a multi-dimensional regenerative model, which is motivated by its relevance to risk models and fluid queueing models. Essentially, the multi-dimensional regenerative process is a process with a random alternating environment, where an independent multi-dimensional fractional Brownian motion (fBm) with trend is assigned at each environment alternating time. We refer to Section 4 for more details. By analysing a related multi-dimensional perturbed random walk, we obtain in Theorem 4.1 the ruin probability of the multi-dimensional regenerative model. This generalizes some of the results in [28] and [40] to the multi-dimensional setting. Note in passing that some related stochastic models with random sampling or resetting have been discussed in the recent literature; see e.g. [9], [24], and [32].

Organization of the rest of the paper. In Section 2 we introduce some notation, recall the definition of multivariate regular variation, and present some preliminary results on the extremes of one-dimensional Gaussian processes. The result for (1.3) is displayed in Section 3, and the ruin probability of the multi-dimensional regenerative model is discussed in Section 4. The proofs are relegated to Sections 5 and 6. Some useful results on multivariate regular variation are discussed in the Appendix.

2. Notation and preliminaries

We shall use some standard notation that is common when dealing with vectors. All the operations on vectors are meant componentwise. For instance, for any given $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $\boldsymbol{xy} = (x_1y_1, \ldots, x_ny_n)$, and write $\boldsymbol{x} > \boldsymbol{y}$ if and only if $x_i > y_i$ for all $1 \le i \le n$. Furthermore, for two positive functions f, h and some $u_0 \in [-\infty, \infty]$, write $f(u) \lesssim h(u)$ or $h(u) \gtrsim f(u)$ if $\limsup_{u\to u_0} f(u)/h(u) \le 1$, write $h(u) \sim f(u)$ if $\lim_{u\to u_0} f(u)/h(u) = 1$, write $f(u) = \mathrm{o}(h(u))$ if $\lim_{u\to u_0} f(u)/h(u) = 0$, and write $f(u) \asymp h(u)$ if $f(u)/h(u)$ is bounded from both below and above for all sufficiently large u. Moreover, $\boldsymbol{Z}_1 \overset{D}{=} \boldsymbol{Z}_2$ means that $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ have the same distribution.

Next, let us recall the definition and some implications of multivariate regular variation. We refer to [21], [22], and [34] for more detailed discussions. Let $\overline{\mathbb{R}}_0^n = \overline{\mathbb{R}}^n \setminus \{\boldsymbol{0}\}$ with $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. An $\mathbb{R}^n$-valued random vector $\boldsymbol{X}$ is said to be *regularly varying* if there exists a non-null Radon measure ν on the Borel σ-field $\mathcal{B}(\overline{\mathbb{R}}_0^n)$ with $\nu(\overline{\mathbb{R}}^n \setminus \mathbb{R}^n) = 0$ such that

$$\frac{\mathbb{P}\{x^{-1}\boldsymbol{X} \in \cdot\}}{\mathbb{P}\{|\boldsymbol{X}| > x\}} \overset{v}{\to} \nu(\,\cdot\,), \quad x \to \infty.$$

Here $|\cdot|$ is any norm in $\mathbb{R}^n$ and $\overset{v}{\to}$ refers to vague convergence on $\mathcal{B}(\overline{\mathbb{R}}_0^n)$. It is known that ν necessarily satisfies the homogeneity property $\nu(sK) = s^{-\alpha}\nu(K)$, $s > 0$, for some $\alpha > 0$ and any Borel set K in $\mathcal{B}(\overline{\mathbb{R}}_0^n)$. In what follows, we say that such a defined $\boldsymbol{X}$ is regularly varying with index α and limiting measure ν. An implication of the homogeneity property of ν is that all the rectangle sets of the form $[\boldsymbol{a}, \boldsymbol{b}] = \{\boldsymbol{x} : \boldsymbol{a} \le \boldsymbol{x} \le \boldsymbol{b}\}$ in $\overline{\mathbb{R}}_0^n$ are ν-continuity sets. Furthermore, we find that $|\boldsymbol{X}|$ is regularly varying at infinity with index α, i.e. $\mathbb{P}\{|\boldsymbol{X}| > x\} \sim x^{-\alpha}L(x)$, $x \to \infty$, with some slowly varying function $L(x)$. Some useful results on multivariate regular variation are discussed in the Appendix.

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In what follows, we review some results on the extremes of one-dimensional Gaussian process with negative drift derived in [19]. Let $X(t)$, $t \geq 0$, be an a.s. continuous centered Gaussian process with stationary increments and $X(0) = 0$, and let $c > 0$ be some constant. We shall present the exact asymptotics of

$$\psi(u) := \mathbb{P}\left\{\sup_{t\geq 0} (X(t) - ct) > u\right\}, \quad u \to \infty.$$

Below are some assumptions that the variance function $\sigma^2(t) = \mathrm{Var}\,(X(t))$ might satisfy:

C1 σ is continuous on $[0, \infty)$ and ultimately strictly increasing;

C2 σ is regularly varying at infinity with index H for some $H \in (0, 1)$;

C3 σ is regularly varying at 0 with index λ for some $\lambda \in (0, 1)$;

C4 σ^2 is ultimately twice continuously differentiable and its first derivative $\dot{\sigma}^2$ and second derivative $\ddot{\sigma}^2$ are both ultimately monotone.

Note that in the above $\dot{\sigma}^2$ and $\ddot{\sigma}^2$ denote the first and second derivative of σ^2, not the square of the derivatives of σ. Henceforth, provided it exists, we let $\overleftarrow{\sigma}$ denote an asymptotic inverse near infinity or zero of σ; recall that it is (asymptotically uniquely) defined by $\overleftarrow{\sigma}(\sigma(t)) \sim \sigma(\overleftarrow{\sigma}(t)) \sim t$. It depends on the context whether $\overleftarrow{\sigma}$ is an asymptotic inverse near zero or infinity.

One known example that satisfies the assumptions **C1–C4** is the fBm $\{B_H(t),\ t \geq 0\}$ with Hurst index $H \in (0, 1)$, i.e. an H-self-similar centered Gaussian process with stationary increments and covariance function given by

$$\mathrm{Cov}\,(B_H(t), B_H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad t, s \geq 0.$$

We introduce the following notation:

$$C_{H,\lambda_1,\lambda_2} = \sqrt{2^{1-1/\lambda_2}\pi}\,\lambda_1 \left(\frac{1}{H}\right)^{1/\lambda_2} \left(\frac{H}{1-H}\right)^{\lambda_1 + H - 1/2 + (1/\lambda_2)(1-H)}.$$

For an a.s. continuous centered Gaussian process $Z(t)$, $t \geq 0$, with stationary increments and variance function σ_Z^2, we define the generalized Pickands constant

$$\mathcal{H}_Z = \lim_{T\to\infty} \frac{1}{T}\,\mathbb{E}\left\{\exp\left(\sup_{t\in[0,T]} (\sqrt{2}Z(t) - \sigma_Z^2(t))\right)\right\}$$

provided both the expectation and the limit exist. When $Z = B_H$, the constant $\mathcal{H}_{B_H}$ is the well-known Pickands constant; see [30]. For convenience, sometimes we also write $\mathcal{H}_{\sigma_Z^2}$ for $\mathcal{H}_Z$. In the following we let $\Psi(\cdot)$ denote the survival function of the $N(0,1)$ distribution. It is known that

$$\Psi(u) = \frac{1}{\sqrt{2\pi}} \int_u^{\infty} \mathrm{e}^{-x^2/2}\,\mathrm{d}x \sim \frac{1}{\sqrt{2\pi}\,u}\mathrm{e}^{-u^2/2}, \quad u \to \infty. \tag{2.1}$$

The following result is derived in Proposition 2 of [19] (here we consider a particular trend function $\phi(t) = ct$, $t \geq 0$).

Proposition 2.1. *Let $X(t)$, $t \geq 0$, be an a.s. continuous centered Gaussian process with stationary increments and $X(0) = 0$. Suppose that **C1–C4** hold. We have the following, as $u \to \infty$.*

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(i) *If* $\sigma^2(u)/u \to \infty$, *then*

$$\psi(u) \sim \mathcal{H}_{B_H} C_{H,1,H} \left(\frac{1-H}{H}\right) \frac{c^{1-H}\sigma(u)}{\overleftarrow{\sigma}(\sigma^2(u)/u)} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(ut/c)}\right).$$

(ii) *If* $\sigma^2(u)/u \to \mathcal{G} \in (0, \infty)$, *then*

$$\psi(u) \sim \mathcal{H}_{(2c^2/\mathcal{G}^2)\sigma^2} \left(\frac{\sqrt{2/\pi}}{c^{1+H}H}\right) \sigma(u) \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(ut/c)}\right).$$

(iii) *If* $\sigma^2(u)/u \to 0$, *then (here we need regularity of* σ *and its inverse at* 0*)*

$$\psi(u) \sim \mathcal{H}_{B_\lambda} C_{H,1,\lambda} \left(\frac{1-H}{H}\right)^{H/\lambda} \frac{c^{-1-H+2H/\lambda}\sigma(u)}{\overleftarrow{\sigma}(\sigma^2(u)/u)} \Psi\left(\inf_{t\geq 0} \frac{u(1+t)}{\sigma(ut/c)}\right).$$

As a special case of the Proposition 2.1 we have the following result (see [19, Corollary 1] or [23]). This will be useful in the proofs below.

Corollary 2.1. *If* $X(t) = B_H(t)$, $t \geq 0$, *the fBm with index* $H \in (0, 1)$, *then as* $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\geq 0} (B_H(t) - ct) > u\right\} \sim K_H \mathcal{H}_{B_H} u^{H+1/H-2} \Psi\left(\frac{c^H u^{1-H}}{H^H(1-H)^{1-H}}\right),$$

with constant

$$K_H = 2^{1/2-1/(2H)} \frac{\sqrt{\pi}}{\sqrt{H(1-H)}} \left(\frac{c^H}{H^H(1-H)^{1-H}}\right)^{1/H-1}.$$

3. Main results

Without loss of generality, we assume that in (1.3) there are n_- coordinates with negative drift, n_0 coordinates without drift, and n_+ coordinates with positive drift, that is,

$$\begin{aligned} c_i &< 0, & i &= 1, \ldots, n_-, \\ c_i &= 0, & i &= n_- + 1, \ldots, n_- + n_0, \\ c_i &> 0, & i &= n_- + n_0 + 1, \ldots, n, \end{aligned}$$

where $0 \leq n_-, n_0, n_+ \leq n$ such that $n_- + n_0 + n_+ = n$. We impose the following assumptions on the standard deviation functions $\sigma_i(t) = \sqrt{\mathrm{Var}(X_i(t))}$ of the Gaussian processes $X_i(t)$, $i = 1, \ldots, n$.

Assumption I. *For* $i = 1, \ldots, n_-$, $\sigma_i(t)$ *satisfies the assumptions* ***C1–C4*** *with the parameters involved indexed by i. For* $i = n_- + 1, \ldots, n_- + n_0$, $\sigma_i(t)$ *satisfies the assumptions* ***C1–C3*** *with the parameters involved indexed by i. For* $i = n_- + n_0 + 1, \ldots, n$, $\sigma_i(t)$ *satisfies the assumptions* ***C1– C2*** *with the parameters involved indexed by i.*

Denote

$$\xi_i := \sup_{t\in[0,1]} B_{H_i}(t), \qquad t_i^* = \frac{H_i}{1-H_i}. \tag{3.1}$$

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Given a Radon measure ν, define

$$\widetilde{\nu}(K) =: \mathbb{E}\{\nu(\boldsymbol{\xi}^{-1/\boldsymbol{H}}K)\}, \quad K \in \mathcal{B}([0, \infty]^n \setminus \{\mathbf{0}\}), \tag{3.2}$$

where

$$\boldsymbol{\xi}^{-1/\boldsymbol{H}}K = \{(\xi_1^{-1/H_1}d_1, \ldots, \xi_n^{-1/H_n}d_n), (d_1, \ldots, d_n) \in K\}.$$

Further, note that for $i = 1, \ldots, n_-$ (where $c_i < 0$), the asymptotic formula, as $u \to \infty$, of

$$\psi_i(u) = \mathbb{P}\left\{\sup_{t\geq 0}(X_i(t) + c_i t) > u\right\} \tag{3.3}$$

is available from Proposition 2.1 under Assumption I.

Below is the principal result of this paper.

Theorem 3.1. *Suppose that $\boldsymbol{X}(t)$, $t \geq 0$, satisfies Assumption I, and $\boldsymbol{\mathcal{T}}$ is a regularly varying random vector with index α and limiting measure ν, and is independent of $\boldsymbol{X}$. Further assume, without loss of generality, that there are $m(\leq n_0)$ positive constants k_i such that $\overleftarrow{\sigma_i}(u) \sim k_i \overleftarrow{\sigma}_{n_-+1}(u)$ for $i = n_- + 1, \ldots, n_- + m$ and $\overleftarrow{\sigma_i}(u) = o(\overleftarrow{\sigma}_{n_-+1}(u))$ for $i = n_- + m + 1, \ldots, n_- + n_0$. With the convention $\prod_{i=1}^{0} = 1$, we have the following.*

(i) *If $n_0 > 0$, then, as $u \to \infty$,*

$$P(u) \sim \widetilde{\nu}\big(\big(\boldsymbol{ka}_0^{1/H_{n_-+1}}, \boldsymbol{\infty}\big]\big)\, \mathbb{P}\{|\boldsymbol{\mathcal{T}}| > \overleftarrow{\sigma}_{n_-+1}(u)\} \prod_{i=1}^{n_-} \psi_i(a_i u),$$

where $\widetilde{\nu}$ and ψ_i are defined in (3.2) and (3.3), respectively, and

$$\boldsymbol{ka}_0^{1/H_{n_-+1}} = (0, \ldots, 0, k_{n_-+1}a_{n_-+1}^{1/H_{n_-+1}}, \ldots, k_{n_-+m}a_{n_-+m}^{1/H_{n_-+1}}, 0, \ldots, 0).$$

(ii) *If $n_0 = 0$, then, as $u \to \infty$,*

$$P(u) \sim \nu((\boldsymbol{a}_1, \boldsymbol{\infty}])\, \mathbb{P}\{|\boldsymbol{\mathcal{T}}| > u\} \prod_{i=1}^{n_-} \psi_i(a_i u),$$

where $\boldsymbol{a}_1 = (t_1^*/|c_1| \ldots, t_{n_-}^*/|c_{n_-}|, a_{n_-+1}/c_{n_-+1}, \ldots, a_n/c_n)$.

Remark 3.1. As a special case, we can obtain from Theorem 3.1 some results for the one-dimensional model. Specifically, let $c > 0$ be some constant; then, as $u \to \infty$,

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} X(t) > u\right\} \sim \mathbb{E}\left\{\left(\sup_{t\in[0,1]} B_H(t)\right)^{\alpha/H}\right\} \mathbb{P}\{\mathcal{T} > \overleftarrow{\sigma}(u)\} \tag{3.4}$$

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} (X(t) - ct) > u\right\} \sim (c(1-H)/H)^{\alpha}\, \mathbb{P}\{\mathcal{T} > u\}\psi(u), \tag{3.5}$$

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}]} (X(t) + ct) > u\right\} \sim c^{\alpha}\, \mathbb{P}\{\mathcal{T} > u\}. \tag{3.6}$$

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Note that (3.4) is derived in Theorem 2.1 of [18], (3.5) is discussed in [11] only for the fBm case. The result in (3.6) seems to be new.

We conclude this section with an interesting example of multi-dimensional subordinate Brownian motion; see e.g. [26].

Example 3.1. For each $i = 0, 1, \ldots, n$, let $\{S_i(t),\ t \geq 0\}$ be an independent α_i-stable subordinator with $\alpha_i \in (0, 1)$, i.e. $S_i(t) \stackrel{D}{=} \mathcal{S}_{\alpha_i}(t^{1/\alpha_i}, 1, 0)$, where $\mathcal{S}_\alpha(\sigma, \beta, d)$ denotes a stable random variable with stability index α, scale parameter σ, skewness parameter β, and drift parameter d. It is known (e.g. [35, Property 1.2.15]) that for any fixed constant $T > 0$,

$$\mathbb{P}\{S_i(T) > t\} \sim C_{\alpha_i,T} t^{-\alpha_i}, \quad t \to \infty,$$

with

$$C_{\alpha_i,T} = \frac{T}{\Gamma(1-\alpha_i)\cos(\pi\alpha_i/2)}.$$

Assume $\alpha_0 < \alpha_i$, for all $i = 1, 2, \ldots, n$. Define an n-dimensional subordinator as

$$\boldsymbol{Y}(t) := (S_0(t) + S_1(t), \ldots, S_0(t) + S_n(t)), \quad t \geq 0.$$

We consider an n-dimensional subordinate Brownian motion with drift defined as

$$\boldsymbol{X}(t) = (B_1(Y_1(t)) + c_1 Y_1(t), \ldots, B_n(Y_n(t)) + c_n Y_n(t)), \quad t \geq 0,$$

where $B_i(t)$, $t \geq 0$, $i = 1, \ldots, n$, are independent standard Brownian motions that are independent of $\boldsymbol{Y}$ and $c_i \in \mathbb{R}$. For any $a_i > 0$, $i = 1, 2, \ldots, n$, $T > 0$ and $u > 0$, define

$$P_B(u) := \mathbb{P}\left\{\cap_{i=1}^n \left(\sup_{t\in[0,T]} (B_i(Y_i(t)) + c_i Y_i(t)) > a_i u\right)\right\}.$$

For illustrative purposes and to avoid further technicalities, we only consider the case where all c_i in the above have the same sign. As an application of Theorem 3.1, we obtain the asymptotic behaviour of $P_B(u)$, $u \to \infty$, as follows.

(i) If $c_i > 0$ for all $i = 1, \ldots, n$, then $P_B(u) \sim C_{\alpha_0,T}(\max_{i=1}^n (a_i/c_i)u)^{-\alpha_0}$.

(ii) If $c_i = 0$ for all $i = 1, \ldots, n$, then $P_B(u) \asymp u^{-2\alpha_0}$.

(iii) If $c_i < 0$ and the density function of $S_i(T)$ is ultimately monotone for all $i = 0, 1, \ldots, n$, then $\ln P_B(u) \sim 2\sum_{i=1}^n (a_i c_i)u$.

The proof of the above is displayed in Section 5.

4. Ruin probability of a multi-dimensional regenerative model

As it is known in the literature that the maximum of random processes over a random interval is relevant to the regenerated models (e.g. [28], [40]), this section is focused on a multi-dimensional regenerative model that is motivated by its applications in queueing theory and ruin theory. More precisely, there are four elements in this model: two sequences of strictly positive random variables, $\{T_i : i \geq 1\}$ and $\{S_i : i \geq 1\}$, and two sequences of n-dimensional processes, $\{\{\boldsymbol{X}^{(i)}(t),\ t \geq 0\} : i \geq 1\}$ and $\{\{\boldsymbol{Y}^{(i)}(t),\ t \geq 0\} : i \geq 1\}$, where $\boldsymbol{X}^{(i)}(t) =$

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$(X_1^{(i)}(t), \ldots, X_n^{(i)}(t))$ and $\boldsymbol{Y}^{(i)}(t) = (Y_1^{(i)}(t), \ldots, Y_n^{(i)}(t))$. We assume that the above four elements are mutually independent. Here T_i, S_i are two successive times representing the random length of the alternating environment (called T-stage and S-stage), and we assume a T-stage starts at time 0. The model grows according to $\{\boldsymbol{X}^{(i)}(t),\ t \ge 0\}$ during the ith T-stage and according to $\{\boldsymbol{Y}^{(i)}(t), t \ge 0\}$ during the ith S-stage.

Based on the above, we define an alternating renewal process with renewal epochs

$$0 = V_0 < V_1 < V_2 < V_3 < \cdots$$

with $V_i = (T_1 + S_1) + \cdots + (T_i + S_i)$, which is the ith environment cycle time. Then the resulting n-dimensional process $\mathbf{Z}(t) = (Z_1(t), \ldots, Z_n(t))$ is defined as

$$\boldsymbol{Z}(t) := \begin{cases} \boldsymbol{Z}(V_i) + \boldsymbol{X}^{(i+1)}(t - V_i) & \text{if } V_i < t \le V_i + T_{i+1}, \\ \boldsymbol{Z}(V_i) + \boldsymbol{X}^{(i+1)}(T_{i+1}) + \boldsymbol{Y}^{(i+1)}(t - V_i - T_{i+1}) & \text{if } V_i + T_{i+1} < t \le V_{i+1}. \end{cases}$$

Note that this is a multi-dimensional regenerative process with regeneration epochs V_i, $i \ge 1$. This is a generalization of the one-dimensional model discussed in [24].

We assume that $\{\{\boldsymbol{X}^{(i)}(t),\ t \ge 0\} \colon i \ge 1\}$ and $\{\{\boldsymbol{Y}^{(i)}(t),\ t \ge 0\} \colon i \ge 1\}$ are independent samples of $\{\boldsymbol{X}(t),\ t \ge 0\}$ and $\{\boldsymbol{Y}(t),\ t \ge 0\}$, respectively, where

$$\begin{aligned} X_j(t) &= B_{H_j}(t) + p_j t, \quad t \ge 0,\ 1 \le j \le n, \\ Y_j(t) &= \widetilde{B}_{\widetilde{H}_j}(t) - q_j t, \quad t \ge 0,\ 1 \le j \le n, \end{aligned}$$

with all the fBms $B_{H_j}, \widetilde{B}_{\widetilde{H}_j}$ being mutually independent and $p_j, q_j > 0$, $1 \le j \le n$. Suppose that (T_i, S_i), $i \ge 1$ are independent samples of (T, S) and T is regularly varying with index $\lambda > 1$. We further assume that

$$\mathbb{P}\{S > x\} = \mathrm{o}(\mathbb{P}\{T > x\}), \quad p_j \mathbb{E}\{T\} < q_j \mathbb{E}\{S\} < \infty,\ 1 \le j \le n. \tag{4.1}$$

For notational simplicity we shall restrict ourselves to the two-dimensional case. The general n-dimensional problem can be analysed similarly. Thus, for the rest of this section and related proofs in Section 6, all vectors (or multi-dimensional processes) are considered to be two-dimensional ones.

We are interested in the asymptotics of the following tail probability:

$$Q(u) := \mathbb{P}\left\{\exists n \ge 1 \colon \sup_{t \in [V_{n-1}, V_n]} Z_1(t) > a_1 u, \sup_{s \in [V_{n-1}, V_n]} Z_2(s) > a_2 u\right\}, \quad u \to \infty,$$

with $a_1, a_2 > 0$. In the fluid queueing context, $Q(u)$ can be interpreted as the probability that both buffers overflow in some environment cycle. In the insurance context, $Q(u)$ can be interpreted as the probability that in some business cycle the two lines of business of the insurer are both ruined (not necessarily at the same time). Similar one-dimensional models have been discussed in the literature; see e.g. [4], [28], and [40].

We introduce the following notation:

$$\boldsymbol{U}^{(n)} = (U_1^{(n)}, U_2^{(n)}) := \boldsymbol{Z}(V_n) - \boldsymbol{Z}(V_{n-1}), \quad n \ge 1,\ \boldsymbol{U}^{(0)} = \mathbf{0}, \tag{4.2}$$

$$\boldsymbol{M}^{(n)} = (M_1^{(n)}, M_2^{(n)}) := \left(\sup_{t \in [V_{n-1}, V_n)} Z_1(t) - Z_1(V_{n-1}), \sup_{s \in [V_{n-1}, V_n)} Z_2(s) - Z_2(V_{n-1})\right), \quad n \ge 1. \tag{4.3}$$

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Then we have

$$Q(u) = \mathbb{P}\left\{\exists n \geq 1 : \sum_{i=1}^{n} U_1^{(i-1)} + M_1^{(n)} > a_1 u, \sum_{i=1}^{n} U_2^{(i-1)} + M_2^{(n)} > a_2 u\right\}.$$

Note that $\boldsymbol{U}^{(n)}$, $n \geq 1$ and $\boldsymbol{M}^{(n)}$, $n \geq 1$ are both (independent and identically distributed) sequences. By the second assumption in (4.1) we have

$$\mathbb{E}\{\boldsymbol{U}^{(1)}\} = (p_1\mathbb{E}\{T\} - q_1\mathbb{E}\{S\}, p_2\mathbb{E}\{T\} - q_2\mathbb{E}\{S\}) =: -\boldsymbol{c} < \boldsymbol{0}, \tag{4.4}$$

which ensures that the event in the above probability is a rare event for large u, i.e. $Q(u) \to 0$, as $u \to \infty$.

It is noted that our question now becomes an exit problem of a *two-dimensional perturbed random walk*. The exit problems of a multi-dimensional random walk have been discussed in many papers, e.g. [21]. However, as far as we know, the multi-dimensional perturbed random walk has not been discussed in the existing literature.

Since T is regularly varying with index $\lambda > 1$, we have that

$$\widetilde{\boldsymbol{T}} := (p_1 T, p_2 T) \tag{4.5}$$

is regularly varying with index λ and some limiting measure μ (whose form depends on the norm $|\cdot|$ that is chosen). We now present the main result of this section, leaving its proof to Section 6.

Theorem 4.1. *Under the above assumptions on regenerative model $\boldsymbol{Z}(t)$, $t \geq 0$, we have that, as $u \to \infty$,*

$$Q(u) \sim u\mathbb{P}\{|\widetilde{\boldsymbol{T}}| > u\} \int_0^\infty \mu((v\boldsymbol{c} + \boldsymbol{a}, \infty])\, dv,$$

where $\boldsymbol{c}$ and $\widetilde{\boldsymbol{T}}$ are given by (4.4) and (4.5), respectively.

Remark 4.1. Consider $|\cdot|$ to be the L^1-norm in Theorem 4.1. We have

$$\mu([\boldsymbol{a}, \infty]) = ((p_1 + p_2) \max(a_1/p_1, a_2/p_2))^{-\lambda},$$

and thus, as $u \to \infty$,

$$Q(u) \sim u\mathbb{P}\{T > u\} \int_0^\infty \max((a_1 + c_1 v)/p_1, (a_2 + c_2 v)/p_2)^{-\lambda}\, dv.$$

5. Proof of main results

This section is devoted to the proof of Theorem 3.1, followed by a proof of Example 3.1.

First we give a result in line with Proposition 2.1. Note that in the proof of the main results in [19], the minimum point t_u^* of the function

$$f_u(t) := \frac{u(1+t)}{\sigma(ut/c)}, \qquad t \geq 0,$$

plays an important role. It has been discussed therein that t_u^* converges, as $u \to \infty$, to $t^* := H/(1-H)$, which is the unique minimum point of $\lim_{u\to\infty} f_u(t)\sigma(u)/u = (1+t)/(t/c)^H$, $t \geq 0$.

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In this sense, t_u^* is asymptotically unique. We have the following corollary of [19], which is useful for the proofs below.

Lemma 5.1. *Let $X(t)$, $t \geq 0$, be an a.s. continuous centered Gaussian process with stationary increments and $X(0) = 0$. Suppose that* **C1–C4** *hold. For any fixed $0 < \varepsilon < t^*/c$, we have, as $u \to \infty$,*

$$\mathbb{P}\left\{\sup_{t\in[0,(t^*/c+\varepsilon)u]} (X(t) - ct) > u\right\} \sim \psi(u),$$

with $\psi(u)$ the same as in Proposition 2.1. Furthermore, for any $\gamma > 0$ we have

$$\lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in[0,(t^*/c-\varepsilon)u]} (X(t) - ct) > u\}}{\psi(u)u^{-\gamma}} = 0.$$

Proof. Note that

$$\mathbb{P}\left\{\sup_{t\in[0,(t^*/c+\varepsilon)u]} (X(t) - ct) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,(t^*+c\varepsilon)]} \frac{X(ut/c)}{1+t} > u\right\}.$$

The first claim follows from [19], as the main interval that determines the asymptotics is in$[0, (t^* + c\varepsilon)]$ (see Lemma 7 and the comments in Section 2.1 therein). Similarly, we have

$$\mathbb{P}\left\{\sup_{t\in[0,(t^*/c-\varepsilon)u]} (X(t) - ct) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,(t^*-c\varepsilon)]} \frac{X(ut/c)}{1+t} > u\right\}.$$

Since t_u^* is asymptotically unique and $\lim_{u\to\infty} t_u^* = t^*$, we can show that, for all u large,

$$\inf_{t\in[0,(t^*-c\varepsilon)]} f_u(t) \geq \rho f_u(t_u^*) = \rho \inf_{t\geq 0} f_u(t)$$

for some $\rho > 1$. Thus, by arguments similar to those in the proof of Lemma 7 of [19] using the Borel inequality, we conclude the second claim. □

The following lemma is crucial for the proof of Theorem 3.1.

Lemma 5.2. *Let $X_i(t)$, $t \geq 0$, $i = 1, 2, \ldots, n_0(< n)$ be independent centered Gaussian processes with stationary increments, and let $\mathcal{T}$ be an independent regularly varying random vector with index α and limiting measure ν. Suppose that all of $\sigma_i(t)$, $i = 1, 2, \ldots, n_0$ satisfy the assumptions* **C1–C3** *with the parameters involved indexed by i, which further satisfy that $\overleftarrow{\sigma_i}(u) \sim k_i \overleftarrow{\sigma_1}(u)$ for some positive constants k_i, $i = 1, 2, \ldots, m \leq n_0$ and $\overleftarrow{\sigma_j}(u) = o(\overleftarrow{\sigma_1}(u))$ for all $j = m + 1, \ldots, n_0$. Then, for any increasing to infinity functions $h_i(u)$, $n_0 + 1 \leq i \leq n$ such that $h_i(u) = o(\overleftarrow{\sigma_1}(u))$, $n_0 + 1 \leq i \leq n$, and any $a_i > 0$,*

$$\mathbb{P}\left\{\cap_{i=1}^{n_0}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_i u\right), \cap_{i=n_0+1}^{n} (\mathcal{T}_i > h_i(u))\right\} \sim \widetilde{\nu}\left(\left(\boldsymbol{ka}_{m,0}^{1/\boldsymbol{H}}, \boldsymbol{\infty}\right]\right) \mathbb{P}\{|\mathcal{T}| > \overleftarrow{\sigma_1}(u)\},$$

where $\widetilde{\nu}$ is defined in (3.2) and $\boldsymbol{ka}_{m,0}^{1/\boldsymbol{H}} = (k_1 a_1^{1/H_1}, \ldots, k_m a_m^{1/H_m}, 0 \ldots, 0)$ with $H_1 = H_2 = \cdots = H_m$.

Proof. We use an argument similar to that in the proof of Theorem 2.1 of [18] to verify our conclusion. For notational convenience, denote

$$H(u) =: \mathbb{P}\left\{\cap_{i=1}^{n_0}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_i u\right), \cap_{i=n_0+1}^{n} (\mathcal{T}_i > h_i(u))\right\}.$$

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We first give an asymptotically lower bound for $H(u)$. Let $G(\boldsymbol{x}) = \mathbb{P}\{\boldsymbol{\mathcal{T}} \le \boldsymbol{x}\}$ be the distribution function of $\boldsymbol{\mathcal{T}}$. Note that, for any constants r and R such that $0 < r < R$,

$$H(u) \ge \mathbb{P}\left\{\cap_{i=1}^{n_0}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_i u\right), \cap_{i=1}^{m}(r\overleftarrow{\sigma_1}(u) \le \mathcal{T}_i \le R\overleftarrow{\sigma_1}(u)), \cap_{i=m+1}^{n}(\mathcal{T}_i > r\overleftarrow{\sigma_1}(u))\right\}$$
$$= \oint_{[r,R]^m\times(r,\infty)^{n-m}} \mathbb{P}\left\{\cap_{i=1}^{n_0}\left(\sup_{t\in[0,\overleftarrow{\sigma_1}(u)t_i]} X_i(t) > a_i u\right)\right\} \mathrm{d}G(\overleftarrow{\sigma_1}(u)t_1, \ldots, \overleftarrow{\sigma_1}(u)t_n)$$
$$= \oint_{[r,R]^m\times(r,\infty)^{n-m}} \prod_{i=1}^{n_0} \mathbb{P}\left\{\sup_{s\in[0,1]} X_i^{u,t_i}(s) > a_i u_i(t_i)\right\} \mathrm{d}G(\overleftarrow{\sigma_1}(u)t_1, \ldots, \overleftarrow{\sigma_1}(u)t_n)$$

holds for sufficiently large u, where

$$X_i^{u,t_i}(s) =: \frac{X_i(\overleftarrow{\sigma_1}(u)t_i s)}{\sigma_i(\overleftarrow{\sigma_1}(u)t_i)}, \quad u_i(t_i) =: \frac{u}{\sigma_i(\overleftarrow{\sigma_1}(u)t_i)},$$
$$s \in [0, 1], (t_1, t_2, \ldots, t_{n_0}) \in [r, R]^m \times (r, \infty)^{n_0-m}.$$

By Lemma 5.2 of [18], we know that, as $u \to \infty$, the processes $X_i^{u,t_i}(s)$ converge weakly in $C([0,1])$ to $B_{H_i}(s)$, uniformly in $t_i \in (r, \infty)$, for $i = 1, 2, \ldots, n_0$. Further, according to the assumptions on $\sigma_i(t)$, Theorems 1.5.2 and 1.5.6 of [8], we find that as $u \to \infty$, $u_i(t_i)$ converges to $k_i^{H_i} t_i^{-H_i}$ uniformly in $t_i \in [r, R]$, for $i = 1, 2, \ldots, m$, and $u_i(t_i)$ converges to 0 uniformly in $t_i \in [r, \infty)$, for $i = m+1, \ldots, n_0$. Then, by the continuous mapping theorem and recalling that ξ_i defined in (3.1) is a continuous random variable (e.g. [39]), we get

$$H(u) \gtrsim \oint_{[r,R]^m\times(r,\infty)^{n-m}} \prod_{i=1}^{m} \mathbb{P}\left\{\sup_{s\in[0,1]} B_{H_i}(s) > a_i k_i^{H_i} t_i^{-H_i}\right\} \mathrm{d}G(\overleftarrow{\sigma_1}(u)t_1, \ldots, \overleftarrow{\sigma_1}(u)t_n) \quad (5.1)$$

$$= \mathbb{P}\left\{\cap_{i=1}^{m}\left(\xi_i^{1/H_i}\mathcal{T}_i > k_i a_i^{1/H_i}\overleftarrow{\sigma_1}(u)\right), \cap_{i=1}^{m}(r\overleftarrow{\sigma_1}(u) \le \mathcal{T}_i \le R\overleftarrow{\sigma_1}(u)), \cap_{i=m+1}^{n}(\mathcal{T}_i > r\overleftarrow{\sigma_1}(u))\right\}$$
$$= J_1(u) - J_2(u),$$

where

$$J_1(u) =: \mathbb{P}\left\{\cap_{i=1}^{m}\left(\xi_i^{1/H_i}\mathcal{T}_i > k_i a_i^{1/H_i}\overleftarrow{\sigma_1}(u)\right), \cap_{i=m+1}^{n}(\mathcal{T}_i > r\overleftarrow{\sigma_1}(u))\right\},$$
$$J_2(u) =: \mathbb{P}\Big\{\cap_{i=1}^{m}\left(\xi_i^{1/H_i}\mathcal{T}_i > k_i a_i^{1/H_i}\overleftarrow{\sigma_1}(u)\right),$$
$$\cap_{i=m+1}^{n}(\mathcal{T}_i > r\overleftarrow{\sigma_1}(u)), \cup_{i=1}^{m}((\mathcal{T}_i < r\overleftarrow{\sigma_1}(u)) \cup (\mathcal{T}_i > R\overleftarrow{\sigma_1}(u)))\Big\}.$$

Putting $\boldsymbol{\eta} = (\xi_1^{1/H_1}, \ldots, \xi_m^{1/H_m}, 1, \ldots, 1)$, then by Lemma A.2 and the continuity of the limiting measure $\widehat{\nu}$ defined therein, we have

$$\lim_{r\to 0}\lim_{u\to\infty} \frac{J_1(u)}{\mathbb{P}\{|\boldsymbol{\mathcal{T}}| > \overleftarrow{\sigma_1}(u)\}} = \widetilde{\nu}\left((\boldsymbol{k}\boldsymbol{a}_{m,0}^{1/\boldsymbol{H}}, \boldsymbol{\infty}]\right). \quad (5.2)$$

Furthermore,

$$J_2(u) \le \sum_{i=1}^{m}\left(\mathbb{P}\left\{\xi_i^{1/H_i}\mathcal{T}_i > k_i a_i^{1/H_i}\overleftarrow{\sigma_1}(u), \mathcal{T}_i < r\overleftarrow{\sigma_1}(u)\right\} + \mathbb{P}\{\mathcal{T}_i > R\overleftarrow{\sigma_1}(u)\}\right).$$

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Then, by the fact that $|\mathcal{T}|$ is regularly varying with index α, and using the same arguments as in the proof of Theorem 2.1 of [18] (see the asymptotic for integral I_4 and (5.14) therein), we conclude that

$$\lim_{r\to 0, R\to\infty} \limsup_{u\to\infty} \frac{J_2(u)}{\mathbb{P}\{|\mathcal{T}| > \overleftarrow{\sigma_1}(u)\}} = 0,$$

which combined with (5.1) and (5.2) yields

$$\lim_{r\to 0, R\to\infty} \liminf_{u\to\infty} \frac{H(u)}{\mathbb{P}\{|\mathcal{T}| > \overleftarrow{\sigma_1}(u)\}} \geq \widetilde{\nu}\big(\big(\boldsymbol{k}\boldsymbol{a}_{m,0}^{1/\boldsymbol{H}}, \boldsymbol{\infty}\big]\big). \tag{5.3}$$

Next we give an asymptotic upper bound for $H(u)$. Note that

$$\begin{aligned} H(u) &\leq \mathbb{P}\left\{\cap_{i=1}^m \left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_i u\right)\right\} \\ &= \mathbb{P}\left\{\cap_{i=1}^m \left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_i u\right), \cap_{i=1}^m (r\overleftarrow{\sigma_1}(u) \leq \mathcal{T}_i \leq R\overleftarrow{\sigma_1}(u))\right\} \\ &\quad + \mathbb{P}\left\{\cap_{i=1}^m \left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_i u\right), \cup_{i=1}^m ((\mathcal{T}_i < r\overleftarrow{\sigma_1}(u)) \cup (\mathcal{T}_i > R\overleftarrow{\sigma_1}(u)))\right\} \\ &=: J_3(u) + J_4(u). \end{aligned}$$

By the same reasoning as that used in the deduction for (5.2), we can show that

$$\lim_{r\to 0, R\to\infty} \limsup_{u\to\infty} \frac{J_3(u)}{\mathbb{P}\{|\mathcal{T}| > \overleftarrow{\sigma_1}(u)\}} \leq \widetilde{\nu}\big(\big(\boldsymbol{k}\boldsymbol{a}_{m,0}^{1/\boldsymbol{H}}, \boldsymbol{\infty}\big]\big). \tag{5.4}$$

Moreover,

$$J_4(u) \leq \sum_{i=1}^m \left(\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_i u,\, \mathcal{T}_i < r\overleftarrow{\sigma_1}(u)\right\} + \mathbb{P}\{\mathcal{T}_i > R\overleftarrow{\sigma_1}(u)\}\right).$$

Thus, by the same arguments as in the proof of Theorem 2.1 of [18] (see the asymptotics for integrals I_1, I_2, I_4 therein), we conclude that

$$\lim_{r\to 0, R\to\infty} \limsup_{u\to\infty} \frac{J_4(u)}{\mathbb{P}\{|\mathcal{T}| > \overleftarrow{\sigma_1}(u)\}} = 0,$$

which together with (5.4) implies that

$$\lim_{r\to 0, R\to\infty} \limsup_{u\to\infty} \frac{H(u)}{\mathbb{P}\{|\mathcal{T}| > \overleftarrow{\sigma_1}(u)\}} \leq \widetilde{\nu}\big(\big(\boldsymbol{k}\boldsymbol{a}_{m,0}^{1/\boldsymbol{H}}, \boldsymbol{\infty}\big]\big). \tag{5.5}$$

Notice that by the assumptions on $\{\overleftarrow{\sigma_i}(u)\}_{i=1}^m$, we in fact have $H_1 = H_2 = \cdots = H_m$. Consequently, combining (5.3) and (5.5) we complete the proof. □

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Proof of Theorem 3.1. In the following we use the convention that $\cap_{i=1}^{0} = \Omega$, the sample space. We first verify the claim for case (i), $n_0 > 0$. For arbitrarily small $\varepsilon > 0$, we have

$$
\begin{aligned}
P(u) \geq \mathbb{P}\Bigg\{ & \cap_{i=1}^{n_-}\left(\sup_{t\in[0,\mathcal{T}_i]} (X_i(t)+c_it) > a_iu,\ \mathcal{T}_i > (t_i^*/|c_i|+\varepsilon)u \right), \cap_{i=n_-+1}^{n_-+n_0}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_iu \right), \\
& \cap_{i=n_-+n_0+1}^{n}\left(\sup_{t\in[0,\mathcal{T}_i]} (X_i(t)+c_it) > a_iu,\ \mathcal{T}_i > \frac{a_i+\varepsilon}{c_i}u \right) \Bigg\} \\
\geq \mathbb{P}\Bigg\{ & \cap_{i=1}^{n_-}\left(\sup_{t\in[0,(t_i^*/|c_i|+\varepsilon)u]} (X_i(t)+c_it) > a_iu,\ \mathcal{T}_i > (t_i^*/|c_i|+\varepsilon)u \right), \\
& \cap_{i=n_-+1}^{n_-+n_0}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_iu \right), \cap_{i=n_-+n_0+1}^{n}\left(X_i\left(\frac{a_i+\varepsilon}{c_i}u\right) > -\varepsilon u,\ \mathcal{T}_i > \frac{a_i+\varepsilon}{c_i}u \right) \Bigg\} \\
= {} & Q_1(u)\times Q_2(u)\times Q_3(u),
\end{aligned}
$$

where

$$
\begin{aligned}
Q_1(u) &:= \mathbb{P}\left\{ \cap_{i=1}^{n_-}\left(\sup_{t\in[0,(t_i^*/|c_i|+\varepsilon)u]} X_i(t)+c_it > a_iu \right) \right\} \\
Q_2(u) &:= \mathbb{P}\Bigg\{ \cap_{i=1}^{n_-}(\mathcal{T}_i > (t_i^*/|c_i|+\varepsilon)u), \\
&\qquad \cap_{i=n_-+1}^{n_-+n_0}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_iu \right), \cap_{i=n_-+n_0+1}^{n}\left(\mathcal{T}_i > \frac{a_i+\varepsilon}{c_i}u \right) \Bigg\}, \\
Q_3(u) &:= \prod_{i=n_-+n_0+1}^{n} \mathbb{P}\left\{ N_i > \frac{-\varepsilon u}{\sigma_i(\frac{a_i+\varepsilon}{c_i}u)} \right\} \to 1, \quad u\to\infty,
\end{aligned}
$$

with N_i, $i = n_- + n_0 + 1, \ldots, n$ being standard normally distributed random variables. By Lemma 5.1, we know, as $u \to \infty$, that

$$Q_1(u) \sim \prod_{i=1}^{n_-} \psi_i(a_iu).$$

Further, according to the assumptions on σ_i and Lemma 5.2, we get

$$\lim_{\varepsilon\to 0}\lim_{u\to\infty} \frac{Q_2(u)}{\mathbb{P}\{|\boldsymbol{\mathcal{T}}| > \overleftarrow{\sigma}_{n_-+1}(u)\}} = \widetilde{\nu}\left(\left(\boldsymbol{k}\boldsymbol{a}_0^{1/H_{n_-+1}}, \boldsymbol{\infty}\right]\right),$$

and thus

$$P(u) \gtrsim \widetilde{\nu}\left(\left(\boldsymbol{k}\boldsymbol{a}_0^{1/H_{n_-+1}}, \boldsymbol{\infty}\right]\right)\mathbb{P}\{|\boldsymbol{\mathcal{T}}| > \overleftarrow{\sigma}_{n_-+1}(u)\}\prod_{i=1}^{n_-}\psi_i(a_iu), \quad u\to\infty.$$

Similarly, we can show that

$$
\begin{aligned}
P(u) &\leq \mathbb{P}\left\{ \cap_{i=1}^{n_-}\left(\sup_{t\in[0,\infty)} X_i(t)+c_it > a_iu \right), \cap_{i=n_-+1}^{n_-+n_0}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > a_iu \right) \right\} \\
&\sim \widetilde{\nu}\left(\left(\boldsymbol{k}\boldsymbol{a}_0^{1/H_{n_-+1}}, \boldsymbol{\infty}\right]\right)\mathbb{P}\{|\boldsymbol{\mathcal{T}}| > \overleftarrow{\sigma}_{n_-+1}(u)\}\prod_{i=1}^{n_-}\psi_i(a_iu), \quad u\to\infty.
\end{aligned}
$$

This completes the proof of case (i).

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Next we consider case (ii), $n_0=0$. Similarly to case (i) we have, for any small $\varepsilon>0$,

$$\begin{aligned}P(u) &\geq \mathbb{P}\Bigg\{\cap_{i=1}^{n_-}\Bigg(\sup_{t\in[0,(t_i^*/|c_i|+\varepsilon)u]}(X_i(t)+c_it)>a_iu,\ \mathcal{T}_i>(t_i^*/|c_i|+\varepsilon)u\Bigg),\\ &\qquad \cap_{i=n_-+1}^{n}\left(X_i\left(\frac{a_i+\varepsilon}{c_i}u\right)>-\varepsilon u,\ \mathcal{T}_i>\frac{a_i+\varepsilon}{c_i}u\right)\Bigg\}\\ &= Q_1(u)\times Q_3(u)\times Q_4(u),\end{aligned}$$

where

$$Q_4(u):=\mathbb{P}\left\{\cap_{i=1}^{n_-}(\mathcal{T}_i>(t_i^*/|c_i|+\varepsilon)u),\ \cap_{i=n_-+1}^{n}\left(\mathcal{T}_i>\frac{a_i+\varepsilon}{c_i}u\right)\right\}.$$

By Lemma A.1, we know that

$$\lim_{\varepsilon\to0}\lim_{u\to\infty}\frac{Q_4(u)}{\mathbb{P}\{|\boldsymbol{\mathcal{T}}|>u\}}=\nu(\boldsymbol{a}_1,\boldsymbol{\infty}],$$

and thus

$$P(u)\gtrsim \nu(\boldsymbol{a}_1,\boldsymbol{\infty}]\,\mathbb{P}\{|\boldsymbol{\mathcal{T}}|>u\}\prod_{i=1}^{n_-}\psi_i(a_iu),\qquad u\to\infty.$$

For the upper bound, we have for any small $\varepsilon>0$

$$P(u)\leq I_1(u)+I_2(u),$$

with

$$\begin{aligned}I_1(u) &:= \mathbb{P}\Bigg\{\cap_{i=1}^{n_-}\Bigg(\sup_{t\in[0,\mathcal{T}_i]}X_i(t)+c_it>a_iu\Bigg),\\ &\qquad \cap_{i=1}^{n_-}(\mathcal{T}_i>(t_i^*/|c_i|-\varepsilon)u),\ \cap_{i=n_-+1}^{n}\Bigg(\sup_{t\in[0,\mathcal{T}_i]}X_i(t)+c_i\mathcal{T}_i>a_iu\Bigg)\Bigg\},\\ I_2(u) &:= \mathbb{P}\Bigg\{\cap_{i=1}^{n_-}\Bigg(\sup_{t\in[0,\mathcal{T}_i]}X_i(t)+c_it>a_iu\Bigg),\\ &\qquad \cup_{i=1}^{n_-}(\mathcal{T}_i\leq(t_i^*/|c_i|-\varepsilon)u),\ \cap_{i=n_-+1}^{n}\Bigg(\sup_{t\in[0,\mathcal{T}_i]}X_i(t)+c_i\mathcal{T}_i>a_iu\Bigg)\Bigg\}.\end{aligned}$$

It follows that

$$\begin{aligned}I_1(u) &\leq \mathbb{P}\Bigg\{\cap_{i=1}^{n_-}\Bigg(\sup_{t\in[0,\infty)}X_i(t)+c_it>a_iu\Bigg),\\ &\qquad \cap_{i=1}^{n_-}(\mathcal{T}_i>(t_i^*/|c_i|-\varepsilon)u),\ \cap_{i=n_-+1}^{n}\Bigg(\sup_{t\in[0,\mathcal{T}_i]}X_i(t)+c_i\mathcal{T}_i>a_iu\Bigg)\Bigg\}\\ &= \prod_{i=1}^{n_-}\psi_i(a_iu)\,\mathbb{P}\Bigg\{\cap_{i=1}^{n_-}(\mathcal{T}_i>(t_i^*/|c_i|-\varepsilon)u),\ \cap_{i=n_-+1}^{n}\Bigg(\sup_{t\in[0,\mathcal{T}_i]}X_i(t)+c_i\mathcal{T}_i>a_iu\Bigg)\Bigg\}.\end{aligned}$$

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Next, for the small chosen $\varepsilon > 0$ we have

$$\begin{aligned}
&\mathbb{P}\left\{\cap_{i=1}^{n_-}(\mathcal{T}_i > (t_i^*/|c_i| - \varepsilon)u),\, \cap_{i=n_-+1}^{n}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) + c_i\mathcal{T}_i > a_i u\right)\right\}\\
&\quad = \mathbb{P}\left\{\cap_{i=1}^{n_-}(\mathcal{T}_i > (t_i^*/|c_i| - \varepsilon)u),\, \cap_{i=n_-+1}^{n}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) + c_i\mathcal{T}_i > a_i u,\ \sup_{t\in[0,\mathcal{T}_i]} X_i(t) \le \varepsilon u\right)\right\}\\
&\qquad + \mathbb{P}\left\{\cap_{i=1}^{n_-}(\mathcal{T}_i > (t_i^*/|c_i| - \varepsilon)u),\right.\\
&\qquad\qquad \left.\cap_{i=n_-+1}^{n}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) + c_i\mathcal{T}_i > a_i u\right),\, \cup_{i=n_-+1}^{n}\left(\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > \varepsilon u\right)\right\}\\
&\quad \le \mathbb{P}\{\cap_{i=1}^{n_-}(\mathcal{T}_i > (t_i^*/|c_i| - \varepsilon)u),\, \cap_{i=n_-+1}^{n}(c_i\mathcal{T}_i > (a_i - \varepsilon)u)\} + \sum_{i=n_-+1}^{n} \mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > \varepsilon u\right\}.
\end{aligned}$$

Furthermore, it follows from Theorem 2.1 of [18] that, for any $i = n_- + 1, \ldots, n$,

$$\mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > \varepsilon u\right\} \sim C_i(\varepsilon)\, \mathbb{P}\{\mathcal{T}_i > \overleftarrow{\sigma_i}(u)\}, \quad u \to \infty,$$

with some constant $C_i(\varepsilon) > 0$. This implies that

$$\sum_{i=n_-+1}^{n} \mathbb{P}\left\{\sup_{t\in[0,\mathcal{T}_i]} X_i(t) > \varepsilon u\right\} = \mathrm{o}(\mathbb{P}\{|\boldsymbol{\mathcal{T}}| > u\}), \quad u \to \infty.$$

Consequently, applying Lemma A.1 and letting $\varepsilon \to 0$, we can obtain the required asymptotic upper bound if we can further show that

$$\lim_{u\to\infty} \frac{I_2(u)}{\prod_{i=1}^{n_-} \psi_i(a_i u)\, \mathbb{P}\{|\boldsymbol{\mathcal{T}}| > u\}} = 0. \tag{5.6}$$

Indeed, we have

$$\begin{aligned}
I_2(u) &\le \sum_{i=1}^{n_-} \mathbb{P}\left\{\cap_{j=1}^{n_-}\left(\sup_{t\in[0,\mathcal{T}_j]} X_j(t) + c_j t > a_j u\right),\, \mathcal{T}_i \le (t_i^*/|c_i| - \varepsilon)u\right\}\\
&\le \sum_{i=1}^{n_-} \prod_{\substack{j=1\\ j\ne i}}^{n_-} \psi_j(a_j u)\, \mathbb{P}\left\{\sup_{t\in[0,(t_i^*/|c_i|-\varepsilon)u]} X_i(t) + c_i t > a_i u\right\}.
\end{aligned} \tag{5.7}$$

Furthermore, by Lemma 5.1 we have that for any $\gamma > 0$

$$\lim_{u\to\infty} \frac{\mathbb{P}\{\sup_{t\in[0,(t_i^*/|c_i|-\varepsilon)u]} X_i(t) + c_i t > a_i u\}}{\psi_i(a_i u) u^{-\gamma}} = 0, \quad i = 1, 2, \ldots, n_-,$$

which together with (5.7) implies (5.6). This completes the proof. □

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Proof of Example 3.1. The proof is based on the following obvious bounds:

$$\begin{aligned} P_L(u) &:= \mathbb{P}\{\cap_{i=1}^n ((B_i(Y_i(T)) + c_i Y_i(T)) > a_i u)\} \\ &\le P_B(u) \\ &\le \mathbb{P}\left\{\cap_{i=1}^n \left(\sup_{t\in[0,Y_i(T)]} (B_i(t) + c_i t) > a_i u \right)\right\} \\ &=: P_U(u). \end{aligned}$$

Since $\alpha_0 < \min_{i=1}^n \alpha_i$, by Lemma A.3 we have that $\boldsymbol{Y}(T)$ is a multivariate regularly varying random vector with index α_0 and the same limiting measure ν as that of $\boldsymbol{S}_0(T) := (S_0(T), \ldots, S_0(T)) \in \mathbb{R}^n$, and further

$$\mathbb{P}\{|\boldsymbol{Y}(T)| > x\} \sim \mathbb{P}\{|\boldsymbol{S}_0(T)| > x\}, \quad x \to \infty.$$

The asymptotics of $P_U(u)$ can be obtained by applying Theorem 3.1. Below we focus on $P_L(u)$.

First, consider case (i), where $c_i > 0$ for all $i = 1, \ldots, n$. We have

$$P_L(u) = \mathbb{P}\{\cap_{i=1}^n ((B_i(1)\sqrt{Y_i(T)} + c_i Y_i(T)) > a_i u)\}.$$

Thus, by Lemma A.3 we obtain

$$P_L(u) \sim \mathbb{P}\{\cap_{i=1}^n (c_i S_0(T) > a_i u)\} \sim C_{\alpha_0,T}\left(\max_{i=1}^n (a_i/c_i)u\right)^{-\alpha_0}, \quad u \to \infty,$$

which is the same as the asymptotic upper bound obtained by using Theorem 3.1(ii).

Next, consider case (ii), where $c_i = 0$ for all $i = 1, \ldots, n$. We have

$$P_L(u) = \mathbb{P}\{\cap_{i=1}^n (B_i(1)\sqrt{Y_i(T)} > a_i u)\} = \frac{1}{2^n}\,\mathbb{P}\{\cap_{i=1}^n (B_i(1)^2 Y_i(T) > (a_i u)^2)\}.$$

Thus, by Lemma A.2, we obtain

$$P_L(u) \asymp u^{-2\alpha_0}, \quad u \to \infty,$$

which is the same as the asymptotic upper bound obtained by using Theorem 3.1(i).

Finally, consider case (iii), where $c_i < 0$ for all $i = 1, \ldots, n$. We have

$$\begin{aligned} P_L(u) &\ge \mathbb{P}\{\cap_{i=1}^n (B_i(Y_i(T)) + c_i Y_i(T) > a_i u,\, Y_i(T) \in [a_i u/|c_i| - \sqrt{u},\, a_i u/|c_i| + \sqrt{u}])\} \\ &\ge \prod_{i=1}^n \left(\min_{t\in[a_i u/|c_i| - \sqrt{u}, a_i u/|c_i| + \sqrt{u}]} \mathbb{P}\{B_1(t) + c_i t > a_i u\} \right) \\ &\quad \times \mathbb{P}\{\cap_{i=1}^n (Y_i(T) \in [a_i u/|c_i| - \sqrt{u},\, a_i u/|c_i| + \sqrt{u}])\}. \end{aligned}$$

Recalling (2.1), we derive that

$$\begin{aligned} &\min_{t\in[a_i u/|c_i| - \sqrt{u}, a_i u/|c_i| + \sqrt{u}]} \mathbb{P}\{B_1(t) + c_i t > a_i u\} \\ &= \min_{t\in[a_i/|c_i| - 1/\sqrt{u}, a_i/|c_i| + 1/\sqrt{u}]} \mathbb{P}\{B_1(1) > (a_i - c_i t)\sqrt{u}/\sqrt{t}\} \\ &\gtrsim \text{Constant} \cdot \frac{1}{\sqrt{u}} \mathrm{e}^{2a_i c_i u + \mathrm{o}(u)}, \quad u \to \infty. \end{aligned}$$

Furthermore,

$$\mathbb{P}\{\cap_{i=1}^n \big(Y_i(T) \in [a_i u/|c_i| - \sqrt{u}, a_i u/|c_i| + \sqrt{u}]\big)\}$$
$$\geq \prod_{i=0}^{n} \mathbb{P}\{S_i(T) \in [a_i u/|2c_i| - \sqrt{u}/2, a_i u/|2c_i| + \sqrt{u}/2]\}. \tag{5.8}$$

Due to the assumptions on the density functions of $S_i(T)$, $i = 0, 1, \ldots, n$, by the Monotone Density Theorem (see e.g. [27]), we know that (5.8) is asymptotically larger than $Cu^{-\beta}$ for some constants $C, \beta > 0$. Therefore

$$\ln P_L(u) \gtrsim 2 \sum_{i=1}^{n} (a_i c_i) u, \quad u \to \infty.$$

The same asymptotic upper bound can be obtained by the fact that

$$\mathbb{P}\left\{\sup_{t>0} (B_i(t) + c_i t) > a_i u\right\} = \mathrm{e}^{2a_i c_i u} \quad \text{for } c_i < 0.$$

This completes the proof. □

6. Proof of Theorem 4.1

We first show one lemma that is crucial for the proof of Theorem 4.1.

Lemma 6.1. *Let $\boldsymbol{U}^{(1)}$, $\boldsymbol{M}^{(1)}$, and $\widetilde{\boldsymbol{T}}$ be given by (4.2), (4.3), and (4.5) respectively. Then $\boldsymbol{U}^{(1)}$ and $\boldsymbol{M}^{(1)}$ are both regularly varying with the same index λ and limiting measure μ as that of $\widetilde{\boldsymbol{T}}$. Moreover,*

$$\mathbb{P}\{|\boldsymbol{U}^{(1)}| > x\} \sim \mathbb{P}\{|\boldsymbol{M}^{(1)}| > x\} \sim \mathbb{P}\{|\widetilde{\boldsymbol{T}}| > x\}, \quad x \to \infty.$$

Proof. First note that, by self-similarity of fBms,

$$\boldsymbol{U}^{(1)} = (X_1^{(1)}(T_1) + Y_1^{(1)}(S_1), X_2^{(1)}(T_1) + Y_2^{(1)}(S_1)) \overset{D}{=} (\widetilde{\boldsymbol{T}} + \boldsymbol{Z}_1 + \boldsymbol{Z}_2 + \boldsymbol{Z}_3),$$

where

$$\boldsymbol{Z}_1 = (B_{H_1}(1)T^{H_1}, B_{H_2}(1)T^{H_2}), \quad \boldsymbol{Z}_2 = (\widetilde{B}_{\widetilde{H}_1}(1)S^{\widetilde{H}_1}, \widetilde{B}_{\widetilde{H}_2}(1)S^{\widetilde{H}_2}), \quad \boldsymbol{Z}_3 = (-q_1 S, -q_2 S).$$

Since every two norms on R^d are equivalent, then by the fact that $H_i, \widetilde{H}_i < 1$ for $i = 1, 2$ and (4.1), we have

$$\max\left(\mathbb{P}\{|(T^{H_1}, T^{H_2})| > x\}, \mathbb{P}\{|(S^{\widetilde{H}_1}, S^{\widetilde{H}_2})| > x\}, \mathbb{P}\{|\boldsymbol{Z}_3| > x\}\right) = \mathrm{o}(\mathbb{P}\{|\widetilde{\boldsymbol{T}}| > x\}), \quad x \to \infty.$$

Thus the claim for $\boldsymbol{U}^{(1)}$ follows directly by Lemma A.3.

Next, note that

$$\boldsymbol{M}^{(1)} \overset{D}{=} \Bigg(\sup_{0\le t\le T+S} (X_1(t)I_{(0\le t<T)} + (X_1(T) + Y_1(t-T))I_{(T\le t<T+S)}),$$
$$\sup_{0\le t\le T+S} (X_2(t)I_{(0\le t<T)} + (X_2(T) + Y_2(t-T))I_{(T\le t<T+S)})\Bigg) =: \boldsymbol{M},$$

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then

$$\boldsymbol{M} \geq (X_1(T), X_2(T)) \stackrel{D}{=} \widetilde{\boldsymbol{T}} + \boldsymbol{Z}_1$$

and

$$\begin{aligned}\boldsymbol{M} &\leq \left(\sup_{0\leq t\leq T} B_{H_1}(t) + p_1 T + \sup_{t\geq 0} Y_1(t), \sup_{0\leq t\leq T} B_{H_2}(t) + p_2 T + \sup_{t\geq 0} Y_2(t) \right) \\ &\stackrel{D}{=} \left(\xi_1 T^{H_1} + \sup_{t\geq 0} Y_1(t), \xi_2 T^{H_2} + \sup_{t\geq 0} Y_2(t) \right) + \widetilde{\boldsymbol{T}},\end{aligned}$$

with ξ_i defined in (3.1). By Corollary 2.1 we know that $\mathbb{P}\{\sup_{t\geq 0} Y_i(t) > x\} = \mathrm{o}(\mathbb{P}\{T > x\})$ as $x \to \infty$. Therefore the claim for $\boldsymbol{M}^{(1)}$ is a direct consequence of Lemmas A.3 and A.4. This completes the proof. □

Proof of Theorem 4.1. First, note that, for any $\boldsymbol{a}, \boldsymbol{c} > \boldsymbol{0}$, by the homogeneity property of μ,

$$\begin{aligned}&\int_0^\infty \mu((v\boldsymbol{c} + \boldsymbol{a}, \boldsymbol{\infty}]) \,\mathrm{d}v \leq \mu((\boldsymbol{a}, \boldsymbol{\infty}]) \\ &\quad + \int_1^\infty v^{-\lambda} \mu((\boldsymbol{c} + \boldsymbol{a}/v, \boldsymbol{\infty}]) \,\mathrm{d}v \leq \mu((\boldsymbol{a}, \boldsymbol{\infty}]) + \frac{1}{\lambda - 1} \mu((\boldsymbol{c}, \boldsymbol{\infty}]).\end{aligned} \tag{6.1}$$

For simplicity we denote $\boldsymbol{W}^{(n)} := \sum_{i=1}^n \boldsymbol{U}^{(i)}$. We consider the lower bound, for which we adopt a standard technique of 'one big jump' (see [28]). Informally speaking, we choose an event on which $\boldsymbol{W}^{(n-1)} + \boldsymbol{M}^{(n)}$, $n \geq 1$, behaves in a typical way up to some time k for which $\boldsymbol{M}^{(k+1)}$ is large. Let δ, ε be small positive numbers. By the Weak Law of Large Numbers, we can choose large $K = K_{\varepsilon,\delta}$ so that

$$\mathbb{P}\{\boldsymbol{W}^{(n)} > -n(1+\varepsilon)\boldsymbol{c} - K\boldsymbol{1}\} > 1 - \delta, \quad n = 1, 2, \ldots.$$

For any $u > 0$, we have

$$\begin{aligned}Q(u) &= \mathbb{P}\{\exists n \geq 1 : \boldsymbol{W}^{(n-1)} + \boldsymbol{M}^{(n)} > \boldsymbol{a}u\} \\ &= \mathbb{P}\{\boldsymbol{M}^{(1)} > \boldsymbol{a}u\} + \sum_{k\geq 1} \mathbb{P}\{\cap_{n=1}^k (\boldsymbol{W}^{(n-1)} + \boldsymbol{M}^{(n)} \not> \boldsymbol{a}u), \boldsymbol{W}^{(k)} + \boldsymbol{M}^{(k+1)} > \boldsymbol{a}u\} \\ &\geq \mathbb{P}\{\boldsymbol{M}^{(1)} > \boldsymbol{a}u\} + \sum_{k\geq 1} \mathbb{P}\Big\{\cap_{n=1}^k (\boldsymbol{W}^{(n-1)} + \boldsymbol{M}^{(n)} \not> \boldsymbol{a}u), \boldsymbol{W}^{(k)} > -k(1+\varepsilon)\boldsymbol{c} - K\boldsymbol{1}, \\ &\qquad\qquad \boldsymbol{M}^{(k+1)} > \boldsymbol{a}u + k(1+\varepsilon)\boldsymbol{c} + K\boldsymbol{1}\Big\} \\ &\geq \mathbb{P}\{\boldsymbol{M}^{(1)} > \boldsymbol{a}\,\mathbf{u}\} \\ &\quad + \sum_{k\geq 1} (1 - \delta - \mathbb{P}\{\cup_{n=1}^k (\boldsymbol{W}^{(n-1)} + \boldsymbol{M}^{(n)} > \boldsymbol{a}u)\})\, \mathbb{P}\{\boldsymbol{M}^{(k+1)} > \boldsymbol{a}u + k(1+\varepsilon)\boldsymbol{c} + K\boldsymbol{1}\} \\ &\geq (1 - \delta - Q(u)) \sum_{k\geq 0} \mathbb{P}\{\boldsymbol{M}^{(1)} > \boldsymbol{a}u + k(1+\varepsilon)\boldsymbol{c} + K\boldsymbol{1}\} \\ &\geq \frac{(1 - \delta - Q(u))}{1+\varepsilon} \int_0^\infty \mathbb{P}\{\boldsymbol{M}^{(1)} > \boldsymbol{a}u + v\boldsymbol{c} + K\boldsymbol{1}\} \,\mathrm{d}v.\end{aligned}$$

For u sufficiently large that $\varepsilon u > K$, we have

$$Q(u) \geq \frac{(1 - \delta - Q(u))}{1+\varepsilon} \int_0^\infty \mathbb{P}\{\boldsymbol{M}^{(1)} > (\boldsymbol{a} + \varepsilon\boldsymbol{1})u + v\boldsymbol{c}\} \,\mathrm{d}v.$$

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Rearranging the above inequality and using a change of variable, we obtain

$$Q(u) \geq \frac{(1-\delta)u \int_0^\infty \mathbb{P}\{\boldsymbol{M}^{(1)} > u(\boldsymbol{a}+\varepsilon\boldsymbol{1}+v\boldsymbol{c})\}\, \mathrm{d}v}{1+\varepsilon+\int_0^\infty \mathbb{P}\{\boldsymbol{M}^{(1)} > (\boldsymbol{a}+\varepsilon\boldsymbol{1})u+v\boldsymbol{c}\}\, \mathrm{d}v},$$

and thus by Lemma 6.1 and Fatou's lemma,

$$\liminf_{u\to\infty} \frac{Q(u)}{u\mathbb{P}\{|\widetilde{\boldsymbol{T}}| > u\}} \geq \frac{1-\delta}{1+\varepsilon} \int_0^\infty \mu((\boldsymbol{a}+\varepsilon\boldsymbol{1}+v\boldsymbol{c}, \boldsymbol{\infty}])\, \mathrm{d}v.$$

Since ε and δ are arbitrary, and by (6.1) the integration on the right-hand side is finite, taking $\varepsilon \to 0$, $\delta \to 0$ and applying the dominated convergence theorem yields

$$\liminf_{u\to\infty} \frac{Q(u)}{u\mathbb{P}\{|\widetilde{\boldsymbol{T}}| > u\}} \geq \int_0^\infty \mu((\boldsymbol{a}+v\boldsymbol{c}, \boldsymbol{\infty}])\, \mathrm{d}v.$$

Next we consider the asymptotic upper bound. Let $y_1, y_2 > 0$ be given. We shall construct an auxiliary random walk $\widetilde{\boldsymbol{W}}^{(n)}$, $n \geq 0$, with $\widetilde{\boldsymbol{W}}^{(0)} = 0$ and $\widetilde{\boldsymbol{W}}^{(n)} = \sum_{i=1}^n \widetilde{\boldsymbol{U}}^{(i)}$, $n \geq 1$, where $\widetilde{\boldsymbol{U}}^{(n)} = (\widetilde{U}_1^{(n)}, \widetilde{U}_2^{(n)})$ is given by

$$\widetilde{U}_i^{(n)} = \begin{cases} M_i^{(n)} & \text{if } M_i^{(n)} > y_1, \\ U_i^{(n)} & \text{if } -y_2 < U_i^{(n)} \leq M_i^{(n)} \leq y_1, \\ -y_2 & \text{if } M_i^{(n)} \leq y_1,\ U_i^{(n)} \leq -y_2, \end{cases} \qquad i = 1, 2.$$

Obviously, $\boldsymbol{W}^{(n)} \leq \widetilde{\boldsymbol{W}}^{(n)}$ for any $n \geq 1$. Furthermore, one can show that

$$M_i^{(n)} \leq \widetilde{U}_i^{(n)} + (y_1 + y_2).$$

Then

$$\boldsymbol{W}^{(n-1)} + \boldsymbol{M}^{(n)} \leq \widetilde{\boldsymbol{W}}^{(n)} + (y_1+y_2)\boldsymbol{1}, \qquad n \geq 1.$$

Thus, for any $\varepsilon > 0$ and sufficiently large u,

$$\begin{aligned} Q(u) &\leq \mathbb{P}\{\exists n \geq 1\colon \widetilde{\boldsymbol{W}}^{(n)} > \boldsymbol{a}u - (y_1+y_2)\boldsymbol{1}\} \\ &\leq \mathbb{P}\{\exists n \geq 1\colon \widetilde{\boldsymbol{W}}^{(n)} > (\boldsymbol{a}-\varepsilon\boldsymbol{1})u\}. \end{aligned}$$

Define $c_{y_1,y_2} = -\mathbb{E}\{\widetilde{\boldsymbol{U}}^{(1)}\}$. Since $\lim_{y_1,y_2\to\infty} \boldsymbol{c}_{y_1,y_2} = \boldsymbol{c}$, we have that for any y_1, y_2 large enough $\boldsymbol{c}_{y_1,y_2} > \boldsymbol{0}$. It follows from Lemmas 6.1 and A.4 that for any $y_1, y_2 > 0$, $\widetilde{\boldsymbol{U}}^{(1)}$ is regularly varying with index λ and limiting measure μ, and $\mathbb{P}\{|\widetilde{\boldsymbol{U}}^{(1)}| > u\} \sim \mathbb{P}\{|\widetilde{\boldsymbol{T}}| > u\}$ as $u \to \infty$. Then, applying Theorem 3.1 and Remark 3.2 of [21], we obtain that

$$\begin{aligned} \mathbb{P}\{\exists n \geq 1\colon \widetilde{\boldsymbol{W}}^{(n)} > (\boldsymbol{a}-\varepsilon\boldsymbol{1})u\} &\sim u\mathbb{P}\{|\widetilde{\boldsymbol{U}}^{(1)}| > u\} \int_0^\infty \mu((\boldsymbol{c}_{y_1,y_2}v + \boldsymbol{a} - \varepsilon\boldsymbol{1}, \boldsymbol{\infty}])\, \mathrm{d}v \\ &\sim u\mathbb{P}\{|\widetilde{\boldsymbol{T}}| > u\} \int_0^\infty \mu((\boldsymbol{c}_{y_1,y_2}v + \boldsymbol{a} - \varepsilon\boldsymbol{1}, \boldsymbol{\infty}])\, \mathrm{d}v. \end{aligned}$$

Consequently, the claimed asymptotic upper bound is obtained by letting $\varepsilon \to 0$, $y_1, y_2 \to \infty$. The proof is complete. □

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Appendix A. Auxiliary results

This section includes some results on the regularly varying random vectors.

Lemma A.1. *Let $\mathcal{T} > \mathbf{0}$ be a regularly varying random vector with index α and limiting measure ν, and let $x_i(u)$, $1 \le i \le n$ be increasing (to infinity) functions such that for some $1 \le m \le n$, $x_1(u) \sim \cdots \sim x_m(u)$, and $x_j(u) = \mathrm{o}(x_1(u))$ for all $j = m+1, \ldots, n$. Then, for any $\boldsymbol{a} > \mathbf{0}$,*

$$\mathbb{P}\{\cap_{i=1}^{n}(\mathcal{T}_i > a_i x_i(u))\} \sim \mathbb{P}\{\cap_{i=1}^{m}(\mathcal{T}_i > a_i x_1(u))\} \sim \nu([\boldsymbol{a}_{m,0}, \boldsymbol{\infty}])\, \mathbb{P}\{|\mathcal{T}| > x_1(u)\}$$

holds as $u \to \infty$, with $\boldsymbol{a}_{m,0} = (a_1, \ldots, a_m, 0, \ldots, 0)$.

Proof. Obviously, for any small enough $\varepsilon > 0$ we find that when u is sufficiently large

$$\begin{aligned}\mathbb{P}\{\cap_{i=1}^{n}(\mathcal{T}_i > a_i x_i(u))\} &\le \mathbb{P}\{\cap_{i=1}^{m}(\mathcal{T}_i > (a_i-\varepsilon) x_1(u)), \cap_{i=m+1}^{n}(\mathcal{T}_i > 0)\}\\ &\sim \nu([\boldsymbol{a}_{-\varepsilon}, \boldsymbol{\infty}])\, \mathbb{P}\{|\mathcal{T}| > x_1(u)\},\end{aligned}$$

where $\boldsymbol{a}_{-\varepsilon} = (a_1 - \varepsilon, \ldots, a_m - \varepsilon, 0, \ldots, 0)$, and

$$\begin{aligned}\mathbb{P}\{\cap_{i=1}^{n}(\mathcal{T}_i > a_i x_i(u))\} &\ge \mathbb{P}\{\cap_{i=1}^{m}(\mathcal{T}_i > (a_i+\varepsilon) x_1(u)), \cap_{i=m+1}^{n}(\mathcal{T}_i > a_i(\varepsilon x_1(u)))\}\\ &\sim \nu([\boldsymbol{a}_{\varepsilon+}, \boldsymbol{\infty}])\, \mathbb{P}\{|\mathcal{T}| > x_1(u)\}\end{aligned}$$

with $\boldsymbol{a}_{\varepsilon+} = (a_1 + \varepsilon, \ldots, a_m + \varepsilon, a_{m+1}\varepsilon, \ldots, a_n\varepsilon)$. Letting $\varepsilon \to 0$, the claim follows by the continuity of $\nu([\boldsymbol{a}_{\varepsilon\pm}, \boldsymbol{\infty}])$ in ε. The proof is complete. □

Lemma A.2. *Let $\mathcal{T}$, a_i, $x_i(u)$, and $\boldsymbol{a}_{m,0}$ be the same as in Lemma A.1. Further, consider $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_n)$ to be a non-negative random vector independent of $\mathcal{T}$ such that $\max_{1\le i\le n} \mathbb{E}\{\eta_i^{\alpha+\delta}\} < \infty$ for some $\delta > 0$. Then*

$$\mathbb{P}\{\cap_{i=1}^{n}(\mathcal{T}_i\eta_i > a_i x_i(u))\} \sim \mathbb{P}\{\cap_{i=1}^{m}(\mathcal{T}_i\eta_i > a_i x_1(u))\} \sim \widehat{\nu}([\boldsymbol{a}_{m,0}, \boldsymbol{\infty}])\, \mathbb{P}\{|\mathcal{T}| > x_1(u)\}$$

holds as $u \to \infty$, where $\widehat{\nu}(K) = \mathbb{E}\{\nu(\boldsymbol{\eta}^{-1}K)\}$, with $\boldsymbol{\eta}^{-1}K = \{(\eta_1^{-1}b_1, \ldots, \eta_n^{-1}b_n), (b_1, \ldots, b_n) \in K\}$ for any $K \in \mathcal{B}([0, \infty]^n \setminus \{\mathbf{0}\})$.

Proof. It follows directly from Lemma 4.6 of [22] (see also Proposition A.1 of [7]) that the second asymptotic equivalence holds. The first claim follows from the same arguments as in Lemma A.1.

Lemma A.3. *Assume $\boldsymbol{X} \in \mathbb{R}^n$ is regularly varying with index α and limiting measure μ, and $\boldsymbol{A}$ is a random $n \times d$ matrix independent of random vector $\boldsymbol{Y} \in \mathbb{R}^d$. If $0 < \mathbb{E}\{\|\boldsymbol{A}\|^{\alpha+\delta}\} < \infty$ for some $\delta > 0$, with $\|\cdot\|$ some matrix norm and*

$$\mathbb{P}\{|\boldsymbol{Y}| > x\} = \mathrm{o}(\mathbb{P}\{|\boldsymbol{X}| > x\}), \quad x \to \infty, \tag{A.1}$$

then $\boldsymbol{X} + \boldsymbol{A}\boldsymbol{Y}$ is regularly varying with index α and limiting measure μ, and

$$\mathbb{P}\{|\boldsymbol{X} + \boldsymbol{A}\boldsymbol{Y}| > x\} \sim \mathbb{P}\{|\boldsymbol{X}| > x\}, \quad x \to \infty.$$

Proof. By Lemma 3.12 of [22], it suffices to show that

$$\mathbb{P}\{|\boldsymbol{A}\boldsymbol{Y}| > x\} = \mathrm{o}(\mathbb{P}\{|\boldsymbol{X}| > x\}), \quad x \to \infty. \tag{A.2}$$

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Defining $g(x)=x^{(\alpha+\delta/2)/(\alpha+\delta)}$, $x\geq 0$, we have

$$\mathbb{P}\{|\boldsymbol{AY}|>x\}\leq \mathbb{P}\{\|\boldsymbol{A}\||\boldsymbol{Y}|>x\}\leq \int_0^{g(x)} \mathbb{P}\{|\boldsymbol{Y}|>x/t\}\,\mathbb{P}\{\|\boldsymbol{A}\|\in \mathrm{d}t\}+\mathbb{P}\{\|\boldsymbol{A}\|>g(x)\}. \tag{A.3}$$

Due to (A.1), for arbitrary $\varepsilon>0$,

$$\int_0^{g(x)} \mathbb{P}\{|\boldsymbol{Y}|>x/t\}\,\mathbb{P}\{\|\boldsymbol{A}\|\in \mathrm{d}t\}\leq \varepsilon \int_0^{g(x)} \mathbb{P}\{|\boldsymbol{X}|>x/t\}\,\mathbb{P}\{\|\boldsymbol{A}\|\in \mathrm{d}t\}$$

holds for large enough x. Furthermore, by Potter's theorem (see e.g. Theorem 1.5.6 of [8]), we have

$$\frac{\mathbb{P}\{|\boldsymbol{X}|>x/t\}}{\mathbb{P}\{|\boldsymbol{X}|>x\}}\leq I_{(t\leq 1)}+2t^{\alpha+\delta}I_{(1<t\leq g(x))},\qquad t\in(0,\,g(x))$$

for sufficiently large x, and thus, by the dominated convergence theorem,

$$\begin{aligned}&\lim_{x\to\infty}\int_0^{g(x)}\frac{\mathbb{P}\{|\boldsymbol{Y}|>x/t\}}{\mathbb{P}\{|\boldsymbol{X}|>x\}}\,\mathbb{P}\{\|\boldsymbol{A}\|\in \mathrm{d}t\}\\ &\quad\leq \lim_{x\to\infty}\int_0^{g(x)}\frac{\varepsilon\mathbb{P}\{|\boldsymbol{X}|>x/t\}}{\mathbb{P}\{|\boldsymbol{X}|>x\}}\,\mathbb{P}\{\|\boldsymbol{A}\|\in \mathrm{d}t\}=\varepsilon\mathbb{E}\{\|\boldsymbol{A}\|^{\alpha}\}.\end{aligned} \tag{A.4}$$

Moreover, Markov's inequality implies that

$$\lim_{x\to\infty}\frac{\mathbb{P}\{\|\boldsymbol{A}\|>g(x)\}}{\mathbb{P}\{|\boldsymbol{X}|>x\}}\leq \lim_{x\to\infty}\frac{\mathbb{E}\{\|\boldsymbol{A}\|^{\alpha+\delta}\}}{g(x)^{\alpha+\delta}\,\mathbb{P}\{|\boldsymbol{X}|>x\}}=0. \tag{A.5}$$

Therefore claim (A.2) follows from (A.3)–(A.5) and the arbitrariness of ε. This completes the proof. □

Lemma A.4. *Assume $\boldsymbol{X},\boldsymbol{Y}\in\mathbb{R}^n$ are regularly varying with the same index α and the same limiting measure μ. Moreover, if $\boldsymbol{X}\geq\boldsymbol{Y}$ and $\mathbb{P}\{|\boldsymbol{X}|>x\}\sim\mathbb{P}\{|\boldsymbol{Y}|>x\}$ as $x\to\infty$, then for any random vector $\boldsymbol{Z}$ satisfying $\boldsymbol{X}\geq\boldsymbol{Z}\geq\boldsymbol{Y}$, $\boldsymbol{Z}$ is regularly varying with index α and limiting measure μ, and $\mathbb{P}\{|\boldsymbol{Z}|>x\}\sim\mathbb{P}\{|\boldsymbol{X}|>x\}$ as $x\to\infty$.*

Proof. We only prove the claim for $n=2$; a similar argument can be used to verify the claim for $n\geq 3$. For any $x>0$, define a measure μ_x as

$$\mu_x(A)\;=:\frac{\mathbb{P}\{x^{-1}\boldsymbol{Z}\in A\}}{\mathbb{P}\{|\boldsymbol{X}|>x\}},\qquad A\in\mathcal{B}(\overline{\mathbb{R}}_0^2).$$

We shall show that

$$\mu_x\xrightarrow{v}\mu,\qquad x\to\infty. \tag{A.6}$$

Given that the above is established, by letting $A=\{\boldsymbol{x}\colon\,|\boldsymbol{x}|>1\}$ (which is relatively compact and satisfies $\mu(\partial A)=0$), we have $\mu_x(A)\to\mu(A)=1$ as $x\to\infty$ and thus $\mathbb{P}\{|\boldsymbol{Z}|>x\}\sim\mathbb{P}\{|\boldsymbol{X}|>x\}$. Furthermore, by replacing the denominator in the definition of μ_x with $\mathbb{P}\{|\boldsymbol{Z}|>x\}$, we conclude that

$$\frac{\mathbb{P}\{x^{-1}\boldsymbol{Z}\in\cdot\}}{\mathbb{P}\{|\boldsymbol{Z}|>x\}}\xrightarrow{v}\mu(\,\cdot\,),\qquad x\to\infty,$$

showing that $\boldsymbol{Z}$ is regularly varying with index α and limiting measure μ.

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Now it remains to prove (A.6). To this end, we define a set $\mathcal{D}$ consisting of all sets in $\overline{\mathbb{R}}_0^2$ that are of the following form:

$$\begin{aligned}
&\text{(a)} \quad (a_1, \infty] \times [a_2, \infty], && a_1 > 0, a_2 \in \mathbb{R},\\
&\text{(b)} \quad [-\infty, a_1] \times (a_2, \infty], && a_1 \in \mathbb{R}, a_2 > 0,\\
&\text{(c)} \quad [-\infty, a_1) \times [-\infty, a_2], && a_1 < 0, a_2 \in \mathbb{R},\\
&\text{(d)} \quad [a_1, \infty] \times [-\infty, a_2), && a_1 \in \mathbb{R}, a_2 < 0.
\end{aligned}$$

Note that every $A \in \mathcal{D}$ is relatively compact and satisfies $\mu(\partial A) = 0$. We first show that

$$\lim_{x\to\infty} \mu_x(A) = \mu(A) \quad \text{for all } A \in \mathcal{D}. \tag{A.7}$$

If $A = (a_1, \infty] \times (a_2, \infty]$ or $A = (a_1, \infty] \times [a_2, \infty]$ with $a_i \in \mathbb{R}$ and at least one $a_i > 0$, $i = 1, 2$, or $A = \overline{\mathbb{R}} \times (a_2, \infty]$ with some $a_2 > 0$, by the order relations of $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$, we have, for any $x > 0$,

$$\frac{\mathbb{P}\{x^{-1}\boldsymbol{Y} \in A\}}{\mathbb{P}\{|\boldsymbol{X}| > x\}} \le \mu_x(A) \le \frac{\mathbb{P}\{x^{-1}\boldsymbol{X} \in A\}}{\mathbb{P}\{|\boldsymbol{X}| > x\}}. \tag{A.8}$$

Letting $x \to \infty$, using the regularity properties as supposed for $\boldsymbol{X}$ and $\boldsymbol{Y}$, and then appealing to Proposition 3.12(ii) of [33], we verify (A.7) for case (a). If $A = [-\infty, a_1] \times (a_2, \infty]$ with some $a_1 \in \mathbb{R}$, $a_2 > 0$, then we have

$$\mu_x(A) = \mu_x(\overline{\mathbb{R}} \times (a_2, \infty]) - \mu_x((a_1, \infty] \times (a_2, \infty]),$$

and thus, by the convergence in case (a),

$$\lim_{x\to\infty} \mu_x(A) = \mu(\overline{\mathbb{R}} \times (a_2, \infty]) - \mu((a_1, \infty] \times (a_2, \infty]) = \mu(A),$$

this validates (A.7) for case (b). If $A = [-\infty, a_1) \times [-\infty, a_2]$ or $A = [-\infty, a_1) \times [-\infty, a_2)$ with $a_i \in \mathbb{R}$ and at least one $a_i < 0$, $i = 1, 2$, or $A = \overline{\mathbb{R}} \times [-\infty, a_2)$ with some $a_2 < 0$, then we get a similar formula to (A.8) with the reverse inequalities. If $A = [a_1, \infty] \times [-\infty, a_2)$ with some $a_1 \in \mathbb{R}$, $a_2 < 0$, then

$$\mu_x(A) = \mu_x(\overline{\mathbb{R}} \times [-\infty, a_2)) - \mu_x([-\infty, a_1) \times [-\infty, a_2)).$$

Therefore, similarly to the proof for cases (a) and (b), one can establish (A.7) for cases (c) and (d).

Next, let f defined on $\overline{\mathbb{R}}_0^2$ be any positive, continuous function with compact support. We see that the support of f is contained in $[\boldsymbol{a}, \boldsymbol{b}]^c$ for some $\boldsymbol{a} < \boldsymbol{0} < \boldsymbol{b}$. Note that

$$\begin{aligned}
[\boldsymbol{a}, \boldsymbol{b}]^c &= (b_1, \infty] \times [a_2, \infty] \cup [-\infty, b_1] \times (b_2, \infty] \cup [-\infty, a_1) \times [-\infty, b_2] \cup [a_1, \infty] \\
&\quad \times [-\infty, a_2) \\
&=: \bigcup_{i=1}^4 A_i,
\end{aligned}$$

where the A_i are sets of the form (a)–(d) respectively, and thus (A.7) holds for these A_i. Therefore

$$\sup_{x>0} \mu_x(f) \le \sup_{z\in\overline{\mathbb{R}}_0^2} f(z) \cdot \sup_{x>0} \mu_x([\boldsymbol{a}, \boldsymbol{b}]^c) \le \sup_{z\in\overline{\mathbb{R}}_0^2} f(z) \cdot \sum_{i=1}^4 \sup_{x>0} \mu_x(A_i) < \infty,$$

which by Proposition 3.16 of [33] implies that $\{\mu_x\}_{x>0}$ is a vaguely relatively compact subset of the metric space consisting of all the non-negative Radon measures on $(\overline{\mathbb{R}}_0^2, \mathcal{B}(\overline{\mathbb{R}}_0^2))$. If μ_0 and μ_0' are two subsequential vague limits of $\{\mu_x\}_{x>0}$ as $x \to \infty$, then by (A.7) we have $\mu_0(A) = \mu_0'(A)$ for any $A \in \mathcal{D}$. Since any rectangle in $\overline{\mathbb{R}}_0^2$ can be obtained from a finite number of sets in $\mathcal{D}$ by operating union, intersection, difference, or complementary, and these rectangles constitute a π-system and generate the σ-field $\mathcal{B}(\overline{\mathbb{R}}_0^2)$, we get $\mu_0 = \mu_0'$ on $\mathcal{B}(\overline{\mathbb{R}}_0^2)$. Consequently (A.6) is valid and thus the proof is complete. □

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References

[1] Arendarczyk, M. (2017). On the asymptotics of supremum distribution for some iterated processes. *Extremes* **20**, 451–474.

[2] Arendarczyk, M. and Dębicki, K. (2011). Asymptotics of supremum distribution of a Gaussian process over a Weibullian time. *Bernoulli* **17**, 194–210.

[3] Arendarczyk, M. and Dębicki, K. (2012). Exact asymptotics of supremum of a stationary Gaussian process over a random interval. *Statist. Prob. Lett.* **82**, 645–652.

[4] Asmussen, S. and Albrecher, H. (2010). *Ruin Probabilities*, 2nd edn (Advanced Series on Statistical Science & Applied Probability **14**). World Scientific, Hackensack, NJ.

[5] Azais, J.-M. and Pham, V. (2019). Geometry of conjunction set of smooth stationary Gaussian fields. Available at arXiv:1909.07090v1.

[6] Barndorff-Nielsen, O. E., Pedersen, J. and Sato, K.-I. (2001). Multivariate subordination, self-decomposability and stability. *Adv. Appl. Prob.* **33**, 160–187.

[7] Basrak, B., Davis, R. A. and Mikosch, T. (2002). Regular variation of GARCH processes. *Stoch. Process. Appl.* **99**, 95–115.

[8] Bingham, N., Goldie, C. and Teugels, J. (1989). *Regular Variation* (Encyclopedia of Mathematics and its Applications **27**). Cambridge University Press.

[9] Constantinescu, C., Delsing, G., Mandjes, M. and Rojas Nandayapa, L. (2020). A ruin model with a resampled environment. *Scand. Actuarial J.* **2020**, 323–341.

[10] Dębicki, K. and Peng, X. (2020). Sojourns of stationary Gaussian processes over a random interval. Available at arXiv:2004.12290.

[11] Dębicki, K., Hashorva, E. and Ji, L. (2014). Tail asymptotics of supremum of certain Gaussian processes over threshold dependent random intervals. *Extremes* **17**, 411–429.

[12] Dębicki, K., Hashorva, E., Ji, L. and Tabiś, K. (2015). Extremes of vector-valued Gaussian processes: exact asymptotics. *Stoch. Process. Appl.* **125**, 4039–4065.

[13] Dębicki, K., Hashorva, E. and Kriukov, N. (2021). Pandemic-type failures in multivariate Brownian risk models. Available at arXiv:2008.07480.

[14] Dębicki, K., Hashorva, E. and Krystecki, K. (2020). Finite-time ruin probability for correlated Brownian motions. Available at arXiv:2004.14015.

[15] Dębicki, K., Hashorva, E. and Wang, L. (2020). Extremes of vector-valued Gaussian processes. *Stoch. Process. Appl.* **130**, 5802–5837.

[16] DĘBICKI, K., JI, L. AND ROLSKI, T. (2020). Exact asymptotics of component-wise extrema of two-dimensional Brownian motion. *Extremes* **23**, 569–602.

[17] DĘBICKI, K., KOSIŃSKI, K., MANDJES, M. AND ROLSKI, T. (2010). Extremes of multi-dimensional Gaussian processes. *Stoch. Process. Appl.* **120**, 2289–2301.

[18] DĘBICKI, K., ZWART, B. AND BORST, S. (2004). The supremum of a Gaussian process over a random interval. *Statist. Prob. Lett.* **68**, 221–234.

[19] DIEKER, A. (2005). Extremes of Gaussian processes over an infinite horizon. *Stoch. Process. Appl.* **115**, 207–248.

[20] HE, H., KEIRSTEAD, W. P. AND REBHOLZ, J. (1998). Double lookbacks. *Math. Finance* **8**, 201–228.

[21] HULT, H., LINDSKOG, F., MIKOSCH, T. AND SAMORODNITSKY, G. (2006). Functional large deviations for multivariate regularly varying random walks. *Ann. Appl. Prob.* **15**, 2651–2680.

[22] JESSEN, A. H. AND MIKOSCH, T. (2006). Regular varying functions. *Publ. Inst. Math.* **80**, 171–192.

[23] JI, L. AND ROBERT, S. (2018). Ruin problem of a two-dimensional fractional Brownian motion risk process. *Stoch. Models* **34**, 73–97.

[24] KELLA, O. AND WHITT, W. (1992). A storage model with a two-state random environment. *Operat. Res.* **40**, S257–S262.

[25] KIM, Y. S. (2012). The fractional multivariate normal tempered stable process. *Appl. Math. Lett.* **25**, 2396–2401.

[26] LUCIANO, E. AND SEMERARO, P. (2010). Multivariate time changes for Lévy asset models: characterization and calibration. *J. Comput. Appl. Math.* **233**, 1937–1953.

[27] MIKOSCH, T. (1999). *Regular Variation, Subexponentiatility and their Applications in Probability Theory* (lecture notes for the workshop 'Heavy tails and queues'). EURANDOM Institute, Eindhoven, The Netherlands.

[28] PALMOWSKI, Z. AND ZWART, B. (2007). Tail asymptotics of the supremum of a regenerative process. *J. Appl. Prob.* **44**, 349–365.

[29] PHAM, V.-H. (2020). Conjunction probability of smooth centered Gaussian processes. *Acta Math. Vietnam.* **45**, 865–874.

[30] PITERBARG, V. (1996). *Asymptotic Methods in the Theory of Gaussian Processes and Fields* (Translations of Mathematical Monographs **148**). American Mathematical Society, Providence, RI.

[31] PITERBARG, V. AND STAMATOVICH, B. (2005). Crude asymptotics of the probability of simultaneous high extrema of two Gaussian processes: the dual action function. *Russian Math. Surveys* **60**, 167–168.

[32] RATANOV, N. (2020). Kac–Lévy processes. *J. Theoret. Prob.* **33**, 239–267.

[33] RESNICK, S. I. (1987). *Extreme Values, Regular variation, and Point Processes*. Springer, London.

[34] RESNICK, S. I. (2007). *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Springer, London.

[35] SAMORODNITSKY, G. AND TAQQ, M. S. (1994). *Stable Non-Gaussian Random Processes*. Chapman & Hall, London.

[36] TAN, Z. AND HASHORVA, E. (2013). Exact asymptotics and limit theorems for supremum of stationary χ-processes over a random interval. *Stoch. Process. Appl.* **123**, 2983–2998.

[37] VAN DER HOFSTAD, R. AND HONNAPPA, H. (2019). Large deviations of bivariate Gaussian extrema. *Queueing Systems* **93**, 333–349.

[38] WORSLEY, K. AND FRISTON, K. (2000). A test for a conjunction. *Statist. Prob. Lett.* **47**, 135–140.

[39] ZAÏDI, N. AND NUALART, D. (2003). Smoothness of the law of the supremum of the fractional Brownian motion. *Electron. Commun. Prob.* **8**, 102–111.

[40] ZWART, B., BORST, S. AND DĘBICKI, K. (2005). Subexponential asymptotics of hybrid fluid and ruin models. *Ann. Appl. Prob.* **15**, 500–517.